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# Definition (Divisors)

Let X be a Riemann surface. A **divisor** on X is a mapping

$$D:X\to\mathbb{Z}$$

such that for any compact subset  $K \subset X$  there are only finitely many points  $x \in K$  such that  $D(x) \neq 0$ .

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# Definition (Divisors of Meromorphic Functions)

Suppose X is a Riemann surface and Y is an open subset of X. For a meromorphic function  $f \in \mathcal{M}(Y)$  and  $a \in Y$  define

$$\operatorname{ord}_{a}(f) = \begin{cases} 0 & \text{if } f \text{ is holomorphic and non-zero at } a \\ k & \text{if } f \text{ has a zero of order k at } a \\ -k & \text{if } f \text{ has a pole of order k at } a \\ \infty & \text{if } f \text{ is identically zero in a n.b.h of } a \end{cases}$$

For any meromorphic function  $f \in \mathscr{M}(X) \setminus \{0\}$ , the mapping  $x \mapsto \operatorname{ord}_x(f)$  is called the **divisor** of f.

# Definition (Divisor of Meromorphic 1 – forms)

For a meromorphic 1-form  $w \in \mathscr{M}^{(1)}(Y)$ . Choose a coordinate n.b.h (U, z) of *a*. Then on  $U \cap Y$  one may write w = fdz, where *f* is a meromorphic function. Set  $\operatorname{ord}_a(w) = \operatorname{ord}_a f$ . For 1-forms  $w \in \mathscr{M}^{(1)}(X) \setminus \{0\}$  the mapping  $x \mapsto \operatorname{ord}_x(w)$  is called the **divisor** of *w*.

# Definition (Divisor of Meromorphic 1-forms)

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#### Proposition

For  $f, g \in \mathscr{M}(X) \setminus \{0\}$  and  $w \in \mathscr{M}^{(1)}(X) \setminus \{0\}$  one has the following relations

$$(fg) = (f) + (g), \ (1/f) = -(f), \ (fw) = (f) + (w).$$

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# Definition

A divisor  $D \in \text{Div}(X)$  is called a **principal divisor** if there exists a function  $f \in \mathcal{M}(X) \setminus \{0\}$  such that D = (f). Two divisors  $D, D' \in \text{Div}(X)$  are said to be **equivalent** if their difference D - D' is a principal divisor.

└─ The Riemann-Roch Theorem └─ Divisors

#### Definition

Let X be a compact Riemann surface. Then for every  $D \in Div(X)$  there are only finitely many  $x \in X$  such that  $D(x) \neq 0$ . Hence one can define a mapping

 $\deg:\operatorname{Div}(X)\to\mathbb{Z}$ 

called the degree, by letting

$$\deg(D):=\sum_{x\in X}D(x).$$

Suppose *D* is a divisor on the Riemann surface *X*. For any open set  $U \subset X$  define  $\mathcal{O}_D(U)$  as follows

$$\mathcal{O}_D(U) := \{ f \in \mathscr{M}(U) | \operatorname{ord}_x(f) \ge -D(x) \text{ for every } x \in U \}.$$

Together with the natural restriction mappings  $\mathcal{O}_D$  is a sheaf (Sheaf of divisor).

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Suppose X is a compact Riemann surface and  $D \in Div(X)$  is a divisor with deg D < 0. Then  $H^0(X, \mathcal{O}_D) = 0$ .

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#### Proof.

Suppose, to the contrary, that there exists an  $f \in H^0(X, \mathcal{O}_D)$  with  $f \neq 0$ . Then  $(f) \geq -D$  and thus

 $\operatorname{ded}(f) \geq -\operatorname{deg} D > 0.$ 

However this contradicts the fact that deg(f) = 0.

Sheaf of Divisors

# The Riemann-Roch Theorem



### Proposition

Let  $p \in X$ . Define for an open set  $U \subset X$ .

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases},$$

where the restriction maps are the obvious homomorphisms. Then

- i.  $H^0(X, \mathbb{C}_p) \cong \mathbb{C}$
- ii.  $H^1(X, \mathbb{C}_p) = 0.$

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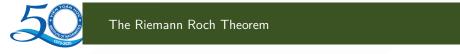
- i.  $H^0(X, \mathbb{C}_p) \cong \mathbb{C}$
- ii.  $H^1(X, \mathbb{C}_p) = 0.$

We also have the following short exact sequence

$$0 o \mathcal{O}_D o \mathcal{O}_{D+P} o \mathbb{C}_P o 0.$$

- The Riemann Roch Theorem

# The Riemann-Roch Theorem



By the long exact sequence theorem, we have

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to \mathbb{C} \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+P}) \to 0.$$

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#### Theorem (Riemann-Roch)

Suppose D is a divisor on a compact Riemann surface X of genus g. Then  $H^0(X, \mathcal{O}_D)$  and  $H^1(X, \mathcal{O}_D)$  are finite dimensional vector spaces and

$$\mathrm{dim} H^0(X,\mathcal{O}_D) - \mathrm{dim} H^1(X,\mathcal{O}_D) = 1 - g + \mathrm{deg} D.$$

L The Riemann Roch Theorem

i. The result holds for the divisor D = 0.

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- ii. Suppose D is a divisor,  $P \in X$ , and D' = D + P. Suppose that the result holds for one of the divisors D, D'. The above exact cohomology sequence can be slit into two short exact sequences. For, let

 $V := \operatorname{Im}(H^0(X, \mathcal{O}_D) \to \mathbb{C})$  $W := \mathbb{C}/V.$ 

Then  $\dim(V) + \dim(W) = 1 = \deg D' - \deg D$  and the sequences

$$\begin{split} 0 &\to H^0(X,\mathcal{O}_D) \to H^0(X,\mathcal{O}_D') \to V \to 0, \\ 0 &\to W \to H^1(X,\mathcal{O}_D) \to H^1(X,\mathcal{O}_{D'}) \to 0 \end{split}$$

are exact. This implies that

 $\dim H^0(X, \mathcal{O}_{D'}) = \dim H^0(X, \mathcal{O}_D) + \dim V$  $\dim H^1(X, \mathcal{O}_D) = \dim H^1(X, \mathcal{O}_{D'}) + \dim W.$ 

- The Riemann Roch Theorem

Therefore,

 $\mathrm{dim} H^0(X,\mathcal{O}_{D'}) - \mathrm{dim} H^1(X,\mathcal{O}_{D'}) - \mathrm{deg} D' = \mathrm{dim} H^0(X,\mathcal{O}_D) - \mathrm{dim} H^1(X,\mathcal{O}_D) - \mathrm{deg} D.$ 

iii. An arbitrary divisor D on X may be written

$$D = P_1 + ... + P_m - P_{m+1} - ... - P_n,$$

where the  $P_i \in X$  are points.

L The Riemann Roch Theorem

The Riemann-Roch Theorem

Canonical Divisor

By Serre's duality, we can identify  $H^1(X, \mathcal{O}_D)$  as  $\mathcal{O}_{K-D}(X)$  with K = (w). The divisor K is called **canonical divisor**.

#### Proposition

The canonical divisor K on a compact Riemann surface of genus g satisfies

 $\deg(K)=2g-2.$ 

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Corollary

If D is a divisor such that  $deg(D) \ge 2g - 1$ , then

$$\dim \mathcal{O}_D(X) = \deg(D) + 1 - g.$$



Maps to Projective Space Given by Meromorphic Functions

# Definition

Let X be a Riemann surface. A map  $\phi : X \to \mathbb{P}^n$  is **holomorphic at a point**  $p \in X$  if there are holomorphic functions  $g_0, ..., g_n$  defined on X near p, not all zero at p, such that  $\phi(x) = [g_0(x) : ... : g_n(x)]$  for x near p. We say  $\phi$  is a **holomorphic map** if it is holomorphic at all points of X.



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Let X be be a Riemann surface. Choose n + 1 meromorphic functions  $f = (f_0, ..., f_n)$  on X, not all identically zero. Define  $\phi_f : X \to \mathbb{P}^n$  by setting

 $\phi_f(p) = [f_0(p) : \ldots : f_n(p)].$ 



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$$\phi_f(p) = [f_0(p) : \ldots : f_n(p)].$$

Note that a priori,  $\phi_f$  is defined at p if

- i. p is not a pole of any  $f_i$
- ii. p is not a zero of every  $f_i$
- iii.  $\phi_f$  is a holomorphic map at all such points *p* where it is defined.

#### Lemma

If the meromorphic functions  $\{f_i\}$  are not all identically zero, then the map  $\phi_f : X \to \mathbb{P}^n$  given above extends to a holomorphic map defined on all of X.

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#### Proof.

Fix a point  $p \in X$ , and let  $n = \min_i \operatorname{ord}_p(f_i)$ . We can choose a n.b.h of p such that no  $f_i$  has a pole other than possibly at p, and there are no common zeroes's to the  $f'_i s$ , other than possibly at p. Hence if we choose a local coordinate z on X centered at p, then every  $f_i(z)$  is holomorphic for z near 0 but  $z \neq 0$ , and there is no z near 0 which is a common root to every  $f_i$ . Hence for  $z \neq 0$ , we have

$$\phi_f(z) = [f_0(z) : \dots : f_n(z)]$$
  
=  $[z^{-n}f_0(z) : \dots : z^{-n}f_n(z)]$   
=  $[g_0(z) : \dots : g_n(z)].$ 

#### Proposition

Let  $\phi: X \to \mathbb{P}^n$  be a holomorphic map. Then there is an (n + 1)-tuple of meromorphic functions  $f = (f_0, ..., f_n)$  on X such that  $\phi = \phi_f$ . Moreover if two (n + 1)-tuples  $f = (f_0, ..., f_n)$  and  $g = (g_0, ..., g_n)$  of meromorphic functions induce the same map, so that  $\phi_f = \phi_g$  as holomorphic maps to  $\mathbb{P}^n$ , then there is a meromorphic function  $\lambda$  on X such that  $g_i = \lambda f_i$  for every *i*.

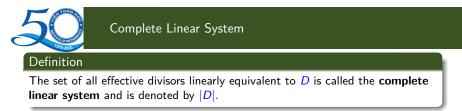
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The above proposition then gives a 1-1 corespondence between the set of holomorphic maps from X to  $\mathbb{P}^n$  and the projective space  $\mathbb{P}^n_{\mathscr{M}(X)}$ 

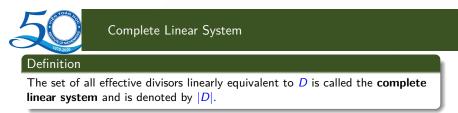
Let The Linear System of a Holomorphic Map

# Divisors and Maps to Projective Space



The Linear System of a Holomorphic Map

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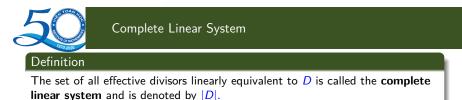


Consider the following map

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\Phi: \mathbb{P}\mathcal{O}_D(X) \to |D| = \{D' | D' \ge 0, D' \sim D\}
f(mod k<sup>*</sup>) \mapsto D + (f)
```

- The Linear System of a Holomorphic Map

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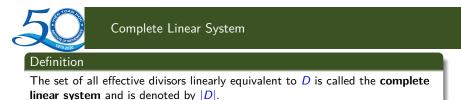
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This map is one to one for if D + (f) = D + (g) then indeed  $f = \lambda g$  and so f = g in  $\mathbb{P}\mathcal{O}_D(X)$  This map is also onto as any element of |D| is D + (f) for some function f. Hence, this map is bijection.

The Linear System of a Holomorphic Map

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This map is one to one for if D + (f) = D + (g) then indeed  $f = \lambda g$  and so f = g in  $\mathbb{P}\mathcal{O}_D(X)$  This map is also onto as any element of |D| is D + (f) for some function f. Hence, this map is bijection.

# Definition

A linear system on a Riemann surface X is a subset of |D| and parametrized by a linear subvariety of  $\mathbb{PO}_D(X)$ .

- The Linear System of a Holomorphic Map

Let  $\phi: X \to \mathbb{P}^n$  be a holomorphic map to projective space. Write  $\phi: [f_0: ...: f_n]$  where each  $f_i$  is a meromorphic function on X. Let  $D = -\min_i(f_i)$  be the inverse of the minimum divisor of the divisors of the functions. Therefore, for  $p \in X$ , we have that -D(p) is the minimum among the orders of the  $f_i$  at p, and so  $-D(p) \leq \operatorname{ord}_p(f_i)$  for each i.

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The linear system  $|\phi|$  defined above is well defined, independent of the choice of the functions  $\{f_i\}$  used to define  $\phi$ .

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#### Definition

Given a holomorphic map  $\phi : X \to \mathbb{P}^n$  with nondegenerate image, the linear system  $|\phi|$  defined above is called the **linear system of the map**  $\phi$ .

## Base Point of Linear Systems

#### Lemma

Let  $X \to \mathbb{P}^n$  be a holomorphic map. Then for every point  $p \in X$  there is a divisor  $E \in |\phi|$  which does not have p in its support. In other words, there is no point of X which is contained in every divisor of the linear system  $|\phi|$ .

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#### Proof.

Fix  $p \in X$ , and write  $\phi = [f_0 : ... : f_n]$  for meromorphic function  $f_i$ . Recall that we define  $D = -\min_i \{(f_i)\}$ . Suppose that the minimum order of the  $f_i's$  at p is k, assume that  $\operatorname{ord}_p(f_j) = p$ . Then D(p) = -k, and  $E = (f_j) + D$  is an element of the linear system  $|\phi|$ . But  $E(p) = \operatorname{ord}_p(f_j) + D(p) = k - k = 0$ , so E does not have p in its support.

Base Point of Linear Systems

#### Definition

Let Q be a linear system on a Riemann surface X. A point p is a **base point** of the linear system Q if every divisor  $E \in Q$  contains p ( $E \ge p$ ). A linear system Q is said to be **base-point-free** if it has no base points.

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#### Proposition

Let *D* be a divisor on a compact Riemann surface *X*. Then a point  $p \in X$  is a base point of the complete linear system |D| if and only if  $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_D(X)$ . Hence |D| is a base-point-free if and only if for every point  $p \in X$ ,  $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_D(X) - 1$ .



Defining a Holomorphic Map via Linear System

#### Proposition

Let  $Q \subset |D|$  be a base-point-free linear system of dimension n on a compact Riemann surface X. Then there is a holomorphic map  $\phi : X \to \mathbb{P}^n$  such that  $Q = |\phi|$ . Moreover  $\phi$  is unique up to the choice of coordinates in  $\mathbb{P}^n$ .



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Therefore we have a 1-1 correspondence between base-point-free linear systems of dimension *n* on *X* and holomorphic map  $\phi : X \to \mathbb{P}^n$  with nondegenerate image, up to linear coordinate changes.

Base Point of Linear Systems

## Divisors and Maps to Projective Space

# 50

## Removing the Base Point

Suppose that the complete linear system |D| has base points. Let  $F = \min\{E|E \in |D|\}$  be the minimum of all of the divisors in the linear system, the divisor F is the largest divisor that occurs in every divisor of |D|. It is obvious that the complete linear system |D - F| then has no base points, and moreover |D| = F + |D - F|

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#### Lemma

If F is the fixed divisor of the complete linear system |D|, then  $\mathcal{O}_{D-F}(X) = \mathcal{O}_D(X)$ .

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#### Lemma

If F is the fixed divisor of the complete linear system |D|, then  $\mathcal{O}_{D-F}(X) = \mathcal{O}_D(X)$ .

#### Proof.

Clearly  $F \ge 0$ , we have that  $D - F \le D$  and so  $\mathcal{O}_{D-F}(X) \subset \mathcal{O}_D(X)$ . To see the reverse inclusion, let  $f \in \mathcal{O}_D(X)$ , so that  $(f) + D \ge 0$ . Therefore  $(f) + D \in |D|$ , and we may write (f) + D = F + D' for some nonnegative divisor D'. Then  $(f) + (D - F) = D' \ge 0$ , so that  $f \in \mathcal{O}_{D-F}(X)$ .



#### Lemma

Let X be a compact Riemann surface, and let D be a divisor on X with |D| base-point-free. Fix a point  $p \in X$ . Then there is a basis  $f_0, ..., f_n$  for  $\mathcal{O}_D$  such that  $\operatorname{ord}_p(f_0) = -D(p)$  and  $\operatorname{ord}_p(f_i) > -D(p)$  for  $i \ge 1$ .

Criteria for 💑 to be an Embedding

#### Lemma

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#### Proof.

Consider the codimension one subspace  $\mathcal{O}_{D-p}(X)$  of  $\mathcal{O}_D(X)$ , and let  $f_1, ..., f_n$  be a basis for  $\mathcal{O}_{D-p}(X)$ . Extend this to a basic for  $\mathcal{O}_D(X)$  by adding a function  $f_0$  in  $\mathcal{O}_D(X) \setminus \mathcal{O}_{D-p}(X)$ . Then  $\operatorname{ord}_p(f_i) \ge -D(p) + 1 > -D(p)$  for every  $i \ge 1$ .

 $\Box Criteria for \phi_D to be an Embedding$ 

#### Proposition

Let X be a compact Riemann surface, and let D be a divisor on X with |D| is base point free. Fix distinct points p and q in X. Then  $\phi_D(p) = \phi_D(q)$  if and only if  $\mathcal{O}_{D-p-q}(X) = \mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$ . Hence  $\phi_D$  if and only if for every pair of distinct points p and q on X, we have  $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$ .

Criteria for  $\phi_D$  to be an Embedding

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We first prove an equivalence. Suppose  $\phi_D(p) = \phi_D(q)$ , and choose a basis for  $\mathbb{P}^n$  satisfying the condition in the above lemma, so that  $\phi_D(p) = \phi_D(q) = [1 : ... : 0]$ , then this condition implies that  $\operatorname{ord}_q(f_0) \leq \operatorname{ord}_q(f_i)$  for all i > 0. This implies that  $f_i \in \mathcal{O}_{D-q}(X)$  for all  $i \geq 1$  as linear independent meromorphic functions, and since D is base-point-free,  $f_1, ..., f_n$  forms a basis for  $\mathcal{O}_{D-q}(X)$ , and therefore  $\mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$ . Similarly, if  $\mathcal{O}_{D-q}(X) = \mathcal{O}_{D-q}(X)$ , then  $\phi_D(p) = \phi_D(q)$ , using the previous basis.

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Let X be a compact Riemann surface, and let D be a divisor on X with |D| is base point free. Fix distinct points p and q in X. Then  $\phi_D(p) = \phi_D(q)$  if and only if  $\mathcal{O}_{D-p-q}(X) = \mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$ . Hence  $\phi_D$  if and only if for every pair of distinct points p and q on X, we have  $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$ .

We first prove an equivalence. Suppose  $\phi_D(p) = \phi_D(q)$ , and choose a basis for  $\mathbb{P}^n$  satisfying the condition in the above lemma, so that  $\phi_D(p) = \phi_D(q) = [1 : ... : 0]$ , then this condition implies that  $\operatorname{ord}_q(f_0) \leq \operatorname{ord}_q(f_i)$  for all i > 0. This implies that  $f_i \in \mathcal{O}_{D-q}(X)$  for all  $i \geq 1$  as linear independent meromorphic functions, and since D is base-point-free,  $f_1, ..., f_n$  forms a basis for  $\mathcal{O}_{D-q}(X)$ , and therefore  $\mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$ . Similarly, if  $\mathcal{O}_{D-q}(X) = \mathcal{O}_{D-q}(X)$ , then  $\phi_D(p) = \phi_D(q)$ , using the previous basis.

This says that every function f in  $\mathcal{O}_D(X)$  with  $\operatorname{ord}_p(f) > -D(p)$  also satisfies  $\operatorname{ord}_q(f) > -D(q)$ . Hence  $\mathcal{O}_{D-p}(X) \subset \mathcal{O}_{D-p-q}(X)$ , since p and q are distinct. This implies that

$$\mathcal{O}_{D-p-q}(X) = \mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X).$$

 $\Box$  Criteria for  $\phi_D$  to be an Embedding

Since |D| is base-point-free, we have that  $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_{D-q}(X) = \dim \mathcal{O}_D(X) - 1$ . Therefore  $\dim \mathcal{O}_{D-p-q}(X)$  is either  $\dim \mathcal{O}_D(X) - 1$  or  $\dim \mathcal{O}_D(X) - 2$ . If  $\phi_D$  is 1 - 1, then by the first part we see that  $\mathcal{O}_{D-p-q}(X)$  is a proper subspace of  $\mathcal{O}_{D-p}(X)$  for all p and q, and so must have dimension equal to  $\dim \mathcal{O}_D(X) - 2$ .

Criteria for  $\phi_D$  to be an Embedding

Since |D| is base-point-free, we have that  $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_{D-q}(X) = \dim \mathcal{O}_D(X) - 1$ . Therefore  $\dim \mathcal{O}_{D-p-q}(X)$  is either  $\dim \mathcal{O}_D(X) - 1$  or  $\dim \mathcal{O}_D(X) - 2$ . If  $\phi_D$  is 1 - 1, then by the first part we see that  $\mathcal{O}_{D-p-q}(X)$  is a proper subspace of  $\mathcal{O}_{D-p}(X)$  for all p and q, and so must have dimension equal to  $\dim \mathcal{O}_D(X) - 2$ . Conversely, if the dimension always does drop by 2, then the tower of subspaces  $\mathcal{O}_{D-p-q}(X) \subset \mathcal{O}_{D-p}(X) \subset \mathcal{O}_D(X)$  must all be distinct for every p and q, so that  $\phi_D$  is 1 - 1.

 $\Box Criteria for \phi_D to be an Embedding$ 

#### Proposition

Let X be a compact Riemann surface, and let D be a divisor on X whose linear system |D| has no base points. Then  $\phi_D$  is a 1-1 holomorphic map and an isomorphism onto its image (which is a holomorphically embedded Riemann surface in  $\mathbb{P}^n$ ), if and only if for every p and q in X, we have  $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$ .

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#### Proof.

We first prove the if statement. In order to define the morphism  $\phi_D = [f_0 : ... : f_n]$ , we choose a basis  $f_2, ..., f_n \in \mathcal{O}_{D-2p}$ , and we let

$$f_1 \in \mathcal{O}_{D-p}(X); f_0 \in \mathcal{O}_D(X) \setminus \mathcal{O}_{D-p}(X).$$

so that  $\operatorname{ord}_{\rho}(f_1) = -D(p) + 1$ , and therefore  $\operatorname{ord}_{\rho}(f_1/f_0) = 1$ , and  $\operatorname{ord}_{\rho}(f_i/f_0) > 2$  for i > 2, so that applying the inverse function theorem, we see that the image of  $\phi_D$  has the local coordinate  $f_1/f_0$  at p, and using injectivity, we get that  $\phi_D$  is an **embedding**.

Criteria for  $\phi_D$  to be an Embedding

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The holomorphic map  $\phi_D$  is an embedding if and only if there is a function in  $\mathcal{O}_{D-p}(X)$  but not in  $\mathcal{O}_{D-2p}(X)$ .

The Riemann-Roch Theorem

Divisors and Maps to Projective Space

## Definition

A divisor *D* such that |D| has no base points and  $\phi_D$  is an embedding is called a **very ample divisor**.



## Definition

Let f be a meromorphic on a Riemann surface X. The function f has **multiplicity one** at a point  $p \in X$  if either f is holomorphic at p and  $\operatorname{ord}_p(f - f(p)) = 1$ , or f has a simple pole at p.



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Algebraic Curve

#### Definition

Let *S* be a set of meromorphic functions on a compact Riemann surface *X*. We say that *S* **separates points** of *X* if for every pair of distinct points *p* and *q* in *X* there is a meromorphic function  $f \in S$  such that  $f(p) \neq f(q)$ . We say that *S* **separates tangents** of *X* if for every point  $p \in X$  there is a meromorphic function  $f \in S$  which has multiplicity one at *p*. A compact Riemann surface *X* is an **algebraic curve** if the field  $\mathcal{M}(X)$  of global meromorphic function function functions separates the points and tangents of *X*.



## Example of Algebraic Curve

## Example

- i. The Riemann sphere  $\mathbb{P}^1$  is an algebraic curve.
- ii. Any complex torus  $\mathbb{C}/L$  is an algebraic curve.
- iii. Any smooth projective plane curve is algebraic curve.
- iv. Any smooth projective curve in  $\mathbb{P}^n$  is an algebraic curve.



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#### Theorem

Every compact Riemann surface is an algebraic curve.

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#### Proof.

First we show that  $\mathscr{M}(X)$  separates the points of X. Fix two points p and q on X, and consider divisor D = (g + 1)p. By Riemann Roch theorem, we see that  $\dim \mathcal{O}_D(X) \ge \deg(D) + 1 - g = 2$ . Hence there is a nonconstant function  $f \in \mathcal{O}_D(X)$ . This function f must have a pole, and the only poles allowed are at p, so f has a pole at p and no other poles. In particular f does not have a pole at q, and f then separates p and q.

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Every compact Riemann surface is an algebraic curve.

#### Proof.

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#### Lemma

If D is a divisor of Riemann surface X, and p some point of X, then we have the inequality  $\dim \mathcal{O}_D(X) \ge \dim \mathcal{O}_{D-p}(X) \ge \dim \mathcal{O}_D(X) - 1$ .

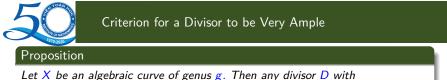
#### Lemma

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#### Lemma

Suppose  $\phi : X \to \mathbb{P}^n$  is a holomorphic map with a smooth projective curve Y as the image. If D is a very ample divisor on X, so that  $\phi_D$  is a holomorphic embedding of X into  $\mathbb{P}^n$ , then

 $\deg(\phi(X)) = \deg(D).$ 



Let X be an algebraic curve of genus g. Then any divisor D with  $deg(D) \ge 2g + 1$  is very ample, that is, the complete linear system |D| has no base points and the associated holomorphic map  $\phi_D$  to projective space is a holomorphic embedding onto a smooth projective curve of degree equal to deg(D).

Classifications of Algebraic Curves of Genus g = 0, 1, 2, 3



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#### Proof.

We need to check that  $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$  for any points p and q on X. Since both D and D - p - q have degree at least 2g - 1, we have that  $H^1(X, \mathcal{O}_D) = H^1(X, \mathcal{O}_{D-p-q}) = 0$ , and

 $\dim \mathcal{O}_D(X) = \deg D + 1 - g$  and  $\dim \mathcal{O}_{D-p-q}(X) = \deg(D-p-q) + 1 - g$ 

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Every Algebraic Curve is Projective

#### Proposition

Every algebraic curve X can be holomorphically embedded into projective space.

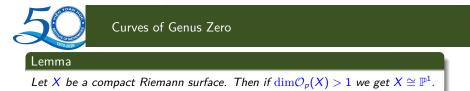
## Every Algebraic Curve is Projective

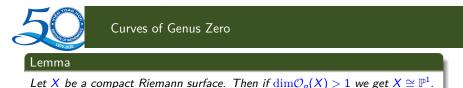
#### Proposition

Every algebraic curve X can be holomorphically embedded into projective space.

## Proof.

Pick any point p and use D = (2g + 1)p.





Corollary

If D is a divisor of degree 2 such that  $\dim \mathcal{O}_D(X) = 2$  on a genus  $g \ge 1$  curve, then |D| is base point free.



Let X be a compact Riemann surface. Then if  $\dim \mathcal{O}_p(X) > 1$  we get  $X \cong \mathbb{P}^1$ .

### Corollary

If D is a divisor of degree 2 such that  $\dim \mathcal{O}_D(X) = 2$  on a genus  $g \ge 1$  curve, then |D| is base point free.

#### Proof.

Suppose *D* is degree 2 and  $\dim \mathcal{O}_D(X) = 2$ . If |D| has a base point, then  $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_D(X)$  so that  $\dim \mathcal{O}_{D-p}(X) = 2$ . Since we can generate |D| with the divisor D = p + q, if *q* is the base point this implies that  $\dim \mathcal{O}_D(X) = 2 > 1$  so  $X \cong \mathbb{P}^1$  by the above lemma, but this contradicts  $g \ge 1$ .



#### Proposition

Let X be an algebraic curve of genus 0. Then X is isomorphic to the Riemann sphere  $\mathbb{P}^1$ .



Curve of Genus Zero are Isomorphic to the Riemann Sphere

#### Proposition

Let X be an algebraic curve of genus 0. Then X is isomorphic to the Riemann sphere  $\mathbb{P}^1$ .

#### Proof.

Fix any point  $p \in X$ . Since the canonical divisor K on X has degree 2g - 2 = -2, then the divisor K - p has degree -3. This is strictly negative, so  $O_{K-D}(X) = 0$  Applying Riemann-Roch to the divisor p, we find that

$$\dim \mathcal{O}_D(X) = \deg(p) + 1 - g + \dim \mathcal{O}_{K-p}(X) = 2.$$



#### Proposition

Every algebraic curve of genus one is isomorphic to a smooth projective plane cubic curve.



Curves of Genus One are Cubic Plane Curves

### Proposition

Every algebraic curve of genus one is isomorphic to a smooth projective plane cubic curve.

#### Proof.

By the criterion for a divisor to be very ample, we see that any divisor of degree 3 is very ample. Since by the Riemann-Roch,  $\dim \mathcal{O}_D(X) = 3$  if  $\deg(D) = 3$ , we see that the holomorphic map  $\phi_D$  would map X to the plane  $\mathbb{P}^2$ . Since  $\deg(\phi_D(X)) = \deg(D) = 3$ , the image is smooth cubic curve.



## Definition

We define a **hyperelliptic curve** to be an compact Riemann surface Y such that there exists some holomorphic map  $F : Y \to \mathbb{P}^1$  that is of degree 2.



Curves of Genus Two are Hyperelliptic

## Definition

We define a **hyperelliptic curve** to be an compact Riemann surface Y such that there exists some holomorphic map  $F : Y \to \mathbb{P}^1$  that is of degree 2.

#### Theorem

Suppose X and Y are Riemann surfaces and  $f : X \to Y$  is proper non-constant holomorphic map. Then there exists a natural number n such that f takes every value  $c \in Y$ , counting multiplicities, n times.



## Proposition

Every algebraic curve Y of genus two is hyperelliptic.



Curves of Genus Two are Hyperelliptic

#### Proposition

Every algebraic curve Y of genus two is hyperelliptic.

#### Proof.

Note that  $\dim \mathcal{O}_{K}(Y) = 2$  and  $\deg(K) = 2$ , we must have K = p + q for some points  $p, q \in Y$ . Let  $f \in \mathcal{O}_{K}(Y)$  be nonconstant. Then f has either one or two poles, which must be at the points p or q. Suppose f has only one pole at p. Then we have that  $f \in \mathcal{O}_{p}(Y)$ , and therefore  $\dim \mathcal{O}_{p}(Y) = 2$ , which implies that Y is isomorphic to  $\mathbb{P}^{1}$ , a contradiction of genus. So f must have either a double pole or two single poles, which give us a morphism  $F : Y \to \mathbb{P}^{1}$  of degree 2.



### Lemma

The canonical linear system |K| on an algebraic curve X of genus  $g \ge 1$  is base-point-free.



#### Lemma

The canonical linear system |K| on an algebraic curve X of genus  $g \ge 1$  is base-point-free.

#### Proof.

Fix a point  $p \in X$ . We must show that  $\mathcal{O}_{K-p}(X) \neq \mathcal{O}_{K}(X)$ , and for this suffice to show that  $\dim \mathcal{O}_{K-P}(X) = \dim \mathcal{O}_{K}(X) - 1 = g - 1$ . Now since  $\dim \mathcal{O}_{p}(X) = 1$ , we have using Riemann-Roch that

$$1 = \dim \mathcal{O}_{\rho}(X) = \mathcal{O}_{K-\rho}(X) + \deg(p) + 1 - g,$$

which gives  $\dim \mathcal{O}_{K-p} = g - 1$  as desired.





# Proof.

By Riemann-Roch theorem, we have

$$\dim \mathcal{O}_{K-p-q}(X) = \deg(K-p-q) + 1 - g + \dim \mathcal{O}_{p+q}(X) = g - 3 + \dim \mathcal{O}_{p+q}(X)$$

so that  $\phi_{\mathcal{K}}$  fails to an embedding if and only if for some points p, q of  $X, \dim \mathcal{O}_{p+q}(X) = 2$   $(\dim \mathcal{O}_{\mathcal{K}-p-q}(X) = \dim \mathcal{O}_{\mathcal{K}}(X) - 2).$ 



## Proof.

By Riemann-Roch theorem, we have

 $\dim \mathcal{O}_{\mathcal{K}-p-q}(X) = \deg(\mathcal{K}-p-q) + 1 - g + \dim \mathcal{O}_{p+q}(X) = g - 3 + \dim \mathcal{O}_{p+q}(X),$ 

so that  $\phi_K$  fails to an embedding if and only if for some points p, q of  $X, \dim \mathcal{O}_{p+q}(X) = 2$   $(\dim \mathcal{O}_{K-p-q}(X) = \dim \mathcal{O}_K(X) - 2)$ . If this happens, then any nonconstant function  $f \in \mathcal{O}_{p+q}(X)$  gives a degree map to the Riemann sphere, and so X is hyperelliptic. Conversely, if X is hyperelliptic and  $\pi : X \to \mathbb{P}^1$  is the degree 2 mapping, then the inverse image divisor p + q of  $\infty$  has degree 2 and  $\dim \mathcal{O}_{p+q}(X) = 2$ .



### Corollary

If X is not hyperelliptic curve, then  $\phi_K$  embeds X in  $\mathbb{P}^{g-1}$  as curve of degree 2g-2.

# Curves of Genus Three

### Corollary

If X is not hyperelliptic curve, then  $\phi_K$  embeds X in  $\mathbb{P}^{g-1}$  as curve of degree 2g-2.

### Corollary

If g = 3 and X is not hyperelliptic, then X is a plane curve of degree 4.





- O. Forster, *Lectures on Riemann Surfaces*, Springer, 1981.
- R. Miranda, Algebraic Curves and Riemann Surfaces, AMS, 1995.

# THANK YOU FOR LISTENING