The Riemann-Roch Theorem

Vo Quoc Bao
Institute of Mathematics of Vietnam,
Hanoi City

August, 2020

Definition (Divisors)

Let X be a Riemann surface. A **divisor** on X is a mapping

$$D : X \to \mathbb{Z}$$

such that for any compact subset $K \subset X$ there are only finitely many points $x \in K$ such that $D(x) \neq 0$.

Definition (Divisors)

Let X be a Riemann surface. A **divisor** on X is a mapping

$$D : X \to \mathbb{Z}$$

such that for any compact subset $K \subset X$ there are only finitely many points $x \in K$ such that $D(x) \neq 0$.

Definition (Divisors of Meromorphic Functions)

Suppose X is a Riemann surface and Y is an open subset of X. For a meromorphic function $f \in \mathscr{M}(Y)$ and $a \in Y$ define

$$\mathrm{ord}_a(f) = \begin{cases} 0 & \text{if } f \text{ is holomorphic and non-zero at } a \\ k & \text{if } f \text{ has a zero of order k at } a \\ -k & \text{if } f \text{ has a pole of order k at } a \\ \infty & \text{if } f \text{ is identically zero in a n.b.h of } a \end{cases}.$$

For any meromorphic function $f \in \mathscr{M}(X) \setminus \{0\}$, the mapping $x \mapsto \mathrm{ord}_x(f)$ is called the **divisor** of f.

Definition (Divisor of Meromorphic $1-$forms)

For a meromorphic $1-$form $w \in \mathscr{M}^{(1)}(Y)$. Choose a coordinate n.b.h (U, z) of a. Then on $U \cap Y$ one may write $w = fdz$, where f is a meromorphic function. Set $\mathrm{ord}_a(w) = \mathrm{ord}_a f$. For $1-$forms $w \in \mathscr{M}^{(1)}(X) \setminus \{0\}$ the mapping $x \mapsto \mathrm{ord}_x(w)$ is called the **divisor** of w.

Definition (Divisor of Meromorphic $1-$forms)

For a meromorphic $1-$form $w \in \mathscr{M}^{(1)}(Y)$. Choose a coordinate n.b.h (U, z) of a. Then on $U \cap Y$ one may write $w = fdz$, where f is a meromorphic function. Set $\mathrm{ord}_a(w) = \mathrm{ord}_a f$. For $1-$forms $w \in \mathscr{M}^{(1)}(X) \setminus \{0\}$ the mapping $x \mapsto \mathrm{ord}_x(w)$ is called the **divisor** of w.

Proposition

For $f, g \in \mathscr{M}(X) \setminus \{0\}$ and $w \in \mathscr{M}^{(1)}(X) \setminus \{0\}$ one has the following relations

$$(fg) = (f) + (g), \ \ (1/f) = -(f), \ \ (fw) = (f) + (w).$$

Definition (Divisor of Meromorphic $1-$forms)

For a meromorphic $1-$form $w \in \mathscr{M}^{(1)}(Y)$. Choose a coordinate n.b.h (U, z) of a. Then on $U \cap Y$ one may write $w = fdz$, where f is a meromorphic function. Set $\mathrm{ord}_a(w) = \mathrm{ord}_a f$. For $1-$forms $w \in \mathscr{M}^{(1)}(X) \setminus \{0\}$ the mapping $x \mapsto \mathrm{ord}_x(w)$ is called the **divisor** of w.

Proposition

For $f, g \in \mathscr{M}(X) \setminus \{0\}$ and $w \in \mathscr{M}^{(1)}(X) \setminus \{0\}$ one has the following relations

$$(fg) = (f) + (g), \quad (1/f) = -(f), \quad (fw) = (f) + (w).$$

Definition

A divisor $D \in \mathrm{Div}(X)$ is called a **principal divisor** if there exists a function $f \in \mathscr{M}(X) \setminus \{0\}$ such that $D = (f)$. Two divisors $D, D' \in \mathrm{Div}(X)$ are said to be **equivalent** if their difference $D - D'$ is a principal divisor.

Definition

Let X be a compact Riemann surface. Then for every $D \in \mathrm{Div}(X)$ there are only finitely many $x \in X$ such that $D(x) \neq 0$. Hence one can define a mapping

$$\deg : \mathrm{Div}(X) \to \mathbb{Z}$$

called the **degree**, by letting

$$\deg(D) := \sum_{x \in X} D(x).$$

Suppose D is a divisor on the Riemann surface X. For any open set $U \subset X$ define $\mathcal{O}_D(U)$ as follows

$$\mathcal{O}_D(U) := \{f \in \mathscr{M}(U) | \mathrm{ord}_x(f) \geq -D(x) \text{ for every } x \in U\}.$$

Together with the natural restriction mappings $\mathcal{O}_D$ is a sheaf (**Sheaf of divisor**).

Suppose D is a divisor on the Riemann surface X. For any open set $U \subset X$ define $\mathcal{O}_D(U)$ as follows

$$\mathcal{O}_D(U) := \{f \in \mathscr{M}(U) | \mathrm{ord}_x(f) \geq -D(x) \text{ for every } x \in U\}.$$

Together with the natural restriction mappings $\mathcal{O}_D$ is a sheaf (**Sheaf of divisor**).

Theorem

Suppose X is a compact Riemann surface and $D \in \mathrm{Div}(X)$ is a divisor with $\deg D < 0$. Then $H^0(X, \mathcal{O}_D) = 0$.

Suppose D is a divisor on the Riemann surface X. For any open set $U \subset X$ define $\mathcal{O}_D(U)$ as follows

$$\mathcal{O}_D(U) := \{f \in \mathscr{M}(U) | \mathrm{ord}_x(f) \geq -D(x) \text{ for every } x \in U\}.$$

Together with the natural restriction mappings $\mathcal{O}_D$ is a sheaf (**Sheaf of divisor**).

Theorem

Suppose X is a compact Riemann surface and $D \in \mathrm{Div}(X)$ is a divisor with $\deg D < 0$. Then $H^0(X, \mathcal{O}_D) = 0$.

Proof.

Suppose, to the contrary, that there exists an $f \in H^0(X, \mathcal{O}_D)$ with $f \neq 0$. Then $(f) \geq -D$ and thus

$$\mathrm{ded}(f) \geq -\deg D > 0.$$

However this contradicts the fact that $\deg(f) = 0$. □

The Riemann-Roch Theorem

Skyscraper Sheaf

Proposition

Let $p \in X$. Define for an open set $U \subset X$.

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases},$$

where the restriction maps are the obvious homomorphisms. Then

i. $H^0(X, \mathbb{C}_p) \cong \mathbb{C}$

ii. $H^1(X, \mathbb{C}_p) = 0$.

The Riemann-Roch Theorem

Skyscraper Sheaf

Proposition

Let $p \in X$. Define for an open set $U \subset X$.

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases},$$

where the restriction maps are the obvious homomorphisms. Then

i. $H^0(X, \mathbb{C}_p) \cong \mathbb{C}$

ii. $H^1(X, \mathbb{C}_p) = 0$.

We also have the following short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \to \mathbb{C}_P \to 0.$$

The Riemann-Roch Theorem

The Riemann Roch Theorem

By the long exact sequence theorem, we have

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to \mathbb{C} \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+P}) \to 0.$$

The Riemann-Roch Theorem

The Riemann Roch Theorem

By the long exact sequence theorem, we have

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to \mathbb{C} \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+P}) \to 0.$$

Theorem (Riemann-Roch)

Suppose D is a divisor on a compact Riemann surface X of genus g. Then $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite dimensional vector spaces and

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D.$$

i. The result holds for the divisor $D = 0$.

i. The result holds for the divisor $D = 0$.

ii. Suppose D is a divisor, $P \in X$, and $D' = D + P$. Suppose that the result holds for one of the divisors D, D'. The above exact cohomology sequence can be slit into two short exact sequences. For, let

$$V := \mathrm{Im}(H^0(X, \mathcal{O}_D) \to \mathbb{C})$$
$$W := \mathbb{C}/V.$$

Then $\dim(V) + \dim(W) = 1 = \deg D' - \deg D$ and the sequences

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_D') \to V \to 0,$$
$$0 \to W \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}) \to 0$$

are exact. This implies that

$$\dim H^0(X, \mathcal{O}_{D'}) = \dim H^0(X, \mathcal{O}_D) + \dim V$$
$$\dim H^1(X, \mathcal{O}_D) = \dim H^1(X, \mathcal{O}_{D'}) + \dim W.$$

Therefore,

$$\dim H^0(X, \mathcal{O}_{D'}) - \dim H^1(X, \mathcal{O}_{D'}) - \deg D' = \dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) - \deg D.$$

iii. An arbitrary divisor D on X may be written

$$D = P_1 + \ldots + P_m - P_{m+1} - \ldots - P_n,$$

where the $P_j \in X$ are points.

The Riemann-Roch Theorem

Canonical Divisor

By Serre's duality, we can identify $H^1(X, \mathcal{O}_D)$ as $\mathcal{O}_{K-D}(X)$ with $K = (w)$. The divisor K is called **canonical divisor**.

Proposition

The canonical divisor K on a compact Riemann surface of genus g satisfies

$$\deg(K) = 2g - 2.$$

The Riemann-Roch Theorem

Canonical Divisor

By Serre's duality, we can identify $H^1(X, \mathcal{O}_D)$ as $\mathcal{O}_{K-D}(X)$ with $K = (w)$. The divisor K is called **canonical divisor**.

Proposition

The canonical divisor K on a compact Riemann surface of genus g satisfies

$$\deg(K) = 2g - 2.$$

Theorem

Suppose X is a compact Riemann surface of genus g and D is a divisor of on X

The Riemann-Roch Theorem

Canonical Divisor

By Serre's duality, we can identify $H^1(X, \mathcal{O}_D)$ as $\mathcal{O}_{K-D}(X)$ with $K = (w)$. The divisor K is called **canonical divisor**.

Proposition

The canonical divisor K on a compact Riemann surface of genus g satisfies

$$\deg(K) = 2g - 2.$$

Theorem

Suppose X is a compact Riemann surface of genus g and D is a divisor of on X

Corollary

If D is a divisor such that $\deg(D) \geq 2g - 1$, then

$$\dim \mathcal{O}_D(X) = \deg(D) + 1 - g.$$

Divisors and Maps to Projective Space

Maps to Projective Space Given by Meromorphic Functions

Definition

Let X be a Riemann surface. A map $\phi : X \to \mathbb{P}^n$ is **holomorphic at a point** $p \in X$ if there are holomorphic functions $g_0, \ldots, g_n$ defined on X near p, not all zero at p, such that $\phi(x) = [g_0(x) : \ldots : g_n(x)]$ for x near p. We say ϕ is a **holomorphic map** if it is holomorphic at all points of X.

Divisors and Maps to Projective Space

Maps to Projective Space Given by Meromorphic Functions

Definition

Let X be a Riemann surface. A map $\phi : X \to \mathbb{P}^n$ is **holomorphic at a point** $p \in X$ if there are holomorphic functions $g_0, \ldots, g_n$ defined on X near p, not all zero at p, such that $\phi(x) = [g_0(x) : \ldots : g_n(x)]$ for x near p. We say ϕ is a **holomorphic map** if it is holomorphic at all points of X.

Let X be be a Riemann surface. Choose $n+1$ meromorphic functions $f = (f_0, \ldots, f_n)$ on X, not all identically zero. Define $\phi_f : X \to \mathbb{P}^n$ by setting

$$\phi_f(p) = [f_0(p) : \ldots : f_n(p)].$$

Divisors and Maps to Projective Space

Maps to Projective Space Given by Meromorphic Functions

Definition

Let X be a Riemann surface. A map $\phi : X \to \mathbb{P}^n$ is **holomorphic at a point** $p \in X$ if there are holomorphic functions $g_0, \ldots, g_n$ defined on X near p, not all zero at p, such that $\phi(x) = [g_0(x) : \ldots : g_n(x)]$ for x near p. We say ϕ is a **holomorphic map** if it is holomorphic at all points of X.

Let X be be a Riemann surface. Choose $n+1$ meromorphic functions $f = (f_0, \ldots, f_n)$ on X, not all identically zero. Define $\phi_f : X \to \mathbb{P}^n$ by setting

$$\phi_f(p) = [f_0(p) : \ldots : f_n(p)].$$

Note that a priori, ϕ_f is defined at p if

i. p is not a pole of any f_i

ii. p is not a zero of every f_i

iii. ϕ_f is a holomorphic map at all such points p where it is defined.

Lemma

If the meromorphic functions $\{f_i\}$ *are not all identically zero, then the map* $\phi_f : X \to \mathbb{P}^n$ *given above extends to a holomorphic map defined on all of* X.

Lemma

If the meromorphic functions $\{f_i\}$ are not all identically zero, then the map $\phi_f : X \to \mathbb{P}^n$ given above extends to a holomorphic map defined on all of X.

Proof.

Fix a point $p \in X$, and let $n = \min_i \mathrm{ord}_p(f_i)$. We can choose a n.b.h of p such that no f_i has a pole other than possibly at p, and there are no common zeroes's to the $f_i' s$, other than possibly at p. Hence if we choose a local coordinate z on X centered at p, then every $f_i(z)$ is holomorphic for z near 0 but $z \neq 0$, and there is no z near 0 which is a common root to every f_i. Hence for $z \neq 0$, we have

$$\begin{aligned}\phi_f(z) &= [f_0(z) : \ldots : f_n(z)] \\ &= [z^{-n} f_0(z) : \ldots : z^{-n} f_n(z)] \\ &= [g_0(z) : \ldots : g_n(z)].\end{aligned}$$

□

Proposition

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map. Then there is an $(n+1)-$tuple of meromorphic functions $f = (f_0, ..., f_n)$ on X such that $\phi = \phi_f$. Moreover if two $(n+1)-$tuples $f = (f_0, ..., f_n)$ and $g = (g_0, ..., g_n)$ of meromorphic functions induce the same map, so that $\phi_f = \phi_g$ as holomorphic maps to $\mathbb{P}^n$, then there is a meromorphic function λ on X such that $g_i = \lambda f_i$ for every i.

Proposition

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map. Then there is an $(n+1)-$tuple of meromorphic functions $f = (f_0, ..., f_n)$ on X such that $\phi = \phi_f$. Moreover if two $(n+1)-$tuples $f = (f_0, ..., f_n)$ and $g = (g_0, ..., g_n)$ of meromorphic functions induce the same map, so that $\phi_f = \phi_g$ as holomorphic maps to $\mathbb{P}^n$, then there is a meromorphic function λ on X such that $g_i = \lambda f_i$ for every i.

The above proposition then gives a $1 - 1$ corespondence between the set of holomorphic maps from X to $\mathbb{P}^n$ and the projective space $\mathbb{P}^n_{\mathscr{M}(X)}$

Divisors and Maps to Projective Space

Complete Linear System

Definition

The set of all effective divisors linearly equivalent to D is called the **complete linear system** and is denoted by $|D|$.

Divisors and Maps to Projective Space

Complete Linear System

Definition

The set of all effective divisors linearly equivalent to D is called the **complete linear system** and is denoted by $|D|$.

Consider the following map

$$\Phi : \mathbb{P}\mathcal{O}_D(X) \to |D| = \{D' | D' \geq 0, D' \sim D\}$$
$$f \pmod{k^*} \mapsto D + (f)$$

Divisors and Maps to Projective Space

Complete Linear System

Definition

The set of all effective divisors linearly equivalent to D is called the **complete linear system** and is denoted by $|D|$.

Consider the following map

$$\Phi : \mathbb{P}\mathcal{O}_D(X) \to |D| = \{D' | D' \geq 0, D' \sim D\}$$
$$f(\mod k^*) \mapsto D + (f)$$

This map is one to one for if $D + (f) = D + (g)$ then indeed $f = \lambda g$ and so $f = g$ in $\mathbb{P}\mathcal{O}_D(X)$ This map is also onto as any element of $|D|$ is $D + (f)$ for some function f. Hence, this map is bijection.

Divisors and Maps to Projective Space

Complete Linear System

Definition

The set of all effective divisors linearly equivalent to D is called the **complete linear system** and is denoted by $|D|$.

Consider the following map

$$\Phi : \mathbb{P}\mathcal{O}_D(X) \to |D| = \{D' | D' \geq 0, D' \sim D\}$$
$$f(\mod k^*) \mapsto D + (f)$$

This map is one to one for if $D + (f) = D + (g)$ then indeed $f = \lambda g$ and so $f = g$ in $\mathbb{P}\mathcal{O}_D(X)$ This map is also onto as any element of $|D|$ is $D + (f)$ for some function f. Hence, this map is bijection.

Definition

A **linear system** on a Riemann surface X is a subset of $|D|$ and parametrized by a linear subvariety of $\mathbb{P}\mathcal{O}_D(X)$.

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map to projective space. Write $\phi : [f_0 : ... : f_n]$ where each f_i is a meromorphic function on X. Let $D = -\min_i(f_i)$ be the inverse of the minimum divisor of the divisors of the functions. Therefore, for $p \in X$, we have that $-D(p)$ is the minimum among the orders of the f_i at p, and so $-D(p) \leq \mathrm{ord}_p(f_i)$ for each i.

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map to projective space. Write $\phi : [f_0 : ... : f_n]$ where each f_i is a meromorphic function on X. Let $D = -\min_i(f_i)$ be the inverse of the minimum divisor of the divisors of the functions. Therefore, for $p \in X$, we have that $-D(p)$ is the minimum among the orders of the f_i at p, and so $-D(p) \leq \mathrm{ord}_p(f_i)$ for each i.
Therefore, $-D \leq (f_i)$ for each i, and we have that $f_i \in \mathcal{O}_D(X)$ for every i.
Hence if we let V_f be the $\mathbb{C}-$linear span of the function $\{f_i\}$, then $V_f \subset \mathcal{O}_D(X)$.

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map to projective space. Write $\phi : [f_0 : ... : f_n]$ where each f_i is a meromorphic function on X. Let $D = -\min_i(f_i)$ be the inverse of the minimum divisor of the divisors of the functions. Therefore, for $p \in X$, we have that $-D(p)$ is the minimum among the orders of the f_i at p, and so $-D(p) \leq \mathrm{ord}_p(f_i)$ for each i.
Therefore, $-D \leq (f_i)$ for each i, and we have that $f_i \in \mathcal{O}_D(X)$ for every i.
Hence if we let V_f be the $\mathbb{C}-$linear span of the function $\{f_i\}$, then $V_f \subset \mathcal{O}_D(X)$.
In conclusion, the set of divisors $|\phi| = \{(g) + D | g \in V_f\}$ forms a linear system on X.

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map to projective space. Write $\phi : [f_0 : ... : f_n]$ where each f_i is a meromorphic function on X. Let $D = -\min_i(f_i)$ be the inverse of the minimum divisor of the divisors of the functions. Therefore, for $p \in X$, we have that $-D(p)$ is the minimum among the orders of the f_i at p, and so $-D(p) \leq \mathrm{ord}_p(f_i)$ for each i. Therefore, $-D \leq (f_i)$ for each i, and we have that $f_i \in \mathcal{O}_D(X)$ for every i. Hence if we let V_f be the $\mathbb{C}-$linear span of the function $\{f_i\}$, then $V_f \subset \mathcal{O}_D(X)$. In conclusion, the set of divisors $|\phi| = \{(g) + D | g \in V_f\}$ forms a linear system on X.

Lemma

The linear system $|\phi|$ defined above is well defined, independent of the choice of the functions $\{f_i\}$ used to define ϕ.

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map to projective space. Write $\phi : [f_0 : ... : f_n]$ where each f_i is a meromorphic function on X. Let $D = -\min_i(f_i)$ be the inverse of the minimum divisor of the divisors of the functions. Therefore, for $p \in X$, we have that $-D(p)$ is the minimum among the orders of the f_i at p, and so $-D(p) \leq \mathrm{ord}_p(f_i)$ for each i. Therefore, $-D \leq (f_i)$ for each i, and we have that $f_i \in \mathcal{O}_D(X)$ for every i. Hence if we let V_f be the $\mathbb{C}-$linear span of the function $\{f_i\}$, then $V_f \subset \mathcal{O}_D(X)$. In conclusion, the set of divisors $|\phi| = \{(g) + D | g \in V_f\}$ forms a linear system on X.

Lemma

The linear system $|\phi|$ defined above is well defined, independent of the choice of the functions $\{f_i\}$ used to define ϕ.

Definition

Given a holomorphic map $\phi : X \to \mathbb{P}^n$ with nondegenerate image, the linear system $|\phi|$ defined above is called the **linear system of the map ϕ**.

Divisors and Maps to Projective Space

Base Point of Linear Systems

Lemma

Let $X \to \mathbb{P}^n$ be a holomorphic map. Then for every point $p \in X$ there is a divisor $E \in |\phi|$ which does not have p in its support. In other words, there is no point of X which is contained in every divisor of the linear system $|\phi|$.

Divisors and Maps to Projective Space

Base Point of Linear Systems

Lemma

Let $X \to \mathbb{P}^n$ be a holomorphic map. Then for every point $p \in X$ there is a divisor $E \in |\phi|$ which does not have p in its support. In other words, there is no point of X which is contained in every divisor of the linear system $|\phi|$.

Proof.

Fix $p \in X$, and write $\phi = [f_0 : ... : f_n]$ for meromorphic function f_i. Recall that we define $D = -\min_i\{(f_i)\}$. Suppose that the minimum order of the $f_i's$ at p is k, assume that $\mathrm{ord}_p(f_j) = p$. Then $D(p) = -k$, and $E = (f_j) + D$ is an element of the linear system $|\phi|$. But $E(p) = \mathrm{ord}_p(f_j) + D(p) = k - k = 0$, so E does not have p in its support. □

Definition

Let Q be a linear system on a Riemann surface X. A point p is a **base point** of the linear system Q if every divisor $E \in Q$ contains p ($E \geq p$). A linear system Q is said to be **base-point-free** if it has no base points.

Definition

Let Q be a linear system on a Riemann surface X. A point p is a **base point** of the linear system Q if every divisor $E \in Q$ contains p ($E \geq p$). A linear system Q is said to be **base-point-free** if it has no base points.

Lemma

A point $p \in X$ is a base point of the linear system $Q \subset |D|$ defined by the vector subspace $V \subset \mathcal{O}_D(X)$ if and only if $V \subset \mathcal{O}_{D-p}(X)$. In particular p is a base point of the complete linear system $|D|$ if and only if $\mathcal{O}_{D-p}(X) = \mathcal{O}_D(X)$.

Definition

Let Q be a linear system on a Riemann surface X. A point p is a **base point** of the linear system Q if every divisor $E \in Q$ contains p ($E \geq p$). A linear system Q is said to be **base-point-free** if it has no base points.

Lemma

A point $p \in X$ is a base point of the linear system $Q \subset |D|$ defined by the vector subspace $V \subset \mathcal{O}_D(X)$ if and only if $V \subset \mathcal{O}_{D-p}(X)$. In particular p is a base point of the complete linear system $|D|$ if and only if $\mathcal{O}_{D-p}(X) = \mathcal{O}_D(X)$.

Proposition

Let D be a divisor on a compact Riemann surface X. Then a point $p \in X$ is a base point of the complete linear system $|D|$ if and only if $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_D(X)$. Hence $|D|$ is a base-point-free if and only if for every point $p \in X, \dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_D(X) - 1$.

Divisors and Maps to Projective Space

Defining a Holomorphic Map via Linear System

Proposition

Let $Q \subset |D|$ be a base-point-free linear system of dimension n on a compact Riemann surface X. Then there is a holomorphic map $\phi : X \to \mathbb{P}^n$ such that $Q = |\phi|$. Moreover ϕ is unique up to the choice of coordinates in $\mathbb{P}^n$.

Divisors and Maps to Projective Space

Defining a Holomorphic Map via Linear System

Proposition

Let $Q \subset |D|$ be a base-point-free linear system of dimension n on a compact Riemann surface X. Then there is a holomorphic map $\phi : X \to \mathbb{P}^n$ such that $Q = |\phi|$. Moreover ϕ is unique up to the choice of coordinates in $\mathbb{P}^n$.

Therefore we have a $1 - 1$ correspondence between base-point-free linear systems of dimension n on X and holomorphic map $\phi : X \to \mathbb{P}^n$ with nondegenerate image, up to linear coordinate changes.

Divisors and Maps to Projective Space

Removing the Base Point

Suppose that the complete linear system $|D|$ has base points. Let $F = \min\{E|E \in |D|\}$ be the minimum of all of the divisors in the linear system, the divisor F is the largest divisor that occurs in every divisor of $|D|$. It is obvious that the complete linear system $|D - F|$ then has no base points, and moreover $|D| = F + |D - F|$

Divisors and Maps to Projective Space

Removing the Base Point

Suppose that the complete linear system $|D|$ has base points. Let $F = \min\{E|E \in |D|\}$ be the minimum of all of the divisors in the linear system, the divisor F is the largest divisor that occurs in every divisor of $|D|$. It is obvious that the complete linear system $|D - F|$ then has no base points, and moreover $|D| = F + |D - F|$

Lemma

If F is the fixed divisor of the complete linear system $|D|$, then $\mathcal{O}_{D-F}(X) = \mathcal{O}_D(X)$.

Divisors and Maps to Projective Space

Removing the Base Point

Suppose that the complete linear system $|D|$ has base points. Let $F = \min\{E | E \in |D|\}$ be the minimum of all of the divisors in the linear system, the divisor F is the largest divisor that occurs in every divisor of $|D|$. It is obvious that the complete linear system $|D - F|$ then has no base points, and moreover $|D| = F + |D - F|$

Lemma

If F is the fixed divisor of the complete linear system $|D|$, then $\mathcal{O}_{D-F}(X) = \mathcal{O}_D(X)$.

Proof.

Clearly $F \geq 0$, we have that $D - F \leq D$ and so $\mathcal{O}_{D-F}(X) \subset \mathcal{O}_D(X)$. To see the reverse inclusion, let $f \in \mathcal{O}_D(X)$, so that $(f) + D \geq 0$. Therefore $(f) + D \in |D|$, and we may write $(f) + D = F + D'$ for some nonnegative divisor D'. Then $(f) + (D - F) = D' \geq 0$, so that $f \in \mathcal{O}_{D-F}(X)$. □

Divisors and Maps to Projective Space

Criteria for ϕ_D to be an Embedding

Lemma

Let X be a compact Riemann surface, and let D be a divisor on X with $|D|$ base-point-free. Fix a point $p \in X$. Then there is a basis $f_0, \ldots, f_n$ for $\mathcal{O}_D$ such that $\mathrm{ord}_p(f_0) = -D(p)$ and $\mathrm{ord}_p(f_i) > -D(p)$ for $i \geq 1$.

Divisors and Maps to Projective Space

Criteria for ϕ_D to be an Embedding

Lemma

Let X be a compact Riemann surface, and let D be a divisor on X with $|D|$ base-point-free. Fix a point $p \in X$. Then there is a basis $f_0, \ldots, f_n$ for $\mathcal{O}_D$ such that $\mathrm{ord}_p(f_0) = -D(p)$ and $\mathrm{ord}_p(f_i) > -D(p)$ for $i \geq 1$.

Proof.

Consider the codimension one subspace $\mathcal{O}_{D-p}(X)$ of $\mathcal{O}_D(X)$, and let $f_1, \ldots, f_n$ be a basis for $\mathcal{O}_{D-p}(X)$. Extend this to a basic for $\mathcal{O}_D(X)$ by adding a function f_0 in $\mathcal{O}_D(X) \setminus \mathcal{O}_{D-p}(X)$. Then $\mathrm{ord}_p(f_i) \geq -D(p) + 1 > -D(p)$ for every $i \geq 1$. □

Proposition

Let X be a compact Riemann surface, and let D be a divisor on X with $|D|$ is base point free. Fix distinct points p and q in X. Then $\phi_D(p) = \phi_D(q)$ if and only if $\mathcal{O}_{D-p-q}(X) = \mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$. Hence ϕ_D if and only if for every pair of distinct points p and q on X, we have
$\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$.

Proposition

Let X be a compact Riemann surface, and let D be a divisor on X with $|D|$ is base point free. Fix distinct points p and q in X. Then $\phi_D(p) = \phi_D(q)$ if and only if $\mathcal{O}_{D-p-q}(X) = \mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$. Hence ϕ_D if and only if for every pair of distinct points p and q on X, we have $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$.

We first prove an equivalence. Suppose $\phi_D(p) = \phi_D(q)$, and choose a basis for $\mathbb{P}^n$ satisfying the condition in the above lemma, so that $\phi_D(p) = \phi_D(q) = [1 : ... : 0]$, then this condition implies that $\mathrm{ord}_q(f_0) \leq \mathrm{ord}_q(f_i)$ for all $i > 0$. This implies that $f_i \in \mathcal{O}_{D-q}(X)$ for all $i \geq 1$ as linear independent meromorphic functions, and since D is base-point-free, $f_1, ..., f_n$ forms a basis for $\mathcal{O}_{D-q}(X)$, and therefore $\mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$. Similarly, if $\mathcal{O}_{D-q}(X) = \mathcal{O}_{D-q}(X)$, then $\phi_D(p) = \phi_D(q)$, using the previous basis.

Proposition

Let X be a compact Riemann surface, and let D be a divisor on X with $|D|$ is base point free. Fix distinct points p and q in X. Then $\phi_D(p) = \phi_D(q)$ if and only if $\mathcal{O}_{D-p-q}(X) = \mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$. Hence ϕ_D if and only if for every pair of distinct points p and q on X, we have $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$.

We first prove an equivalence. Suppose $\phi_D(p) = \phi_D(q)$, and choose a basis for $\mathbb{P}^n$ satisfying the condition in the above lemma, so that $\phi_D(p) = \phi_D(q) = [1 : \ldots : 0]$, then this condition implies that $\mathrm{ord}_q(f_0) \leq \mathrm{ord}_q(f_i)$ for all $i > 0$. This implies that $f_i \in \mathcal{O}_{D-q}(X)$ for all $i \geq 1$ as linear independent meromorphic functions, and since D is base-point-free, $f_1, \ldots, f_n$ forms a basis for $\mathcal{O}_{D-q}(X)$, and therefore $\mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X)$. Similarly, if $\mathcal{O}_{D-q}(X) = \mathcal{O}_{D-q}(X)$, then $\phi_D(p) = \phi_D(q)$, using the previous basis.

This says that every function f in $\mathcal{O}_D(X)$ with $\mathrm{ord}_p(f) > -D(p)$ also satisfies $\mathrm{ord}_q(f) > -D(q)$. Hence $\mathcal{O}_{D-p}(X) \subset \mathcal{O}_{D-p-q}(X)$, since p and q are distinct. This implies that

$$\mathcal{O}_{D-p-q}(X) = \mathcal{O}_{D-p}(X) = \mathcal{O}_{D-q}(X).$$

Since $|D|$ is base-point-free, we have that $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_{D-q}(X) = \dim \mathcal{O}_D(X) - 1.$ Therefore $\dim \mathcal{O}_{D-p-q}(X)$ is either $\dim \mathcal{O}_D(X) - 1$ or $\dim \mathcal{O}_D(X) - 2.$ If ϕ_D is $1-1,$ then by the first part we see that $\mathcal{O}_{D-p-q}(X)$ is a proper subspace of $\mathcal{O}_{D-p}(X)$ for all p and $q,$ and so must have dimension equal to $\dim \mathcal{O}_D(X) - 2.$

Since $|D|$ is base-point-free, we have that $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_{D-q}(X) = \dim \mathcal{O}_D(X) - 1.$ Therefore $\dim \mathcal{O}_{D-p-q}(X)$ is either $\dim \mathcal{O}_D(X) - 1$ or $\dim \mathcal{O}_D(X) - 2.$ If ϕ_D is $1-1,$ then by the first part we see that $\mathcal{O}_{D-p-q}(X)$ is a proper subspace of $\mathcal{O}_{D-p}(X)$ for all p and $q,$ and so must have dimension equal to $\dim \mathcal{O}_D(X) - 2.$
Conversely, if the dimension always does drop by $2,$ then the tower of subspaces $\mathcal{O}_{D-p-q}(X) \subset \mathcal{O}_{D-p}(X) \subset \mathcal{O}_D(X)$ must all be distinct for every p and $q,$ so that ϕ_D is $1-1.$

Proposition

Let X be a compact Riemann surface, and let D be a divisor on X whose linear system $|D|$ has no base points. Then ϕ_D is a $1-1$ holomorphic map and an isomorphism onto its image (which is a holomorphically embedded Riemann surface in $\mathbb{P}^n$), if and only if for every p and q in X, we have $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2.$

Proposition

Let X be a compact Riemann surface, and let D be a divisor on X whose linear system $|D|$ has no base points. Then ϕ_D is a $1-1$ holomorphic map and an isomorphism onto its image (which is a holomorphically embedded Riemann surface in $\mathbb{P}^n$), if and only if for every p and q in X, we have $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2.$

Proof.

We first prove the if statement. In order to define the morphism $\phi_D = [f_0 : ... : f_n]$, we choose a basis $f_2, ..., f_n \in \mathcal{O}_{D-2p}$, and we let

$$f_1 \in \mathcal{O}_{D-p}(X); f_0 \in \mathcal{O}_D(X) \setminus \mathcal{O}_{D-p}(X).$$

so that $\mathrm{ord}_p(f_1) = -D(p) + 1$, and therefore $\mathrm{ord}_p(f_1/f_0) = 1$, and $\mathrm{ord}_p(f_i/f_0) > 2$ for $i > 2$, so that applying the inverse function theorem, we see that the image of ϕ_D has the local coordinate f_1/f_0 at p, and using injectivity, we get that ϕ_D is an **embedding**.

Proposition

Let X be a compact Riemann surface, and let D be a divisor on X whose linear system $|D|$ has no base points. Then ϕ_D is a $1-1$ holomorphic map and an isomorphism onto its image (which is a holomorphically embedded Riemann surface in $\mathbb{P}^n$), if and only if for every p and q in X, we have $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$.

Proof.

We first prove the if statement. In order to define the morphism $\phi_D = [f_0 : ... : f_n]$, we choose a basis $f_2, ..., f_n \in \mathcal{O}_{D-2p}$, and we let

$$f_1 \in \mathcal{O}_{D-p}(X); f_0 \in \mathcal{O}_D(X) \setminus \mathcal{O}_{D-p}(X).$$

so that $\mathrm{ord}_p(f_1) = -D(p) + 1$, and therefore $\mathrm{ord}_p(f_1/f_0) = 1$, and $\mathrm{ord}_p(f_i/f_0) > 2$ for $i > 2$, so that applying the inverse function theorem, we see that the image of ϕ_D has the local coordinate f_1/f_0 at p, and using injectivity, we get that ϕ_D is an **embedding**.

The holomorphic map ϕ_D is an embedding if and only if there is a function in $\mathcal{O}_{D-p}(X)$ but not in $\mathcal{O}_{D-2p}(X)$. □

Definition

A divisor D such that $|D|$ has no base points and ϕ_D is an embedding is called a **very ample divisor**.

Algebraic Curve

Algebraic Curve

Definition

Let f be a meromorphic on a Riemann surface X. The function f has **multiplicity one** at a point $p \in X$ if either f is holomorphic at p and $\mathrm{ord}_p(f - f(p)) = 1$, or f has a simple pole at p.

Algebraic Curve

Algebraic Curve

Definition

Let f be a meromorphic on a Riemann surface X. The function f has **multiplicity one** at a point $p \in X$ if either f is holomorphic at p and $\mathrm{ord}_p(f - f(p)) = 1$, or f has a simple pole at p.

Definition

Let S be a set of meromorphic functions on a compact Riemann surface X. We say that S **separates points** of X if for every pair of distinct points p and q in X there is a meromorphic function $f \in S$ such that $f(p) \neq f(q)$. We say that S **seperates tangents** of X if for every point $p \in X$ there is a meromorphic fucntion $f \in S$ which has multiplicity one at p. A compact Riemann surface X is an **algebraic curve** if the field $\mathscr{M}(X)$ of global meromorphic function functions separates the points and tangents of X.

Algebraic Curve

Example of Algebraic Curve

Example

i. The Riemann sphere $\mathbb{P}^1$ is an algebraic curve.
ii. Any complex torus $\mathbb{C}/L$ is an algebraic curve.
iii. Any smooth projective plane curve is algebraic curve.
iv. Any smooth projective curve in $\mathbb{P}^n$ is an algebraic curve.

Algebraic Curve

Example of Algebraic Curve

Example

i. The Riemann sphere $\mathbb{P}^1$ is an algebraic curve.
ii. Any complex torus $\mathbb{C}/L$ is an algebraic curve.
iii. Any smooth projective plane curve is algebraic curve.
iv. Any smooth projective curve in $\mathbb{P}^n$ is an algebraic curve.

Theorem

Every compact Riemann surface is an algebraic curve.

Theorem

Every compact Riemann surface is an algebraic curve.

Theorem

Every compact Riemann surface is an algebraic curve.

Proof.

First we show that $\mathscr{M}(X)$ separates the points of X. Fix two points p and q on X, and consider divisor $D = (g+1)p$. By Riemann Roch theorem, we see that $\dim \mathcal{O}_D(X) \geq \deg(D) + 1 - g = 2$. Hence there is a nonconstant function $f \in \mathcal{O}_D(X)$. This function f must have a pole, and the only poles allowed are at p, so f has a pole at p and no other poles. In particular f does not have a pole at q, and f then separates p and q.

Theorem

Every compact Riemann surface is an algebraic curve.

Proof.

First we show that $\mathscr{M}(X)$ separates the points of X. Fix two points p and q on X, and consider divisor $D = (g+1)p$. By Riemann Roch theorem, we see that $\dim \mathcal{O}_D(X) \geq \deg(D) + 1 - g = 2$. Hence there is a nonconstant function $f \in \mathcal{O}_D(X)$. This function f must have a pole, and the only poles allowed are at p, so f has a pole at p and no other poles. In particular f does not have a pole at q, and f then separates p and q. Secondly we show that $\mathscr{M}(X)$ separates the tangents of X. Fix a point $p \in X$, and consider the divisor $D_n = np$. For large $n, \dim \mathcal{O}_{D_n}(X) = n + 1 - g$; hence there are functions in $\mathcal{O}_{D_{n+1}}(X)$ which are not in $\mathcal{O}_{D_n}(X)$ for large n. This implies that for large n, there are functions f_n with a pole of order exactly n at p and no other poles. The ratio f_n/f_{n+1} then has a simple zero at p. □

Lemma

If D is a divisor of Riemann surface X, and p some point of X, then we have the inequality $\dim \mathcal{O}_D(X) \geq \dim \mathcal{O}_{D-p}(X) \geq \dim \mathcal{O}_D(X) - 1.$

Lemma

If D is a divisor of Riemann surface X, and p some point of X, then we have the inequality $\dim \mathcal{O}_D(X) \geq \dim \mathcal{O}_{D-p}(X) \geq \dim \mathcal{O}_D(X) - 1.$

Lemma

Suppose $\phi : X \to \mathbb{P}^n$ is a holomorphic map with a smooth projective curve Y as the image. If D is a very ample divisor on X, so that ϕ_D is a holomorphic embedding of X into $\mathbb{P}^n$, then

$$\deg(\phi(X)) = \deg(D).$$

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Criterion for a Divisor to be Very Ample

Proposition

Let X be an algebraic curve of genus g. Then any divisor D with $\deg(D) \geq 2g + 1$ is very ample, that is, the complete linear system $|D|$ has no base points and the associated holomorphic map ϕ_D to projective space is a holomorphic embedding onto a smooth projective curve of degree equal to $\deg(D)$.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Criterion for a Divisor to be Very Ample

Proposition

Let X be an algebraic curve of genus g. Then any divisor D with $\deg(D) \geq 2g + 1$ is very ample, that is, the complete linear system $|D|$ has no base points and the associated holomorphic map ϕ_D to projective space is a holomorphic embedding onto a smooth projective curve of degree equal to $\deg(D)$.

Proof.

We need to check that $\dim \mathcal{O}_{D-p-q}(X) = \dim \mathcal{O}_D(X) - 2$ for any points p and q on X. Since both D and $D - p - q$ have degree at least $2g - 1$, we have that $H^1(X, \mathcal{O}_D) = H^1(X, \mathcal{O}_{D-p-q}) = 0$, and

$$\dim \mathcal{O}_D(X) = \deg D + 1 - g \quad \text{and} \quad \dim \mathcal{O}_{D-p-q}(X) = \deg(D - p - q) + 1 - g$$

□

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Every Algebraic Curve is Projective

Proposition

Every algebraic curve X can be holomorphically embedded into projective space.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Every Algebraic Curve is Projective

Proposition

Every algebraic curve X can be holomorphically embedded into projective space.

Proof.

Pick any point p and use $D = (2g + 1)p$. □

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Zero

Lemma

Let X be a compact Riemann surface. Then if $\dim \mathcal{O}_p(X) > 1$ we get $X \cong \mathbb{P}^1$.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Zero

Lemma

Let X be a compact Riemann surface. Then if $\dim \mathcal{O}_p(X) > 1$ we get $X \cong \mathbb{P}^1$.

Corollary

If D is a divisor of degree 2 such that $\dim \mathcal{O}_D(X) = 2$ on a genus $g \geq 1$ curve, then $|D|$ is base point free.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Zero

Lemma

Let X be a compact Riemann surface. Then if $\dim \mathcal{O}_p(X) > 1$ we get $X \cong \mathbb{P}^1$.

Corollary

If D is a divisor of degree 2 such that $\dim \mathcal{O}_D(X) = 2$ on a genus $g \geq 1$ curve, then $|D|$ is base point free.

Proof.

Suppose D is degree 2 and $\dim \mathcal{O}_D(X) = 2$. If $|D|$ has a base point, then $\dim \mathcal{O}_{D-p}(X) = \dim \mathcal{O}_D(X)$ so that $\dim \mathcal{O}_{D-p}(X) = 2$. Since we can generate $|D|$ with the divisor $D = p + q$, if q is the base point this implies that $\dim \mathcal{O}_D(X) = 2 > 1$ so $X \cong \mathbb{P}^1$ by the above lemma, but this contradicts $g \geq 1$. □

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curve of Genus Zero are Isomorphic to the Riemann Sphere

Proposition

Let X be an algebraic curve of genus 0. Then X is isomorphic to the Riemann sphere $\mathbb{P}^1$.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curve of Genus Zero are Isomorphic to the Riemann Sphere

Proposition

Let X be an algebraic curve of genus 0. Then X is isomorphic to the Riemann sphere $\mathbb{P}^1$.

Proof.

Fix any point $p \in X$. Since the canonical divisor K on X has degree $2g - 2 = -2$, then the divisor $K - p$ has degree -3. This is strictly negative, so $\mathrm{O}_{K-D}(X) = 0$ Applying Riemann-Roch to the divisor p, we find that

$$\dim \mathcal{O}_D(X) = \deg(p) + 1 - g + \dim \mathcal{O}_{K-p}(X) = 2.$$

□

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus One are Cubic Plane Curves

Proposition

Every algebraic curve of genus one is isomorphic to a smooth projective plane cubic curve.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus One are Cubic Plane Curves

Proposition

Every algebraic curve of genus one is isomorphic to a smooth projective plane cubic curve.

Proof.

By the criterion for a divisor to be very ample, we see that any divisor of degree 3 is very ample. Since by the Riemann-Roch, $\dim \mathcal{O}_D(X) = 3$ if $\deg(D) = 3$, we see that the holomorphic map ϕ_D would map X to the plane $\mathbb{P}^2$. Since $\deg(\phi_D(X)) = \deg(D) = 3$, the image is smooth cubic curve. □

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Two are Hyperelliptic

Definition

We define a **hyperelliptic curve** to be an compact Riemann surface Y such that there exists some holomorphic map $F : Y \to \mathbb{P}^1$ that is of degree 2.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Two are Hyperelliptic

Definition

We define a **hyperelliptic curve** to be an compact Riemann surface Y such that there exists some holomorphic map $F : Y \to \mathbb{P}^1$ that is of degree 2.

Theorem

Suppose X and Y are Riemann surfaces and $f : X \to Y$ is proper non-constant holomorphic map. Then there exists a natural number n such that f takes every value $c \in Y$, counting multiplicities, n times.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Two are Hyperelliptic

Proposition

Every algebraic curve Y of genus two is hyperelliptic.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Two are Hyperelliptic

Proposition

Every algebraic curve Y of genus two is hyperelliptic.

Proof.

Note that $\dim \mathcal{O}_K(Y) = 2$ and $\deg(K) = 2$, we must have $K = p + q$ for some points $p, q \in Y$. Let $f \in \mathcal{O}_K(Y)$ be nonconstant. Then f has either one or two poles, which must be at the points p or q. Suppose f has only one pole at p. Then we have that $f \in \mathcal{O}_p(Y)$, and therefore $\dim \mathcal{O}_p(Y) = 2$, which implies that Y is isomorphic to $\mathbb{P}^1$, a contradiction of genus. So f must have either a double pole or two single poles, which give us a morphism $F : Y \to \mathbb{P}^1$ of degree 2. □

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

The Canonical Map

Lemma

The canonical linear system $|K|$ on an algebraic curve X of genus $g \geq 1$ is base-point-free.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

The Canonical Map

Lemma

The canonical linear system $|K|$ on an algebraic curve X of genus $g \geq 1$ is base-point-free.

Proof.

Fix a point $p \in X$. We must show that $\mathcal{O}_{K-p}(X) \neq \mathcal{O}_K(X)$, and for this suffice to show that $\dim \mathcal{O}_{K-P}(X) = \dim \mathcal{O}_K(X) - 1 = g - 1$.
Now since $\dim \mathcal{O}_p(X) = 1$, we have using Riemann-Roch that

$$1 = \dim \mathcal{O}_p(X) = \mathcal{O}_{K-p}(X) + \deg(p) + 1 - g,$$

which gives $\dim \mathcal{O}_{K-p} = g - 1$ as desired. □

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curve of Genus Three

Proposition

Let $g \geq 3$, then ϕ_K is an embedding if and only if X is not hyperelliptic.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curve of Genus Three

Proposition

Let $g \geq 3$, then ϕ_K is an embedding if and only if X is not hyperelliptic.

Proof.

By Riemann-Roch theorem, we have

$$\dim \mathcal{O}_{K-p-q}(X) = \deg(K-p-q)+1-g+\dim \mathcal{O}_{p+q}(X) = g-3+\dim \mathcal{O}_{p+q}(X),$$

so that ϕ_K fails to an embedding if and only if for some points p, q of X, $\dim \mathcal{O}_{p+q}(X) = 2$ ($\dim \mathcal{O}_{K-p-q}(X) = \dim \mathcal{O}_K(X) - 2$).

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curve of Genus Three

Proposition

Let $g \geq 3$, then ϕ_K is an embedding if and only if X is not hyperelliptic.

Proof.

By Riemann-Roch theorem, we have

$$\dim \mathcal{O}_{K-p-q}(X) = \deg(K-p-q)+1-g+\dim \mathcal{O}_{p+q}(X) = g-3+\dim \mathcal{O}_{p+q}(X),$$

so that ϕ_K fails to an embedding if and only if for some points p, q of $X, \dim \mathcal{O}_{p+q}(X) = 2$ $(\dim \mathcal{O}_{K-p-q}(X) = \dim \mathcal{O}_K(X) - 2)$.
If this happens, then any nonconstant function $f \in \mathcal{O}_{p+q}(X)$ gives a degree map to the Riemann sphere, and so X is hyperelliptic. Conversely, if X is hyperelliptic and $\pi : X \to \mathbb{P}^1$ is the degree 2 mapping, then the inverse image divisor $p + q$ of ∞ has degree 2 and $\dim \mathcal{O}_{p+q}(X) = 2$. □

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Three

Corollary

If X is not hyperelliptic curve, then ϕ_K embeds X in $\mathbb{P}^{g-1}$ as curve of degree $2g - 2$.

Classifications of Algebraic Curves of Genus $g = 0, 1, 2, 3$

Curves of Genus Three

Corollary

If X is not hyperelliptic curve, then ϕ_K embeds X in $\mathbb{P}^{g-1}$ as curve of degree $2g - 2$.

Corollary

If $g = 3$ and X is not hyperelliptic, then X is a plane curve of degree 4.

References

References I

O. Forster, *Lectures on Riemann Surfaces*, Springer, 1981.

R. Miranda, *Algebraic Curves and Riemann Surfaces*, AMS, 1995.

THANK YOU FOR LISTENING