# Proper Morphisms

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# Content

# 1. Universally closed morphism

| **Local on the base** | **Stable under composition** | **Stable under base change** |
|---|---|---|
| surjective | surjective | surjective |
| open immersion | open immersion | open immersion |
| closed immersion | closed immersion | closed immersion |
| immersion | immersion | immersion |
| closed | closed | |
| injective | injective | |
| open | open | |
| bijective | bijective | |
| homeomorphism | homeomorphism | |

## Definition

Let **P** be a property of morphisms of schemes. We say that a morphism $f\colon X \to S$ is **universally P** if it satisfies **P** and for any morphism $S' \to S$ the corresponding morphism $X \times_S S' \to S'$ also satisfies **P**.

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

- Let **P** be a property that is not necessarily stable under base change and let **P'** be the property "universally **P**". Then **P'** is stable under base change.
- If **P** is stable under composition (resp. local on the base), the same is true for **P'**.

| **Local on the base** | **Stable under composition** | **Stable under base change** |
|---|---|---|
| closed | closed | |
| injective | injective | |
| universally closed | universally closed | universally closed |
| universally injective | universally injective | universally injective |

# Morphism of finite type

**Definition**

A morphism $f : X \to Y$ is called of finite type, if $f$ is quasi-compact and the following equivalent conditions are satisfied:

1. For every affine open subscheme $\mathcal{V}$ of $Y$ and every affine open subscheme $\mathcal{U}$ of $f^{-1}(\mathcal{V})$, the $\mathcal{O}_X(\mathcal{U})$ is finitely generated algeba over $\mathcal{O}_Y(\mathcal{V})$.
2. There exist a covering $Y = \bigcup_I \mathcal{V}_i$ by open affine subschemes $\mathcal{V}_i = Spec\ A_i$, and for each $i$ a covering $f^{-1}(\mathcal{V}_i) = \bigcup_J \mathcal{U}_{ij}$ by open affine subschemes $\mathcal{U}_{ij} = Spec\ B_{ij}$ of $X$, such that for all $i, j$ the $B_{ij}$ is finitely generated algebra over $A_i$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{U} & \xrightarrow{f|_{\mathcal{U}}} & \mathcal{V} \end{array} \qquad\qquad \mathcal{O}_X(\mathcal{U}) \longleftarrow \mathcal{O}_Y(\mathcal{V})$$

## Proposition

1. *Morphisms of finite type are local on the base.*
2. *Morphisms of finite type are stable under base change.*
3. *Morphisms of finite type are stable under composition.*

## 2. Proper morphisms

**Definition**

A continuous function $f\colon X \to Y$ is called **proper** if inverse image under $f$ of every quasi-compact subset of $Y$ is quasi-compact.

**Definition**

A continuous function $f\colon X \to Y$ is called **proper** if the mapping $f \times 1_Z\colon X \times Z \to Y \times Z$ is closed, for every topological space $Z$.

*(N. Bourbaki, General topology, Chapters 1, 10.1 definition 1.)*

**Proposition**

*Let $f\colon X \to Y$ be a continuous map. Suppose $X$ is Hausdorff and $Y$ is loaclly compact. Then $f$ is proper of and only if $f \times 1_Z\colon X \times Z \to Y \times Z$ is closed, for every topological space $Z$.*

## Proposition

*A topological space $X$ is quasi-compact if and only if the mapping $X \to \bullet$ is proper, where $\bullet$ denotes a space consisting of a single point.*

## Definition (Proper morphism)

A morphism of schemes $f : X \to Y$ is called **proper** (or $X$ is called proper over $Y$) if $f$ is separated, universally closed and of finite type.

# Example

Let $k$ be a field. Consider morphism $\mathbb{A}^1_k \to Spec\ k$ where $\mathbb{A}^1_k = Spec\ k[x]$ is affine line.
This morphism is not proper. Indeed, $\mathbb{A}^1_k \to Spec\ k$ is separated, of finite type, and closed. However,

$$\begin{array}{ccc} \mathbb{A}^1_k \times_k \mathbb{A}^1_k & \longrightarrow & \mathbb{A}^1_k \\ \downarrow & \square & \downarrow \\ \mathbb{A}^1_k & \longrightarrow & Spec\ k \end{array}$$

$$\begin{array}{rcl} Spec\ k[x,y] = \mathbb{A}^2_k & \longrightarrow & \mathbb{A}^1_k \\ V(xy-1) & \longmapsto & \mathbb{A}^1_k - \{0\} \ \text{ is not closed} \end{array}$$

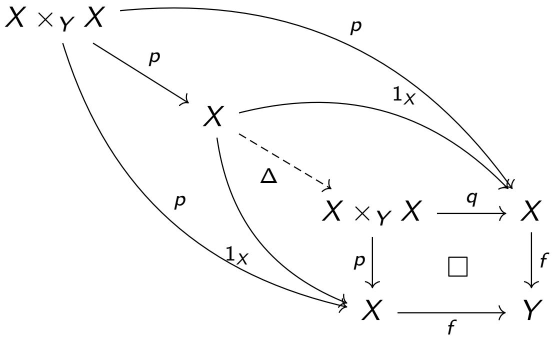
# 3. Some properties of proper morphisms

**Proposition**

1. *Any closed immersion is proper.*
2. *Proper morphisms are local on the base.*
3. *Proper morphisms are stable under base change.*
4. *Proper morphisms are stable under composition.*

**Proof:**

(1) <u>*Fact* :</u> Any closed immersion is monomorphism.




Since $f$ is monomorphism, we have $p = q$.

$$p \circ \Delta = 1_X, \ \Delta \circ p = 1_{X \times_Y X}$$

We get $\Delta$ is isomorphism and hence $f : X \to Y$ is separated.
As being closed immersion is stable under base change, $f : X \to Y$ is universally closed.

## Definition

A morphism $f: X \to Y$ is called of finite type, if $f$ is quasi-compact and the following equivalent conditions are satisfied:

1. For every affine open subscheme $\mathcal{V}$ of $Y$ and every affine open subscheme $\mathcal{U}$ of $f^{-1}(\mathcal{V})$, the $\mathcal{O}_X(\mathcal{U})$ is finitely generated algeba over $\mathcal{O}_Y(\mathcal{V})$.
2. There exist a covering $Y = \bigcup_I \mathcal{V}_i$ by open affine subschemes $\mathcal{V}_i = Spec\ A_i$, and for each $i$ a covering $f^{-1}(\mathcal{V}_i) = \bigcup_J \mathcal{U}_{ij}$ by open affine subschemes $\mathcal{U}_{ij} = Spec\ B_{ij}$ of $X$, such that for all $i, j$ the $B_{ij}$ is finitely generated algebra over $A_i$.

- A closed subset of a quasi-compact topological space is quasi-compact.
- Every closed subscheme of $Spec\ A$ is of the form $Spec\ A/I$ for some ideal $I$ of $A$.

## Corollary

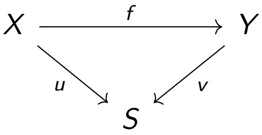
1. *If $f : X \to Y$ is a morphism of $S$-schemes such that $X$ is proper over $S$ and such that $Y$ is separated over $S$. Then $f$ is proper.*
2. *Let $f : X \to Y$ be a surjective morphism of $S$-schemes. Suppose that $Y$ is separated of finite type over $S$ and that $X$ is proper over $S$. Then $Y$ is proper over $S$.*

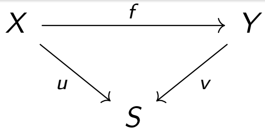
## Proposition (Topological version)

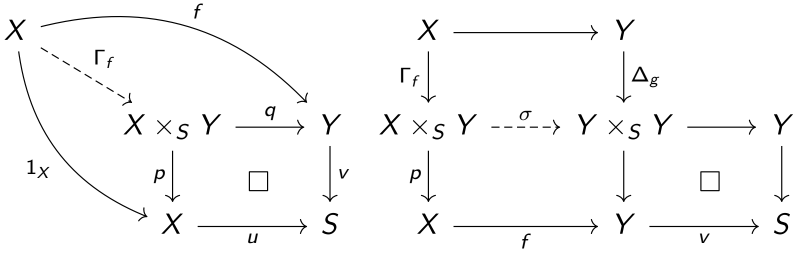
*Let $f : X \to Y$ be a continuous map such that $X$ is quasi-compact and $Y$ is Hausdorff. Then $f$ is proper.*

(1)

$$X \xrightarrow{f} Y$$






As being a proper morphism is stable under base change, $q$ is proper morphism.
Furthermore, as being a closed immersion is stable under base change, $\Gamma_f$ is closed immersion and hence proper. This proves (1).

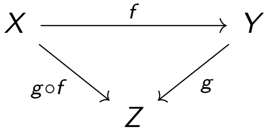
## Corollary

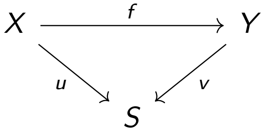
1. *If $f : X \to Y$ is a morphism of $S$-schemes such that $X$ is proper over $S$ and such that $Y$ is separated over $S$. Then $f$ is proper.*
2. *Let $f : X \to Y$ be a surjective morphism of $S$-schemes. Suppose that $Y$ is separated of finite type over $S$ and that $X$ is proper over $S$. Then $Y$ is proper over $S$.*

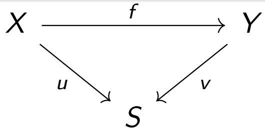
## Proposition (Topological version)

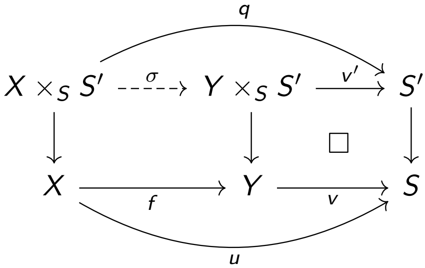
*If $g \circ f$ is proper and $f$ is surjective, then $g$ is proper.*

(2)






Since $X \to S$ is closed and $X \to Y$ is surjective, $Y \to S$ is closed. As being surjective is stable under base change, $X \times_S S' \to Y \times_S S'$ is surjective. Furthermore, $q$ is closed as $u$ is universally closed. Thus, $Y \times_S S' \to S$ is closed. Therefore, $Y \to S$ is universally closed and hence proper. □

## Proposition

*Let $f : X \to Y$ be a $S$-morphism of $S$-schemes. Suppose $X, Y$ are separated and of finite type over $S$. Then $f$ is proper if and only if $f \times 1_{S'} : X \times_S S' \to Y \times_S S'$ is closed, for every $S$-scheme $S'$.*

*(A.Grothendieck, J.A.Dieudonné, EGA II, corollary 5.4.8.)*

## Proposition (Topological version)

*Let $f : X \to Y$ be a continuous map. Suppose $X$ is Hausdorff and $Y$ is loaclly compact. Then $f$ is proper of and only if $f \times 1_Z : X \times Z \to Y \times Z$ is closed, for every topological space $Z$.*

$$X \xrightarrow{\;f\;} Y$$

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & S & \end{array}$$

# 4. Valuative criterion of properness

**Theorem (Valuative criterion for properness - general version)**

*Let $f : X \to Y$ be a morphism of schemes. Then the following assertions are equivalent.*

1. *$f$ is proper.*
2. *$f$ is quasi-separated, of finite type and for all diagrams below where $A$ is a valuation ring and $K = Frac(A)$ there exists a unique lift of $u$.*

$$\begin{array}{ccc} Spec\ K & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & \overset{\overline{u}}{\nearrow} & \downarrow{\scriptstyle f} \\ Spec\ A & \xrightarrow[v]{} & Y \end{array}$$

For many applications the following Noetherian version is useful.

## Theorem (Valuative criterion for properness - Noetherian version)

*Let $f : X \to Y$ be a morphism of finite type and suppose that $X$ is Noetherian. Then the following assertions are equivalent.*

1. *$f$ is proper.*
2. *For all diagrams below where $A$ is a valuation ring and $K = Frac(A)$ there exists a unique lift of $u$.*

$$\begin{array}{ccc} Spec\ K & \xrightarrow{\ u\ } & X \\ {\scriptstyle i}\downarrow & \overset{\overline{u}}{\nearrow} & \downarrow{\scriptstyle f} \\ Spec\ A & \xrightarrow[\ v\ ]{} & Y \end{array}$$

**Application:** $\mathbb{P}^n_S \to S$ is proper, for every scheme $S$. Here $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$.

THANK YOU FOR YOUR LISTENING!