Modular Curves as Riemann Surfaces

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November, 2020

Topology

Topology on Modular Curves

Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup. The corresponding modular curve is the set of orbits

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The natural surjection

$$\pi : \mathbb{H} \to Y(\Gamma), \qquad \pi(\tau) = \Gamma\tau$$

gives $Y(\Gamma)$ the quotient topology.

Here are some claims:

1. π is an open mapping.
   Because for every U open in $\mathbb{H}$, one has

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5. $Y(\Gamma)$ is second countable.

Theorem (1)

*The action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ is **properly discontinuous**, i.e, given any $t_1, t_2 \in \mathbb{H}$ there exist neighborhoods U_1 of t_1 and U_2 of t_2 in $\mathbb{H}$ such that*

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Corollary

For any congruence subgroup Γ of $SL_2(\mathbb{Z})$, the modular curve $Y(\Gamma)$ is Hausdorff.

Proof.

Let $\pi(\tau_1) \neq \pi(\tau_2)$ be 2 distinct points in $Y(\Gamma)$. Take neighborhoods U_1 of τ_1, U_2 of τ_2 as in the previous theorem. Since $\gamma(\tau_1) \neq \tau_2$ for all $\gamma \in \Gamma$, then $\Gamma(U_1) \cap U_2 = \varnothing$ in $\mathbb{H}$. This implies $\pi(U_1) \cap \pi(U_2) = \varnothing$. □

To prove the theorem 1, let U_1' (resp. U_2') be any neighborhood of t_1 (resp. t_2) with compact closure in $\mathbb{H}$.

Lemma (1)

The inequality

$$\sup\{Im(\gamma(t)) : \gamma \in SL_2(\mathbb{Z}) \textit{ has bottom row } (c,d), t \in U_1'\} < \inf\{Im(t) : t \in U_2'\}$$

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Remark This lemma implies that:
If $\gamma \in SL_2(\mathbb{Z})$ satisfying $\gamma(U_1') \cap U_2' \neq \varnothing$ then the bottom row of γ has only finitely many choices.

Proof of lemma 1.

Observe that

$$Im(\gamma(t)) = \frac{Im(t)}{|ct+d|^2} \leq \frac{Im(t)}{c^2 Im(t)^2}, \forall \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}), \forall t \in U_1'.$$

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So as $c \to \infty$, $Im(\gamma(t)) \to 0$. Let $2\varepsilon = \inf\{Im(t) : t \in U_2'\} > 0$, there exists $N > 0$ such that whenever $|c| > N$, $Im(\gamma(t)) < \varepsilon$ for all $t \in U_1'$. It implies that the inequality holds true for all integer pairs (c, d) with $\gcd(c, d) = 1$ and $|c| > N$.

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Now suppose $|c| \leq N$, then $|ct|$ is bounded. Then $Im(\gamma(t)) \to 0$ as $d \to \infty$. So there exists $M > 0$ such that whenever $|d| > M$, $Im(\gamma(t)) < \varepsilon$ for all $t \in U_1'$. This means the inequality holds true for all integer pairs (c, d) with $\gcd(c, d) = 1$, $|c| \leq N$ and $|d| > M$.
From above, we deduces that the inequality holds true whenever either $|c| > N$ or $|d| > M$. □

Lemma (2)

For an integer pair (c, d) *with* $\gcd(c, d) = 1$*, the number of*

$$\gamma \in SL_2(\mathbb{Z}) \text{ with bottom row } (c, d) \text{ such that } \gamma(U_1') \cap U_2' \neq \varnothing$$

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Observe that the set of matrices $\gamma \in SL_2(\mathbb{Z})$ with bottom row (c, d) are

$$\left\{ \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} : k \in \mathbb{Z} \right\}$$

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where (a, b) is any particular pair such that $ad - bc = 1$. Thus

$$\gamma(U_1') \cap U_2' = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} U_1' + k \right) \cap U_2'$$

is empty for all but finitely many γ with bottom row (c, d). □

Then $F = \{\gamma \in SL_2(\mathbb{Z}) : \gamma(U_1') \cap U_2' \neq \varnothing, \gamma(t_1) \neq t_2\}$ is a finite set.

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For each $\gamma \in F$, there exists disjoint neighborhoods $U_{1,\gamma}$ of $\gamma(t_1)$ and $U_{2,\gamma}$ of t_2 in $\mathbb{H}$.
Define

$$U_1 = U_1' \cap \left(\bigcap_{\gamma \in F} \gamma^{-1}(U_{1,\gamma}) \right)$$

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Take any $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma(U_1) \cap U_2 \neq \varnothing$. To show $\gamma(t_1) = t_2$, it suffices to show $\gamma \notin F$. If $\gamma \in F$, then

$$\gamma^{-1}(U_{1,\gamma}) \supset U_1 \quad \text{and} \quad U_{2,\gamma} \supset U_2,$$

so $U_{1,\gamma} \cap U_{2,\gamma} \supset \gamma(U_1) \cap U_2 \neq \varnothing$, contradiction.

To summarize

1. $\pi : \mathbb{H} \to Y(\Gamma)$ is open.
2. $Y(\Gamma)$ is connected.
3. $Y(\Gamma)$ is Hausdorff.
4. $Y(\Gamma)$ is second countable.

Charts

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For a point $\pi(t) \in Y(\Gamma)$ where $t \in \mathbb{H}$, consider 2 cases:

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is a homeomorphism.
Then we can define

$$\varphi : \pi(U) \to U \subset \mathbb{C}, \qquad \varphi = (\pi|_U)^{-1}$$

as a local coordinate at $\pi(t)$.

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Example

Consider $\Gamma = SL_2(\mathbb{Z})$, $t = i$ and $\gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\gamma(i) = \frac{-1}{i} = i$.

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Let $\delta = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \in GL_2(\mathbb{C})$. We know that δ is a conformal map from $\mathbb{H}$ to the unit disc $\mathbb{D}$ which sends i to 0. By direct computation we have

$$\delta \cdot \gamma \cdot \delta^{-1} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

which acts as 180-degree rotation about 0 in the unit disc. We observe that any neighborhood about 0 contains pair of $\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$-equivalent points.

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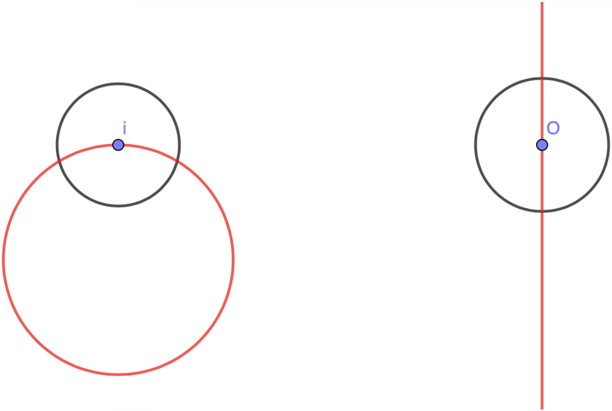
Then we deduce that any neighborhood of i contains pairs of γ-equivalent points. Thus cannot biject to a neighborhood of $\pi(i)$ in $Y(SL_2(\mathbb{Z}))$.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Definition

For each $t \in \mathbb{H}$, the **isotropy subgroup** of t is

$$\Gamma_t = \{\gamma \in \Gamma : \gamma(t) = t\}.$$

A point $t \in \mathbb{H}$ is an **elliptic point** for Γ if Γ_t is nontrivial as a group of transformations. The corresponding point $\pi(t)$ on $Y(\Gamma)$ is also called elliptic.

Proposition

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Thus each point $t \in \mathbb{H}$ has an associated positive integer

$$h_t = \left| \frac{\{\pm I\}\Gamma_t}{\{\pm I\}} \right| = \begin{cases} |\Gamma_t|/2 & \text{if } -I \in \Gamma_t, \\ |\Gamma_t| & \text{if } -I \notin \Gamma_t. \end{cases}$$

This h_t is called the **period** of t, and $h_t > 1$ only for elliptic points. h_t correctly counts the t-fixing transformations.

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1. If $t \in \mathbb{H}$ and $\gamma \in SL_2(\mathbb{Z})$ then

   the period of t under $\Gamma =$ the period of $\gamma(t)$ under $\gamma\Gamma\gamma^{-1}$.

   In particular, h_t depends only on Γt. So the period is well-defined on $Y(\Gamma)$.

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   In particular, h_t depends only on Γt. So the period is well-defined on $Y(\Gamma)$.

2. If Γ is normal in $SL_2(\mathbb{Z})$ then all points of $Y(\Gamma)$ over a point of $Y(SL_2(\mathbb{Z}))$ have the same period.

   $$\text{If } SL_2(\mathbb{Z})t_1 = SL_2(\mathbb{Z})t_2 \text{ then } \Gamma t_1 \text{ and } \Gamma t_2 \text{ have the same period.}$$

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2. The isotropy subgroup of 0 in the conjugated transformation group is the conjugate of the isotropy subgroup of t, i.e

$$(\delta_t\{\pm I\}\Gamma\delta_t^{-1})_0/\{\pm I\} = \delta_t(\{\pm I\}\Gamma_t/\{\pm I\})\delta_t^{-1},$$

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5. The map δ_t is "straightening" neighborhoods of t to neighborhoods of 0 in the sense that after the map, equivalent points are spaced apart by fixed angles.

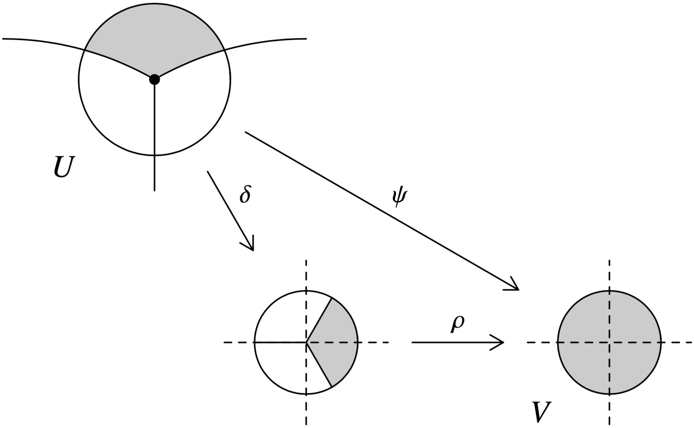

Figure 2.2. Local coordinates at an elliptic point

Now given any point $\pi(t) \in Y(\Gamma)$, take a neighborhood U of t such that

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Such a neighborhood exists by theorem 1, and has no elliptic points except possibly t.

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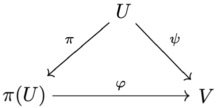
To see this, observe that

$$\pi(t_1) = \pi(t_2) \Leftrightarrow t_1 \in \Gamma t_2 \Leftrightarrow t_1 \in \Gamma_t t_2 \Leftrightarrow \delta(t_1) \in (\delta\Gamma_t\delta^{-1})(\delta(t_2)) \Leftrightarrow \delta(t_1) = \mu_h^d(\delta(t_2)),$$

for some integer d, $\mu_h = e^{2\pi i/h}$ since $\delta\Gamma_t\delta^{-1}$ is a cyclic transformation group of h rotations. So

$$\pi(t_1) = \pi(t_2) \Leftrightarrow (\delta(t_1))^h = (\delta(t_2))^h \Leftrightarrow \psi(t_1) = \psi(t_2).$$

Thus there exists an injection $\varphi : \pi(U) \to V$ such that the diagram



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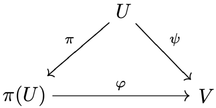


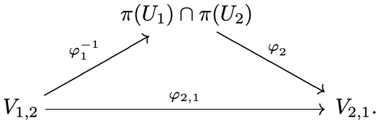
commutes. Also φ surjects since ψ surjects, and $\varphi : \pi(U) \to V$ is a homeomorphism. So φ is a local coordinate and $\pi(U)$ is a coordinate neighborhood about $\pi(t)$ in $Y(\Gamma)$.

Charts

Holomorphicity of Transition Maps

Given overlapping $\pi(U_1)$ and $\pi(U_2)$. Let

$$V_{1,2} = \varphi_1(\pi(U_1) \cap \pi(U_2)), \quad V_{2,1} = \varphi_2(\pi(U_1) \cap \pi(U_2)), \quad \varphi_{2,1} = \varphi_2 \circ \varphi_1^{-1}|_{V_{1,2}}.$$



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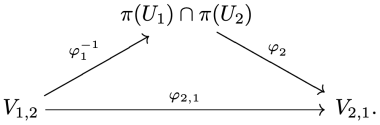

For each $x \in \pi(U_1) \cap \pi(U_2)$ it suffices to check holomorphy in some neighborhood of $\varphi_1(x)$ in $V_{1,2}$.

Write $x = \pi(t_1) = \pi(t_2)$ with $t_1 \in U_1, t_2 \in U_2$ and $t_2 = \gamma(t_1)$ for some $\gamma \in \Gamma$. Let $U_{1,2} = U_1 \cap \gamma^{-1}(U_2)$, then $\pi(U_{1,2})$ is a neighborhood of x and so $\varphi_1(\pi(U_{1,2}))$ is a neighborhood of $\varphi_1(x)$ in $V_{1,2}$.

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We'll prove for the case $\varphi_1(x) = 0$. So the first straightening map is $\delta_1 = \delta_{t_1}$. Let $q = \varphi_1(x') \in \varphi_1(\pi(U_{1,2}))$, one has

$$q = \varphi_1(\pi(t')) = \psi_1(t') = (\delta_1(t'))^{h_1}, \quad \text{for some } t' \in U_{1,2}$$

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Let $\tilde{t_2} \in U_2$ be the point such that $\psi(\tilde{t_2}) = 0$ and let h_2 be its period. Then

$$\begin{aligned}
\varphi_{2,1}(q) &= \varphi_2(x') \\
&= \varphi_2(\pi(t')) \\
&= \varphi_2(\pi(\gamma(t')) \\
&= \psi_2(\gamma(t')) \quad \text{which is defined since } \gamma(t') \in U_2 \\
&= (\delta_2(\gamma(t')))^{h_2} \\
&= ((\delta_2\gamma\delta_1^{-1})(\delta_1(t')))^{h_2} \\
&= ((\delta_2\gamma\delta_1^{-1})(q^{1/h_1}))^{h_2}.
\end{aligned}$$

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We have the following diagrams

$$0 \xmapsto{\delta_1^{-1}} t_1 \xmapsto{\gamma} t_2 \xmapsto{\delta_2} 0 \qquad , \qquad \infty \xmapsto{\delta_1^{-1}} \bar{t_1} \xmapsto{\gamma} \bar{t_2} \xmapsto{\delta_2} \infty.$$

This shows $\delta_2\gamma\delta_1^{-1} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ for some nonzero $\alpha, \beta \in \mathbb{C}$.

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The formula for the transition map in this case is

$$\varphi_{2,1}(q) = \left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (q^{1/h}) \right)^h = \left(\frac{\alpha}{\beta} \right)^h q$$

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The general case is quite similar.

Elliptic Points

Elliptic Points

In this section we will show the remaining proposition: the isotropy subgroup Γ_t is finite and cyclic. Then we will discover some properties of elliptic points of a congruence subgroup Γ. It turns out that the set of elliptic points is quite "small".

Example

Consider the case $Y(1) = SL_2(\mathbb{Z}) \setminus \mathbb{H}$. Let $\mathcal{D}$ be the set

$$\mathcal{D} = \{t \in \mathbb{H} : |Re(t)| \leq 1/2, |t| \geq 1\}.$$

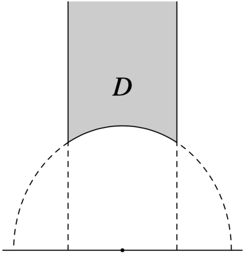

Figure 2.3. The fundamental domain for $SL_2(\mathbf{Z})$

Lemma

The map $\pi : \mathcal{D} \to Y(1)$ *surjects, where* π *is the natural projection* $\pi(t) = SL_2(\mathbb{Z})t$.

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Lemma

Suppose $t_1 \neq t_2$ are distinct points in $\mathcal{D}$ such that $t_2 = \gamma(t_1)$ for some $\gamma \in SL_2(\mathbb{Z})$. Then either

1. $Re(t_1) = \pm 1/2$ *and* $t_2 = \mp 1$, *or*
2. $|t_1| = 1$ *and* $t_2 = -1/t_1$.

Returning to elliptic points, suppose $t \in \mathbb{H}$ is fixed by a nontrivial transformation $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. Then

$$at + b = ct^2 + dt.$$

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Since $t \in \mathbb{H}$, we can show that $c \neq 0$ and $|a + d| < 2$. Then $a + d \in \{-1, 0, 1\}$. The characteristic polynomial of γ is given by

$$(a - x)(d - x) - bc = x^2 - (a + d)x + 1.$$

So the characteristic polynomial is $x^2 + 1$ or $x^2 \pm x + 1$.

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Then γ has order $1, 2, 3, 4$ or 6 as a matrix. Observe that orders 1 and 2 give the identity transformations. The following proposition will discribe all nontrivial fixing transformations.

Proposition

Let $\gamma \in SL_2(\mathbb{Z})$.

1. *If* γ *has order* 3 *then* γ *is conjugate to* $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^{\pm 1}$ *in* $SL_2(\mathbb{Z})$.
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4. *For each elliptic point* t *of* $SL_2(\mathbb{Z})$ *the isotropy subgroup* $SL_2(\mathbb{Z})_t$ *is finite cyclic.*

Corollary

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. The modular curve $Y(\Gamma)$ has finitely many elliptic points. For each elliptic point t of Γ, the isotropy subgroup Γ_t is finite cyclic.

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$$SL_2(\mathbb{Z}) = \bigsqcup_{j=1}^{d} \Gamma\gamma_j$$

then the set of elliptic points of $Y(\Gamma)$ is a subset of

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For each $t \in \mathbb{H}$, observe that

$$\Gamma_t \text{ is a subgroup of } SL_2(\mathbb{Z})_t.$$

Then Γ_t is finite cyclic. □

Cusps

Compactify a Modular Curve

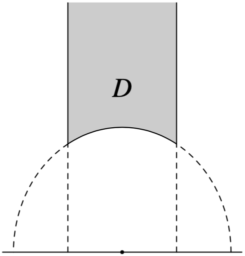

Figure 2.3. The fundamental domain for $SL_2(\mathbf{Z})$

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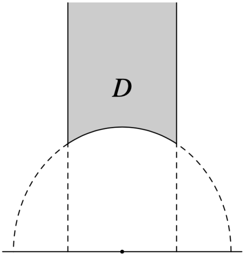

Figure 2.3. The fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$

The picture suggests that the modular curve $Y(\Gamma)$ can be compactified by adjoining all the cusps.

Let $\mathcal{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and take the extended quotient

$$X(\Gamma) = \Gamma \setminus \mathcal{H}^* = Y(\Gamma) \cup \Gamma \setminus (\mathbb{Q} \cup \{\infty\}).$$

The points Γs in $\Gamma \setminus \mathbb{Q} \cup \{\infty\}$ are also called the **cusps** of $X(\Gamma)$.

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Remark The action of Γ on $\mathbb{Q} \cup \{\infty\}$ is induced from the action of $GL_2^+(\mathbb{Q})$ (the group of 2×2 matrices with positive determinant and rational entries) on $\mathbb{Q} \cup \{\infty\}$ given by

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1. $SL_2(\mathbb{Z})$ acts transitively on $\mathbb{Q} \cup \{\infty\}$.
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Lemma

The modular curve $X(1) = SL_2(\mathbb{Z}) \setminus \mathcal{H}^*$ has one cusp. For any subgroup Γ of $SL_2(\mathbb{Z})$ the modular curve $X(\Gamma)$ has finitely many cusps.

Cusps

Topology on $X(\Gamma)$

The usual topology on $\mathcal{H}^*$ contains too many points of $\mathbb{Q} \cup \{\infty\}$ in each neighborhood to make the quotient $X(\Gamma)$ Hausdorff. So we need to define a new topology on $\mathcal{H}^*$.

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Adjoin to the usual open sets in $\mathbb{H}$ more sets in $\mathcal{H}^*$ to serve a base of neighborhoods of the cusps, the sets

$$\alpha(\mathcal{N}_M \cup \{\infty\}) : M > 0, \alpha \in SL_2(\mathbb{Z}),$$

and take the resulting topology on $\mathcal{H}^*$.

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Adjoin to the usual open sets in $\mathbb{H}$ more sets in $\mathcal{H}^*$ to serve a base of neighborhoods of the cusps, the sets

$$\alpha(\mathcal{N}_M \cup \{\infty\}) : M > 0, \alpha \in SL_2(\mathbb{Z}),$$

and take the resulting topology on $\mathcal{H}^*$.
Remark Under this topology, each $\gamma \in SL_2(\mathbb{Z})$ is a homeomorphism of $\mathcal{H}^*$.

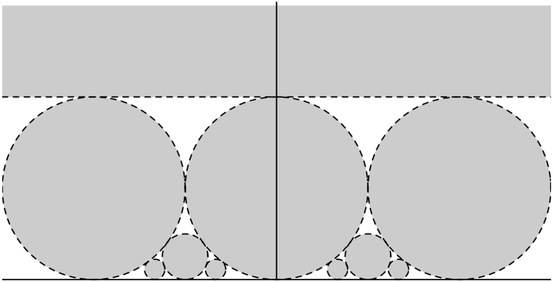

Figure 2.5. Neighborhoods of ∞ and of some rational points

Giving $X(\Gamma)$ the quotient topology and extending natural projection $\pi : \mathcal{H}^* \to X(\Gamma)$, we have:

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2. $x_1 = \Gamma s_1, x_2 = \Gamma t_2$ where $s_1 \in \mathbb{Q} \cup \{\infty\}, t_2 \in \mathbb{H}$: Let U_2 be any neighborhood of t_2 in $\mathbb{H}$ with compact closure K. We have the inequality

$$Im(\gamma(t)) \leq \max\{Im(t), 1/Im(t)\} \quad \text{for } t \in \mathbb{H} \text{ and } \gamma \in SL_2(\mathbb{Z}).$$

   This implies for M large enough, $SL_2(\mathbb{Z})K \cap \mathcal{N}_M = \varnothing$.
   Let $\alpha \in SL_2(\mathbb{Z})$ such that $s_1 = \alpha(\infty)$, then $\alpha(\mathcal{N}_M \cup \{\infty\})$ is a neighborhood of s_1 and $\alpha(\mathcal{N}_M \cup \{\infty\}) \cap U_2 = \varnothing$.

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3. $x_1 = \Gamma s_1, x_2 = \Gamma s_2$ where $s_1, s_2 \in \mathbb{Q} \cup \{\infty\}$: Let $\alpha_1, \alpha_2 \in SL_2(\mathbb{Z})$ such that $s_1 = \alpha_1(\infty), s_2 = \alpha_2(\infty)$.
   Let $U_1 = \alpha_1(\mathcal{N}_2 \cup \{\infty\}), U_2 = \alpha_2(\mathcal{N}_2 \cup \{\infty\})$. Then we claim that $\pi(U_1)$ and $\pi(U_2)$ are disjoint.

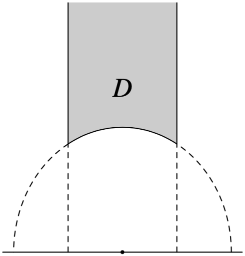
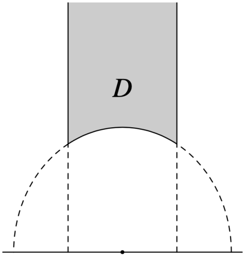


Figure 2.3. The fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$

To see this, suppose that $\exists \gamma \in \Gamma : \gamma\alpha_1(t_1) = \alpha_2(t_2)$,then $\alpha_2^{-1}\gamma\alpha_1$ maps t_1 to t_2.



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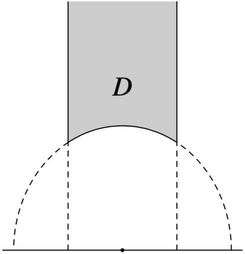
$$\alpha_2^{-1}\gamma\alpha_1 = \pm \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \quad \text{for some } m \in \mathbb{Z}.$$



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Thus $\alpha_2^{-1}\gamma\alpha_1$ fixes ∞, consequently $\gamma(s_1) = s_2$, contradiction. Then $X(\Gamma)$ is Hausdorff.

Suppose $\mathcal{H}^* = O_1 \cup O_2$ is a disjoint union of open subsets. Intersect with the connected sed $\mathbb{H}$ to conclude that $O_1 \supset \mathbb{H}$ and so $O_2 \subset \mathbb{Q} \cup \{\infty\}$. But then O_2 is not open unless it is empty. Thus $\mathcal{H}^*$ is connected and so is its continuous image $X(\Gamma)$.

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Note that

$$\mathcal{H}^* = SL_2(\mathbb{Z})\mathcal{D}^* = \bigcup_j \Gamma\gamma_j(\mathcal{D}^*), \text{ where the } \gamma_j \text{ are coset representatives.}$$

Then

$$X(\Gamma) = \bigcup_j \pi(\gamma_j(\mathcal{D}^*)).$$

Since each γ_j is continuous, π is continuous and $[SL_2(\mathbb{Z}) : \Gamma] < \infty$, $X(\Gamma)$ is compact.

Cusps

Charts about Cusps

For each cusp $s \in \mathbb{Q} \cup \{\infty\}$, define the **width** of s to be

$$h_s = \left| \frac{SL_2(\mathbb{Z})_s}{\{\pm I\}\Gamma_s} \right|.$$

This notion is dual to the period of an elliptic point, being inversely propotional to the size of an isotropy subgroup.

Claims:

1. If $s \in \mathbb{Q} \cup \{\infty\}$ and $\gamma \in SL_2(\mathbb{Z})$ then

$$\text{the width of } \gamma(s) \text{ under } \gamma\Gamma\gamma^{-1} = \text{the width of } s \text{ under } \Gamma.$$

   In particular, the width h_s depends only on Γs, making the width is well-defined on $X(\Gamma)$.

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$$h_s = \left| \frac{SL_2(\mathbb{Z})_\infty}{(\delta\{\pm I\}\Gamma\delta^{-1})_\infty} \right|.$$
Moreover,
$$(\delta\{\pm I\}\Gamma\delta^{-1})_\infty = \{\pm I\} \left\langle \begin{bmatrix} 1 & h_s \\ 0 & 1 \end{bmatrix} \right\rangle.$$

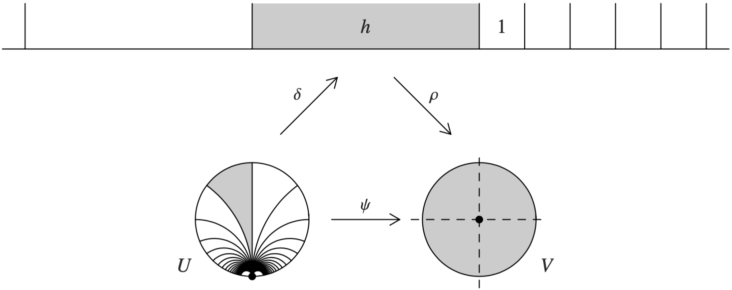
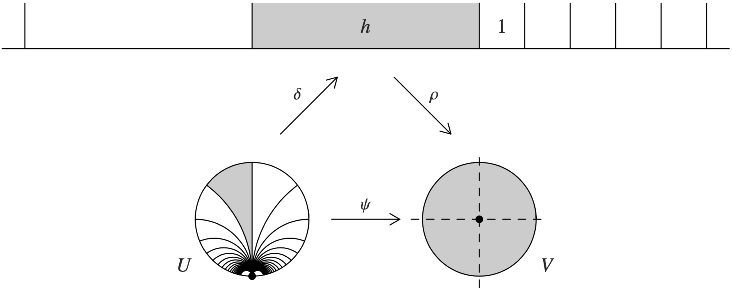


Figure 2.6. Local coordinates at a cusp



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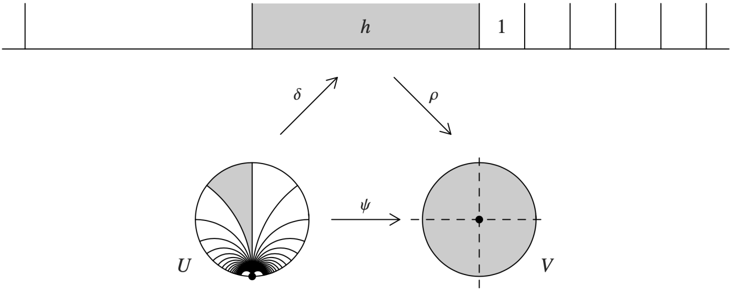

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Define $U = U_s = \delta^{-1}(\mathcal{N}_2 \cup \{\infty\})$ and define $\psi = \rho \circ \delta$, where $\rho = e^{2\pi i z/h}$, $h = h_s$. Let $V = im\psi$ then V is an open subset of $\mathbb{C}$, we have

$$\psi : U \to V, \qquad \psi(t) = e^{2\pi i \delta(t)/h}.$$

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for some $\gamma \in \Gamma$. Since $\delta(t_1)$ and $\delta(t_2)$ both lie in $\mathcal{N}_2 \cup \{\infty\}$, $\delta\gamma\delta^{-1}$ must be a translation. So

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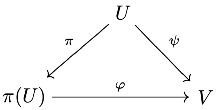
Then

$$\begin{aligned} \pi(t_1) = \pi(t_2) &\Leftrightarrow \delta(t_1) = \delta(t_2) + mh \text{ for some } m \in \mathbb{Z} \\ &\Leftrightarrow \psi(t_1) = \psi(t_2). \end{aligned}$$

Therefore, there exists a bijection $\varphi : \pi(U) \to V$ such that the following diagram commutes



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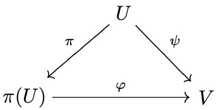

The coordinate neighborhood about $\pi(s)$ in $X(\Gamma)$ is $\pi(U)$, and the coordinate map is $\varphi : \pi(U) \to V$, a homeomorphism.

Cusps

Holomorphicity of Transition Maps

It suffices to consider 2 following cases.

Case 1 Suppose $U_1 \subset \mathbb{H}$ has the corresponding straightening map $\delta_1 = \delta_{t_1} \in GL_2(\mathbb{C})$ where t_1 has period h_1 and suppose $U_2 = \delta_2^{-1}(\mathcal{N}_2 \cup \{\infty\})$.

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For each $x \in \pi(U_1) \cap \pi(U_2)$, write $x = \pi(\tilde{t}_1) = \pi(t_2)$ for $\tilde{t}_1 \in U_1, t_2 \in U_2$.

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Let $U_{1,2} = U_1 \cap \gamma^{-1}(U_2)$, then $\varphi_1(\pi(U_{1,2}))$ is a neighborhood of $\varphi_1(x)$.
For any $q = \varphi_1(x') \in \varphi_1(\pi(U_{1,2}))$, the formula is

$$\varphi_{2,1}(q) = \exp(2\pi i \delta_2 \gamma \delta_1^{-1}(q^{1/h_1})/h_2).$$

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If $h_1 = 1$: OK.
If $h_1 > 1$, then $t_1 \notin U_{1,2}$, else the point $\delta_2(\gamma(t_1)) \in \mathcal{N}_2$ is also an elliptic point for Γ, which is contradiction since $\mathcal{N}_2$ contains no elliptic points. Then $t_1 \notin U_{1,2}$ so $0 \notin \varphi_1(\pi(U_{1,2}))$. The transition map is holomorphic.

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Since $\delta_1\gamma\delta_2^{-1}$ moves some point in $\mathcal{N}_2 \cup \{\infty\}$ to another, it must be a translation $\pm \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$.
In this case $\gamma(s_1) = s_2$, so $h_1 = h_2 = h$. Using this, we can compute

$$\varphi_{2,1}(q) = e^{2\pi i m/h} q.$$

This is clearly holomorphic.

To summarize, for any congruence subgroup Γ of $SL_2(\mathbb{Z})$ the extended quotient $X(\Gamma)$ is a compact Riemann surface.

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Problems:

1. Compute the genus of $X(\Gamma)$.
2. Study the meromorphic functions and differentials on $X(\Gamma)$.

Modular Curves and Modularity

Modular Curves and Modularity

Theorem (Modularity Theorem)

Let E be a complex elliptic curve with $j(E) \in \mathbb{Q}$. Then for some positive integer N there exists a surjective holomorphic function of compact Riemann surfaces from the modular curve $X_0(N)$ to the elliptic curve E,

$$X_0(N) \longrightarrow E.$$

THANK YOU FOR LISTENING