# An introduction to sheaves

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# Outline

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Let $O(X)$ denote the category where objects are open sets in $X$, and arrows are inclusions. A presheaf (of abelian groups) on $X$ is a functor $\mathscr{F} : O(X)^{\mathrm{op}} \to \mathbf{Ab}$, i.e. the following data.

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1. for open sets $W \subseteq V \subseteq U$ in $X$, $\mathrm{res}_{V \to W} \circ \mathrm{res}_{U \to V} = \mathrm{res}_{U \to W}$,
2. for each open set $U$ in $X$, $\mathrm{res}_{U \to U} = \mathrm{id}_{\mathscr{F}(U)}$.

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2. If $A$ is an abelian group, the constant presheaf $\underline{A}^{\mathrm{pre}}$ associated to $A$ is given by $U \mapsto A$, with restrictions being the identity $A \to A$.

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4. Let $A$ be a commutative ring and $X = \text{Spec}(A)$ with the Zariski topology. Its structure (pre-)sheaf $\mathscr{O}_X$ is given by $X \setminus V(I) \mapsto \varinjlim_{f \in I} A_f$, where, for each ideal $I$ of $A$, $V(I) = \{\mathfrak{p} \in \text{Spec}(A) : I \subseteq \mathfrak{p}\}$.

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## Definition

Let $\mathscr{F}, \mathscr{G}$ be presheaves on $X$. A morphisms of presheaves $\mathscr{F} \to \mathscr{G}$ is a natural transformation $\varphi : \mathscr{F} \to \mathscr{G}$, i.e. a collection $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ of group homomorphisms, where $U \subseteq X$ are open, such that the diagram

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- We have the category $\mathbf{PSh}(X)$ of presheaves on $X$. For each open set $U$, there is a section functor $\Gamma(U, -) : \mathbf{PSh}(X) \to \mathbf{Ab}$.

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If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on $X$, we define

1. the direct sum presheaf $\mathscr{F} \oplus \mathscr{G}$, given by $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$,
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- $\varphi$ is an isomorphism iff it is bijective.

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4. Let $X = \text{Spec}(A)$ for a commutative ring $A$. For each $\mathfrak{p} \in X$, $\mathscr{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$.

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A presheaf $\mathscr{F}$ on $X$ is a sheaf if for any open set $U$ in $X$, and any open cover $U = \bigcup_{i \in I} U_i$ of $U$, the following conditions are satisfied.

1. (Locality) If $s \in \mathscr{F}(U)$ such that $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$.
2. (Gluability) Given $s_i \in \mathscr{F}(U_i)$ for each $i \in I$, such that for $i, j \in I$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there is $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for $i \in I$.

The category $\mathbf{Sh}(X)$ of sheaves on $X$ is a full subcategory of $\mathbf{PSh}(X)$.

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4. The constant sheaf $\underline{A}$ is $U \mapsto \{\text{locally constant functions } U \to A\}$.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on $X$. For each $x \in X$, we have a homomorphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$, given by $\varphi_x(s_x) = \varphi(s)_x$, which is functorial.

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Injectivity : Assume that $\varphi$ is injective. Fix $x \in X$. If $U, V \ni x$ are open set, $s \in \mathscr{F}(U)$ and $t \in \mathscr{F}(V)$ such that $\varphi_x(s_x) = \varphi_x(t_x)$, then $\varphi(s)_x = \varphi(t)_x$, meaning $\varphi(s)|_W = \varphi(t)|_W$ in an open neighborhood $W \subseteq U \cap V$ of $x$, i.e. $\varphi(s|_W) = \varphi(t|_W)$. Thus, $s|_W = t|_W$, implying $s_x = t_x$, hence $\varphi_x$ is injective.

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Bijectivity : Assume now that $\varphi$ is surjective. Fix $x \in X$. Let $U \ni x$ be open and $t \in \mathscr{G}(U)$. Then there is some section $s \in \mathscr{F}(U)$ with $\varphi(s) = t$. It follows that $\varphi_x(s_x) = \varphi(s)_x = t_x$, i.e. $\varphi_x$ is surjective.

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Conversely, assume that $\varphi_x$ is bijective for all $x \in X$. Let $U$ be open and $t \in \mathscr{G}(U)$. For any $x \in U$, there is an open neighborhood $V_x \subseteq U$ of $x$ and a section $r^x \in \mathscr{F}(V_x)$ such that $\varphi(r^x)_x = \varphi_x(r^x_x) = t_x$, meaning $\varphi(r^x|_{U_x}) = \varphi(r^x)|_{U_x} = t|_{U_x}$ in some open neighborhood $U_x \subseteq V_x$ of $x$. We obtain an open cover $\{U_x\}_{x \in U}$ of $U$. Let $s^x := r^x|_{U_x}$. For $x, y \in U$, one has $\varphi(s^x|_{U_x \cap U_y}) = \varphi(r^x)|_{U_x \cap U_y} = t|_{U_x \cap U_y} = \varphi(r^y)|_{U_x \cap U_y} = \varphi(s^y|_{U_x \cap U_y})$. Now, $\varphi$ is injective since it is injective stalkwise, so $s^x|_{U_x \cap U_y} = s^y|_{U_x \cap U_y}$. We can then glue the $s^x$ $(x \in U)$ into a section $s \in \mathscr{F}(U)$. Clearly, $\varphi(s) = t$. □

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on $X$. Then $\mathscr{F} \oplus \mathscr{G}$ and $\ker \varphi$ are sheaves, making $\mathbf{Sh}(X)$ an additive category.

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**Definition**

Let $\mathscr{F}$ be a presheaf on $X$. For each open set $U$, let $\widehat{\mathscr{F}}(U)$ be the set of tuples $(s^x) \in \prod_{x \in U} \mathscr{F}_x$ such that every point $x \in U$ admits an open neighborhood $V \subseteq U$ and a section $t \in \mathscr{F}(V)$ such that $s^y = t_y$ for all $y \in V$. Then $\widehat{\mathscr{F}} : U \mapsto \widehat{\mathscr{F}}(U)$ defines a sheaf on $X$, called the sheafification of $\mathscr{F}$.

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There is a morphism of presheaves $i : \mathscr{F} \to \widehat{\mathscr{F}}$, $i(s) = (s_x)_{x \in U}$ for $s \in \mathscr{F}(U)$.

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## Proposition

- The morphism $i : \mathscr{F} \to \widehat{\mathscr{F}}$ induces isomorphisms at the level of stalks.
- If $\mathscr{G}$ is a sheaf, any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ factors uniquely through $i$.

$$\begin{tikzcd} \mathscr{F} \arrow[rr, "\forall \varphi"] \arrow[d, "i"'] & & \mathscr{G} \\ \widehat{\mathscr{F}} \arrow[urr, dashed, "\exists ! \widehat{\varphi}"'] & & \end{tikzcd}$$

Let $x \in X$.

The projections $\mathrm{pr}_{U \to x} : \widehat{\mathscr{F}}(U) \hookrightarrow \prod_{y \in U} \mathscr{F}_y \to \mathscr{F}_x$ for open sets $U \ni x$ induce a homomorphism $j_x : \widehat{\mathscr{F}}_x \to \mathscr{F}_x$. For open sets $U \supseteq V \ni x$, we have the commutative diagram on the right. Now, $\mathrm{pr}_{V \to x} \circ i(V)$ is the map $s \mapsto s_x$, which is the same as the vertical arrow $\mathscr{F}(V) \to \mathscr{F}_x$. The same holds for $U$, which implies $j_x \circ i_x = \mathrm{id}_{\mathscr{F}_x}$ by universal property of direct limit. Hence $i_x$ is injective.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "i(U)"] \arrow[d, "\mathrm{res}_{U \to V}"] & \widehat{\mathscr{F}}(U) \arrow[d, "\mathrm{res}_{U \to V}"] \arrow[ddr, bend left, "\mathrm{pr}_{U \to x}"] & \\
\mathscr{F}(V) \arrow[r, "i(V)"'] \arrow[d] & \widehat{\mathscr{F}}(V) \arrow[d] \arrow[dr, "\mathrm{pr}_{V \to x}"] & \\
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Now, any element of $\widehat{\mathscr{F}}_x$ is the germ $s_x$ of a tuple $s = (s^y) \in \widehat{\mathscr{F}}(U)$, where $U \ni x$ is open. By definition, there is an open neighborhood $V \subseteq U$ of $x$ and a section $t \in \mathscr{F}(V)$ such that $t_y = s^y$ for all $y \in V$, i.e. $i(t)|_V = s|_V$. It follows that $i_x(t_x) = i(t)_x = s_x$, hence $i_x$ is surjective. $\square$

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We conclude that $i_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism. In particular, if $\mathscr{F}$ is a sheaf, $i : \mathscr{F} \to \widehat{\mathscr{F}}$ is an isomorphism.

Given a sheaf $\mathscr{G}$ and a morphism $\varphi : \mathscr{F} \to \mathscr{G}$.
For any $U$ open, the product

$$\prod_{x \in U} \varphi_x : \prod_{x \in U} \mathscr{F}_x \to \prod_{x \in U} \mathscr{G}_x$$

induces a homomorphism $\widetilde{\varphi}(U) : \widehat{\mathscr{F}}(U) \to \widehat{\mathscr{G}}(U)$, which in turn gives a morphism of sheaves $\widetilde{\varphi} : \widehat{\mathscr{F}} \to \widehat{\mathscr{G}}$, making the diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[d, "i"'] & \mathscr{G} \arrow[d, "i", "\simeq"'] \\
\widehat{\mathscr{F}} \arrow[r, "\widetilde{\varphi}"'] \arrow[ur, dashed, "\widehat{\varphi}"] & \widehat{\mathscr{G}}
\end{tikzcd}$$

Since the right vertical arrow is an isomorphism, $\widetilde{\varphi}$ lifts into a morphism $\widehat{\varphi} : \widehat{\mathscr{F}} \to \mathscr{G}$ such that $\varphi = \widehat{\varphi} \circ i$. Finally, if $\varphi' : \widehat{\mathscr{F}} \to \mathscr{G}$ is such that $\varphi' \circ i = \varphi = \widehat{\varphi} \circ i$. Taking stalks at each $x \in X$ (which is functorial), and cancelling out the isomorphism $i_x$, gives $\widehat{\varphi}_x = \varphi'_x$. Therefore $\widehat{\varphi} = \varphi'$, showing uniqueness. $\square$

If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on $X$, the sheafification $\widehat{\operatorname{coker} \varphi}$ is the cokernel in the category $\mathbf{Sh}(X)$, making $\mathbf{Sh}(X)$ an abelian category.

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**Proposition**

$\varphi$ is an epimorphism in $\mathbf{Sh}(X)$ iff $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective for all $x \in X$.

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Assume that $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective for al $x \in X$. Let $\psi : \mathscr{G} \to \mathscr{H}$ is a morphism of sheaves with $\psi \circ \varphi = 0$. For any $x \in X$, $\psi_x \circ \varphi_x = (\psi \circ \varphi)_x = 0$, therefore $\psi_x = 0$ since $\varphi_x$ is surjective. It follows that $\psi = 0$.

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**Proposition**

$\varphi$ is an epimorphism in $\mathbf{Sh}(X)$ iff $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective for all $x \in X$.

Conversely, assume that $\varphi$ is an epimorphism. Let $x \in X$ and $A := \operatorname{coker}\varphi_x$. Define a morphism $\psi : \mathscr{G} \to \operatorname{skysc}_x(A)$ as follows. For an open set $U$, let $\psi(U)$ be the composition $\mathscr{G}(U) \to \mathscr{G}_x \to A$ if $x \in U$, and $\psi(U) = 0$ otherwise. If $x \in U$, the composition is $\mathscr{F}_x \to \mathscr{G}_x \to A$ is 0, and the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "\varphi(U)"'] & \mathscr{F}_x \arrow[d, "\varphi_x"] & \\
\mathscr{G}(U) \arrow[r] & \mathscr{G}_x \arrow[r] & A
\end{tikzcd}$$

commutes. It follows that $(\psi \circ \varphi)(U) = \psi(U) \circ \varphi(U) = 0$ in all cases, i.e. $\psi \circ \varphi = 0$, which implies $\psi = 0$. Since $\psi_x$ is precisely the projection $\mathscr{G}_x \to A$, we have $\operatorname{coker}\varphi_x = A = 0$, i.e. $\varphi_x$ is surjective. $\square$

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on $X$. The image presheaf $\operatorname{im}\varphi$ given by $U \mapsto \operatorname{im}\varphi(U)$ is the image of $\varphi$ in the category $\mathbf{PSh}(X)$. Its sheafification $\widehat{\operatorname{im}\varphi}$ is the image of $\varphi$ in the category $\mathbf{Sh}(X)$.

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**Proposition**

A sequence of sheaves $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$ on $X$ is exact iff for all $x \in X$, the sequence $\mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x \xrightarrow{\psi_x} \mathscr{H}_x$ is exact.

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Assume that $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$ on $X$ is exact, i.e. the inclusion $\psi \circ \varphi = 0$, and the inclusion $\operatorname{im}\varphi \to \ker\psi$ induces an isomorphism $\widehat{\operatorname{im}\varphi} \xrightarrow{\simeq} \ker\psi$. Let $x \in X$. Recall that the sheafification $\operatorname{im}\varphi \to \widehat{\operatorname{im}\varphi}$ induces isomorphisms on stalks, so that we have $\operatorname{im}\varphi_x = (\operatorname{im}\varphi)_x \simeq \widehat{\operatorname{im}\varphi}_x \simeq \ker\psi_x$.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on $X$. The image presheaf $\operatorname{im}\varphi$ given by $U \mapsto \operatorname{im}\varphi(U)$ is the image of $\varphi$ in the category $\mathbf{PSh}(X)$. Its sheafification $\widehat{\operatorname{im}\varphi}$ is the image of $\varphi$ in the category $\mathbf{Sh}(X)$.

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Conversely, assume that $\ker\psi_x = \operatorname{im}\varphi_x$ for all $x \in X$. In particular, $(\psi\circ\varphi)_x = \psi_x \circ \varphi_x = 0$, so $\psi\circ\varphi = 0$, i.e. there is an inclusion $\operatorname{im}\varphi \to \ker\psi$. It induces a morphism $\widehat{\operatorname{im}\varphi} \to \ker\psi$, which is an isomorphism since we have isomorphisms $\ker\psi_x = \operatorname{im}\varphi_x = (\operatorname{im}\varphi)_x \simeq \widehat{\operatorname{im}\varphi}_x$. □

## Example

Let $X = \mathbb{C}$, $\mathscr{H}$ the sheaf of holomorphic functions and $\mathscr{H}^\times$ the sheaf of non-vanishing holomorphic functions (on open subsets of $\mathbb{C}$). The first is a sheaf of $\mathbb{C}$-vector spaces, whereas the second is a sheaf of (multiplicative) abelian groups. Let $d : \mathscr{H} \to \mathscr{H}$ denote the derivation. A holomorphic function on an open set $U$ in $\mathbb{C}$ has derivative 0 iff its is constant on each connected component of $U$. We have an exact sequence of sheaves of $\mathbb{C}$-vector spaces,

$$0 \to \underline{\mathbb{C}} \to \mathscr{H} \xrightarrow{d} \mathscr{H} \to 0.$$

In fact, taking stalk at any point $x \in \mathbb{C}$ yields an exact sequence of $\mathbb{C}$-vector spaces

$$0 \to \mathbb{C} \to \mathbb{C}\{z\} \xrightarrow{\partial/\partial z} \mathbb{C}\{z\} \to 0.$$

For $U$ open in $\mathbb{C}$, we only have an exact sequence

$$0 \to \underline{\mathbb{C}}(U) \to \mathscr{H}(U) \xrightarrow{d} \mathscr{H}(U),$$

since surjectivity of $d$ fails for $U = \mathbb{C}^\times$ ($z \mapsto 1/z$ admits no primitive).

## Example

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$$0 \to \underline{2\pi i\mathbb{Z}} \to \mathscr{H} \xrightarrow{\exp} \mathscr{H}^\times \to 1.$$

In fact, taking stalk at any point $x \in \mathbb{C}$ yields an exact sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathbb{C}\{z\} \xrightarrow{\exp} \mathbb{C}\{z\}^\times \to 1.$$

For $U$ open in $\mathbb{C}$, we only have an exact sequence

$$0 \to \underline{2\pi i\mathbb{Z}}(U) \to \mathscr{H}(U) \xrightarrow{\exp} \mathscr{H}^\times(U),$$

since surjectivity of exp fails for $U = \mathbb{C}^\times$ ($z \mapsto z$ admits no logarithm).

Let $f : X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a sheaf on $Y$. We want to transport $\mathscr{G}$ to a sheaf on $X$ using $f$.

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**Definition**

For each open set $U$ in $X$, let $f^{-1}\mathscr{G}(U)$ be the set of tuples $(s^x) \in \prod_{x \in U} \mathscr{G}_{f(x)}$ such that for every point $x \in U$, there are open sets $W$ in $X$, $V$ in $Y$ with $x \in W \subseteq U$, $f(W) \subseteq V$, and a section $t \in \mathscr{G}(V)$ such that for every $y \in W$, one has $s^y = t_{f(y)}$. Then $f^{-1}\mathscr{G} : U \mapsto f^{-1}\mathscr{G}(U)$ defines a sheaf on $X$, called pullback or inverse image sheaf of $\mathscr{G}$ by $f$.

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For each open set $V$ in $Y$, there is an adjunction map

$$\mathrm{adj}(V) : \mathscr{G}(V) \to f^{-1}\mathscr{G}(f^{-1}(V)), \qquad s \mapsto (s_{f(x)})_{x\in f^{-1}(V)}.$$

If $f(x) \in V$, composing with $f^{-1}\mathscr{G}(f^{-1}(V)) \to (f^{-1}\mathscr{G})_x$ yields a homomorphism $\mathscr{G}(V) \to (f^{-1}\mathscr{G})_x$. Fix $x$ and let $V$ varies through the open neighborhoods of $f(x)$, we get a homomorphism

$$\mathrm{adj}_x : \mathscr{G}_{f(x)} \to (f^{-1}\mathscr{G})_x.$$

**Proposition**

We have an isomorphism $\mathrm{adj}_x : \mathscr{G}_{f(x)} \xrightarrow{\simeq} (f^{-1}\mathscr{G})_x$.

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As $U_{\text{open}} \ni x$ varies, the projections $\mathrm{pr}_{U\to f(x)} : f^{-1}\mathscr{G}(U) \hookrightarrow \prod_{y\in U} \mathscr{G}_{f(y)} \to \mathscr{G}_{f(x)}$, induce a map $p_x : (f^{-1}\mathscr{G})_x \to \mathscr{G}_{f(x)}$. For open sets $V \supseteq V' \ni f(x)$, we have the commutative diagram on the right (where $U = f^{-1}(V)$ and $U' = f^{-1}(V')$). Now, $\mathrm{pr}_{U'\to f(x)} \circ \mathrm{adj}(V)$ is the map $s \mapsto s_{f(x)}$, which is the same as the vertical arrow $\mathscr{G}(V') \to \mathscr{G}_{f(x)}$. The same holds for $V$, thus $p_x \circ \mathrm{adj}_x = \mathrm{id}_{\mathscr{G}_{f(x)}}$. Hence $\mathrm{adj}_x$ is injective.

$$
\begin{tikzcd}
\mathscr{G}(V) \arrow[r, "\mathrm{adj}(V)"] \arrow[d, "\mathrm{res}_{V\to V'}"] & f^{-1}\mathscr{G}(U) \arrow[d, "\mathrm{res}_{U\to U'}"] \arrow[rdd, bend left, "\mathrm{pr}_{U\to f(x)}"] & \\
\mathscr{G}(V') \arrow[r, "\mathrm{adj}(V')"'] \arrow[d] & f^{-1}\mathscr{G}(U') \arrow[d] \arrow[rd, "\mathrm{pr}_{U'\to f(x)}"] & \\
\mathscr{G}_{f(x)} \arrow[r, "\mathrm{adj}_x"'] & (f^{-1}\mathscr{G})_x \arrow[r, "p_x"'] & \mathscr{G}_{f(x)}
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$$\begin{tikzcd}
\mathscr{G}(V) \arrow[r, "\mathrm{adj}(V)"] \arrow[d, "\mathrm{res}_{V\to V'}"] & f^{-1}\mathscr{G}(U) \arrow[d, "\mathrm{res}_{U\to U'}"] \arrow[ddr, bend left, "\mathrm{pr}_{U\to f(x)}"] & \\
\mathscr{G}(V') \arrow[r, "\mathrm{adj}(V')"'] \arrow[d] & f^{-1}\mathscr{G}(U') \arrow[d] \arrow[dr, "\mathrm{pr}_{U'\to f(x)}"] & \\
\mathscr{G}_{f(x)} \arrow[r, "\mathrm{adj}_x"'] & (f^{-1}\mathscr{G})_x \arrow[r, "p_x"'] & \mathscr{G}_{f(x)}
\end{tikzcd}$$

As for surjectivity, take the germ $s_x$ of a tuple $s = (s^y) \in f^{-1}\mathscr{G}(U)$, $U_{\text{open}} \ni x$. By definition, there are open sets $x \in W \subseteq U$ in $X$, $V \supseteq f(W)$ in $Y$ and a section $t \in \mathscr{G}(V)$ such that for every $y \in W$, one has $s^y = t_{f(y)}$. Therefore, $\mathrm{adj}(V)(t)|_W = s|_W$, so $\mathrm{adj}(V)(t)_x = s_x$. A closer look into the above diagram yields $\mathrm{adj}_x(t_{f(x)}) = \mathrm{adj}(V)(t)_x = s_x$. $\square$

The construction $f^{-1}$ is functorial : given a morphism $\psi : \mathscr{G} \to \mathscr{G}'$ of sheaves on $Y$, the products $\prod_{x \in U} \psi_{f(x)} : \prod_{x \in U} \mathscr{G}_{f(x)} \to \prod_{x \in U} \mathscr{G}'_{f(x)}$ (for each $U$ open in $X$), induce a morphism $f^{-1}\psi : f^{-1}\mathscr{G} \to f^{-1}\mathscr{G}'$ of sheaves on $X$.

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**Proposition**

The functor $f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ is exact.

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**Proposition**

The functor $f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ is exact.

Clearly $f^{-1}$ is additive. If $0 \to \mathscr{G} \to \mathscr{G}' \to \mathscr{G}'' \to 0$ is an exact sequence of sheaves on $Y$, we have a commutive diagram in $\mathbf{Ab}$ for each $x \in X$,

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{G}_{f(x)} \arrow[r] \arrow[d, "\simeq"] & \mathscr{G}'_{f(x)} \arrow[r] \arrow[d, "\simeq"] & \mathscr{G}''_{f(x)} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & (f^{-1}\mathscr{G})_x \arrow[r] & (f^{-1}\mathscr{G}')_x \arrow[r] & (f^{-1}\mathscr{G}'')_x \arrow[r] & 0
\end{tikzcd}$$

with the (isomorphic) vertical arrows being adjunctions. But exactness can be verified stalkwise. $\square$

Let $\mathscr{F}$ be a sheaf on $X$.

**Example**

1. If $i : U \hookrightarrow X$ is an inclusion of an open set, then $i^{-1}\mathscr{F}$ is canonically isomorphic to the restriction sheaf $\mathscr{F}|_U$ on $U$ given by $V \mapsto \mathscr{F}(V)$.

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**Definition**

If $f : X \to Y$ is continuous, we define the pushfoward, or direct image sheaf $f_*\mathscr{F}$ on $Y$ by $V \mapsto \mathscr{F}(f^{-1}(V))$.

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If $f : X \to Y$ is continuous, we define the pushfoward, or direct image sheaf $f_*\mathscr{F}$ on $Y$ by $V \mapsto \mathscr{F}(f^{-1}(V))$.

Indeed, this gives rise to an additive functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

**Proposition**

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Given an exact sequence $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{F}' \xrightarrow{\psi} \mathscr{F}''$, we show the exactness of

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For all $V$ open in $Y$, $f_*\varphi(V) = \varphi(f^{-1}(V)) : f_*\mathscr{F} \to f_*\mathscr{F}'$ is injective.

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For all $V$ open in $Y$, $f_*\varphi(V) = \varphi(f^{-1}(V)) : f_*\mathscr{F} \to f_*\mathscr{F}'$ is injective.
To show exactness at $f_*\mathscr{F}'$, first notice that the image $\operatorname{im}\varphi$ of an injective morphism $\varphi : \mathscr{F} \to \mathscr{F}'$ of sheaves is a sheaf. Indeed, the morphism of presheaves $\varphi : \mathscr{F} \to \operatorname{im}\varphi$ is bijective, hence is an isomorphism, thus $\operatorname{im}\varphi$ is a sheaf, so is $\operatorname{im}(f_*\varphi)$. Exactness at $\mathscr{F}'$ says that $\operatorname{im}\varphi = \ker\psi$. If $V$ is open in $Y$,

$$\operatorname{im}(f_*\varphi(V)) = \operatorname{im}\varphi(f^{-1}(V)) = \ker\psi(f^{-1}(V)) = \ker f_*(\psi(V)),$$

i.e. $\operatorname{im}(f_*\varphi) = \ker(f_*\psi)$ as desired. $\square$

**Theorem**

For any continuous map $f : X \to Y$, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \simeq \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}),$$

natural in $\mathscr{F} \in \mathbf{Sh}(X)$ and $\mathscr{G} \in \mathbf{Sh}(Y)$, i.e. the functor $f^{-1}$ is left adjoint to $f_*$.

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natural in $\mathscr{F} \in \mathbf{Sh}(X)$ and $\mathscr{G} \in \mathbf{Sh}(Y)$, i.e. the functor $f^{-1}$ is left adjoint to $f_*$.

Recall the adjunction maps $\mathrm{adj}(V) : \mathscr{G}(V) \to f^{-1}\mathscr{G}((f^{-1}(V)) = f_*f^{-1}\mathscr{G}(V)$, $s \mapsto (s_{f(x)})_{x \in f^{-1}(V)}$ (for each $V$ open in $Y$). They fit together to form the adjunction morphism $\mathrm{adj} : \mathscr{G} \to f_*f^{-1}\mathscr{G}$. Define a map $\eta = \eta_{\mathscr{G},\mathscr{F}} : \mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F})$ by $\varphi \mapsto f_*\varphi \circ \mathrm{adj}$.

$$\eta : \mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}), \qquad \eta(\varphi) = f_*\varphi \circ \mathrm{adj}\,.$$

$\eta$ is injective. In fact, for each $x \in X$, taking limit of the commutative square

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[r, "\eta(\varphi)(V)"] \arrow[d, "\mathrm{adj}(V)"'] & f_*\mathscr{F}(V) \arrow[d, "="] \\
f_*f^{-1}\mathscr{G}(V) = f^{-1}\mathscr{G}(f^{-1}(V)) \arrow[r, "f_*\varphi(V)=\varphi(f^{-1}(V))"] & \mathscr{F}(f^{-1}(V)),
\end{tikzcd}$$

as $V_{\text{open}} \ni f(x)$ varies, yields a commutative square

$$\begin{tikzcd}
\mathscr{G}_{f(x)} \arrow[r, "\eta(\varphi)_x"] \arrow[d, "\mathrm{adj}_x"'] & (f_*\mathscr{F})_{f(x)} \arrow[d] \\
(f^{-1}\mathscr{G})_x \arrow[r, "\varphi_x"] & \mathscr{F}_x.
\end{tikzcd}$$

Since $\mathrm{adj}_x$ is an isomorphism, if $\eta(\varphi) = 0$, we would have $\varphi_x = 0$ for all $x \in X$, which implies $\varphi = 0$.

Let $\psi : \mathscr{G} \to f_*\mathscr{F}$ be given. For each $x \in X$, let $b_x : (f_*\mathscr{F})_{f(x)} \to \mathscr{F}_x$ be the map induced by $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V)) \to \mathscr{F}_x$, $V_{\text{open}} \ni f(x)$. It is given by $b_x(s_{f(x)}) = s_x$ for any $s \in f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$. For each $U$ open in $X$, the product $\prod_{x\in U}(b_x \circ \psi_{f(x)}) : \prod_{x\in U} \mathscr{G}_{f(x)} \to \prod_{x\in U} \mathscr{F}_x$ takes $f^{-1}\mathscr{G}(U)$ to $\widehat{\mathscr{F}}(U) \simeq \mathscr{F}(U)$. These maps form a morphism $\varphi : f^{-1}(\mathscr{G}) \to \mathscr{F} \xrightarrow{\simeq} \mathscr{F}$. Let $t \in \mathscr{G}(V)$, $V$ open in $Y$. The diagram

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[r, "\mathrm{adj}(V)"'] & f^{-1}\mathscr{G}(f^{-1}(V)) \arrow[r] \arrow[d, "="] \arrow[rr, bend left, "\varphi(f^{-1}(V))"] & \widehat{\mathscr{F}}(f^{-1}(V)) \arrow[r, "\simeq"'] & \mathscr{F}(f^{-1}(V)) \arrow[d, "="'] \\
 & f_*f^{-1}\mathscr{G}(V) \arrow[rr, "f_*\varphi(V)"] & & f_*\mathscr{F}(V)
\end{tikzcd}$$

commutes for all $V$ open in $Y$. Let $s \in \mathscr{G}(V)$. For any $x \in f^{-1}(V)$, $b_x(\psi_x(s_{f(x)})) = b_x(\psi(s)_{f(x)}) = \psi(s)_x$. Hence, under the composition of the top row, $s \mapsto (s_{f(x)})_{x\in f^{-1}(V)} \mapsto (\psi(s)_x)_{x\in f^{-1}(V)} \mapsto \psi(s)$, i.e. $\eta(\varphi) = f_*\varphi \circ \mathrm{adj} = \psi$, so $\eta$ is surjective.

Finally, we verify naturality of $\eta = \eta_{\mathscr{G},\mathscr{F}}$ in $\mathscr{G}$ and $\mathscr{F}$, that is, for morphisms $\theta : \mathscr{G}' \to \mathscr{G}$ and $\psi : \mathscr{F} \to \mathscr{F}'$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \arrow[r, "\eta_{\mathscr{G},\mathscr{F}}"'] \arrow[d, "\psi\circ - \circ f^{-1}\theta"] & \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F}) \arrow[d, "f_*\psi\circ - \circ\theta"'] \\
\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathscr{G}', \mathscr{F}') \arrow[r, "\eta_{\mathscr{G}',\mathscr{F}'}"] & \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}', f_*\mathscr{F}').
\end{tikzcd}$$

Let $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$ be any morphism. We have to show that

$$f_*\psi \circ f_*\varphi \circ \mathrm{adj} \circ \theta = f_*(\psi \circ \varphi \circ f^{-1}\theta) \circ \mathrm{adj}\,.$$

It suffices to show that $\mathrm{adj} \circ \theta = f_* f^{-1}\theta \circ \mathrm{adj}$. This is a direct computation. Take $s \in \mathscr{G}'(V)$, $V$ open in $Y$, then

$$f_* f^{-1}\theta(\mathrm{adj}(s)) = f^{-1}\theta((s_{f(x)})_{x\in f^{-1}(V)}) = (\theta(s)_{f(x)})_{x\in f^{-1}(V)} = \mathrm{adj}(\theta(s))$$

as desired. $\square$

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The category $\mathbf{Sh}(X)$ has enough injectives.

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Let $\mathscr{F}$ be any sheaf on $X$. For each $x \in X$, embed $\mathscr{F}_x$ into a divisible group $D_x$. Define the sheaf $\mathscr{D}$ on $X$ by $U \mapsto \prod_{x \in U} D_x$, with restrictions being projections. The composition $\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x \hookrightarrow \prod_{x \in U} D_x$ is injective for each $U$ open in $X$, hence $\mathscr{F} \hookrightarrow \mathscr{D}$.

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$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{G} \arrow[r, "\varphi"'] \arrow[d, "\psi"'] & \mathscr{H} \arrow[dl, dashed, "\theta"] \\
& \mathscr{D} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
0 \arrow[r] & \mathscr{G}_x \arrow[r, "\varphi_x"'] \arrow[d, "\psi_x"'] & \mathscr{H}_x \arrow[d, "\theta^x"] \\
& \mathscr{D}_x \arrow[r, "j_x"] & D_x.
\end{tikzcd}
$$

## Proposition

The category $\mathbf{Sh}(X)$ has enough injectives.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{G} \arrow[r, "\varphi"'] \arrow[d, "\psi"'] & \mathscr{H} \arrow[dl, dashed, "\theta"] \\ & \mathscr{D} & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \mathscr{G}_x \arrow[r, "\varphi_x"'] \arrow[d, "\psi_x"'] & \mathscr{H}_x \arrow[d, "\theta^x"] \\ & \mathscr{D}_x \arrow[r, "j_x"] & D_x . \end{tikzcd}$$

For each $x \in X$, the projections $\mathscr{D}(U) \to D_x$ (for $U_{\text{open}} \ni x$) induce a homomorphism $j_x : \mathscr{D}_x \to D_x$, under which the stalk $s_x$ of each tuple $s = (s^y) \in \mathscr{D}(U)$ maps to $s^x$. Now, $\varphi_x : \mathscr{G}_x \to \mathscr{H}_x$ is injective, so we can lift $j_x \circ \psi_x$ into a map $\theta^x : \mathscr{H}_x \to D_x$. For each $U$ open, let $\theta(U)$ be the composition $\mathscr{H}(U) \to \prod_{x \in U} \mathscr{H}_x \xrightarrow{\prod_{x\in U} \theta^x} \mathscr{D}(U)$. These maps form a morphism $\theta : \mathscr{H} \to \mathscr{D}$ of sheaves. Furthermore, if $s \in \mathscr{G}(U)$,

$$\theta(\varphi(s)) = (\theta^x(\varphi(s)_x))_{x \in U} = (j_x(\psi(s)_x))_{x \in U} = \psi(s)$$

(the last equality comes from comparison on each coordinate $x \in U$ of $\psi(s)$, and the definition of $j_x$), so $\theta \circ \varphi = \psi$.

**Proposition**

The section functors $\Gamma(U, -) : \mathbf{Sh}(X) \to \mathbf{Ab}$, $U$ open in $X$, are left exact.

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Let $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$ be an exact sequence of sheaves on $X$. We want to show that $0 \to \mathscr{F}(U) \xrightarrow{\varphi(U)} \mathscr{G}(U) \xrightarrow{\psi(U)} \mathscr{H}(U)$ is exact.

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Clearly, $\varphi(U)$ is injective and $\psi(U) \circ \varphi(U) = (\psi \circ \varphi)(U) = 0$.

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Clearly, $\varphi(U)$ is injective and $\psi(U) \circ \varphi(U) = (\psi \circ \varphi)(U) = 0$.
Hence, it suffices to show that $\ker \psi(U) \subseteq \operatorname{im} \varphi(U)$. Take any global section $s \in \mathscr{G}(U)$ that is sent to 0 in $\mathscr{H}(U)$ by $\psi(U)$. For each $x \in U$,

$$0 \to \mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x \xrightarrow{\psi_x} \mathscr{H}_x$$

is exact, and $\psi_x(t_x) = \psi(t)_x = 0$, so there is an open neighborhood $U_x \subseteq U$ of $x$ and a section $s^x \in \mathscr{F}(U_x)$ with $\varphi(s^x) = t|_{U_x}$. For $x, y \in X$,

$$\varphi(s^x|_{U_x \cap U_y}) = \varphi(s^x)|_{U_x \cap U_y}) = t|_{U_x \cap U_y} = \varphi(s^y)|_{U_x \cap U_y}) = \varphi(s^y|_{U_x \cap U_y}),$$

so $s^x|_{U_x \cap U_y} = s^y|_{U_x \cap U_y}$ since $\varphi$ is injective. It follows that the sections $s^x$ can be glued together into a section $s \in \mathscr{F}(U)$. Obviously, $\varphi(s) = t$. $\square$

**Definition**

Let $\mathscr{F}$ be a sheaf on $X$. For $i \geqslant 0$, the $i$-th cohomology group of $X$ with coefficient in $\mathscr{F}$ is $\mathrm{H}^i(X, \mathscr{F}) := R^i\Gamma(X, -)(\mathscr{F})$.

To compute, take any injective resolution $0 \to \mathscr{F} \xrightarrow{\delta^0} \mathscr{D}^0 \xrightarrow{\delta^1} \mathscr{D}^1 \xrightarrow{\delta^2} \cdots$ of $\mathscr{F}$, then apply $\Gamma(X, -)$ to obtain a cochain complex of abelian groups,

$$0 \xrightarrow{d^0} \mathscr{D}^0(X) \xrightarrow{d^1} \mathscr{D}^1(X) \xrightarrow{d^2} \cdots,$$

where $d^0 = 0$ and $d^i = \delta^i(X)$ for $i \geqslant 1$. Accordingly, $\mathrm{H}^i(X, \mathscr{F}) := \dfrac{\ker d^{i+1}}{\operatorname{im} d^i}$.

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**Definition**

A sheaf $\mathscr{D}$ on $X$ is acyclic if $\mathrm{H}^i(X, \mathscr{D}) = 0$ for all $i > 0$.

Injective sheaves are acyclic. A standard fact from homological algebra [nla] is

**Theorem (de Rham-Weil isomorphism theorem)**

*The groups $\mathrm{H}^i(X, \mathscr{F})$, $i \geqslant 0$, can be computed using acyclic resolutions.*

**Definition**

A sheaf $\mathscr{F}$ on $X$ is flabby or flasque its restrictions are all surjective.

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**Proposition**

If $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \to 0$ is a short exact sequence of sheaves on $X$ with $\mathscr{F}$ flabby, then $\psi(U) : \mathscr{G}(U) \to \mathscr{H}(U)$ is surjective for all $U$ open in $X$.

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Let there be $t \in \mathscr{H}(U)$. The set of sections $s$ of $\mathscr{G}(V)$ (for some $V_{\text{open}} \subseteq U$) such that $\psi(s) = t|_V$ is nonempty since $\psi$ is surjective on stalks. Take such a section $s \in \mathscr{G}(V)$ that is maximal for the restriction order. We claim that $V = U$, for if there exists $x \in U - V$, then there is a section $r \in \mathscr{G}(W)$, $U \supseteq W_{\text{open}} \ni x$ with $\psi(r)_x = \psi_x(r_x) = t_x$. Squeeze $W$ if necessary to get $\psi(r) = t|_W$. Now, $\psi(s|_{V\cap W}) = t|_{V\cap W} = \psi(r|_{V\cap W})$, so $s|_{V\cap W} = r|_{V\cap W} + \varphi(v)$, where $v \in \mathscr{F}(V \cap W)$, by exactness. Since $\mathscr{F}$ is flabby, one can extend $v$ to a section $w \in \mathscr{F}(W)$ and replace $r$ by $r - \varphi(w)$. Now $s|_{V\cap W} = r|_{V\cap W}$, so we can glue them together into a section $s' \in \mathscr{G}(V \cup W)$ with $\psi(s') = t|_{V\cup W}$, contradicting the maximality of $s$. $\square$

## Corollary

If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is a short exact sequence of sheaves with $\mathscr{F}$ and $\mathscr{G}$ flabby, then $\mathscr{H}$ is flabby.

**Corollary**

If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is a short exact sequence of sheaves with $\mathscr{F}$ and $\mathscr{G}$ flabby, then $\mathscr{H}$ is flabby.

In fact, for $V \subseteq U$ open in $X$, we have a commutative diagram with exact rows, and the middle vertical arrow is surjective.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}(U) \arrow[r] \arrow[d, "\mathrm{res}_{V \to U}"] & \mathscr{G}(U) \arrow[r] \arrow[d, "\mathrm{res}_{V \to U}"] & \mathscr{H}(U) \arrow[r] \arrow[d, "\mathrm{res}_{V \to U}"] & 0 \\
0 \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{G}(V) \arrow[r] & \mathscr{H}(V) \arrow[r] & 0.
\end{tikzcd}$$

□

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0 \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{G}(V) \arrow[r] & \mathscr{H}(V) \arrow[r] & 0.
\end{tikzcd}$$ $\square$

## Example

Every sheaf $\mathscr{F}$ can be embedded into a flabby sheaf (Godement construction) $C^0\mathscr{F}$, given by $U \mapsto \prod_{x\in U} \mathscr{F}_x$, and restrictions being projections. The embedding $\mathscr{F} \hookrightarrow C^0\mathscr{F}$ is indeed given by taking stalks.

## Proposition

If $\mathscr{F}$ is an injective sheaf, it is flabby.

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Let $\varphi : \mathscr{F} \to \mathscr{G}$ be an embedding, with $\mathscr{G}$ flabby. Since $\mathscr{F}$ is injective, the identity $\mathscr{F} \to \mathscr{F}$ lifts into a retraction $\psi : \mathscr{G} \to \mathscr{F}$ of $\varphi$.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r, "\varphi"'] \arrow[d, "="'] & \mathscr{G} \arrow[dl, dashed, "\psi"] \\
 & \mathscr{F} &
\end{tikzcd}$$

In particular, $\psi(U)$ is surjective for any $U$ open in $X$. When $V \subseteq U$ are open, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\mathrm{res}_{U\to V}"'] \arrow[d, "\psi(U)"'] & \mathscr{G}(V) \arrow[d, "\psi(V)"] \\
\mathscr{F}(U) \arrow[r, "\mathrm{res}_{U\to V}"] & \mathscr{F}(V)
\end{tikzcd}$$

with surjective vertical arrows and upper horizontal arrow. It follows that the restriction $\mathrm{res}_{U\to V} : \mathscr{F}(U) \to \mathscr{F}(V)$ is also surjective.

**Proposition**

If $\mathscr{F}$ is a flabby sheaf, it is acyclic (therefore, sheaf cohomology can be computed using flabby resolution).

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Embed $\mathscr{F}$ into an injective sheaf $\mathscr{G}$ and let $\mathscr{H}$ denotes the cokernel (in $\mathbf{Sh}(X)$) of $\mathscr{F} \hookrightarrow \mathscr{G}$. We have a short exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ of flabby sheaves. Consider the long exact sequence in cohomology

$$0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \to \mathrm{H}^1(X, \mathscr{F}) \to \mathrm{H}^1(X, \mathscr{G}) \to \mathrm{H}^1(X, \mathscr{F}) \to \cdots .$$

Let us proceed to prove by induction on $i \geqslant 1$ that $\mathrm{H}^i(X, \mathscr{F}) = 0$ for any flabby sheaf $\mathscr{F}$. Indeed, for $i \geqslant 1$, since $\mathscr{G}$ is injective, the first and last groups in

$$\mathrm{H}^i(X, \mathscr{G}) \to \mathrm{H}^i(X, \mathscr{H}) \to \mathrm{H}^{i+1}(X, \mathscr{F}) \to \mathrm{H}^{i+1}(X, \mathscr{G})$$

vanish, so $\mathrm{H}^{i+1}(X, \mathscr{F}) \simeq \mathrm{H}^i(X, \mathscr{H})$. Now, $\mathscr{G}(X) \to \mathscr{H}(X)$ is surjective, so the map $\mathscr{H}(X) \to \mathrm{H}^1(X, \mathscr{F})$ is 0. But $\mathrm{H}^1(X, \mathscr{G}) = 0$, thus $\mathrm{H}^1(X, \mathscr{F}) = 0$. The inductive step is trivial. □

## Example

Let $\mathscr{F}$ be a sheaf on $X$. Consider the Godement construction $0 \to \mathscr{F} \xrightarrow{\delta^0} C^0\mathscr{F}$. We can extend this exact sequence to the right by setting $C^i\mathscr{F} := C^0\widehat{\text{coker}\,\delta^{i-1}}$ and letting $\delta^i$ be the composition $C^{i-1}\mathscr{F} \to \widehat{\text{coker}\,\delta^{i-1}} \to C^i\mathscr{F}$ for $i \geqslant 1$.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r, "\delta^0"] & C^0\mathscr{F} \arrow[r, "\delta^1"] \arrow[d] & C^1\mathscr{F} \arrow[r, "\delta^2"] \arrow[d] & \cdots . \\
& & \widehat{\text{coker}\,\delta^0} \arrow[ur] & \widehat{\text{coker}\,\delta^1} \arrow[ur] &
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$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r, "\delta^0"] & C^0\mathscr{F} \arrow[r, "\delta^1"] \arrow[d] & C^1\mathscr{F} \arrow[r, "\delta^2"] \arrow[d] & \cdots . \\
 & & \widehat{\operatorname{coker}\delta^0} \arrow[ur] & \widehat{\operatorname{coker}\delta^1} \arrow[ur] &
\end{tikzcd}$$

One obtains a flabby resolution of $\mathscr{F}$, called its Godement canonical resolution. Applying $\Gamma(X, -)$ yields

$$0 \xrightarrow{d^0} C^0\mathscr{F}(X) \xrightarrow{d^1} C^1\mathscr{F}(X) \xrightarrow{d^2} \cdots ,$$

allowing one to compute $\mathrm{H}^i(X, \mathscr{F}) := \ker d^{i+1} / \operatorname{im} d^i$. Historically, this was Godement's original definition of the groups $\mathrm{H}^i(X, \mathscr{F})$.

## Example

Let $X$ be a smooth manifold. The sheaves of smooth differential forms on open sets of $X$ form a flabby [Lee13, Lemma 2.26] resolution of the constant sheaf $\underline{\mathbb{R}}$,

$$0 \to \underline{\mathbb{R}} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots$$

(where $d$ is the exterior derivates, and exactness follows from Poincaré lemma). Hence, $\mathrm{H}^i(X, \underline{\mathbb{R}}) \simeq \mathrm{H}^i_{\mathsf{dR}}(X)$, where the later denote de Rham cohomology.

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(where $d$ is the exterior derivates, and exactness follows from Poincaré lemma). Hence, $\mathrm{H}^i(X, \underline{\mathbb{R}}) \simeq \mathrm{H}^i_{\mathrm{dR}}(X)$, where the later denote de Rham cohomology. On the other hand, let $\mathrm{C}^i_{\mathrm{sing}}(-, \mathbb{R})$ be the presheaves of singular cochains with real coefficients. There is an exact sequence in of presheaves,

$$0 \to \underline{\mathbb{R}}^{\mathrm{pre}} \xrightarrow{\delta} \mathrm{C}^0_{\mathrm{sing}}(-, \mathbb{R}) \xrightarrow{\delta} \mathrm{C}^1_{\mathrm{sing}}(-, \mathbb{R}) \to \cdots ,$$

where $\delta$ denotes the coboundary maps (because it is exact at the level of stalks, by taking limit through a contractible fundamental system of neighborhoods). Their sheafifications $\mathscr{C}^i$ are flabby, and give a resolution $\underline{\mathbb{R}} \to \mathscr{C}^\bullet$. It follows that $\mathrm{H}^i(X, \underline{\mathbb{R}}) \simeq \mathrm{H}^i(\mathscr{C}^\bullet(X)) \simeq \mathrm{H}^i_{\mathrm{sing}}(X, \mathbb{R})$ (the later isomorphisms are subtle, see [Cib05, Prosition 2.1] for details). This proves de Rham theorem.

Thank you for your attention !

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