The Integration of Differential Forms

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Outline

Notations

Suppose $U \subset \mathbb{C}$ is an open subset.

- Denote by $\mathscr{E}(U)$ the $\mathbb{C}$-algebra of all those functions $f : U \to \mathbb{C}$ which are infinitely differentiable (with respect to the real coordinates).
- Denote by $\mathscr{O}(U)$ the $\mathbb{C}$-algebra of all holomorphic functions on U.

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- Denote by $\mathscr{O}(U)$ the $\mathbb{C}$-algebra of all holomorphic functions on U.

Suppose X is a Riemann surface and $Y \subset X$ is an open subset.

- Denote by $\mathscr{E}(Y)$ the set of all functions $f : Y \to \mathbb{C}$ such that for every chart $z : U \to V \subset \mathbb{C}$ on X with $U \subset Y$ there exists a function $\widetilde{f} \in \mathscr{E}(V)$ with $f|U = \widetilde{f} \circ z$.
- Denote by $\mathscr{E}^{(1)}(Y)$ the vector space of differentiable 1-forms and by $\Omega(Y)$ the vector space of holomorphic 1-forms on Y.
- Denote by $\mathscr{E}^{(2)}(Y)$ the vector space of differentiable 2-forms on Y.

Outline

1. Differential 1-Forms

2. Differential 2-Forms

Suppose X is a Riemann surface and $\omega \in \mathscr{E}^{(1)}(X)$. We consider a piece-wise continuously differentiable curve in X, i.e. a continuous mapping

$$c : [0,1] \to X$$

for which there exists a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

of the interval $[0,1]$ and charts $(U_k, z_k), z_k = x_k + iy_k, k = 1, \ldots, n$, such that $c([t_{k-1}, t_k]) \subset U_k$ and the functions

$$x_k \circ c : [t_{k-1}, t_k] \to \mathbb{R},\ y_k \circ c : [t_{k-1}, t_k] \to \mathbb{R}$$

have continuous first order derivatives.

Definition of integration of 1-forms

On U_k we can write $\omega = f_k\,dx_k + g_k\,dy_k$, where the functions f_k, g_k are differentiable. Set

$$\int_c \omega := \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(f_k(c(t)) \frac{dx_k(c(t))}{dt} + g_k(c(t)) \frac{dy_k(c(t))}{dt} \right) dt.$$

Theorem 1

Suppose X is a Riemann surface, $c : [0,1] \to X$ is a piece-wise continuously differentiable curve and $F \in \mathscr{E}(X)$. Then

$$\int_c dF = F(c(1)) - F(c(0)).$$

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Some properties of primitives:

- A differential form which has a primitive is closed.
- If F is a primitive of ω and $c \in \mathbb{C}$, then $F + c$ is also a primitive of ω.
- If $dF = 0$ then F is a constant.

The Local Existence of Primitives

Consider $U := \{z \in \mathbb{C} : |z| < r, r > 0\}$ and $\omega \in \mathscr{E}^{(1)}(U)$. The differential form ω may be written

$$\omega = f\,dx + g\,dy; \quad f, g \in \mathscr{E}(U),$$

where x, y are the usual real coordinates on $\mathbb{R}^2 \cong \mathbb{C}$.

The Local Existence of Primitives: Closed forms

Assume ω is *closed*. We have

$$d\omega = df \wedge dx + dg \wedge dy = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy,$$

this is equivalent to $\partial g/\partial x = \partial f/\partial y$. We will prove that the following integral is a primitive of ω:

$$F(x,y) := \int_0^1 (f(tx,ty)x + g(tx,ty)y)dt, \text{ for } (x,y) \in U.$$

We can see directly that F is infinitely differentiable. We have to prove that $df = \omega$, i.e. $(\partial F/\partial x) = f$ and $(\partial F/\partial y) = g$. Differentiating we get

$$\frac{\partial F(x,y)}{\partial x} = \int_0^1 \left(\frac{\partial f}{\partial x}(tx,ty)tx + \frac{\partial g}{\partial x}(tx,ty)ty + f(tx,ty)\right) dt.$$

Since

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y} \text{ and } \frac{d}{dt}f(tx,ty) = \frac{\partial f}{\partial x}(tx,ty)x + \frac{\partial f}{\partial y}(tx,ty)y,$$

one then has

$$\begin{aligned}\frac{\partial F(x,y)}{\partial x} &= \int_0^1 \left(t\frac{d}{dt}f(tx,ty) + f(tx,ty)\right) dt \\ &= \int_0^1 \frac{d}{dt}(tf(tx,ty))dt = f(x,y).\end{aligned}$$

Similarly $\partial F/\partial y = g$. This proves that $dF = \omega$.

The Local Existence of Primitives: Holomorphic forms

Assume that ω is *holomorphic*, i.e.,

$$\omega = f\,dz \text{ with } f \in \mathscr{O}(U).$$

Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be the Taylor series expansion of f. Then defining

$$F(z) := \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}$$

gives us a function $F \in \mathscr{O}(U)$ such that $dF = \omega$.

Theorem 2

Suppose X is a Riemann surface and $\omega \in \mathscr{E}^{(1)}(X)$ is a closed differential form. Then there exist a covering map $p : \widehat{X} \to X$ with $\widehat{X}$ connected, and a primitive $F \in \mathscr{E}(\widehat{X})$ of the differential form $p^*\omega$.

Corollaries

1. Suppose X is a Riemann surface, $\pi : \widetilde{X} \to X$ its universal covering and $\omega \in \mathscr{E}^{(1)}(X)$ a closed differential form. Then there exists a primitive $f \in \mathscr{E}(X)$ of $\pi^*\omega$.
2. On a simply connected Riemann surface X every closed differential form $\omega \in \mathscr{E}^{(1)}(X)$ has a primitive $F \in \mathscr{E}(X)$.

Theorem 3

Suppose X is a Riemann surface, $\pi : \widetilde{X} \to X$ its universal covering. Suppose $\omega \in \mathscr{E}^{(1)}(X)$ a closed differential form and $F \in \mathscr{E}(X)$ is a primitive of $p^*\omega$. If $c : [0,1] \to X$ is a piece-wise continuously differentiable curve and $\hat{c} : [0,1] \to \widetilde{X}$ is a lifting of c then

$$\int_c \omega = F(\hat{c}(1)) - F(\hat{c}(0)).$$

Theorem 4

Suppose X is a Riemann surface and $\omega \in \mathscr{E}^{(1)}(X)$ is a closed differential form.

a. If $a, b \in X$ are two points and $u, v : [0,1] \to X$ are two homotopic curves from a to b, then
$$\int_u \omega = \int_v \omega.$$

b. If $u, v : [0,1] \to X$ are two closed curves which are free homotopic, then
$$\int_u \omega = \int_v \omega.$$

Periods

Suppose X is a Riemann surface and $\omega \in \mathscr{E}^{(1)}(X)$ is a closed differential form. Then by theorem 4 one can define the integral

$$a_\sigma := \int_\sigma \omega, \quad \sigma \in \pi_1(X).$$

These integrals are called the **periods** of ω. Clearly

$$\int_{\sigma\cdot\tau} \omega = \int_\sigma \omega + \int_\tau \omega \text{ for } \sigma, \tau \in \pi_1(X).$$

We get a homomorphism $\pi_1(X) \to \mathbb{C}$. This homomorphism is called the **period homomorphism** associated to the closed differential form ω.

Example

Suppose $X = \mathbb{C}^*$; $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$. A generator of $\pi_1(\mathbb{C}^*)$ is represented by the curve $u : [0,1] \to \mathbb{C}^*$, $u(t) = e^{2\pi i t}$. Let $\omega := (dz/z)$, where z is the canonical coordinate. Then

$$\int_u \omega = \int_u \frac{dz}{z} = 2\pi i.$$

Hence the period homomorphism of ω is

$$\mathbb{Z} \to \mathbb{C}, \qquad n \mapsto 2\pi i n.$$

Summands of Automorphy

Suppose X is a Riemann surface and $p : \widetilde{X} \to X$ is its universal covering. We know that $G := \mathrm{Deck}(\widetilde{X}/X) \cong \pi_1(X)$.
If $\sigma \in G$ and $f : \widetilde{X} \to \mathbb{C}$ is a function, then we can define $\sigma f : \widetilde{X} \to \mathbb{C}$ by $\sigma f := f \circ \sigma^{-1}$. A function $f : \widetilde{X} \to \mathbb{C}$ is called **additively automorphic with constant summands of automorphy**, if there exist constants $a_\sigma \in \mathbb{C}, \sigma \in G$, such that

$$f - \sigma f = a_\sigma \text{ for every } \sigma \in G.$$

The constants a_σ are called the **summands of automorphy** of f. The correspondence $\sigma \mapsto a_\sigma$ is a group homomorphism $\mathrm{Deck}(\widetilde{X}/X) \to \mathbb{C}$.

Theorem 5

Suppose X is a Riemann surface and $p : \widetilde{X} \to X$ is its universal covering.

a. If $\omega \in \mathscr{E}^{(1)}(X)$ is a closed differential form and $F \in \mathscr{E}(\widetilde{X})$ is a primitive of $p^*\omega$, then F is additively automorphic with constant summands of automorphy. Its summands of automorphy $a_\sigma, \sigma \in \mathrm{Deck}(\widetilde{X}/X)$, are exactly the periods of ω with respect to the isomorphism $\pi_1(X) \cong \mathrm{Deck}(\widetilde{X}/X)$.
b. Conversely suppose $F \in \mathscr{E}(\widetilde{X})$ is an additively automorphic function with constant summands of automorphy. Then there exists precisely one closed differential form $\omega \in \mathscr{E}^{(1)}(X)$ such that $dF = p^*\omega$.

Example

Suppose $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is a lattice in $\mathbb{C}$. Let $X := \mathbb{C}/\Gamma$.
The canonical quotient map $\pi : \mathbb{C} \to X$ is also the universal covering map and $\mathrm{Deck}(\mathbb{C}/\Gamma)$ is the group of all translations by vectors $\gamma \in \Gamma$.
Consider the identity map $z : \mathbb{C} \to \mathbb{C}$. Then z is additively automorphic under the action of $\mathrm{Deck}(\mathbb{C}/X)$ with summands of automorphy $a_\gamma = \gamma, \gamma \in \Gamma$. Hence dz is invariant under covering transformations. Thus there exists a holomorphic differential form $\omega \in \Omega(X)$ such that $p^*\omega = dz$ and whose periods are exactly the elements of the lattice Γ.

Theorem

Suppose X is a Riemann surface. A closed differential form $\omega \in \mathscr{E}^{(1)}(X)$ has a primitive $f \in \mathscr{E}(X)$ iff all the periods of ω are zero.

Corollary

Suppose X is a compact Riemann surface and $\omega_1, \omega_2 \in \Omega(X)$ are two holomorphic differential forms which define the same period homomorphism $\pi_1(X) \to \mathbb{C}$. Then $\omega_1 = \omega_2$.

Outline

The integration of differential 2-forms on the complex plane

Suppose $U \subset \mathbb{C}$ is open and $\omega \in \mathscr{E}^{(2)}(U)$. Then ω may be written

$$\omega = f\,dx \wedge dy = \frac{i}{2} f\,dz \wedge d\bar{z}, \quad \text{where } f \in \mathscr{E}(U).$$

Assume that f vanishes outside of a compact subset of U. Then define

$$\iint\limits_{U} \omega := \iint\limits_{U} f(x, y) dx\,dy.$$

Suppose V is another open subset of $\mathbb{C}$ and $\phi : V \to U$ is a biholomorphic mapping. Then

$$\iint\limits_{U} \omega = \iint\limits_{V} \phi^* \omega.$$

The integration of differential 2-forms on a Riemann surface

Suppose X is a Riemann surface. The **support** of a differential form ω on X is defined as the closed set

$$\operatorname{Supp}(\omega) := \overline{\{a \in X : \omega(a) \neq 0\}}.$$

The integration of differential forms is defined in two steps.

- Suppose $\phi : U \to V$ is a chart on X and $\omega \in \mathscr{E}^{(2)}(X)$ is a differential form whose support is compact and contained in U. We define

$$\iint_X \omega := \iint_U \omega := \iint_V (\phi^{-1})^* \omega.$$

- Suppose $\omega \in \mathscr{E}^{(2)}(X)$ is an arbitrary differential form with compact support. There exists finitely many charts $\phi_k : U_k \to V_k, k = 1, \ldots, n$ such that

$$\mathrm{Supp}(\omega) \subset \bigcup_{k=1}^{n} U_k.$$

We can find functions $f_k \in \mathscr{E}(X)$ with the following properties (partitions of unity):

1. $\mathrm{Supp}(f_k) \subset U_k$.
2. $\sum_{k=1}^{n} f_k(x) = 1$ for every $x \in \mathrm{Supp}(\omega)$.

Then $f_k\omega$ is a differential form whose support is compact and contained in U and $\omega = \sum_{k=1}^{n} f_k\omega$. Define

$$\iint_X \omega := \sum_{k=1}^{n} \iint_X f_k\omega.$$

Stokes' Theorem (Theorem 6-3, G.Springer, Introduction to Riemann Surfaces)

Suppose S is a Riemann surface. Let G be a regular region on S and ω be a differential 1-form on $\overline{G}$. Then

$$\iint_{\partial G} \omega = \iint_{G} d\omega.$$

Stokes' Theorem on the complex plane

Suppose $U \subset \mathbb{C}$ is open and $A \subset U$ is a compact subset with smooth boundary ∂A. Then for every differential form $\omega \in \mathscr{E}^{(1)}(U)$

$$\iint\limits_A d\omega = \int_{\partial A} \omega.$$

Theorem 7

Suppose X is a Riemann surface and $\omega \in \mathscr{E}^{(1)}(X)$ is a differential form with compact support. Then

$$\iint\limits_X d\omega = 0.$$

Residue Theorem

Suppose X is a compact Riemann surface and $a_1, \dots, a_n$ are distinct points in X. Let $X' := X \setminus \{a_1, \dots, a_n\}$. Then for every holomorphic 1-form $\omega \in \Omega(X')$ we have

$$\sum_{k=1}^{n} \operatorname{Res}_{a_k}(\omega) = 0.$$

Proof

Choose disjoint coordinate neighborhoods (U_k, z_k) of the a_k. We assume that $z_k(a_k) = 0$ and $z_k(U_k) \subset \mathbb{C}$ is a disk. For every k choose a function f_k with compact support $\mathrm{Supp}(f_k) \subset U_k$ such that there exists an open $U_k' \subset U_k$ of a_k with $f_k|U_k' = 1$. Set $g := 1 - (f_1 + \cdots + f_k)$. Then $g|U_k' = 0$. Thus $g\omega \in \mathscr{E}^{(1)}(X)$. From Theorem 7

$$\iint_X d(g\omega) = 0.$$

Proof

Since ω is holomorphic, $d\omega = 0$ on X'. On $U_k' \cap X'$ we have $f_k\omega = \omega$ and thus $d(f_k\omega) = 0$. Hence $d(f_k\omega) \in \mathscr{E}^{(2)}(X)$ whose support is a compact subset of $U_k \setminus \{a_k\}$. Now $d(g\omega) = -\sum d(f_k\omega)$ implies

$$\sum_{k=1}^{n} \iint_X d(f_k\omega) = 0.$$

Hence we only have to show

$$\iint_X d(f_k\omega) = -2\pi i \operatorname{Res}_{a_k}(\omega).$$

Proof

Since the support of $d(f_k\omega)$ is contained in U_k, we only integrate over U_k. We may identify U_k with the unit disk. There exists $0 < \epsilon < R < 1$ such that

$$\operatorname{Supp}(f_k) \subset \{|z_k| < R\} \text{ and } f_k|\{|z_k| \leq \epsilon\} = 1.$$

But then

$$\iint_X d(f_k\omega) = \iint_{\epsilon \leq |z_k| \leq R} d(f_k\omega) = \int_{|z_k|=R} f_k\omega - \int_{|z_k|=\epsilon} f_k\omega$$
$$= -\int_{|z_k|=\epsilon} \omega = -2\pi i \operatorname{Res}_{a_k}(\omega)$$

by the Residue Theorem in the complex plane.

Corollary

Any non-constant meromorphic function f on a compact Riemann surface X has, counting multiplicities, as many zeros as poles.

Thank you for listening!