Meromorphic Functions with Prescribed Princple Parts

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Introduction

The Classical Version of Mittag-Leffler's Theorem

Theorem (Mittag-Leffler)

Let Ω be an open subset of $\mathbb{C}$. Suppose $A \subset \Omega$ has no limit point in Ω, and to each $a \in A$ we associate a rational function:

$$P_a(z) = \sum_{j=1}^{m(a)} c_{j,a}(z-a)^{-j}.$$

Then there exists a meromorphic function $f \in \mathcal{M}(\Omega)$ whose principal part at each $a \in A$ is P_a and which has no other poles in Ω.

Mittag-Leffler's Problem on Compact Riemann Surfaces

Mittag-Leffler Distributions of Meromorphic Functions

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Mittag-Leffler's Problem on Compact Riemann Surfaces

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Definition

A *Mittag-Leffler distribution* (with respect to $\mathfrak{U}$) is meant a cochain $\mu = (f_i) \in C^0(\mathfrak{U}, \mathcal{M})$ such that $f_j - f_i \in \mathcal{O}(U_i \cap U_j)$ for all $i, j \in I$, i.e, $\delta\mu \in Z^1(\mathfrak{U}, \mathcal{O})$.

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Definition

A *solution* of μ is a global meromorphic function $f \in \mathcal{M}(X)$ which has the same principle parts as μ, i.e, $f|U_i - f_i \in \mathcal{O}(U_i)$ for every $i \in I$.

Mittag-Leffler's Problem on Compact Riemann Surfaces

The Existence and Uniqueness of Solutions

Remark If $f_1, f_2 \in \mathcal{M}(X)$ are two solutions of μ, then $f_1 - f_2$ is holomorphic on X and thus constant. Hence the solution is unique up to an additive constant.

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($\Rightarrow$) Suppose $f \in \mathcal{M}(X)$ is a solution of $\mu = (f_i)$. Set $g_i := f_i - f \in \mathcal{O}(U_i)$.

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($\Leftarrow$) Suppose $[\delta\mu] = 0$, then $\delta\mu \in B^1(\mathfrak{U}, \mathcal{O})$. There exists a cochain $(g_i) \in C^0(\mathfrak{U}, \mathcal{O})$ such that $f_j - f_i = g_j - g_i$ on $U_i \cap U_j$.

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($\Leftarrow$) Suppose $[\delta\mu] = 0$, then $\delta\mu \in B^1(\mathfrak{U}, \mathcal{O})$. There exists a cochain $(g_i) \in C^0(\mathfrak{U}, \mathcal{O})$ such that $f_j - f_i = g_j - g_i$ on $U_i \cap U_j$.
Let $h_i = f_i - g_i \in \mathcal{M}(U_i)$, then there exists a meromorphic function $h \in \mathcal{M}(X)$ with $h|U_i = f_i - g_i$. Easily check that h is a solution of μ.

□

Corollary

1. *On the Riemann sphere* $\mathbb{P}^1$ *every Mittag-Leffler distribution has a solution.*
2. *On every compact Riemann surface of genus* $g \geq 1$ *there are Mittag-Leffler distributions which have no solution.*

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Proof.

On every compact Riemann surface X one has $H^1(X, \mathcal{M}) = 0$.
Given any cohomology class $\xi \in H^1(X, \mathcal{O})$, represented by $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$. There exists $\mu = (g_i) \in C^0(\mathfrak{U}, \mathcal{M})$ such that $(f_{ij}) = \delta\mu$, i.e, $f_{ij} = g_j - g_i$ on $U_i \cap U_j$ for all i, j, which implies that μ is a Mittag-Leffler distribution and $\xi = [\delta\mu]$. □

Let $\omega \in \Omega(X)$ be a holomorphic 1-form. Then $\omega\mu \in C^0(\mathfrak{U}, \mathcal{M}^{(1)})$ is a Mittag-Leffler distribution of 1-forms, and the residue $Res(\omega\mu)$ is defined.

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$$\begin{aligned}
\mu \text{ has a solution } &\Leftrightarrow [\delta\mu] = 0 \text{ in } H^1(X, \mathcal{O}).\\
&\Leftrightarrow \lambda([\delta\mu]) = 0,\ \forall \lambda \in H^1(X, \mathcal{O})^*.\\
&\Leftrightarrow < \omega, [\delta\mu] >= 0,\ \forall \omega \in \Omega(X) \text{ (by Serre's Duality)}.\\
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Remark If $\omega_1, ..., \omega_g$ is a basis of $\Omega(X)$, then μ has a solution if and only if $Res(\omega_k\mu) = 0 \ \forall k = 1, 2, ..., g$.

Mittag-Leffler's Problem on Compact Riemann Surfaces

Application to Doubly Periodic Functions.

Let $\gamma_1, \gamma_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ and

$$P := \{t_1\gamma_1 + t_2\gamma_2 : 0 \leq t_1 < 1, 0 \leq t_2 < 1\}.$$

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Given these prescribed principal parts at the points $a_1, ..., a_n \in P$:

$$f_j(z) = \sum_{k=-r_j}^{-1} c_k^{(j)}(z - a_j)^k, \text{ for } j = 1, 2, ..., n.$$

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Then there exists a function $f \in \mathcal{M}(\mathbb{C})$ doubly periodic with respect to Γ and having poles with the prescribed principal parts at $a_1, ..., a_n$ if and only if

$$\sum_{j=1}^{n} c_{-1}^{(j)} = 0.$$

Proof.

Any function doubly periodic with respect to Γ can be considered as a function on $X = \mathbb{C}/\Gamma$. The prescribed principle parts then give rise to a Mittag-Leffler distribution μ on X. Then such a function f exists if and only if μ has a solution, or equivalently, $Res(\omega\mu) = 0$ for every $\omega \in \Omega(X)$.

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Note that $dim\Omega(X) = 1$, it suffices to show that $Res(\omega\mu) = 0$ for some $\omega \in \Omega(X) \setminus \{0\}$.
Choose ω induced by the 1-form dz on $\mathbb{C}$, i.e, $p^*\omega = dz$ ($p : \mathbb{C} \to X$ is the projection). One has:

$$Res(\omega\mu) = \sum_{j=1}^{n} Res_{p(a_j)}(\omega\mu) = \sum_{j=1}^{n} Res_{a_j}(f_j dz) = \sum_{j=1}^{n} c_{-1}^{(j)}.$$

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(The second identity is given by this observation: If $p : (Y, y) \to (X, x)$ is a holomorphic covering map then $Res_x(\omega) = Res_y(p^*\omega)$ for ω is a holomorphic 1-form on a punctured neighborhood of x). □

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Remark The above argument implies that there are no doubly periodic meromorphic functions having precisely one pole of order one in any period parallelogram.

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3. To define the notion of **Weierstrass points** of a Riemann surface X, we need some preliminaries.

Mittag-Leffler's Problem on Compact Riemann Surfaces

The Wronskian Determinant

Definition

Suppose $f_1, ..., f_g$ are holomorphic functions on a domain $U \subset \mathbb{C}$. Define:

$$W(f_1, ..., f_g) := det \begin{bmatrix} f_1 & f_2 & \dots & f_g \\ f_1' & f_2' & \dots & f_g' \\ \vdots & \vdots & & \vdots \\ f_1^{(g-1)} & f_2^{(g-1)} & \dots & f_g^{(g-1)} \end{bmatrix}.$$

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Remark By induction on g, one can show that if the functions $f_1, ..., f_g$ are linearly independent then $W(f_1, ..., f_g) \not\equiv 0$.

Suppose X is a compact Riemann surface of genus $g \geq 1$ and $\omega_1, ..., \omega_g$ is a basis of $\Omega(X)$. On a coordinate neighborhood (U, z) one can write $\omega_k = f_k dz$. Define:

$$W_z(\omega_1, ..., \omega_g) := W(f_1, ..., f_g),$$

where the derivatives of the functions f_k are taken with respect to z.

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Theorem (Change of Coordinates)

If (U, z) and $(\tilde{U}, \tilde{z})$ are two coordinate neighborhoods on X, then on $U \cap \tilde{U}$ one has:

$$W_z(\omega_1, \ldots, \omega_g) = \left(\frac{d\tilde{z}}{dz}\right)^N W_{\tilde{z}}(\omega_1, \ldots, \omega_g), \text{ where } N = \frac{g(g+1)}{2}.$$

Proof.

Set $\psi := \frac{d\tilde{z}}{dz} \in \mathcal{O}(U \cap \tilde{U})$ and define the functions f_k and $\tilde{f}_k$ on $U \cap \tilde{U}$ by

$$\omega_k = f_k dz = \tilde{f}_k d\tilde{z}.$$

Then $f_k = \psi \tilde{f}_k$.

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Then $f_k = \psi \tilde{f}_k$. By induction on m one can prove that:

$$\frac{d^m f_k}{dz^m} = \psi^{m+1} \frac{d^m \tilde{f}_k}{d\tilde{z}^m} + \sum_{\mu=0}^{m-1} \varphi_{m\mu} \frac{d^\mu \tilde{f}_k}{d\tilde{z}^\mu},$$

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where $\varphi_{m\mu} \in \mathcal{O}(U \cap \tilde{U})$ and independent of k. Then:

$$det \left(\frac{d^m f_k}{dz^m} \right)_{0 \le m \le g-1}^{1 \le k \le g} = det \left(\psi^{m+1} \frac{d^m \tilde{f}_k}{d\tilde{z}^m} \right)_{0 \le m \le g-1}^{1 \le k \le g}.$$

The result follows. □

Remark If $\tilde{\omega}_1, ..., \tilde{\omega}_g$ is another basis of $\Omega(X)$, then there exist constants $c_{jk} \in \mathbb{C}$ such that $\omega_j = \sum_k c_{jk}\tilde{\omega}_k$. Let $c = det(c_{jk})$, easily see that $c \neq 0$ and:

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Definition

A point $P \in X$ is called a **Weierstrass point**, if for a basis $\omega_1, ..., \omega_g$ of $\Omega(X)$ and a coordinate neighborhood (U, z) of P, $W_z(\omega_1, ..., \omega_g)$ has a zero at P. The order of this zero is called the weight of the Weierstrass point P.

Theorem

Suppose X is a compact Riemann surface of genus g and P is a point of X. Then there exists a non-constant meromorphic function $f \in \mathcal{M}(X)$ which has a pole of order $\leq g$ at P and is holomorphic on $X \setminus \{P\}$ if and only if P is a Weierstrass point.

Proof.

Suppose $\omega_1, ..., \omega_g$ is a basis of $\Omega(X)$ and (U, z) is a coordinate neighborhood of P with $z(P) = 0$. Near P one can write $\omega_k = \sum_{j=0}^{\infty} a_{kj} z^j dz$, $1 \leq k \leq g$.

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Our desired function f has a principal part at P of the form:

$$h = \sum_{j=0}^{g-1} \frac{c_j}{z^{1+j}}, (c_0, \ldots, c_{g-1}) \neq (0, \ldots, 0).$$

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Consider the following Mittag-Leffler distribution:

$$\mu = (h, 0) \in C^0(\mathfrak{U}, \mathcal{M}) \text{ with } \mathfrak{U} = (U, X \setminus \{P\}).$$

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One has $Res(\omega_k \mu) = Res_P(\omega_k h) = \sum_{j=0}^{g-1} a_{kj} c_j$. Thus μ has a (non-trivial) solution if and only if:

$$det \begin{bmatrix} a_{1,0} & a_{2,0} & \cdots & a_{g,0} \\ a_{1,1} & a_{2,1} & \cdots & a_{g,1} \\ \vdots & \vdots & & \vdots \\ a_{1,(g-1)} & a_{2,(g-1)} & \cdots & a_{g,(g-1)} \end{bmatrix} = 0.$$

Or equivalently, $W_z(\omega_1, ..., \omega_g)(P) = 0$. □

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Theorem

The number of Weierstrass points, counted according to their weights, is $(g-1)g(g+1)$.

Proof.

Firstly, cover X by coordinate neighborhoods $(U_i, z_i)_{i \in I}$. On each $U_i \cap U_j$ define $\psi_{ij} = dz_j/dz_i \in \mathcal{O}^*(U_i \cap U_j)$.

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$$\Rightarrow deg(D) = N deg(D_1) = \frac{g(g+1)}{2}(2g-2) = (g-1)g(g+1).$$

Corollary

Every compact Riemann surface X of genus $g \geq 2$ admits a holomorphic covering mapping $f : X \to \mathbb{P}^1$ having at most g sheets. In particular every compact Riemann surface of genus 2 is hyperelliptic.

THANK YOU FOR LISTENING