Theorem on Formal Functions

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Theorem 1

Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $Y' \subset Y$ be a closed subscheme defined by $\mathcal{I}$ and $X' = X \times_Y Y'$. If $\mathcal{F}$ is a coherent sheaf on X, then there is a canonical isomorphism

$$(R^p f_* \mathcal{F})^\wedge \simeq \varprojlim R^p f_* \mathcal{F}_k$$

where

1. *The left term is $\varprojlim R^p f_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y / \mathcal{I}^k$;*
2. $\mathcal{F}_k = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X / f^* \mathcal{I}^k$.

Higher direct image

Theorem 2

Let $f : X \to Y$ be a separated and quasi-compact morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for every $p \geq 0$, there exists a unique quasi-coherent sheaf $R^p f_* \mathcal{F}$ such that for every affine open subset V of Y, we have

$$R^p f_* \mathcal{F}(V) = H^p(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}).$$

Reduce to Y affine

Since the assertion is local on Y, we can take $U = \operatorname{Spec} A$ an affine open subset of Y. Then $Y' \cap U$ is given by an ideal I of A and

$$R^p f_* \mathcal{F}|_U = (H^p(f^{-1}(U), \mathcal{F}))^\sim$$

We have to prove that

$$\varprojlim (H^p(f^{-1}(U), \mathcal{F}) \otimes A/I^n)^\sim \simeq \varprojlim (H^p(f^{-1}(U), \mathcal{F}/\mathcal{I}^n\mathcal{F}))$$

where $\mathcal{I}$ is the pullback of the coherent ideal associated to I.
Since projective limit of quasi-coherent sheaf is still quasi-coherent, one just have to compare the global section of both side.

Case Y affine

Theorem 3 (EGA III 4.1.7)

Let X be a proper scheme over $\operatorname{Spec} A$, for A Noetherian. Let $I \subset A$ be an ideal whose pullback $\mathcal{I}$ is a sheaf of ideals on X. Fix a coherent sheaf $\mathcal{F}$ on X and define as before the sheaves $\mathcal{F}_k = \mathcal{F}/\mathcal{I}^k\mathcal{F}$. Then the natural morphism

$$H^p(X, \mathcal{F})^\wedge \to \varprojlim H^p(X, \mathcal{F}_k)$$

Proof of Theorem 3: I

For $k \leq k'$, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}^{k'} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_{k'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}^{k} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_{k} & \longrightarrow & 0
\end{array}$$

Hence a diagram of long exact sequence

$$\begin{array}{ccccccc}
H^p(X, \mathcal{I}^{k'}\mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}_{k'}) & \longrightarrow & H^{p+1}(X, \mathcal{I}^{k'}\mathcal{F}) \\
\downarrow & & \| & & \downarrow & & \downarrow \\
H^p(X, \mathcal{I}^{k}\mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}_{k}) & \longrightarrow & H^{p+1}(X, \mathcal{I}^{k}\mathcal{F})
\end{array}$$

Proof of Theorem 3: II

Fix p. For every k, denote

$$\begin{aligned} H &= H^p(X, \mathcal{F}) \\ H_k &= H^p(X, \mathcal{F}_k) \\ R_k &= \operatorname{Im}\left(H^p(X, \mathcal{I}^k\mathcal{F}) \to H^p(X, \mathcal{F})\right) \\ Q_k &= \operatorname{Im}\left(H^p(X, \mathcal{F}_k) \to H^{p+1}(X, \mathcal{I}^k\mathcal{F})\right) \end{aligned}$$

We get an inverse system of exact sequences

$$0 \to R_k \to H \to H_k \to Q_k \to 0$$

Proof of Theorem 3: III

Hence

$$0 \to H/R_k \to H_k \to Q_k \to 0$$

We will prove that:

1. R_k form an *I-adic filtration* for H, so $\varprojlim H/R_k = \widehat{H}$.
2. Morphisms between Q_k's are eventually zero.

Then taking inverse limits yields

$$0 \to \widehat{H} \to \varprojlim H_k \to 0$$

Proof of Theorem 3: IV

First prove R_k form an *I-adic filtration* for H. Clearly $IR^k \subset R_{k+1}$ for all k, since if $x \in I$, the map

$$x : \mathcal{I}^k \mathcal{F} \to \mathcal{I}^{k+1} \mathcal{F}$$

induce the following

$$H^p(X, \mathcal{I}^k \mathcal{F}) \xrightarrow{x} H^p(X, \mathcal{I}^{k+1} \mathcal{F}) \to H^p(X, \mathcal{F})$$

We will need the following results to prove the other inclusion.

The finiteness theorem(s)

Theorem 4

If $f : X \to Y$ is a proper morphism, with Y locally Noetherian, and $\mathcal{F}$ a coherent sheaf on X, then, for all $p \geq 0$, the sheaves $R^p f_* \mathcal{F}$ are coherent.

We need a graded variant of the above theorem (EGA III 3.3.1):

Theorem 5

Let $f : X \to Y$ be a proper morphism of Noetherian schemes and let $\mathcal{S}$ be a quasi-coherent, finitely generated algebra over $\mathcal{O}_Y$. Then if $\mathcal{F}$ is a quasi-coherent sheaf on X which is a finitely generated $f^*(\mathcal{S})$, the higher direct image $R^p f_* \mathcal{F}$ is finitely generated over $\mathcal{S}$.

Proof of Theorem 3 (continue) I

Consider the **blowup algebra** $A \oplus I \oplus I^2 \oplus \dots$, which is a finitely generated A-algebra, and its sheafish version $\mathcal{S}$ on $\operatorname{Spec} A$.

$$f^*(\mathcal{S}) = \mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots$$

Now the quasi-coherent sheaf $\mathcal{F} \oplus \mathcal{I}\mathcal{F} \oplus \dots$ is f.g. over $f^*(\mathcal{S})$. It follows that

$$H^p(X, \mathcal{F}) \oplus H^p(X, \mathcal{I}\mathcal{F}) \oplus \dots$$

is finitely generated over $A \oplus I \oplus \dots$ By finite generation, its follow that

$$IH^p(X, \mathcal{I}^k\mathcal{F}) = H^p(X, \mathcal{I}^{k+1}\mathcal{F})$$

for all large k (Atiyah's Lemma 10.8). The image of these are R_k, so $IR_k = R_{k+1}$ for large k.

Proof of Theorem 3 (continue) II

Now we deal with the Q_k. Recall the definition

$$Q_k = \text{Ker}\left(H^{p+1}(X, \mathcal{I}^k\mathcal{F}) \to H^{p+1}(X, \mathcal{F})\right)$$

or

$$Q_k = \text{Im}\left(H^p(X, \mathcal{F}_k) \to H^{p+1}(X, \mathcal{I}^k\mathcal{F})\right)$$

By the first definition, there are canonical map $Q_k \times I \to Q_{k+1}$, we find that $Q = \bigoplus Q_k$ is a module over $A \oplus I \oplus \dots$. In fact, Q is a submodule of $\bigoplus H^{p+1}(X, \mathcal{I}^k\mathcal{F})$. This is f.g. by Theorem 5, so is Q. However, from the second definition Q_k is annihilated by I^k. It follows that there is a high power N such that $I^N \oplus I^{N+1} \oplus \dots$ annihilates the entire Q.

Proof of Theorem 3 (continue) III

Now the map $Q_{k+r} \to Q_k$ is induced by the usual map

$$H^{p+1}(X, \mathcal{I}^{k+r}\mathcal{F}) \to H^{p+1}(X, \mathcal{I}^{k}\mathcal{F})$$

We have a sequence

$$Q_k \otimes_A I^r \to Q_{k+r} \to Q_k$$

By finite generation, the first map is surjective for large r and the composite is zero for $r \geq N$ by the above reasoning.
It follows that $Q_{k+r} \to Q_k$ is zero if r sufficiently large.
That completes the proof. □

Geometric version

The following is an example of Theorem 1.

Theorem 6

Let $f : X \to Y$ be a proper morphism of Noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X. Suppose furthermore that $y \in Y$. Then there is a canonical isomorphism between

$$((R^p f_* \mathcal{F})_y)^\wedge \simeq \varprojlim H^p(X_y, \mathcal{F}/\mathfrak{m}_y^k \mathcal{F})$$

where the first term is the completion of the stalk $(R^p f_* \mathcal{F})_y$ at the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}_y$.

First, make the flat base change $\operatorname{Spec} \mathcal{O}_y \to Y$. The formation of higher direct images commutes with flat base change (Qing Liu, Corollary 2.27), and consequently we can reduce to the case where $Y = \operatorname{Spec} \mathcal{O}_y$. As a result, y is a closed point. In this case, the result is a direct application of Theorem 1.

The following is a corollary of Theorem 6.

Corollary 7

Let $f : X \to Y$ be a proper morphism whose fibers are of dimension $\leq r$. Then $R^p f_* \mathcal{F}$ for any coherent sheaf $\mathcal{F}$ if $p > r$.

By Grothendieck's Vanishing Theorem, for all $y \in Y$ and $p > r$ the second term in Theorem 6 is zero. Therefore all the stalks of $R^p f_* \mathcal{F}$ is zero.

Corollary 8 (Zariski's connectedness principle)

Let $f : X \to Y$ be a proper morphism of Noetherian schemes, and assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then $f^{-1}(y)$ is connected, for every $y \in Y$.

Assume that $f^{-1}(y) = X' \cup X''$ and let $\mathcal{F}_k = \mathcal{O}_X/\mathfrak{m}_y^k\mathcal{O}_X$. Then

$$H^0(f^{-1}(y), \mathcal{F}_k) = H^0(X', \mathcal{F}_k) \oplus H^0(X'', \mathcal{F}_k)$$

By Theorem 6

$$\widehat{\mathcal{O}}_y = (f_*\mathcal{O}_X)_y^\wedge = \varprojlim H^0(f^{-1}(y), \mathcal{F}_k)$$

Therefore the local ring $\widehat{\mathcal{O}}$ is the sum of two other rings, which absurd. (See Hartshorne III.11.3 or Atiyah, Exercise 1.12, 1.22).

Zariski's Main Theorem

Theorem 9 (Zariski's Main Theorem)

Let $f : X \to Y$ be a birational proper morphism of Noetherian integral schemes, and assume that Y is normal. Then for every $y \in Y, f^{-1}(y)$ is connected.

By the previous result, we have only to verify that $f_*\mathcal{O}_X = \mathcal{O}_Y$. The question is local so we can assume $Y = \operatorname{Spec} A$. Then $B = \Gamma(Y, f_*\mathcal{O}_X)$ is f.g. A-module. But A and B are integral domains with the same quotient field, and A is integrally closed, so we must have $A = B$. Thus $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Stein factorization

Theorem 10

Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Then one can factor f into $g \circ f'$, where $f' : X \to Y'$ is a proper morphism with connected fibers, and $g : Y' \to Y$ is a finite morphism.