Flat morphism

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A map of rings $A \rightarrow B$ is called flat, if it is a homomorphism that makes B a flat $A-$module.

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Flatness is preserved by change of base ring

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An A-module M is flat if and only if M_p is a flat A_p-module for all prime ideals p.

A finitely generated module M over a local noetherian ring is flat if and only if it is free.

Flat quasi-coherent sheaf

We say that a quasi-coherent sheaf $\mathscr{F}$ on a scheme X is flat at $p \in X$ if $\mathscr{F}_p$ is a flat $\mathscr{O}_{X,p}$-module. We say that $\mathscr{F}$ is flat over X if it is flat at all $p \in X$.

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We say that a morphim of schemes $f : X \to Y$ is flat at $p \in X$ if $\mathscr{O}_{X,p}$ is a flat $\mathscr{O}_{Y,f(p)}-$module. We say that f is flat if it is flat at all $p \in X$.

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Remark

A morphism $f : X \to Y$ is flat if and only if the pull-back functor $f^* : Qcoh(Y) \to Qcoh(X)$ is exact.

Base change

Let $f : X \to Y$ be a morphism, let $\mathscr{F}$ be an quasi-coherent sheaf on X which is flat over Y, and let $g : Y' \to Y$ be any morphism. Let $X' = X \times_Y Y'$, and let $\mathscr{F}' = p_1^*(\mathscr{F})$. Then $\mathscr{F}'$ is flat over Y'.

$$
\begin{array}{ccc}
(p')^*\mathscr{F} & & \mathscr{F} \\
\Big| & & \Big| \\
X \times_Y Y' & \xrightarrow{p'} & X \\
\Big\downarrow & & \Big\downarrow f \\
Y' & \xrightarrow{p} & Y
\end{array}
$$

Base change

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We may assume all the schemes involved are affine:
$X = Spec(B)$, $Y = Spec(A)$, $Y' = Spec(A')$ and $X' = Spec(B')$ where $B' = B \underset{A}{\otimes} A'$.

$\mathscr{F} = \widetilde{M}$, where M is a $B-$module M and $p_1^* \mathscr{F} = M \underset{B}{\otimes} B'$.

Since M is flat over A,

$$M \underset{B}{\otimes} B' = M \underset{B}{\otimes} B \underset{A}{\otimes} A' = M \underset{A}{\otimes} A' \text{ is flat over } A'.$$

Propositon

A map of rings $\phi : A \to B$ is flat if and only if the corresponding map of schemes $f : Spec(B) \to Spec(A)$ is flat.

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A map of rings $\phi : A \to B$ is flat if and only if the corresponding map of schemes $f : Spec(B) \to Spec(A)$ is flat.

Suppose ϕ is flat. Given $p \in Spec(B)$. Then for all primes ideal q of A, where $q = \phi^{-1}(p)$, B_q is flat over A_q. As B_p is flat over B_q, it follows that B_p is flat over A_q.

Conversely, suppose that the sequence of A-modules $0 \to M \to N$ is exact. Let K be the kernel of $M \underset{A}{\otimes} B \to N \underset{A}{\otimes} B$, i.e we have an exact sequence

$$0 \to K \to M \underset{A}{\otimes} B \to N \underset{A}{\otimes} B$$

Given any $p \in Spec(B)$, we have an exact sequence

$$0 \to K \underset{B}{\otimes} B_p \to (M \underset{A}{\otimes} B) \underset{B}{\otimes} B_p \to (N \underset{A}{\otimes} B) \underset{B}{\otimes} B_p$$

As $(M \underset{A}{\otimes} B) \underset{B}{\otimes} B_p \cong M \underset{A}{\otimes} B_p$ and $A \to A_q \to B_p$ is flat where $q = f(p)$, we have $K_p \cong K \underset{B}{\otimes} B_p = 0$ for every $p \in Spec(B)$. It follows that $K = 0$. Hence B is flat over A.

The composite of two flat morphisms is flat

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Let X be a noetherian scheme, and $\mathscr{F}$ a coherent $\mathscr{O}_X-$ module. Then $\mathscr{F}$ is flat if and only if it is locally free.

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Proposition

If X is scheme and $p \in X$, the natural morphism $Spec\mathscr{O}_{X,p} \to X$ is flat.

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Proposition

If X is scheme and $p \in X$, the natural morphism $Spec\mathscr{O}_{X,p} \to X$ is flat.

Let $p \in U = Spec(R) \subset X$ be any open affine neighbourhood containing x.
Then, $Spec\mathscr{O}_{X,p} = Spec(R_p) \to Spec(R) \subset X$ is flat
since $R \to R_p$ is flat.

Euler characteristic

Let X be a projective scheme over field k and $\mathscr{F} \in Coh(X)$.
We define the Euler characteristic of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum (-1)^i . dim_k H^i(X, \mathscr{F})$$

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Hilbert polynomial

Let X be a projective scheme over a field k, let $\mathscr{O}_x(1)$ be a very ample invertible sheaf on X over k, and $\mathscr{F} \in Coh(X)$.
Then there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that

$$\chi(\mathscr{F}(n)) = P(n)$$

for all $n \in \mathbb{Z}$.

Theorem

Let $f : X \to Y$ be a projective morphism of locally Noetherian schemes, and $\mathscr{F} \in Coh(X)$ is flat over Y.
Then $\chi(X_y, \mathscr{F}_y)$ is a locally constant function of $y \in Y$

Proposition

If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $R-$modules and M'' is flat,
then $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact for any N.

Proposition

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then $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact for any N.

We have a long exact sequence:

$$\begin{array}{ccccccccc} Tor_1(M'', N) & \longrightarrow & M' \otimes N & \longrightarrow & M \otimes N & \longrightarrow & M'' \otimes N & \longrightarrow & 0 \\ \| & & & & & & & & \\ 0 & & & & & & & & \end{array}$$

Proposition

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R-$modules.

1. If M' and M'' are flat, then M is flat
2. If M and M'' are flat, then M' is flat.

Proposition

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1. For any N, we have an exact sequence

$$\begin{array}{ccccc} Tor_1(M', N) & \longrightarrow & Tor_1(M, N) & \longrightarrow & Tor_1(M'', N) \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

Therefore $Tor_1(M, N) = 0$ for any N, i.e M is flat.

Corollary

Let $0 \to M_0 \to M_1 \to \ldots \to M_n \to 0$ be an exact sequence of $R-$modules. Then

- if each M_i is flat, then tensoring $- \underset{R}{\otimes} N$ with any N is exact
- if the M_i are flat for $i \neq 0$, then M_0 is also flat.

Corollary

Let $0 \to M_0 \to M_1 \to ... \to M_n \to 0$ be an exact sequence of $R-$modules. Then

- if each M_i is flat, then tensoring $-\underset{R}{\otimes} N$ with any N is exact
- if the M_i are flat for $i \neq 0$, then M_0 is also flat.

Splitting this sequence into short exact sequences and using the previous results and induction.

Theorem

Let $f : X \to Y$ be a projective morphism of locally Noetherian schemes, and $\mathscr{F} \in Coh(X)$ is flat over Y.
Then $\chi(X_y, \mathscr{F}_y)$ is a locally constant function of $y \in Y$

We may assume $Y = SpecB$ where B is Noetherian.
We may assume that $X = \mathbb{P}^n_B$.

Take r large enough so that $\mathscr{F}(m)$ has vanishing higher cohomology for $m > r$.

We compute $H^i(X, \mathscr{F}(m))$ by $\check{C}ech$ cohomology using the standard open affine cover $\mathfrak{U}$ of X. Then

$$H^i(X, \mathscr{F}(m)) = h^i(C^{\cdot}(\mathfrak{U}, \mathscr{F}(m))$$

Therefore we have an exact sequence of $B-$module

$$0 \to H^0(X, \mathscr{F}(m)) \to C^0(\mathfrak{U}, \mathscr{F}(m)) \to ... \to C^n(\mathfrak{U}, \mathscr{F}(m)) \to 0$$

The terms C^i are all flat.
Thus, the term $H^0(X, \mathscr{F}(m)) = H^0(Y, f_*\mathscr{F})$ is flat.
But $f_*\mathscr{F}$ is also coherent, and hence it is locally free sheaf of finite rank.

This exact sequence is still exact after tensoring with $k(y)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(Y, f_*\mathscr{F}(m)) \otimes k(y) & \longrightarrow & \dots C^n(\mathfrak{U}, \mathscr{F}(m)) \otimes k(y) & \longrightarrow & 0 \\ & & & & \| & & \\ 0 & \longrightarrow & H^0(X_y, \mathscr{F}_y(m)) & \longrightarrow & \dots C^n(\mathfrak{U}_y, \mathscr{F}_y(m)) & \longrightarrow & 0 \end{array}$$

Thus $H^0(X_y, \mathscr{F}_y(m)) = \chi(X_y, \mathscr{F}_y(m))$ and the Hilbert polynomial of X_y is $P_t(m) = dim_{k(y)} H^0(X_y, \mathscr{O}_{X_y}(m))$ for $m > r$.

We also have $H^0(Y, f_*\mathscr{F}(m)) \otimes k(y) \cong H^0(X_y, \mathscr{F}_y(m))$.

So the Hilbert polynomial

$$P_y(m) = dim_{k(y)} H^0(X_y, \mathscr{O}_{X_y}(m))$$

is the rank at y of a locally free sheaf, which is a locally constant funtion of $y \in Y$.

Example

Define $\phi : k[b] \to k[x]$ by $\phi(b) = x^2$. Let $f : X = \mathbb{A}^1_k \to Y = \mathbb{A}^1_k$ be the map induced by ϕ. Then f is flat.

We have

$$\begin{array}{ccc} k[x] \underset{k[b]}{\otimes} k[b]/(b-t) \cong k[x]/(x^2-t) & \longleftarrow & k[b]/(b-t) \\ \uparrow & & \uparrow \\ k[x] & \longleftarrow & k[b] \end{array}$$

So the fiber of f over $y = (b-t)$ is $Spec(k[x]/(x^2-t))$

And $\chi(X_y, (\mathcal{O}_X)_y) = dim_k H^0(X_y, \mathcal{O}_{X_y}) = dim_k(k[x]/(x^2-t)) = 2.$

Example

Define $\phi : k[x,y,z]/(xz - y^2) \to k[x,y]$ by $\phi(\bar{g}) = g(x^2, xy, y^2)$.
Let $v : X = \mathbb{A}^2_k \to Y = Spec(k[x,y,z]/(xz - y^2) \subset \mathbb{A}^3_k$ be the Veronese map induced by ϕ.
The fiber of $y = (a,b,c) \in Y$ is

$$X_y = Spec(k[x,y]/(x^2 - a, xy - b, y^2 - c))$$

If $a = b = c = 0$, $\chi(X_y, (\mathscr{O}_X)_y) = dim_k(k[x,y]/(x^2, xy, y^2) = 3$.
However if $a \neq 0$ and $b = c = 0$,

$$\chi(X_y, (\mathscr{O}_X)_y) = dim_k(k[x,y]/(x^2 - a, xy, y^2) = 2$$

Therefore v is not flat.

Proposition

Suppose $f : X \to Y$ be a morphism of locally Noetherian schemes.
Let $x \in X$ and $y = f(x)$.
Then

$$codim_X x \leq codim_Y y + codim_{f^{-1}(y)} x$$

Definition

An $R-$module M is faithfully flat if a sequence

$$0 \to N' \to N \to N'' \to 0$$

is exact if and only if

$$0 \to N' \underset{R}{\otimes} M \to N \underset{R}{\otimes} M \to N'' \underset{R}{\otimes} M \to 0$$

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is exact if and only if

$$0 \to N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$$

is exact.

Proposition

A flat $R-$module M is faithfully flat if and only if $M \otimes N = 0$ implies $N = 0$.

Proposition

M is faithfully flat if and only if it is flat and M/mM is nonzero for every maximal ideal m in R.

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Proposition

A flat, local map $f : A \to B$ of local rings is faithfully flat.

Suppose that M is a non-zero $A-$module such that $M \otimes B = 0$.

Pick an injection $A/I \hookrightarrow M$ where I is a proper ideal of A. Since B is flat over A, it follows that

$$B/f(I)B \cong A/I \otimes B \hookrightarrow M \otimes B = 0$$

However,

$$f(I) \subset f(m_A) \subset m_B$$

, which is a contradiction.

Definition

A map of schemes $f : X \to Y$ is faithfully flat if it is flat and surjective.

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Proposition

A map of affine schemes $f : Spec(A) \to Spec(B)$ is faithfully flat if and only if the corresponding map ϕ on rings is faithfully flat.

Definition

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Proposition

A map of affine schemes $f : Spec(A) \to Spec(B)$ is faithfully flat if and only if the corresponding map ϕ on rings is faithfully flat.

Suppose $B \to A$ is faithfully flat.
The fiber $f^{-1}(y) = Spec(A \otimes_B k(y))$ is nonzero since $k(y)$ is nonzero.

Definition

A map of schemes $f : X \to Y$ is faithfully flat if it is flat and surjective.

Proposition

A map of affine schemes $f : Spec(A) \to Spec(B)$ is faithfully flat if and only if the corresponding map ϕ on rings is faithfully flat.

Suppose $B \to A$ is faithfully flat.
The fiber $f^{-1}(y) = Spec(A \bigotimes_B k(y))$ is nonzero since $k(y)$ is nonzero.
Conversely, suppose f is flat and surjective.Then for every $P \in Spec(B)$, there exists $Q \in Spec(A)$ such that $f(Q) = P$.
Then A/PA surjects onto $A/QA \neq 0$.
Thus $B \to A$ is faithfully flat.

Going down theorem for flat morphisms

Let $f : X \to Y$ be a flat map and $f(x) = y$.
If y' is a generalization of y,
then there exists a generalization x' of x such that $f(x') = y'$.
In other words, $f(Spec\mathscr{O}_{X,x}) = Spec\mathscr{O}_{Y,f(x)}$.

Going down theorem for flat morphisms

Let $f : X \to Y$ be a flat map and $f(x) = y$.
If y' is a generization of y,
then there exists a generization x' of x such that $f(x') = y'$.
In other words, $f(Spec\mathscr{O}_{X,x}) = Spec\mathscr{O}_{Y,f(x)}$.

Consider the commutative diagram

$$\begin{array}{ccc} Spec(\mathscr{O}_{X,x}) & \longrightarrow & X \\ \downarrow{\scriptstyle f_x} & & \downarrow{\scriptstyle f} \\ Spec(\mathscr{O}_{Y,y}) & \longrightarrow & Y \end{array}$$

Since f_x is surjective, it follows that $f(Spec\mathscr{O}_{X,x}) = Spec\mathscr{O}_{Y,f(x)}$.

Going down theorem for flat morphisms

Let $f : X \to Y$ be a flat map and $f(x) = y$.
If y' is a generization of y,
then there exists a generization x' of x such that $f(x') = y'$.
In other words, $f(Spec\mathscr{O}_{X,x}) = Spec\mathscr{O}_{Y,f(x)}$.

This allows us to prove the " $\geq$ "

Theorem

Suppose $f : X \to Y$ be a flat morphism of locally Noetherian schemes. Let $x \in X$ and $y = f(x)$.
Then

$$codim_X x = codim_Y y + codim_{f^{-1}(y)} x$$

In particular, if X and Y are irreducible varieties, then the fibers of f have dimension $dimX - dimY$

We have a chain of irreducible closed subset

$$y \subset \bar{y} \subset \bar{y_1} \subset ... \subset \bar{y_r} \subset Y$$

By the going down theorem, this can be lifted to X

$$x \subset \bar{x} \subset \bar{x_1} \subset ... \subset \bar{x_r} \subset X$$

The inverse image of a chain in $f^{-1}(y)$ gives a chain in X

$$x \subset \bar{x} \subset \bar{z_1} \subset ... \subset \bar{z_t}$$

The concatenation of these two chains gives a chain in X

$$x \subset \bar{x} \subset ... \subset \bar{z_1} \subset ... \subset \bar{x_1} \subset ... \subset \bar{x_r} \subset ... \subset X$$

Example

The blow-up of $\mathbb{A}_k^2$ at O is defined as :

$$B_0\mathbb{A} = V(xu - yt) \subset \mathbb{A}_k^2 \times \mathbb{P}$$

where the affine coordinates are $(x; y)$ and the projective coordinates are $[t : u]$.
The map $\pi : B_0\mathbb{A}_k^2 \to \mathbb{A}_k^2$ is the restriction to $B_0\mathbb{A}^2$ of the projection map $\mathbb{A}_k^2 \times \mathbb{P} \to \mathbb{A}_k^2$.
π is an isomorphism on the complement of $\pi^{-1}(O)$ and $\pi^{-1}(O) \cong \mathbb{P}$.
Therefore the dimension of $\pi^{-1}(p)$ jumps at $p = 0$.
Hence $\pi : B_0\mathbb{A}_k^2 \to \mathbb{A}_k^2$ is not flat.

Proposition

A flat morphism $f : X \to Y$ of finite type of noertherian schemes is open.

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Proposition

Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X | f \text{ is flat at } x\}$ is an open subset of X

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