The Exact Cohomology Sequence on Riemann surfaces

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Sheaf homomorphism

Definition

$\mathscr{F}, \mathscr{G}$ are sheaves of abelian groups on the topological space X. A *sheaf homomorphism* $\alpha : \mathscr{F} \to \mathscr{G}$ is a family of group homomorphisms:

$$\alpha_u : \mathscr{F}(U) \to \mathscr{G}(U)$$

which compatible with the restriction homomorphism:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\alpha_U} & \mathscr{G}(U) \\ \downarrow{\scriptstyle restr.} & & \downarrow{\scriptstyle restr.} \\ \mathscr{F}(V) & \xrightarrow{\alpha_V} & \mathscr{G}(V) \end{array}$$

is commutative for every $V \subset U \subset X$.

Sheaf homomorphism

Example 1

$\mathscr{E}, \mathscr{E}^{(1)}, \mathscr{E}^{(2)}$ are the sheaves of differentiable functions, 1-forms and 2-forms on a Riemann surface X. The exterior derivative d on functions and differentiable forms induces sheaf homomorphism.

$$d : \mathscr{E} \to \mathscr{E}^{(1)}$$

$$d : \mathscr{E}^{(1)} \to \mathscr{E}^{(2)}$$

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For $V \subset U \subset X$, we define:
$d_U : \mathscr{E}(U) \to \mathscr{E}^{(1)}(U)$: $f \mapsto df$, $(df)(a) := d_a f \in T_a^{(1)}$, for $a \in U$.
$d_V : \mathscr{E}(V) \to \mathscr{E}^{(1)}(V)$: $f \mapsto df$, $(df)(a) := d_a f \in T_a^{(1)}$, for $a \in V$.
$\mathscr{E}(U) \xrightarrow{restr.} \mathscr{E}(V) : f \mapsto f|_V$, $\mathscr{E}^{(1)}(U) \xrightarrow{restr.} \mathscr{E}^{(1)}(V) : df \mapsto df|_V$.

Sheaf homomorphism

d_U is a vector space homomorphism since
$d_U(zf + z'g)(a) = d(zf + z'g)(a) = d_a(zf + z'g) =$
$(zf + z'g)(a) - (zf + z'g) = (zf(a) - zf) + (z'g(a) - z'g) =$
$zd_a(f) + z'd_a(g) = zd_U(f)(a) + z'd_U(g)(a)$, for every
$z, z' \in \mathbb{C}, f, g \in \mathscr{E}(U)$
For $f \in \mathscr{E}(U) : restr. \circ d_U(f) = d_V \circ restr.(f)$ since
$restr. \circ d_U(f)(a) = d_a(f) = d_V \circ restr.(f)(a)$ for every $a \in V$.

Sheaf homomorphism

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$zd_a(f) + z'd_a(g) = zd_U(f)(a) + z'd_U(g)(a)$, for every
$z, z' \in \mathbb{C}, f, g \in \mathscr{E}(U)$
For $f \in \mathscr{E}(U) : restr. \circ d_U(f) = d_V \circ restr.(f)$ since
$restr. \circ d_U(f)(a) = d_a(f) = d_V \circ restr.(f)(a)$ for every $a \in V$.
Hence the diagram

$$\begin{array}{ccc} \mathscr{E}(U) & \xrightarrow{d_U} & \mathscr{E}^{(1)}(U) \\ \big\downarrow{\scriptstyle restr.} & & \big\downarrow{\scriptstyle restr.} \\ \mathscr{E}(V) & \xrightarrow{d_V} & \mathscr{E}^{(1)}(V) \end{array}$$

is commutative.

Sheaf homomorphism

Example 2

$\mathscr{O}, \mathscr{O}^*$ are sheaf of homomorphic functions and multiplicative sheaf of homomorphic functions with value in $\mathbb{C}^*$ on Riemann surface X.
U :open$\subset X$, we define $ex_U : \mathscr{O}(U) \to \mathscr{O}^*(U) : f \mapsto \exp(2\pi i f)$.
$ex : \mathscr{O} \to \mathscr{O}^*$ is a sheaf homomorphism.

Sheaf homomorphism

Example 2

$\mathscr{O}, \mathscr{O}^*$ are sheaf of homomorphic functions and multiplicative sheaf of homomorphic functions with value in $\mathbb{C}^*$ on Riemann surface X. U :open$\subset X$, we define $ex_U : \mathscr{O}(U) \to \mathscr{O}^*(U) : f \mapsto \exp(2\pi i f)$. $ex : \mathscr{O} \to \mathscr{O}^*$ is a sheaf homomorphism.

It is easy to check that ex_U is a group homomorphism. Since $restr. \circ ex_U(f) = \exp(2\pi i f)|_V = \exp(2\pi i f|_V) = ex_V \circ restr$, the diagram

$$\begin{array}{ccc} \mathscr{O}(U) & \xrightarrow{ex_U} & \mathscr{O}^*(U) \\ \downarrow{\scriptstyle restr.} & & \downarrow{\scriptstyle restr.} \\ \mathscr{O}(V) & \xrightarrow{ex_V} & \mathscr{O}^*(V) \end{array}$$

is commutative.

The Kernel of a Sheaf Homomorphism

Definition

$\mathscr{F}, \mathscr{G}$: sheaves on a topological space X and $\alpha : \mathscr{F} \to \mathscr{G}$ is a sheaf homomorphism. U : open $\subset X$. Let

$$\mathscr{K}(U) := \ker(\mathscr{F}(U) \to \mathscr{G}(U))$$

The family of groups $\{\mathscr{K}(U)\}_{U:open}$ together with the natural restriction is a sheaf $\mathscr{K}$. It is called kernel of α and is denoted by $\mathscr{K} = \ker \alpha$.

The Kernel of a Sheaf Homomorphism

Examples

1. $\mathcal{O} = \ker(\mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1})$.
   Proof: For $f \in \mathcal{E}(U)$, $a \in U$, $d''f(a) = d''_a f = \frac{\partial f}{\partial \bar{z}}(a) d_a \bar{z}$.
   $d''f = 0 \Leftrightarrow d''f(a) = 0 \forall a \in U \Leftrightarrow \frac{\partial f}{\partial \bar{z}}(a) = 0 \forall a \in U \Leftrightarrow f \in \mathcal{O}(U)$. □

The Kernel of a Sheaf Homomorphism

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1. $\mathscr{O} = \ker(\mathscr{E} \xrightarrow{d''} \mathscr{E}^{0,1})$.
   Proof: For $f \in \mathscr{E}(U), a \in U$, $d''f(a) = d''_a f = \frac{\partial f}{\partial \bar{z}}(a) d_a \bar{z}$.
   $d''f = 0 \Leftrightarrow d''f(a) = 0 \forall a \in U \Leftrightarrow \frac{\partial f}{\partial \bar{z}}(a) = 0 \forall a \in U \Leftrightarrow f \in \mathscr{O}(U)$. □
2. $\Omega = \ker(\mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^{(2)})$
   Proof: By Theorem 9.16, every closed 1-form $\omega \in \mathscr{E}^{1,0}(U)$ is holomorphic,i.e., $d\omega = 0$ implies $\omega \in \Omega(U)$, we get $\Omega \supset \ker(\mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^{(2)})$. Also by Theorem 9.16, every holomorphic 1-form $\omega \in \Omega(U)$ is closed, i.e., $d\omega = 0$ and using the obvious fact that $\Omega \subset \mathscr{E}^{1,0}$ we get $\Omega \subset \ker(\mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^{(2)})$. □

The Image of a Sheaf Homomorphism

Definition

$\alpha : \mathscr{F} \to \mathscr{G}$ is a sheaf homomorphism on the topological space X. We define:

$$\mathscr{B}(U) := \mathrm{Im}(\mathscr{F}(U) \xrightarrow{\alpha} \mathscr{G}(U))$$

$\{\mathscr{B}(U)\}_{U:open}$ defines a presheaf $\mathscr{B}$. $\mathscr{B}$ in general is not a sheaf.

The Image of a Sheaf Homomorphism

Counter-example

Consider the sheaf homomorphism: $ex : \mathscr{O} \to \mathscr{O}^*$ on space $\mathbb{C}^*$. Let $U_1 = \mathbb{C}^* - \mathbb{R}_-$ and $U_2 = \mathbb{C}^* - \mathbb{R}_+$. We define $f_k \in \mathscr{O}^*(U_k)$ by $f_k(z) = z$ for every $z \in U_k$, $k = 1, 2$. Since U_k is simply connected, there exists $h_k \in \mathscr{O}(U_k)$ s.t $f_k = \exp(h_k)$. Hence, $f_k = ex_{U_k}(h_k/(2\pi i))$, i.e.,

$$f_k \in \operatorname{Im}(\mathscr{O}(U_k) \xrightarrow{ex} \mathscr{O}^*(U_k))$$

and $f_1 = f_2$ on $U_1 \cap U_2$ but there is no

$$f \in \operatorname{Im}(\mathscr{O}(\mathbb{C}^*) \xrightarrow{ex} \mathscr{O}^*(\mathbb{C}^*))$$

s.t $f|_{U_k} = f_k$ since $\log(z) \notin \mathscr{O}(\mathbb{C}^*)$.

Exact sequences

Definition

$\alpha : \mathscr{F} \to \mathscr{G}$ is a sheaf homomorphism on the topological space X. For each $x \in X$, there is an induced homomorphism of the stalks $\alpha_x : \mathscr{F}_x \to \mathscr{G}_x$.

A sequence $\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ is called *exact* if for any $x \in X$, the sequence

$$\mathscr{F}_x \xrightarrow{\alpha_x} \mathscr{G}_x \xrightarrow{\beta_x} \mathscr{H}_x$$

is exact. A sequence

$$\mathscr{F}_1 \xrightarrow{\alpha_1} \mathscr{F}_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} \mathscr{F}_n$$

is called exact if the sequence $\mathscr{F}_k \xrightarrow{\alpha_k} \mathscr{F}_{k+1} \xrightarrow{\alpha_{k+1}} \mathscr{F}_2$ is exact for every $1 \leq k \leq n-2$.

Exact sequence

Lemma 1

$\alpha : \mathscr{F} \to \mathscr{G}$ is a sheaf monomorphism on the topological space X. Then for every open $U \subset X$, the mapping $\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.

Fact

$\alpha : \mathscr{F} \to \mathscr{G}$ is a sheaf epimorphism on the topological space X. It is not generally true that $\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is surjective.

We consider $ex : \mathscr{O} \to \mathscr{O}^*$ as a counter example, $ex_x : \mathscr{O}_x \to \mathscr{O}_x^*$ is surjective since every non-vanishing function locally has a logarithm (at least in a simply-connected domain) but $ex : \mathscr{O}(\mathbb{C}^*) \to \mathscr{O}^*(\mathbb{C}^*)$ is not surjective.

Exact sequence

Lemma 2

$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ is an exact sequence of sheaves on topological space X. For every open $U \subset X$ the sequence

$$0 \to \mathscr{F}(U) \xrightarrow{\alpha} \mathscr{G}(U) \xrightarrow{\beta} \mathscr{H}(U)$$

is exact.

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is exact.

Proof:
Step 1: By lemma 1, $0 \to \mathscr{F}(U) \xrightarrow{\alpha} \mathscr{G}(U)$ is exact.

Exact sequence

Lemma 2

$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ is an exact sequence of sheaves on topological space X. For every open $U \subset X$ the sequence

$$0 \to \mathscr{F}(U) \xrightarrow{\alpha} \mathscr{G}(U) \xrightarrow{\beta} \mathscr{H}(U)$$

is exact.

Proof:
Step 1: By lemma 1, $0 \to \mathscr{F}(U) \xrightarrow{\alpha} \mathscr{G}(U)$ is exact.
Step 2: Proving $\operatorname{Im} \alpha \subset \ker \beta$:
Since the sequence of stalks $\mathscr{F}_x \xrightarrow{\alpha_x} \mathscr{G}_x \xrightarrow{\beta_x} \mathscr{H}_x$ is exact for every $x \in U$. It means for $f \in \mathscr{F}(U)$ there exists a neighbor $V_x \subset U$ of x s.t $\beta \circ \alpha(f)|_{V_x} = 0$. By sheaf axiom I, since $U = \bigcup_{x \in U} V_x$, $\beta \circ \alpha(f) = 0$.

Exact sequence

Step 3: Proving $\operatorname{Im}\alpha \supset \ker\beta$
Suppose $g \in \ker\beta_U$. Since for every $x \in U$ $\ker\beta_x = \operatorname{Im}\alpha_x$, there exists a open covering $(V_i)_{i\in I}$ of U s.t $\alpha(f_i) = g|_{V_i}$ for every $i \in I$.
On $V_i \cap V_j$, $\alpha(f_i - f_j) = \alpha(f_i) - \alpha(f_j) = (g - g)|_{V_i \cap V_j} = 0$.
By lemma 1, α is a monomorphism then
$\alpha : \mathscr{F}(V_i \cap V_j) \to \mathscr{G}(V_i \cap V_j)$ is injective. Hence, $f_i - f_j = 0$ or $f_i = f_j$ on $V_i \cap V_j$. By sheaf axiom II, there exists $f \in \mathscr{F}(U)$ s.t $f_i = f|_{V_i}$. By sheaf axiom I, since
$\alpha(f)|_{V_i} = \alpha(f|_{V_i}) = \alpha(f_i) = g|_{V_i}$, $\alpha(f) = g$ on U. □

Exact sequence

Examples

$0 \to \mathscr{O} \to \mathscr{E} \xrightarrow{d''} \mathscr{E}^{0,1} \to 0$ is exact.

Exact sequence

Examples

$0 \to \mathscr{O} \to \mathscr{E} \xrightarrow{d''} \mathscr{E}^{0,1} \to 0$ is exact.

Proof: Since $\mathscr{O}_x = \ker(\mathscr{E}_x \xrightarrow{d''} \mathscr{E}_x^{0,1})$ for every x, we only need to show that $\mathscr{E} \xrightarrow{d''} \mathscr{E}^{0,1} \to 0$ is exact.

For a fixed $x \in X$, (U, z) is a neighborhood coordinate of x, WLOG we can assume that $z(x) = 0$. For some $\omega = g d\bar{z} \in \mathscr{E}^{0,1}(Y \cap U)$, $Y \ni x$, $g \in \mathscr{E}(Y \cap U)$. There exists $V_x^R := \{z \in \mathbb{C} : |z| < R\}$ s.t $V_x^R \subset z(Y \cap U)$ (it also means $x \in z^{-1}(V_x^R) \subset Y \cap U$). By Dolbeault's lemma, $\exists f \in \mathscr{E}(V_x^R) : \frac{\partial f}{\partial \bar{z}} = g \circ z^{-1} \in \mathscr{E}(V_x^R)$. It implies $\frac{\partial f \circ z}{\partial \bar{z}} = g$ on $z^{-1}(V_x^R)$ and $f \circ z \in \mathscr{E}(z^{-1}(V_x^R))$. Hence,

$$d''(f \circ z)|_{z^{-1}(V_x^R)} = d''(f \circ z|_{z^{-1}(V_x^R)}) = \frac{\partial f \circ z}{\partial \bar{z}} d\bar{z} = g d\bar{z} = \omega$$

$\square$

Exact sequence

Example

1. Similarly we get

$$0 \to \Omega \to \mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^{(2)} \to 0$$

is exact.

Exact sequence

Example

1. Similarly we get
$$0 \to \Omega \to \mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^{(2)} \to 0$$
is exact.
2. $0 \to \mathbb{C} \to \mathscr{E} \xrightarrow{d} \mathscr{L} \to 0$ is exact.
Where $\mathscr{L} := \ker(\mathscr{E}^{(1)} \xrightarrow{d} \mathscr{E}^{(2)})$.
3. $0 \to \mathbb{C} \to \mathscr{O} \xrightarrow{d} \Omega \to 0$ is exact.
4. $0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{ex} \mathscr{O}^* \to 0$ is exact.

Cohomology homomorphism

Definition

Any homomorphism $\alpha : \mathscr{F} \to \mathscr{G}$ induces homomorphism:

$$\alpha_0 : H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G})$$

$$\alpha_1 : H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G})$$

Where $\alpha_0 := \alpha_X : \mathscr{F}(X) \to \mathscr{G}(X)$ and α_1 is constructed by: $\mathcal{U} = (U_i)_{i \in I}$ is an open covering of X, consider $\alpha_{\mathcal{U}} : C^1(\mathcal{U}, \mathscr{F}) \to C^1(\mathcal{U}, \mathscr{G})$, $(f_{ij}) \to (\alpha(f_{ij}))$. $\alpha_{\mathcal{U}}$ induces: $\overline{\alpha}_{\mathcal{U}} : H^1(\mathcal{U}, \mathscr{F}) \to H^1(\mathcal{U}, \mathscr{G})$. The family of $\overline{\alpha}_{\mathcal{U}}$ together make $\alpha_1 : H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G})$.

The Connecting Homomorphism

Definition

$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ is an exact sequence of sheaves on topological space X. We will define the *connecting homomorphism*

$$\delta^* : H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F})$$

$h \in H^0(X, \mathscr{H}) = \mathscr{H}(X)$. Since all $\beta_x : \mathscr{G}_x \to \mathscr{H}_x$ are surjective, there exists a open covering $\mathcal{U} = (U_i)$ of X and a cochain $g_u \in C^0(\mathcal{U}, \mathscr{G})$ s.t $\beta(g_i) = h|_{U_i}$. Hence $\beta(g_i - g_j) = 0$ on $U_i \cap U_j$. By lemma 2, there exists $f_{ij} \in \mathscr{F}(U_i \cap U_j)$ s.t $\alpha(f_{ij}) = g_j - g_i$. It implies $\alpha(f_{ij} + f_{jk} - f_{ik}) = 0$ on $U_i \cap U_j \cap U_k$. By lemma 1, α is injective on $U_i \cap U_j \cap U_k$, we get $f_{ik} = f_{ij} + f_{jk}$, i.e., $(f_{ij}) \in Z^1(\mathcal{U}, \mathscr{F})$. Finally, we define $\delta^* h \in H^1(X, \mathscr{F})$ is the class represent by (f_{ij}).

Exact Cohomology Sequence

Theorem

The exact sequence of sheaves on topological space X

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

induces the exact sequence of cohomology groups

$$0 \to H^0(X, \mathscr{F}) \xrightarrow{\alpha^0} H^0(X, \mathscr{G}) \xrightarrow{\beta^0} H^0(X, \mathscr{H}) \xrightarrow{\delta^*}$$

$$\xrightarrow{\delta^*} H^1(X, \mathscr{F}) \xrightarrow{\alpha^1} H^1(X, \mathscr{G}) \xrightarrow{\beta^1} H^1(X, \mathscr{H})$$

Exact Cohomology Sequence

Proof:

Step 1: We have already proved the exactness at $H^0(X, \mathscr{F})$ and $H^0(X, \mathscr{G})$ by proving lemma 2.

Step 2: Proving the exactness at $H^0(X, \mathscr{H})$.

Suppose $h = \beta^0(g)$, $g \in H^0(X, \mathscr{G})$. For a open covering $\mathcal{U} = (U_i)$, we have $\beta(g|_{U_i}) = h|_{U_i}$. This implies, by the definition of δ^*, $\delta^* h = \overline{(f_{ij})} \in H^1(X, \mathscr{F})$ where $f_{ij} = g|_{U_j} - g|_{U_i} = 0$ on $U_i \cap U_j$ for all i, j. Hence, $\delta^* h = 0$, i.e., $\operatorname{Im} \beta^0 \subset \ker \delta^*$.

Suppose cocycles $(f_{ij}) \in Z^1(\mathcal{U}, \mathscr{F})$ is a representation of $\delta^* h$. Assume that $\delta^* h = 0$ then $(f_{ij}) \in B^1(\mathcal{U}, \mathscr{F})$,i.e., there exists $(f_i) \in C^0(\mathcal{U}, \mathscr{F})$ s.t $f_{ij} = f_j - f_i$ on $U_i \cap U_j$. For $(g_i) \in C^0(\mathcal{U}, \mathscr{G})$ s.t $\beta(g_i) = h|_{U_i}$, set $\overline{g}_i := g_i - \alpha(f_i)$. It is easy to check that $\overline{g}_i = \overline{g}_j$ on $U_i \cap U_j$. Thus they are the restriction of $g \in \mathscr{G}(X) = H^0(X, \mathscr{G})$. We can also check that $\beta(g) = h$. Hence, $\ker \delta^* \subset \operatorname{Im} \beta^0$.

Exact Cohomology Sequence

Step 3: Proving the exactness at $H^1(X, \mathscr{F})$.
For some $\delta^* h$, take $(f_{ij}) \in Z^1(\mathcal{U}, \mathscr{F})$ is a representation of it. We have $\overline{\alpha}_{\mathcal{U}}(\overline{(f_{ij})}) = 0$ since $\alpha_{\mathcal{U}}((f_{ij})) = (\alpha(f_{ij})) = (g_j - g_i) \in B^1(\mathcal{U}, \mathscr{G})$. Thus $\alpha^1 \delta^* h = 0$, i.e., $\operatorname{Im} \delta^* \subset \ker \alpha^1$.
Suppose $\xi \in \ker \alpha^1$ represented by $(f_{ij}) \in Z^1(\mathcal{U}, \mathscr{F})$. Since $\alpha^1 \xi = 0$, there exists $(g_i) \in C^0(\mathcal{U}, \mathscr{G})$ s.t $\alpha(f_{ij}) = g_j - g_i$ on $U_i \cap U_j$. This implies

$$0 = \beta(\alpha(f_{ij})) = \beta(g_j) - \beta(g_i)$$

on $U_i \cap U_j$. Hence we can take $h \in \mathscr{H}(X) = H^0(X, \mathscr{H})$ s.t $h|_{U_i} = \beta(g_i)$, i.e., $\delta^* h = \xi$ or overall $\ker \alpha^1 \subset \operatorname{Im} \delta^*$.

Exact Cohomology Sequence

Step 4: Proving the exactness at $H^1(X,\mathscr{G})$.
Take a representation$(f_{ij}) \in C^1(\mathcal{U},\mathscr{F})$ of $\xi \in H^1(X,\mathscr{F})$, we have $\beta(\alpha(f)_{ij}) = 0$ on every $U_i \cap U_j$. Hence $\operatorname{Im}\alpha^1 \subset \ker\beta^1$.
Suppose $\eta \in \ker\beta^1$ is represented by $(g_{ij}) \in Z^1(\mathcal{U},\mathscr{G})$. By its definition, there exists $(h_i) \in C^0(\mathcal{U},\mathscr{H})$ s,t $\beta(g_{ij}) = h_j - h_i$. For every $x \in U_{\tau x} \subset X$, since $\mathscr{G}_x \xrightarrow{\beta_x} \mathscr{H}_x \to 0$ is exact, there exist a neighborhood $V_x \subset U_{\tau x}$ of x and $g_x \in \mathscr{G}(V_x)$ s.t $\beta(g_x) = h_{\tau x}|_{V_x}$.
Let $\mathcal{B} = (V_x)$ is a open covering of X and $\overline{g}_{xy} := g_{\tau x \tau y}|_{V_x \cap V_y}$. We can check that $(\overline{g}_{xy}) \in Z^1(\mathcal{B},\mathscr{G})$ is also a representation of η. Let $\psi_{xy} := \overline{g}_{xy} - g_y + g_x$, (ψ_{xy}) is also a representation of η and $\beta(\psi_{xy}) = 0$. Since the sequence
$\mathscr{F}(V_x \cap V_y) \xrightarrow{\alpha} \mathscr{G}(V_x \cap V_y) \xrightarrow{\beta} \mathscr{H}(V_x \cap V_y)$ is exact, there exists $(f_{xy}) \in Z^1(\mathcal{B},\mathscr{F})$ s.t $\alpha^1(f_{xy}) = \psi_{xy}$. Hence $\alpha(\overline{(f_{xy})}) = \eta$, i.e., $\ker\beta^1 \subset \operatorname{Im}\alpha^1$. $\square$

Consequences and Applications

Consequence 1

$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ is exact sequence of sheaves on topological space X s.t $H^1(X, \mathscr{G}) = 0$. Then

$$H^1(X, \mathscr{F}) \cong \mathscr{H}(X)/\beta\mathscr{G}(X)$$

Consequences and Applications

Dolbeault's theorem

Let X be a Riemann surface. We have:

1. $H^1(X, \mathscr{O}) \cong \mathscr{E}^{0,1}(X)/d''\mathscr{E}(X)$
2. $H^1(X, \Omega) \cong \mathscr{E}^2(X)/d\mathscr{E}^{1,0}(X)$.

Consequences and Applications

Dolbeault's theorem

Let X be a Riemann surface. We have:

1. $H^1(X, \mathscr{O}) \cong \mathscr{E}^{0,1}(X)/d''\mathscr{E}(X)$
2. $H^1(X, \Omega) \cong \mathscr{E}^2(X)/d\mathscr{E}^{1,0}(X)$.

Proof:

1. Using Consequence 1 and the facts
   $H^1(X, \mathscr{E}) = 0$
   $0 \to \mathscr{O} \to \mathscr{E} \xrightarrow{d''} \mathscr{E}^{0,1} \to 0$ is exact.
2. Using Consequence 1 and the fact
   $H^1(X, \mathscr{E}^{1,0}) = 0$
   $0 \to \Omega \to \mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^{(2)} \to 0$ is exact.

The deRham Groups

Definition

On Riemann surface X, every exact 1-form is closed but every closed 1-form is not exact in general. Consider the quotient group:

$$Rh^1(X) := \frac{\ker(\mathscr{E}^{(1)}(X) \xrightarrow{d} \mathscr{E}^{(2)}(X))}{\operatorname{Im}(\mathscr{E}(X) \xrightarrow{d} \mathscr{E}^{(1)}(X))}$$

$Rh^1(X)$ is called the 1st deRham group of X.

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$Rh^1(X)$ is called the 1st deRham group of X.

Fact

$Rh^1(X) = 0$ if every closed 1-from is exact, i.e., it has a primitive. For an example, if X is simply connected then $Rh^1(X) = 0$.

The deRham groups

deRham's Theorem

Let X be a Riemann surface. Then

$$H^1(X, \mathbb{C}) \cong Rh^1(X)$$

.

The deRham groups

deRham's Theorem

Let X be a Riemann surface. Then

$$H^1(X, \mathbb{C}) \cong Rh^1(X)$$

.

Prove: Using Consequence 1 on the exact sequence:

$$0 \to \mathbb{C} \to \mathscr{E} \xrightarrow{d} \mathscr{L} := \ker(\mathscr{E}^{(1)} \xrightarrow{d} \mathscr{E}^{(2)}) \to 0$$

□