Etale Morphism

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k:= algebraic closed field.

Definition (M1)

Let W and V be nonsingular algebraic varieties over k. A regular map $\phi : W \to V$ is said to be **etale** at $Q \in W$ if the map $\mathrm{d}\phi : \mathrm{Tgt}_Q(W) \to \mathrm{Tgt}_{\phi(Q)}(V)$ on tangent spaces is an isomorphism, and ϕ is said to be **etale** if it is etale at every point of W.

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Arcording to the Inverse function theorem, a smooth map $f : M \to N$ is a local diffeomorphism if and only if the derivative $Df_p : T_pM \to T_{f(p)}N$ is a linear isomorphism for all points $p \in M$.

Etale morphism of nonsingular algebraic varieties

Tangent Spaces

Definition (Milner, AG)

Let $W \subset \mathbb{A}^m$ be an algebraic subset of $\mathbb{A}^m$, and let $\mathfrak{a} = I(W)$. The tangent space $\mathrm{Tgt}_Q(W)$ to W at a point $Q = (a_1, ..., a_m)$ of W is the subspace of the vector space with origin Q cut out by the linear equations

$$\sum_{i=1}^{m} \frac{\delta F}{\delta X_i}|_Q (X_i - a_i) = 0 \quad F \in \mathfrak{a}.$$

Consider the regular map

$$\phi : \mathbb{A}^m \to \mathbb{A}^n, \ \ (P_1(a_1, ..., a_m), ..., P_n(a_1, ..., a_m)).$$

We think of ϕ being given by the equations

$$Y_i = P_i(X_1, ..., X_m), \ i = 1, ..., n.$$

It corresponds to the map of rings $\phi^* : k[Y_1, ..., Y_n] \to k[X_1, ..., X_m]$ sending Y_i to $P_i(X_1, ..., X_m), i = 1, ..., n$. Let $a \in \mathbb{A}^m$, and let $= \phi(a)$. Define $(d\phi)_a : \mathrm{Tgt}_a(\mathbb{A}^m) \to \mathrm{Tgt}_b(\mathbb{A}^n)$ to the map such that

$$(dY_i)_b \circ (d\phi)_a = \sum \frac{\delta P_i}{\delta X_j}|_a (dX_j)_a.$$

Proposition (Milner, Etale Cohomology)

Let $\phi : U \to V$ be a regular map, where U and V both equal A^m. Then ϕ is etale at $(a_1, ..., a_m)$ if and only if the Jacobian matrix $\left(\frac{\delta(X_i \circ \phi)}{\delta Y_j} \mid_{(a_1,\ldots,a_m)} \right)$ is nonsingular. (Here X_i is the $i-$th coordinate function on V and Y_j is the jth coordinate function on U.)

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Example

Consider the map $x \mapsto x^n : \mathrm{A}^1 \to \mathrm{A}^1$. Since $\dfrac{dX^n}{dX} = nX^{n-1}$. We see from that the map os etale at no point of A^1 if the characteristic of k divides n, and that otherwise it is etale at all $x \neq 0$.

Etale morphism of arbitrary varieties

Tangent Cone

Definition (Milner, Etale Cohomology)

Let $\phi : W \to V$ be a regular map of varieties over k. Then ϕ is said to be **etale** at $Q \in W$ if it induces an isomorphism $C_{\phi(Q)}(V) \to C_Q(W)$ of tangent cones (as k-algebra).

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Definition (Milner, Algebraic Geometry)

Let V be an algebraic subset of $\mathbb{A}^m$, and let $\mathfrak{a} = I(V)$. Assume that $P = (0, ..., 0) \in V$. Define $\mathfrak{a}_*$ to be the ideal generated by the polynomial F_* for $F \in a$, where F_* is the leading form of F. The **geometric tangent cone** at P, $C_P(V)$ is $V(\mathfrak{a}_*)$, and the **tangent cone** is the pair $(V(\mathfrak{a}_*), k[X_1, ..., X_n]/a_*)$.

Example

The tangent cone at the origin to the curve

$$V: \; Y^2 = X^3$$

is defined by the eqution

$$Y^2 = 0.$$

Thus it is the line $Y = 0$ with multiplicity 2. The map $t \mapsto (t^2, t^3) : \mathbb{A}^1 \to V$ is not etale at the origin because the map

$$x \mapsto 0, \; y \mapsto 0 : k[x, y]/y^2 \to k[t]$$

is defines on the tangent cones is not an isomorphism.

Definition (Milner, Etale Cohomology)

Let A be a local ring with maximal ideal $\mathfrak{m}$. The **associated graded ring** is

$$\mathrm{gr}(A) = \sum_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}.$$

The multiplication on $\mathrm{gr}(A)$ is induced by the multiplication

$$a, b \mapsto ab : \mathfrak{m}^i \times \mathfrak{m}^j \to \mathfrak{m}^{i+j}.$$

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Proposition (Milner, Algebraic Geometry)

The map $k[X_1, .., X_n]/\mathfrak{a}_* \to \mathrm{gr}(\mathcal{O}_P)$ sending the class of X_i in $k[X_1, .., X_n]/\mathfrak{a}_*$ to the class of X_i in $\mathrm{gr}(\mathcal{O}_P)$ is an isomorphism.

Proposition (Atyah-Mc.Donald, Commutative Algebra)

Let $\alpha : A \to B$ be a local homorphism of local rings. Then α induces an isomorphism on the associated ring rings if and only if it induces an isomorphism on the completions.

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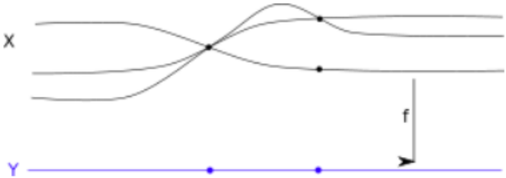
Definition (Milner, Etale Cohomology)

Let $\phi : W \to V$ be a regular map of varieties over a field k. We say that ϕ is etale at $w \in W$ if, for some algebraic closure k^{al} of k, $\phi_{k^{al}} : W_{k^{al}} \to V_{k^{al}}$ is etale at points of $W_{k^{al}}$ mapping to w.

Unramified morphism

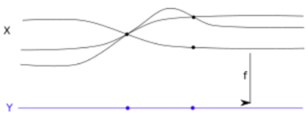
Motivation

In this section, we use the material from Youcis, https://ayoucis.wordpress.com/2014/04/06/unramified-morphisms/
To understand the idea behind unramified morphisms, let us look at a picture (taken from Wikipedia)



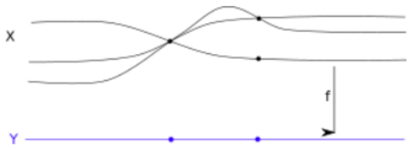
Unramified morphism

Open Diagonal



Figure

Let us suppose that we have a morphism $f : X \to Y$. Considering the fiber product $X \times_Y X$ and diagonal $\Delta_X \subset X \times_Y X$. Now, a point of ramification, call it c is one where two sheets come together. In other words, taking a sequence of points (x_n, x_n') in $X \times_Y X \setminus \Delta_X$, where x_n is in one of the sheets coming together and x_n' is in other, whose limit is precisely the point (c, c).



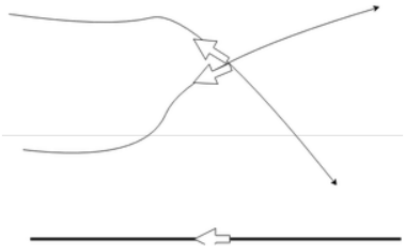
Figure

This give us one way to define ramifiedness, i.e., $X \times_Y X \setminus \Delta_X$ is not closed. Thus, if we want to force this to not happen, we merely want to specify that $X \times_Y X \setminus \Delta_X$ is closed, that is, Δ_X is open.

Unramified morphism

Unicity of Tangent Vector Lifts

Let's consider, once again, a morphism $f : X \to Y$. Let's fix a point $y \in Y$, and a point $x \in X$ which is in the fiber $f^{-1}(y)$. Given a tangent vector p of X based at x we can push it forward, through f, to a tangent vector p of X based at y. Or, in other terms, we have the derivative map $df_x : T_xX \to T_yY$.



Unramified morphism

Compact Riemann Surfaces

Let's recall that if $f : X \to Y$ is a non-constant map of compact Riemann surfaces, then we define the *ramification index* of a point $p \in X$ as follows. There exists a unique integer $e_p \geq 1$ such that, up to a change of coordinates, f looks like $z \mapsto z^{e_p}$ near p. Intuitely, this says that near p, f always looks like a e_p–to–1 cover of its image.

Unramified morphism

Meromorphic Functions

Let us denote the meromophic function fields of X and Y by $M(X)$ and $M(Y)$ respectively. We know that our map $f : X \to Y$ induces a morphism $f^* : M(Y) \to M(X)$. For our point $p \in X$ we have the subring $\mathcal{O}_p \subset M(X)$ consisting of those meromorphic functions which are holomorphic at p. If $q := f(p)$, then we similarly have the subring $\mathcal{O}_q \subset M(Y)$. In conclusion, the mapping $M(Y) \to M(X)$ induces a mapping $\mathcal{O}_q \to \mathcal{O}_p$. Also, we have $\mathfrak{m}_q\mathcal{O}_p = \mathfrak{m}_p^{e_p}$. In conclusion, we would get that $f : X \to Y$ is unramified at p if and only if $\mathfrak{m}_q\mathcal{O}_p = \mathfrak{m}_p$.

Unramified morphism

Unramifiedness for the Extension of Residue Fields

Let $f : X \to Y$ is morphism of schemes, with $y = f(x)$ and $[k(x) : k(y)] = n$ To make hidden points become seen, we need to move to the geometric setting. Or, in other words, we need to base change our situation to $\overline{k(y)}$ where all points become visible.

Unramified morphism

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Let $f : X \to Y$ is morphism of schemes, with $y = f(x)$ and $[k(x) : k(y)] = n$ To make hidden points become seen, we need to move to the geometric setting. Or, in other words, we need to base change our situation to $\overline{k(y)}$ where all points become visible.
But, the base change of $k(x)$ to $\overline{k(y)}$ is merely $k(x) \otimes_{k(y)} \overline{k(y)}$. So, we would want to say that $k(x)/k(y)$ is unramified if and only if $k(x) \otimes_{k(y)} \overline{k(y)}$ has n points. But, this happes precisely when $k(x)/k(y)$ is separable.

Definition (Morphism of finite type, Stacks)

Let $f : X \to S$ be a morphism of schemes.

i. We say that $f : X \to Y$ is of **finite type** at $x \in X$ if there exists an affine open neighborhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset Y$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.

ii. We say that f is **locally of finite type** if it is of finite type at every point of X.

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Theorem (Milner, Etale Cohomology)

Let $f : X \to Y$ be locally of finite type and let $x \in X$. Then, the following are equivalent:

i. *If $y = f(x)$, then $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$, and $k(x)/k(y)$ is a finite separable extension.*

ii. *The quality $(\Omega_{X/Y})_x = 0$ holds.*

iii. *There is a neighborhood U of x for which the restriction of the diagonal map $\Delta : X \to X \times_Y X$ is an open immersion.*

Lemma (Qing Liu)

Let $B = A[T_1, ..., T_n]$, *let* $F \in B$, *and let* $C = B/FB$. *Then*

$$\Omega_{C/A} = (\sum_{1 \leq i \leq n} C dT_i)/CdF,$$

where $dF = \Sigma_i (\delta F/\delta T_i) dT_i$.

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where $dF = \Sigma_i(\delta F/\delta T_i)dT_i$.

Example

Let us first consider the natural example of a map $\mathbb{A}_k^n \to \mathbb{A}_k^m$, where k is a field, and $n > m$. To check this map whether unramified or not, we need to compute $\Omega_{\mathbb{A}^n/\mathbb{A}^m} = \widetilde{\Omega_{A/B}}$, where $B = k[x_1, ..., x_n]$ and $A = k[x_1, ..., x_m]$. Since B is free $A-$algebra of rank $n-m$, it follows from the basis theory of differentials that $\Omega_{B/A}$ is a free $B-$module of rank $n-m$.

Unramified morphism

Field Example

Example

Let $f : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ where L/K is a finite field extension. In particular, if L/K is finite and separable, then f is unramified. To prove this, we must merely show that $\Omega_{L/K} = 0$. But, since L/K is finite separable, the primitive element theorem tells us that $L = K[x]/(p(x))$ for some irreducible, separable $p(x) \in K[x]$. So,

$$\Omega_{L/K} = Ldx/Ld(p(x)) = Ldx/Lp(x)'dx$$

which, if we write $L = K[x]/(p(x))$ is the same as $K[x]/(p(x), p(x)')$. But, this is equal to zero since p is separable so that $(p(x), p(x)') = K[x]$.

Unramified morphism

Relation to Number theory

Just to recall, suppose that $R \subset S$ is an integral extension of Dedekind domains. Then, for any non-zero prime $\mathfrak{p} \in \mathrm{Spec}(R)$, we can look at its extension to S, defined to be $\mathfrak{p}S$. Note that $\mathfrak{p}S$ is a proper ideal of S. Indeed, since $R \subset S$ is an integral extension, the Lying over Theorem implies that there is some prime $\mathfrak{P} \in \mathrm{Spec}(S)$ restring to $\mathfrak{p}$. Thus, $\mathfrak{p}S \subset \mathfrak{P}$. Thus, since S is a Dedekind domain, we can factor $\mathfrak{p}S$ as a finite product of primes

$$\mathfrak{p}S = \mathfrak{P}_1^{e_1} \dots \mathfrak{P}_n^{e_n}.$$

We claim that $R \subset S$ is unramified at a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ if and only if $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is unramified at all prime $\mathfrak{P} \in \mathrm{Spec}(S)$ which lie above $\mathfrak{p}$. Indeed, note that for any prime $\mathfrak{P}_i$ lying above $\mathfrak{p}$, we have that

$$\begin{aligned}
\mathfrak{p}S_{\mathfrak{P}_i} &= (\mathfrak{p}S)_{\mathfrak{P}_i} \\
&= (\mathfrak{P}_1^{e_1} \dots \mathfrak{P}_n^{e_n})_{\mathfrak{P}_i} \\
&= (\mathfrak{P}_1 S_{\mathfrak{P}_i})^{e_1} \dots (\mathfrak{P}_n S_{\mathfrak{P}_i})^{e_n} \\
&= (\mathfrak{P}_i S_{\mathfrak{P}_i})^{e_i}.
\end{aligned}$$

But, $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ being unramified (in algebraic geometry sense) at $\mathfrak{P}_i$, is equivalent to $\mathfrak{p}S_{\mathfrak{P}_i} = \mathfrak{P}_i S_{\mathfrak{P}_i}$. So, we see that being unramified at $\mathfrak{P}_i$ is equivalent to $e_i = 1$.

Definition

A morphism $\phi : Y \to X$ of schemes is etale if it is flat and unramifed.

A homomorphism of rings $f : A \to B$ is etale if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is etale. Equivalently, it is etale if

i. B is finitely generated $A-$algebra
ii. B is a flat $A-$algebra
iii. for all maximal ideas $\mathfrak{n}$ of $B, B_{\mathfrak{n}}/f(\mathfrak{p})B_{\mathfrak{n}}$ is a finite separable field extension of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

Proposition

For a regular map $\phi : Y \to X$ of varieties over an algebraically closed field, the definition of etale in scheme agrees with that in classical.

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Lemma

Let $\phi : A \to B$ be a local homomorphism of local rings. If

i. *ϕ is injective*
ii. *the map on residue fields $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism.*
iii. *ϕ is unramified, and*
iv. *B is finite $A-$algebra,*

then ϕ is an isomorphism.

Etale Morphism of Schemes

Jacobian Conjecture

Every etale map $\phi : \mathbb{A}^n \to \mathbb{A}^n$ is an isomorphism. If write $\phi = (P_1, ..., P_n)$, then this becomes the statement:

$$\text{if } \det\left(\frac{\delta P_i}{\delta X_j}(a)\right) \text{ is never zero (for } a \in k^n\text{), then } \phi \text{ has a inverse.}$$

THANK YOU FOR LISTENING