Differential Forms on Riemann surfaces

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Cauchy-Riemann equations

The Cauchy-Riemann equations on a pair of real-valued functions of two real variables $u(x, y)$ and $v(x, y)$ are the two equations:

1. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
2. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Wirtinger derivatives:

Consider the complex plane $\mathbb{C} \equiv \mathbb{R}^2$, the Wirtinger derivatives are defined as the following linear partial differential operators of first order:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

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Fact

Given a complex function $f(z = x + iy) = u(x, y) + iv(x, y)$, u and v satisfy the C-R equations iff $\frac{\partial f}{\partial \bar{z}} = 0$.

Theorem

U: an open subset of $\mathbb{C}$, a complex function $f : U \to \mathbb{C}$, $f(z = x + iy) = u(x, y) + iv(x, y)$, continuous on U and exists partial derivatives respect to x and y on U then f is holomorphic iff u and v satisfy C-R equations.

Denote $\mathscr{E}(U)$ be the $\mathbb{C}-$algebra of all those function $f : U \to \mathbb{C}$ which are infinitely differentiable with respect to the real coordinates x and y.

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Fact

Vector space $\mathcal{O}(U)$ of holomorphic functions on U is the kernel of the mapping $(\partial/\partial\bar{z}) : \mathscr{E}(U) \to \mathscr{E}(U)$.

Define differentiable function on Riemann surfaces

Definition

X: a Riemann surface, Y: open $\subset X$. Denote by $\mathscr{E}(Y)$ by the $\mathbb{C}-$algebra of all functions $f : Y \to \mathbb{C}$ s.t for every chart $z : U \to V \subset \mathbb{C}$ on X with U: open $\subset Y$ there exists a function $\tilde{f} \in \mathscr{E}(V)$ with $f|U = \tilde{f} \circ z$.

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Fact

$\{\mathscr{E}(Y)\}_{Y:open \subset X}$ together with the natural restriction mappings one gets the sheaf $\mathscr{E}$ of differentiable functions on the Riemann surface X.

Differential operators

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Just like differentiable complex functions, differentiable functions on a Riemann surface are always infinitely differentiable.

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Definition

If $(U, z = x + iy)$ is a coordinate neighborhood on X, then the differential operators

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}} : \mathscr{E}(U) \to \mathscr{E}(U)$$

can be define in the obvious way by writing $f|U = \tilde{f} \circ z$.

Function germs vanish at a to first and second order

$a \in X$: Riemann surface. $\mathscr{E}_a$: germs of all differentiable functions at the point a. Denote $\mathfrak{m}_a \subset \mathscr{E}_a$ the vector subspace of all function germs which vanish at the point a.

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Functions vanish to second orders

A function germ $\phi \in \mathfrak{m}_a$ is said to vanish to second order if it can be represented by a function f such that, with respect to a coordinate neighborhood $(U, z = x + iy)$ of a, one has

$$\frac{\partial f}{\partial x}(a) = \frac{\partial f}{\partial y}(a) = 0$$

We denote by $\mathfrak{m}_a^2 \subset \mathfrak{m}_a$ the vector subspace of all function germs which vanish at a to second order.

Algebra note

Note

If we consider $\mathscr{E}_a$ as a ring then $\mathfrak{m}_a$ is a maximal ideal of $\mathscr{E}_a$ and $\mathscr{E}_a/\mathfrak{m}_a \equiv \mathbb{C}$.

$\mathfrak{m}_a^2 \supset \{\sum f_k g_k : f_k, g_k \in \mathfrak{m}_a\}$.

Cotangent vector space

Definition

We define the quotient vector space $T_a^{(1)} := \frac{\mathfrak{m}_a}{\mathfrak{m}_a^2}$ as the *cotangent space* of X at the point a.

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a$\in U$: open $\subset X$. $f \in \mathscr{E}(U)$. The *differential* $d_a f \in T_a^{(1)}$ is defined as the element:

$$d_a f := (f - f(a)) \mod \mathfrak{m}_a^2$$

Theorem

$a \in X$: a Riemann surface. $(U, z = x + iy)$: a coordinate neighborhood of a. Then

1. $d_a x$ and $d_a y$ form a basis of $T_a^{(1)}$.
2. $d_a z$ and $d_a \overline{z}$ form a basis of $T_a^{(1)}$ as well.
3. If f is a function differentiable in a neighorhood of a, the

$$d_a f = \frac{\partial f}{\partial x}(a) d_a x + \frac{\partial f}{\partial y}(a) d_a y = \frac{\partial f}{\partial z}(a) d_a z + \frac{\partial f}{\partial \overline{z}}(a) d_a \overline{z}$$

Cotangent Vectors of Type $(1, 0)$ and $(0, 1)$

(U, z) and (U', z') are two different coordinate neighborhoods of $a \in X$. Then $\frac{\partial z'}{\partial z}(a) = c \in \mathbb{C}$, $\frac{\partial \overline{z'}}{\partial z}(a) = \overline{c}$ and $\frac{\partial z'}{\partial \overline{z}}(a) = \frac{\partial \overline{z'}}{\partial z}(a) = 0$.
Which means $d_a z' = c d_a z$ and $d_a \overline{z'} = \overline{c} d_a \overline{z}$.
Thus one-dim vector subspaces of $T_a^{(1)}$ are independent of the choice of local coordinate (U, z).

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Thus one-dim vector subspaces of $T_a^{(1)}$ are independent of the choice of local coordinate (U, z).

We define: $T_a^{1,0} := \mathbb{C} d_a z$, $T_a^{0,1} := \mathbb{C} d_a \overline{z}$
$T_a^{(1)} = T_a^{0,1} \oplus T_a^{1,0}$
If f is a differentiable function in a neighborhood of a, we can write $d_a f = d'_a f + d''_a f$, $d'_a f \in T_a^{1,0}$, $d''_a f \in T_a^{0,1}$
Then $d'_a f = \frac{\partial f}{\partial z}(a) d_a z$, $d'_a f = \frac{\partial f}{\partial \overline{z}}(a) d_a \overline{z}$.

Differential form of degree one

Definition

Y: open $\subset X$: Riemann surface. We define *differential form of degree one* (or just 1-*form*) on Y by the mapping

$$\omega : Y \to \bigcup_{a \in Y} T_a^{(1)}$$

with $\omega(a) \in T_a^{(1)}$ for every $a \in Y$.
If $\omega(a) \in_a^{1,0}$ or $\omega(a) \in_a^{0,1}$ for every $a \in Y$, then ω is said to be of type $(1,0)$ or type $(0,1)$ respectively.

Differentiable and holomorphic 1-forms

Definition

Y : open $\in X$: Riemann surface. A 1-form ω on Y is called *differentiable* (or holomorphic) if for every chart (U, z), ω may be writen:

$$\omega = fdz + gd\bar{z} \text{ on } U \cap Y \text{ where } f, g \in \mathscr{E}(U \cap Y)$$

respectively

$$\omega = fdz \text{ on} U \cap Y \text{ where } f \in \mathcal{O}(U \cap Y)$$

Differentiable and holomorphic 1-form

Notation

For U : open $\in X$, we denote:
$\mathscr{E}^{(1)}(U)$ the vector space of differentiable 1-forms on U.
$\mathscr{E}^{1,0}(U)$ the vector space of differentiable 1-forms on U of type $(1,0)$.
$\mathscr{E}^{0,1}(U)$ the vector space of differentiable 1-forms on U of type $(0,1)$.
$\Omega(U)$ the vector space of holomorphic 1-forms.

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$\Omega(U)$ the vector space of holomorphic 1-forms.

Fact

$\mathscr{E}^{(1)}$,$\mathscr{E}^{1,0}$,$\mathscr{E}^{0,1}$ and Ω with the natural restriction mappings are sheaves of vector spaces over X.

Singularity of 1-forms

Y : open $\subset X$: Riemann surface, $a \in Y$, ω : holomorphic 1-form on $Y - \{a\}$. (U, z): coordinate neighborhood of a such that $U \subset Y$ and $z(a) = 0$.
We can write $\omega = fdz$, $f \in \mathcal{O}(U - \{a\})$. Let

$$f = \sum_{n=-\infty}^{n=\infty} c_n z^n$$

be the Laurent series expansion of f about a respect to z.

Singularity of 1-forms

Notation

If $c_n < 0$ for every $n < 0$ then ω may be holomorphically continued to all of Y and a is called a removable singularity of w.
If there exist $k < 0$ s.t $c_k \neq 0$ and $c_n = 0$ for every $n < k$ then ω has a *pole* of kth order at a
If there are many infinity $k < 0$ s.t $n_k \neq 0$ then w has an essential singularity at a.

The Residue

Definition

The coefficient c_{-1} is called the *residue* of ω at a and denoted by

$$c_{-1} = Res_a(\omega)$$

We will prove that this definition is well-define by proving this lemma:

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Lemma

The residue is independent of the choice of chart (U, z).

Proving the lemma

Step 1: Let V be an open neighborhood of a. If g is holomorphic on $V - \{a\}$, we define dg by $(dg)(a) := d_a g$. We will prove that $Res_a(dg) = 0$ thus it is independent of the choice of chart.
(U, z): coordinate neighborhood of a with $z(a) = 0$. The Laurent series expansion of g about a:

$$g = \sum_{n=-\infty}^{\infty} c_n z^n$$

Then

$$dg = (\sum_{n=-\infty}^{\infty} n c_n z^{n-1}) dz$$

This implies $Res_a(dg) = 0$.

Proving the lemma

Step 2: If ϕ is a holomorphic function on V which has a zero of first order at a, then $Res_a(\phi^{-1}d\phi) = 1$ thus it is independent of the choice of chart.
(U, z): coordinate neighborhood of a with $z(a) = 0$. $\exists h :$ holomorphic at a and $h(a) \neq 0$ s.t $\phi = zh$. Then $d\phi = hdz + zdh$ and

$$\frac{d\phi}{\phi} = \frac{hdz + zdh}{zh} = \frac{dz}{z} + \frac{dh}{h}$$

Since $h(a) \neq 0$, $\frac{1}{h}dh$ is holomorphic at a and $Res_a(1/hdh) = 0$. This implies

$$Res_a(\frac{d\phi}{\phi}) = Res_a(\frac{dz}{z}) = 1$$

.

Proving the lemma

Step 3: For $\omega = fdz$ where $f = \sum_{n=-\infty}^{\infty} c_n z^n$. Let

$$g := \sum_{n=-\infty}^{-2} \frac{c_n z^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{c_n z^{n+1}}{n+1}$$

Then $\omega = dg + c_{-1} z^{-1} dz$. From step 1, $Res_a dg = 0$, from step 2, $Res_a z^{-1} dz = 1$ for every choice of the chart z. This means $Res_a(\omega) = c_{-1}$ is independent of the chart.

Meromorphic DF

Definition

A 1-form ω on an open subset Y of a Riemann surface is said to be a meromorphic differential form on Y if there exists an open subset $Y' \subset Y$ such that the following hold:

1. ω is a holomorphic 1-form on Y'
2. $Y - Y'$ consists of only isolated points
3. ω has a pole at every point $a \in Y - Y'$

Meromorphic DF

Notation

Let $\mathscr{M}^{(1)}(Y)$ denote the set of all meromorphic 1-form on Y.
The $\mathscr{M}^{(1)}(X)$ is also called *abelian differentials*.
An abelian differential ω is said to be of the first kind if it is holomorphic everywhere, of the second kind if $Res_a(\omega) = 0$ for every a as its pole, of the third kind otherwise.

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Fact

$\mathscr{M}^{(1)}$ together with the natural algebraic operations and the usual restriction mapping is a sheaf.

The Exterior Product

Definition

V: vector space over $\mathbb{C}$, we define $\Lambda^2 V$ the vector space over $\mathbb{C}$ whose elements are finite sums of elements of the form $v_1 \wedge v_2$ for every $v_1, v_2 \in V$ s.t:

1. $(v_1 + v_2) \wedge v_3 = v_1 \wedge v_3 + v_2 \wedge v_3$
2. $(\lambda v_1) \wedge v_2 = \lambda(v_1 \wedge v_2)$
3. $v_1 \wedge v_2 = -v_2 \wedge v_1$

for every $v_1, v_2, v_3 \in V$ and $\lambda \in \mathbb{C}$.

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for every $v_1, v_2, v_3 \in V$ and $\lambda \in \mathbb{C}$.

Fact

Since $v \wedge v = 0$ for every $v \in V$, $\{v_i \wedge v_j\}_{i<j}$ is a basis of $\Lambda^2 V$ for $\{v_i\}_{1 \leq i \leq n}$ is a basis of V.

The Exterior Product

Definition

$$T_a^{(2)} := \Lambda^2 T_a^{(1)}$$

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Fact

$T_a^{(2)}$ is one dimensinal vector space over $\mathbb{C}$.
Let $(U, z = x + iy)$ be a coordinate neighborhood of a then
$T_a^{(2)} = \mathbb{C} d_a x \wedge d_a y = \mathbb{C} d_a z \wedge d_a \bar{z}$ since $d_a z \wedge d_a \bar{z} = -2i d_a x \wedge d_a y$.

2-forms on Riemann surfaces

Definiton

Y : open $\subset X$: Riemann surface. A *2-form* on Y is a mapping

$$\omega : Y \to \bigcup_{a \in Y} T_a^{(2)}$$

where $\omega(a) \in T_a^{(2)}$.
The form ω is called differentiable on Y if for every chart (U, z) on X, ω can be writen $\omega = f dz \wedge d\bar{z}$ on $(U \cap Y)$ with $f \in \mathscr{E}(U \cap Y)$.
Denote $\mathscr{E}^{(2)}(Y)$ the vector space of all differentiable 2-forms on Y.

2-forms on Riemann surfaces

Fact

For $\omega_1, \omega_2 \in \mathscr{E}^{(1)}(Y)$, we can define the 2-form $\omega_1 \wedge \omega_2 \in \mathscr{E}^{(2)}(Y)$ by

$$(\omega_1 \wedge \omega_2)(a) := \omega_1(a) \wedge \omega_2(a)$$

For $f \in \mathscr{E}(Y), \omega \in \mathscr{E}^{(1)}(Y)$, we can define the 2-form $f\omega \in \mathscr{E}^{(2)}(Y)$ by

$$(f\omega)(a) := f(a)\omega(a)$$

Exterior Differentiation of Forms

Definition

Y: open $\subset X$: Riemann surface. A 1-form ω can be writen as a finite sum

$$\omega = \sum f_k dg_k$$

in a neighborhood (U, z) of $a \in U$. We define:

$$d\omega := \sum df_k \wedge dg_k,$$

$$d'\omega := \sum d'f_k \wedge dg_k,$$

$$d''\omega := \sum d''f_k \wedge dg_k.$$

Exterior Differentiation of Forms

Lemma

This definition is independent of the representation $\omega = \sum f_k dg_k$ thus it makes sense.

Exterior Differentiation of Forms

Lemma

This definition is independent of the representation $\omega = \sum f_k dg_k$ thus it makes sense.

Proof:
We will prove it for the operator d:
For a coordinate neighborhood $(U, z = x + iy)$, suppose $\omega = \sum f_k dg_k = \sum \tilde{f}_k d\tilde{g}_k$, we have to show that:

$$\sum df_k \wedge dg_k = \sum d\tilde{f}_k \wedge d\tilde{g}_k$$

Exterior Differentiation of Forms

We have:

1. $\sum f_k \frac{\partial g_k}{\partial x} = \sum \tilde{f}_k \frac{\partial \tilde{g}_k}{\partial x}$
2. $\sum f_k \frac{\partial g_k}{\partial y} = \sum \tilde{f}_k \frac{\partial \tilde{g}_k}{\partial y}$

since $dg_k = \frac{\partial g_k}{\partial x} dx + \frac{\partial g_k}{\partial y} dy$ and $d\tilde{g}_k = \frac{\partial \tilde{g}_k}{\partial x} dx + \frac{\partial \tilde{g}_k}{\partial y} dy$
Taking partial derivatives with respect to x and y on □1 and □2 and subtracting yields:

$$\sum \frac{\partial f_k}{\partial y}\frac{\partial g_k}{\partial x} - \frac{\partial f_k}{\partial x}\frac{\partial g_k}{\partial y} = \sum \frac{\partial \tilde{f}_k}{\partial y}\frac{\partial \tilde{g}_k}{\partial x} - \frac{\partial \tilde{f}_k}{\partial x}\frac{\partial \tilde{g}_k}{\partial y}$$

This implies $\sum df_k \wedge dg_k = \sum d\tilde{f}_k \wedge d\tilde{g}_k$. □

Elementary Properties

Properties

U : open $\subset X$: Riemann surface, $f \in \mathscr{E}(U), \omega \in \mathscr{E}^{(1)}(U)$. Then

1. $ddf = d'd'f = d''d''f = 0$.
2. $d\omega = d'\omega + d''\omega$.
3. $d(f\omega) = df \wedge \omega + fd\omega$.
4. $d'(f\omega) = d'f \wedge \omega + fd'\omega$.
5. $d''(f\omega) = d''f \wedge \omega + fd''\omega$.
6. $d'd''f = -d''d'f$
7. $d'd''f = \frac{\partial^2 f}{\partial z \partial \overline{z}} dz \wedge d\overline{z} = \frac{1}{2i}(\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y})$.

A differentiable function f define on a open subset U of a Riemann surface X is called *harmonic* if $d'd''f = 0$.

Closed and exact forms

Definition

Y : open $\subset X$: Riemann surface. A differentiable 1-form $\omega \in \mathscr{E}^{(1)}(Y)$ is called *closed* if $d\omega = 0$ and *exact* if there exists $f \in \mathscr{E}(Y)$ such that $\omega = df$.

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Definition

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Fact

Every exact form is closed since $ddf = 0$. The converse is not true in general.

Closed and exact forms

Theorem

Y: open $\subset X$: Riemann surface. The following hold:

1. Every holomorphic 1-form $\omega \in \Omega(Y)$ is closed.
2. Every closed 1-form $\omega \in \mathscr{E}^{1,0}(Y)$ is holomorphic.

Closed and exact forms

Theorem

Y: open $\subset X$: Riemann surface. The following hold:

1. Every holomorphic 1-form $\omega \in \Omega(Y)$ is closed.
2. Every closed 1-form $\omega \in \mathscr{E}^{1,0}(Y)$ is holomorphic.

Proof: Implies easily from the result $d\omega = 0 \iff \partial f/\partial \bar{z} = 0$.

Closed and exact forms

Theorem

Y: open $\subset X$: Riemann surface. The following hold:

1. Every holomorphic 1-form $\omega \in \Omega(Y)$ is closed.
2. Every closed 1-form $\omega \in \mathscr{E}^{1,0}(Y)$ is holomorphic.

Proof: Implies easily from the result $d\omega = 0 \iff \partial f/\partial \bar{z} = 0$.

Consequence

If u is a harmonic function then $d'u$ is a holomorphic 1-form since $dd'u = d''d'u = 0$.

The Pull-Back of DF

Definition of Pull-Back mapping

X, Y : Riemann surfaces, $F : X \to Y$: holomorphic mapping. For every U : open $\subset Y$, F induces a homomorphism:

$$F^* : \mathscr{E}(U) \to \mathscr{E}(F^{-1}(U)), \; f \mapsto f \circ F.$$

and

$$F^* : \mathscr{E}^{(k)}(U) \to \mathscr{E}^{(k)}(F^{-1}(U))$$

for $k = 1, 2$ defined by

$$F^*(\sum f_k dg_k) = \sum (F^* f_k) d(F^* g_k)$$

$$F^*(\sum f_k dg_k \wedge h_k) = \sum (F^* f_k) d(F^* g_k)(F^* h_k)$$

The Pull-Back of DF

Properties

For $f \in \mathscr{E}(U)$, $\omega \in \mathscr{E}^{(1)}(U)$:

1. $F^*(df) = d(F^*f)$
2. $F^*(d\omega) = d(F^*\omega)$

and the corresponding formulas for d' and d''.

The Pull-Back of DF

Properties

For $f \in \mathscr{E}(U)$, $\omega \in \mathscr{E}^{(1)}(U)$:

1. $F^*(df) = d(F^*f)$
2. $F^*(d\omega) = d(F^*\omega)$

and the corresponding formulas for d' and d''.

Consequence

If $f \in \mathscr{E}(U)$ is harmonic, then F^*f is also harmonic since $d'd''(F^*f) = d'(F^*d''f) = F^*(d''f) = 0$.

Thank you for listening