The Coverings of Riemann Surfaces

Hoang Son Tran
Institute of Mathematics of Hanoi

May, 2020

Homotopy of Curves

Homotopy of Curves

Definition

Let X be a topological space. Two curves u,v from a to b in X are called homotopic if there exists a continuous mapping $A : I \times I \to X$ with the following properties:

(i) $A(t,0) = u(t)\ \forall t \in I$

(ii) $A(t,1) = v(t)\ \forall t \in I$

(iii) $A(0,s) = a, A(1,s) = b\ \forall s \in I$

Denote: $u \sim v$

Homotopy of Curves

Homotopy of Curves

Definition

Let X be a topological space. Two curves u,v from a to b in X are called homotopic if there exists a continuous mapping $A : I \times I \to X$ with the following properties:

(i) $A(t, 0) = u(t)\ \forall t \in I$

(ii) $A(t, 1) = v(t)\ \forall t \in I$

(iii) $A(0, s) = a, A(1, s) = b\ \forall s \in I$

Denote: $u \sim v$

Remark By setting $u_s(t) := A(t, s)$ we have a family $(u_s)_{s\in I}$ of curves from a to b, called a deformation of u into v.

Homotopy of Curves

Homotopy of Curves

Theorem

The notion of homotopy is an equivalence relation on the set of all curves from a to b.

From now, for any curve u in X we denote its homotopy class by $cl(u)$.

Homotopy of Curves

Homotopy of Curves

Definition

Let a, b, c be 3 points in X. Suppose u is a curve from a to b and v is a curve from b to c. Define:

(i) The product curve: $(u \cdot v)(t) := \begin{cases} u(2t) & \text{if } t \in [0, \frac{1}{2}] \\ v(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$

(ii) The inverse curve: $u^{-}(t) := u(1-t)\ \forall t \in [0,1]$.

(iii) The constant curve at a: $u_0(t) := a\ \forall t \in [0,1]$.

Homotopy of Curves

Homotopy of Curves

Theorem

Let $a, b, c, d \in X$. Suppose u (resp. v, w) is a curve from a to b (resp. from b to c, from c to d) and u_0 (resp. v_0) is the constant curve at a (resp. b). Then:

(i) $u_0 \cdot u \sim u \sim u \cdot v_0$.

(ii) $u \cdot u^- \sim u_0$.

(iii) $(u \cdot v) \cdot w \sim u \cdot (v \cdot w)$.

Homotopy of Curves

The Fundamental Group

Theorem (The fundamental group)

The set $\pi_1(X, a)$ of homotopy classes of closed curves in X with initial point and end point a forms a group under the operation induced by the product of curves. This group is called the fundamental group of X with base point a.

Homotopy of Curves

The Fundamental Group

Theorem (Dependence of the base point)

If there exists a curve w joining a and b then $\pi_1(X, a) \cong \pi_1(X, b)$.

Homotopy of Curves

The Fundamental Group

Theorem (Dependence of the base point)

If there exists a curve w joining a and b then $\pi_1(X, a) \cong \pi_1(X, b)$.

Hint: The isomorphism $f : \pi_1(X, a) \to \pi_1(X, b)$ is given by

$$f(cl(u)) := cl(w^{-} \cdot u \cdot w).$$

Homotopy of Curves

The Fundamental Group

Theorem (Dependence of the base point)

If there exists a curve w joining a and b then $\pi_1(X, a) \cong \pi_1(X, b)$.

Hint: The isomorphism $f : \pi_1(X, a) \to \pi_1(X, b)$ is given by

$$f(cl(u)) := cl(w^{-} \cdot u \cdot w).$$

Remark When X is arcwise connected, the fundamental group $\pi_1(X, a)$ is independent of the base point so we just write $\pi_1(X)$.

Homotopy of Curves

Functorial Behavior

Suppose $f : X \to Y$ is a continuous mapping. If u is a curve in X then $f \circ u$ is a curve in Y. Moreover, if $u \sim u'$ in X then $f \circ u \sim f \circ u'$ in Y. Hence f induces a homomorphism:

$$f_* : \pi_1(X, a) \to \pi_1(Y, f(a))$$

Moreover, if $g : Y \to Z$ is another continuous mapping, one has:

$$(g \circ f)_* = g_* \circ f_*$$

The Coverings of Riemann Surfaces

Ramification points of a holomorphic mapping

Recall

Let $p : Y \to X$ be a non-constant holomorphic mapping. Suppose $p(y_0) = x_0$, then there exists charts (U, ϕ) of X, (V, ψ) of Y such that the map $F := \phi \circ p \circ \psi^{-1}$ has the form $F(z) = z^k$, for some interger $k \geq 1$.

The Coverings of Riemann Surfaces

Ramification points of a holomorphic mapping

Recall

Let $p : Y \to X$ be a non-constant holomorphic mapping. Suppose $p(y_0) = x_0$, then there exists charts (U, ϕ) of X, (V, ψ) of Y such that the map $F := \phi \circ p \circ \psi^{-1}$ has the form $F(z) = z^k$, for some interger $k \geq 1$.

Definition

A point y_0 is called a ramification point (or branch point) of p if the number k in the local description is greater or equal to 2.
The number k is called the ramification order of the ramification point y_0.
If the map p has no branch points, then we call it unbranched.

Theorem

Let $p : Y \to X$ be a non-constant holomorphic map. Then p is unbranched if and only if p is a local homeomorphism.

Theorem

Let $p : Y \to X$ be a non-constant holomorphic map. Then p is unbranched if and only if p is a local homeomorphism.

Theorem

Suppose X is a Riemann surface, Y is a Hausdorff space and $p : Y \to X$ is a local homeomorphism. Then there is a unique complex structure on Y such that p is holomorphic. (p is even locally biholomorphic).

The Coverings of Riemann Surfaces

The Topological Coverings

Definition

(i) Let $p : Y \to X$ be a continuous map between topological spaces. For each $x \in X$, the set $p^{-1}(x)$ is called the fiber of p over x. The map p is said to be discrete if every fiber of p is a discrete subset of Y.

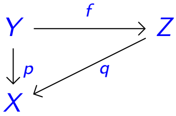
The Coverings of Riemann Surfaces

The Topological Coverings

Definition

(i) Let $p : Y \to X$ be a continuous map between topological spaces. For each $x \in X$, the set $p^{-1}(x)$ is called the fiber of p over x. The map p is said to be discrete if every fiber of p is a discrete subset of Y.

(ii) If $q : Z \to X$ is another continuous map, then a map $f : Y \to Z$ is called a fiber preserving map if $p = q \circ f$.



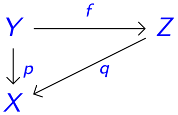
The Coverings of Riemann Surfaces

The Topological Coverings

Definition

(i) Let $p : Y \to X$ be a continuous map between topological spaces. For each $x \in X$, the set $p^{-1}(x)$ is called the fiber of p over x. The map p is said to be discrete if every fiber of p is a discrete subset of Y.

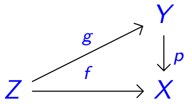
(ii) If $q : Z \to X$ is another continuous map, then a map $f : Y \to Z$ is called a fiber preserving map if $p = q \circ f$.



Remark A non-constant holomorphic map $p : Y \to X$ between 2 Riemann surfaces is discrete.

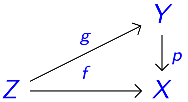
Definition (The Lifting of Mappings)

Suppose X, Y, Z are topological spaces, $p : Y \to X$ and $f : Z \to X$ are continuous maps. Then by a lifting of f with respect to p is meant a continuous mapping $g : Z \to Y$ such that $f = p \circ g$.



Definition (The Lifting of Mappings)

Suppose X, Y, Z are topological spaces, $p : Y \to X$ and $f : Z \to X$ are continuous maps. Then by a lifting of f with respect to p is meant a continuous mapping $g : Z \to Y$ such that $f = p \circ g$.



Theorem (Holomorphicity of Lifting)

Suppose X, Y, Z are Riemann surfaces, $p : Y \to X$ is an unbranched holomorphic map and $f : Z \to X$ is any holomorphic map. Then every lifting $g : Z \to Y$ of f is holomorphic.

Theorem (Uniqueness of Lifting)

Let X, Y be Hausdorff spaces and $p : Y \to X$ be a local homeomorphism. Suppose Z is connected and $f : Z \to X$ is a continuous mapping. If $g_1, g_2 : Z \to Y$ are 2 liftings of f and $g_1(z_0) = g_2(z_0)$ for some point $z_0 \in Z$ then $g_1 = g_2$.

Theorem (Uniqueness of Lifting)

Let X, Y be Hausdorff spaces and $p : Y \to X$ be a local homeomorphism. Suppose Z is connected and $f : Z \to X$ is a continuous mapping. If $g_1, g_2 : Z \to Y$ are 2 liftings of f and $g_1(z_0) = g_2(z_0)$ for some point $z_0 \in Z$ then $g_1 = g_2$.

Sketch of the proof: Consider $T = \{z \in Z : g_1(z) = g_2(z)\}$.

Theorem (Uniqueness of Lifting)

Let X, Y be Hausdorff spaces and $p : Y \to X$ be a local homeomorphism. Suppose Z is connected and $f : Z \to X$ is a continuous mapping. If $g_1, g_2 : Z \to Y$ are 2 liftings of f and $g_1(z_0) = g_2(z_0)$ for some point $z_0 \in Z$ then $g_1 = g_2$.

Sketch of the proof: Consider $T = \{z \in Z : g_1(z) = g_2(z)\}$.
T is closed: since T is the preimage of the diagonal $\Delta \subset Y \times Y$ under the continuous mapping $(g_1, g_2) : Z \to Y \times Y$.

Theorem (Uniqueness of Lifting)

Let X, Y be Hausdorff spaces and $p : Y \to X$ be a local homeomorphism. Suppose Z is connected and $f : Z \to X$ is a continuous mapping. If $g_1, g_2 : Z \to Y$ are 2 liftings of f and $g_1(z_0) = g_2(z_0)$ for some point $z_0 \in Z$ then $g_1 = g_2$.

Sketch of the proof: Consider $T = \{z \in Z : g_1(z) = g_2(z)\}$.
T is closed: since T is the preimage of the diagonal $\Delta \subset Y \times Y$ under the continuous mapping $(g_1, g_2) : Z \to Y \times Y$.
T is open: Take $z_0 \in T$, let $y_0 := g_1(z_0) = g_2(z_0)$.
$p|_V : V \to U$ is a homeomorphism (V is a neighborhood of y_0). Let $\phi : U \to V$ be its inverse.
Let $W := g_1^{-1}(V) \cap g_2^{-1}(V)$. Since $p \circ g_1|_W = p \circ g_2|_W$ then $g_1|_W = g_2|_W$. So T is open.

The Coverings of Riemann Surfaces

Lifting of Curves

Suppose X, Y are Hausdorff spaces and $p : Y \to X$ is a local homeomorphism.

If a curve $u : I \to X$ can be lifted to a curve $\hat{u} : I \to Y$ then $\hat{u}$ is uniquely determined once the lifting of a point is specified, for example: the initial point.

The Coverings of Riemann Surfaces

Lifting of Curves

Suppose X, Y are Hausdorff spaces and $p : Y \to X$ is a local homeomorphism.
If a curve $u : I \to X$ can be lifted to a curve $\hat{u} : I \to Y$ then $\hat{u}$ is uniquely determined once the lifting of a point is specified, for example: the initial point.

Theorem (Lifting of Homotopic Curves)

Let $a, b \in X$ and $\hat{a} \in p^{-1}(a)$.
Suppose $(u_s)_{0 \le s \le 1}$ is a homotopy from u_0 to u_1 in X ((u_s) are curves from a to b). If each u_s can be lifted to a curve $\hat{u}_s$ with initial point $\hat{a}$, then $(\hat{u}_s)_{0 \le s \le 1}$ is a homotopy from $\hat{u}_0$ to $\hat{u}_1$ in Y.

Sketch of the proof:

- Based on this fact: If $q : V \to U$ is a homeomorphism then every continuous map $f : Z \to U$ can be lifted to the map $g : Z \to V$ given by $g := q^{-1} \circ f$.
- Setting $A(t,s) := u_s(t)$ and $\hat{A}(t,s) := \hat{u}_s(t)$. We shall prove that $\hat{A}$ is continuous.
- Take U a neighborhood of a and V a neighborhood of $\hat{a}$ such that $p|_V : V \to U$ is a homeomorphism. Let $\phi : U \to V$ be its inverse.
- $A(\{0\} \times I) = a$ then there exists $\varepsilon \geq 0$ such that $A([0,\varepsilon] \times I) \subset U$. In other words, every curve $u_s|_{[0,\varepsilon]}$ is in U so that it can be lifted to the curve $\phi \circ u_s|_{[0,\varepsilon]}$. By uniqueness of lifting, $\hat{u}_s|_{[0,\varepsilon]} = \phi \circ u_s|_{[0,\varepsilon]}$. Thus $\hat{A} = \phi \circ A$ on $[0,\varepsilon] \times I$.

Definition

A continuous map $p : Y \to X$ is said to have the curve lifting property if the following holds: for every curve u in X with initial point a, for every point $\hat{a}$ in the fiber $p^{-1}(a)$, there exists a lifting $\hat{u}$ of u such that its initial point is $\hat{a}$.

Definition

A continuous map $p : Y \to X$ is said to have the curve lifting property if the following holds: for every curve u in X with initial point a, for every point $\hat{a}$ in the fiber $p^{-1}(a)$, there exists a lifting $\hat{u}$ of u such that its initial point is $\hat{a}$.

Definition (Covering maps)

A continuous map $p : Y \to X$ is called a covering map if for every $x \in X$ there exists an open neighborhood U of x such that $p^{-1}(U)$ is a union of disjoint open sets in Y, each of which is mapped homeomorphically onto U by p.

The special open neighborhoods U of x given in the definition are called **evenly covered neighborhoods**. The homeomorphic copies of U are called the **sheets** over U.

Examples

- Let $k \in \mathbb{N}, k \geq 2$ and $p_k : \mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^k$.
  Suppose $a \in \mathbb{C}^*$ and $b \in p_k^{-1}(a)$.
  Since p_k is a local homeomorphism there exists open neighborhoods V_0 of b, U of a such that $p_k|_{V_0} : V_0 \to U$ is a homeomorphism.
  Let ω be a k^{th} primitive root of unity. Then:

$$p_k^{-1}(U) = V_0 \cup \omega V_0 \cup \ldots \cup \omega^{(k-1)} V_0$$

.

Examples

- Let $k \in \mathbb{N}, k \geq 2$ and $p_k : \mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^k$.
  Suppose $a \in \mathbb{C}^*$ and $b \in p_k^{-1}(a)$.
  Since p_k is a local homeomorphism there exists open neighborhoods V_0 of b, U of a such that $p_k|_{V_0} : V_0 \to U$ is a homeomorphism.
  Let ω be a k^{th} primitive root of unity. Then:
  $$p_k^{-1}(U) = V_0 \cup \omega V_0 \cup \ldots \cup \omega^{(k-1)} V_0$$
  .
- $\exp : \mathbb{C} \to \mathbb{C}^*$.
  Suppose $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ such that $\exp(b) = a$.
  Since $\exp$ is a local homeomorphism there exists open neighborhoods V_0 of b, U of a such that $\exp|_{V_0} : V_0 \to U$ is a homeomorphism.
  Then:
  $$\exp^{-1}(U) = \bigcup_{n \in \mathbb{Z}} (V_0 + 2\pi i n)$$
  .

Theorem

Every covering map $p : Y \to X$ *has the curve lifting property.*

Theorem

Every covering map $p : Y \to X$ has the curve lifting property.

Sketch of the proof: Suppose $u : [0, 1] \to X$ is a curve and $y_0 \in Y$ with $p(y_0) = u(0)$.
Because of the compactness of $[0, 1]$ there exists a partition:

$$0 = t_0 < t_1 < ... < t_n = 1$$

and evenly covered neighborhoods $U_k \subset X, k = 1, 2, .., n$ such that $u([t_{k-1}, t_k]) \subset U_k$.
Note that $p^{-1}(U_k) = \bigcup_{j \in J} V_{kj}$, where each $p|_{V_{kj} \to U_k}$ is a homeomorphism.
In each U_k the curve $u([t_{k-1}, t_k])$ can be lifted to the curve $\hat{u}([t_{k-1}, t_k])$ with $\hat{u}(t_{k-1})$ is some point in the fiber $p^{-1}(u(t_{k-1}))$.

Suppose X, Y are Hausdorff spaces and $p : Y \to X$ is a covering map.

Theorem

If X is pathwise connected then for any 2 points $x_0, x_1 \in X$ the sets $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same cardinality.

Suppose X, Y are Hausdorff spaces and $p : Y \to X$ is a covering map.

Theorem

If X is pathwise connected then for any 2 points $x_0, x_1 \in X$ the sets $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same cardinality.

Proof.

Choose a curve u in X joining x_0 to x_1. Construct $\varphi : p^{-1}(x_0) \to p^{-1}(x_1)$ in the following way:
If $y \in p^{-1}(x_0)$ then there exists a lifting $\hat{u}$ of u with $\hat{u}(0) = y$. Let $\varphi(y) := \hat{u}(1) \in p^{-1}(x_1)$. The uniqueness of lifting implies that φ is bijective. □

Definition

The cardinality of $p^{-1}(x)$ for $x \in X$ is called the number of sheets of the covering p.

Definition

The cardinality of $p^{-1}(x)$ for $x \in X$ is called the number of sheets of the covering p.

Example

- $z \mapsto z^k$ is a k-sheeted covering.
- $\exp : \mathbb{C} \to \mathbb{C}^*$ has an infinite number of sheets.

Suppose X, Y are Hausdorff spaces and $p : Y \to X$ is a covering map. If Z is another topological space and $f : Z \to X$ is a continuous map. Question: Under which conditions of Z, the map f can be lifted to a continuous map $g : Z \to Y$?

Suppose X, Y are Hausdorff spaces and $p : Y \to X$ is a covering map. If Z is another topological space and $f : Z \to X$ is a continuous map. Question: Under which conditions of Z, the map f can be lifted to a continuous map $g : Z \to Y$?

Theorem (Lifting of Mappings)

If Z is simply connected, pathwise connected and locally pathwise connected, then for every choice of points $z_0 \in Z$ and $y_0 \in Y$ with $f(z_0) = p(y_0)$ there exists precisely one lifting $\hat{f} : Z \to Y$ such that $\hat{f}(z_0) = y_0$.

Suppose X, Y are Hausdorff spaces and $p : Y \to X$ is a covering map. If Z is another topological space and $f : Z \to X$ is a continuous map. Question: Under which conditions of Z, the map f can be lifted to a continuous map $g : Z \to Y$?

Theorem (Lifting of Mappings)

If Z is simply connected, pathwise connected and locally pathwise connected, then for every choice of points $z_0 \in Z$ and $y_0 \in Y$ with $f(z_0) = p(y_0)$ there exists precisely one lifting $\hat{f} : Z \to Y$ such that $\hat{f}(z_0) = y_0$.

Sketch of the proof: Suppose $z \in Z$ is an arbitrary point and $u : I \to Z$ is a curve from z_0 to z. Then $v := f \circ u$ is a curve in X with initial point $f(z_0)$ and end point $f(z)$.
Let $\hat{v} : I \to Y$ be the unique lifting of v which has initial point y_0. Define $\hat{f}(z) := \hat{v}(1) \in Y$. This definition is independent of the choice of the curve u, hence $\hat{f}$ is a well-defined mapping from Z to Y. Also by construction $f = p \circ \hat{f}$.

Need to show that $\hat{f}$ is continuous.
Let $z \in Z, x = f(z), y = \hat{f}(z)$ and V be an open neighborhood of y. It suffices to show that there is an open neighborhood W of z such that $\hat{f}(W) \subset V$.

Need to show that $\hat{f}$ is continuous.
Let $z \in Z, x = f(z), y = \hat{f}(z)$ and V be an open neighborhood of y. It suffices to show that there is an open neighborhood W of z such that $\hat{f}(W) \subset V$.
Without loss of generality, we can assume that $p|_V : V \to U$ is a homeomorphism, let $\varphi : U \to V$ be its inverse.
Since f is continuous, there exists W' open neighborhood of z such that $f(W') \subset U$. Z is locally pathwise connected, there exists a pathwise connected subset W of W' containing z.

Need to show that $\hat{f}$ is continuous.
Let $z \in Z, x = f(z), y = \hat{f}(z)$ and V be an open neighborhood of y. It suffices to show that there is an open neighborhood W of z such that $\hat{f}(W) \subset V$.
Without loss of generality, we can assume that $p|_V : V \to U$ is a homeomorphism, let $\varphi : U \to V$ be its inverse.
Since f is continuous, there exists W' open neighborhood of z such that $f(W') \subset U$. Z is locally pathwise connected, there exists a pathwise connected subset W of W' containing z.
Now we claim that $\hat{f}(W) \subset V$. Let $z' \in W$ be an arbitrary point and u' be a curve from z to z' which lies entirely in W. Then the curve $v' = f \circ u'$ lies entirely in U, we let $\hat{v}' := \varphi \circ v'$ be a lifting of v' with initial point y.
Note that $\hat{v} \cdot \hat{v}'$ is the lifting of $v \cdot v' = f \circ (u \cdot u')$ with initial point y_0.
Thus

$$\hat{f}(z') = (\hat{v} \cdot \hat{v}')(1) = \hat{v}'(1) \in V$$

.

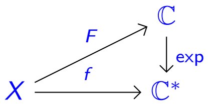
Need to show that $\hat{f}$ is continuous.
Let $z \in Z, x = f(z), y = \hat{f}(z)$ and V be an open neighborhood of y. It suffices to show that there is an open neighborhood W of z such that $\hat{f}(W) \subset V$.
Without loss of generality, we can assume that $p|_V : V \to U$ is a homeomorphism, let $\varphi : U \to V$ be its inverse.
Since f is continuous, there exists W' open neighborhood of z such that $f(W') \subset U$. Z is locally pathwise connected, there exists a pathwise connected subset W of W' containing z.
Now we claim that $\hat{f}(W) \subset V$. Let $z' \in W$ be an arbitrary point and u' be a curve from z to z' which lies entirely in W. Then the curve $v' = f \circ u'$ lies entirely in U, we let $\hat{v}' := \varphi \circ v'$ be a lifting of v' with initial point y.
Note that $\hat{v} \cdot \hat{v}'$ is the lifting of $v \cdot v' = f \circ (u \cdot u')$ with initial point y_0. Thus

$$\hat{f}(z') = (\hat{v} \cdot \hat{v}')(1) = \hat{v}'(1) \in V$$

. **Remark** The theorem still holds when p is just a local homeomorphism with curve lifting property.

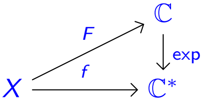
Example (The Logarithm of a Function)

Suppose X is a simply connected Riemann surface and $f : X \to \mathbb{C}^*$ is a nowhere vanishing holomorphic function on X.
We want to find the logarithm of f,i.e, find a holomorphic function $F : X \to \mathbb{C}$ such that $\exp(F) = f$. But this just means that F is a lifting of f with respect to the covering $\exp : \mathbb{C} \to \mathbb{C}^*$.



Example (The Logarithm of a Function)

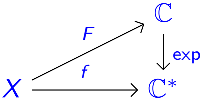
Suppose X is a simply connected Riemann surface and $f : X \to \mathbb{C}^*$ is a nowhere vanishing holomorphic function on X.
We want to find the logarithm of f,i.e, find a holomorphic function $F : X \to \mathbb{C}$ such that $\exp(F) = f$. But this just means that F is a lifting of f with respect to the covering $\exp : \mathbb{C} \to \mathbb{C}^*$.



By the theorem above, for each $x_0 \in X$ and $c \in \mathbb{C}$ such that $\exp(c) = f(x_0)$ there exists a lifting $F : X \to \mathbb{C}$ with $F(x_0) = c$. Since $\exp$ and f are holomorphic, F is also holomorphic. Moreover, any other solution of the problem differs from F by an additive constant $2\pi in, n \in \mathbb{Z}$.

Example (The Logarithm of a Function)

Suppose X is a simply connected Riemann surface and $f : X \to \mathbb{C}^*$ is a nowhere vanishing holomorphic function on X.
We want to find the logarithm of f,i.e, find a holomorphic function $F : X \to \mathbb{C}$ such that $\exp(F) = f$. But this just means that F is a lifting of f with respect to the covering $\exp : \mathbb{C} \to \mathbb{C}^*$.



By the theorem above, for each $x_0 \in X$ and $c \in \mathbb{C}$ such that $\exp(c) = f(x_0)$ there exists a lifting $F : X \to \mathbb{C}$ with $F(x_0) = c$. Since $\exp$ and f are holomorphic, F is also holomorphic. Moreover, any other solution of the problem differs from F by an additive constant $2\pi in, n \in \mathbb{Z}$.
When X is a simply connected domain in $\mathbb{C}^*$, we let $j : X \to \mathbb{C}^*$ be the canonical injection. Then every lifting of j with respect to $\exp$ is a branch of the function log on X.

Theorem

Suppose X is a manifold, Y is a Hausdorff space and $p : Y \to X$ is a local homeomorphism with the curve lifting property. Then p is a covering map.

Theorem

Suppose X is a manifold, Y is a Hausdorff space and $p : Y \to X$ is a local homeomorphism with the curve lifting property. Then p is a covering map.

Proof.

Take $x \in X$ and let $p^{-1}(x) = \{y_j : j \in J\}$.
Let U be an open ball centered at x and $f : U \to X$ be the canonical injection. By "Lifting of Mappings Theorem", for every $j \in J$ there exists a lifting $\hat{f}_j : U \to Y$ such that $\hat{f}_j(x) = y_j$.
Let $V_j := \hat{f}_j(U)$, we can easily show that:

$$p^{-1}(U) = \bigcup_{j \in J} V_j$$

□

Definition

- A locally compact space is a Hausdorff space such that every point has a compact neighborhood.
- A proper mapping is a continuous mapping between 2 locally compact spaces such that the preimage of every compact set is compact

Definition

- A locally compact space is a Hausdorff space such that every point has a compact neighborhood.
- A proper mapping is a continuous mapping between 2 locally compact spaces such that the preimage of every compact set is compact

Remark Suppose X, Y are locally compact spaces and $p : Y \to X$ is a proper, discrete map. Then:

- p is a closed map.
- For every $x \in X$, the set $p^{-1}(x)$ is finite.
- If V is a neighborhood of $p^{-1}(x)$, then there exists a neighborhood U of x with $p^{-1}(U) \subset V$.

Theorem

If X, Y are locally compact spaces and $p : Y \to X$ is a proper local homeomorphism, then p is a covering map.

Proof.

Suppose $x \in X$ is arbitrary and let $p^{-1}(x) = \{y_1, ..., y_n\}$.
For every $1 \leq j \leq n$ there exists an open neighborhood W_j of y_j and an open neighborhood U_j of x such that $p|W_j \to U_j$ is a homeomorphism.
We may assume that W_j are pairwise disjoint.
Since $W_1 \cup ... \cup W_n$ is a neighborhood of $p^{-1}(x)$, there exists an open neighborhood $U \subset U_1 \cap ... \cap U_n$ of x with $p^{-1}(U) \subset W_1 \cup ... \cup W_n$.
Let $V_j := W_j \cap p^{-1}(U)$, then the V_j are disjoint open sets and

$$p^{-1}(U) = V_1 \cup ... \cup V_n$$

and all the mappings $p|V_j \to U$ are homeomorphism. □

The Coverings of Riemann Surfaces

Proper Holomorphic Mappings

Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a proper, non-constant holomorphic mapping.

Remark The set A of branch points of f is closed and discrete. Thus $B := f(A)$ is also closed and discrete. B is called the set of **critical values** of f.

Let $Y' := Y \setminus B$ and $X' := X \setminus f^{-1}(B)$. Then X', Y' are Riemann surfaces and $f|_{X'} : X' \to Y'$ is a proper, non-constant, unbranched holomorphic map. Then $f|_{X'}$ is a covering map, it has a well-defined number of sheets n. In other words, for every value $y \in Y'$ is taken exactly n times on X' by the map f.

We want to extent this statement to the critical values $b \in B$ as well. It leads us to the notion of the ramification order of a point.
For $x \in X$ we denote by $v(f,x)$ the ramification order of the point x (it means that f takes the value $f(x)$ at the point x with multiplicity $v(f,x)$). We will say that f takes the value $c \in Y$, counting multiplicities, m times on X if

$$m = \sum_{x \in p^{-1}(c)} v(f,x).$$

Theorem

Suppose X, Y are Riemann surfaces and $f : Y \to X$ is a proper, non-constant holomorphic mapping. Then there exists a natural number n such that f takes every value $c \in Y$, counting multiplicities, n times.

Proof.

Let n be the number of sheets of the unbranched covering $f|_{X'} : X' \to Y'$. Suppose $b \in B$ is a critical point.
Let $p^{-1}(b) = \{x_1, x_2, ..., x_r\}$ and $v(f, x_j) =: k_j$. We only need to show that

$$k_1 + k_2 + ... + k_r = n$$

By the "Local Behavior Theorem", there exists disjoint open neighborhoods U_j of x_j and open neighborhoods V_j of b such that: For each j, for every point $c \in V_j \setminus \{b\}$ the set $f^{-1}(c) \cap U_j$ consists of exactly k_j points.
Because f is proper, we can find a neighborhood $V \subset V_1 \cap V_2 \cap ... \cap V_r$ of b such that $f^{-1}(V) \subset U_1 \cup U_2 \cup ... \cup U_r$. Then for every point $c \in V \cap Y'$ we have $f^{-1}(c)$ consists of $k_1 + k_2 + ... + k_r$ points. On the other hand, $\forall c \in Y'$ the cardinality of $f^{-1}(c)$ is equal to n. Thus

$$n = k_1 + k_2 + ... + k_n.$$

□

Corollary

On any compact Riemann surface X every non-constant meromorphic function has as many zeros as poles, where each is counted according to multiplicities.

Corollary

There does not exists any meromorphic function on the complex torus $\mathbb{C}/\Gamma$ which has a single pole of first order.

Riemann-Hurwitz Theorem

The Euler Characteristic

Let S be a compact differentiable orientable surface. Denote by Δ the standard triangle of vertices $(0,0), (1,0)$ and $(0,1)$ in $\mathbb{R}^2$. We call a topological triangle of S the image of Δ by a homeomorphism $\varphi : \Delta \to S$. The images of vertices and edges by φ will be called vertices and edges of the triangle $\varphi(\Delta)$

Riemann-Hurwitz Theorem

The Euler Characteristic

Let S be a compact differentiable orientable surface. Denote by Δ the standard triangle of vertices $(0,0),(1,0)$ and $(0,1)$ in $\mathbb{R}^2$. We call a topological triangle of S the image of Δ by a homeomorphism $\varphi : \Delta \to S$. The images of vertices and edges by φ will be called vertices and edges of the triangle $\varphi(\Delta)$

A triangulation of S is a finite family of triangles $\Delta_i \subset S$ whose union is equal to S such that the intersection of two different triangles is either empty, a common edge or a common vertex.

Riemann-Hurwitz Theorem

The Euler Characteristic

Let S be a compact differentiable orientable surface. Denote by Δ the standard triangle of vertices $(0,0),(1,0)$ and $(0,1)$ in $\mathbb{R}^2$. We call a topological triangle of S the image of Δ by a homeomorphism $\varphi : \Delta \to S$. The images of vertices and edges by φ will be called vertices and edges of the triangle $\varphi(\Delta)$

A triangulation of S is a finite family of triangles $\Delta_i \subset S$ whose union is equal to S such that the intersection of two different triangles is either empty, a common edge or a common vertex.

Theorem (Radó)

Any compact surface admits a triangulation.

Riemann-Hurwitz Theorem

The Euler Characteristic

Theorem

For every triangulation of S, the number of vertices, minus the number of edges and plus the number of triangles, does not depend on the triangulation. This number, denoted by χ, is called the Euler characteristic of S. Moreover, we also have $\chi = 2 - 2g$ where g is the genus of the surface.

Riemann-Hurwitz Theorem

Riemann-Hurwitz formula

Let X, Y be compact Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic mapping.
Since X is compact, f is proper and hence it is a branched covering. We call the number n defined as above the degree of f.
For each ramification point $x \in X$, let $o(x)$ be its ramification order.

Theorem (Riemann-Hurwitz)

$$\chi(X) = n\chi(Y) - \sum(o(x) - 1)$$

THANK YOU FOR LISTENING