A Finiteness Theorem

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Definition

$$L^2(D, \mathcal{O}) := \{f \in \mathcal{O}(D) : \|f\|_{L^2(D)} < \infty\}$$

Remark If $vol(D) < \infty$ then for every bounded function $f \in \mathcal{O}(D)$, one has:

$$\|f\|_{L^2(D)} \leq \sqrt{vol(D)}\,\|f\|_D$$

where $\|f\|_D := sup\{|f(z)| : z \in D\}$.

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This inner product makes $L^2(D, \mathcal{O})$ a unitary space and has a well-defined notion of orthogonality.

Consider the case $D = B = B(a, r)$. Let:

$$\psi_n(z) := (z - a)^n, n \in \mathbb{N}.$$

Then the family $\{\psi_n\}_{n\in\mathbb{N}}$ forms an orthogonal system in $L^2(B, \mathcal{O})$.

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If $f \in L^2(B, \mathcal{O})$, $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$, it follows from Pythagoras that

$$\|f\|^2_{L^2(B)} = \sum_{n=0}^{\infty} \frac{\pi r^{2n+2}}{n+1} |c_n|^2.$$

Theorem

Suppose $D \subset \mathbb{C}$ *is open,* $r > 0$ *and*

$$D_r := \{z \in \mathbb{C} : B(z, r) \subset D\}.$$

Then for every $f \in L^2(D, \mathcal{O})$ *one has*

$$\|f\|_{D_r} \leq \frac{1}{\sqrt{\pi} r} \|f\|_{L^2(D)} .$$

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Proof.

Let $a \in D_r$ and $f(z) = \sum c_n (z - a)^n$. One has:

$$|f(a)| = |c_0| \leq \frac{1}{\sqrt{\pi} r} \|f\|_{L^2(B(a,r))} \leq \frac{1}{\sqrt{\pi} r} \|f\|_{L^2(D)} .$$

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Remark If $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(D, \mathcal{O})$ then the sequence converges uniformly on every compact subset of D. Thus the limit function is holomorphic on D. Hence $L^2(D, \mathcal{O})$ is a Hilbert space.

Lemma

Suppose $D' \Subset D$ *are open subsets of* $\mathbb{C}$. *Then given any* $\varepsilon > 0$, *there exists a closed vector subspace* $A \subset L^2(D, \mathcal{O})$ *of finite codimension such that*

$$\|f\|_{L^2(D')} \leq \varepsilon \|f\|_{L^2(D)}, \forall f \in A.$$

Proof.

There exists $r > 0$ and finitely many points $a_1, ..., a_k \in D$ with the following properties:

i. $B(a_j, r) \subset D$ for $j = 1, 2, .., k$,

ii. $D' \subset \bigcup_{j=1}^{k} B(a_j, r/2)$.

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Let A be the closed vector subspace of $L^2(D, \mathcal{O})$ consists all functions f vanishing at every point a_j at least to order n $(n \in \mathbb{N})$. One can easily see that $codim(A) \leq kn$.

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$$f(z) = \sum_{m=n}^{\infty} c_m (z - a_j)^m.$$

For every $\rho \leq r$ one has:

$$\|f\|^2_{L^2(B(a_j,\rho))} = \sum_{m=n}^{\infty} \frac{\pi \rho^{2n+2}}{m+1} |c_m|^2.$$

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It follows that:

$$\|f\|_{L^2(B(a_j,r/2))} \leq 2^{-n-1} \|f\|_{L^2(B(a_j,r))}, \forall j = 1, ..., k.$$

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Then one has:

$$\begin{aligned} \|f\|_{L^2(D')} &\leq \sum_{j=1}^{k} \|f\|_{L^2(B(a_j,r/2))} \\ &\leq 2^{-n-1} \sum_{j=1}^{k} \|f\|_{L^2(B(a_j,r))} \\ &\leq 2^{-n-1} k \|f\|_{L^2(D)} . \end{aligned}$$

Choose n large enough gives us our desire result. □

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Suppose $\mathfrak{B} = (V_i, z_i)_{1 \leq i \leq n}$ is a finite family of charts on X. Let $|\mathfrak{B}| := V_1 \cup \ldots \cup V_n$, we can define the cochain groups $C^0(\mathfrak{B}, \mathcal{O})$ and $C^1(\mathfrak{B}, \mathcal{O})$ on the space $|\mathfrak{B}|$.

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If $\mathfrak{W} = (W_i)_{1 \le i \le n}$ is another family such that $W_i \subset V_i$ (resp. $W_i \Subset V_i$) $\forall i$, we'll write $\mathfrak{W} < \mathfrak{B}$ (resp. $\mathfrak{W} \ll \mathfrak{B}$) for short.

Consider $\mathfrak{U}^* = (U_i^*, z_i)_{1 \leq i \leq n}$ where $z_i(U_i^*) \subset \mathbb{C}$ are disks $\forall i = 1, ..., n$. Suppose $\mathfrak{U} < \mathfrak{U}^*$, define L^2-norms on $C^0(\mathfrak{U}, \mathcal{O}), C^1(\mathfrak{U}, \mathcal{O})$ as follow:

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i. For $\eta = (f_i) \in C^0(\mathfrak{U}, \mathcal{O})$ let:

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ii. For $\xi = (f_{ij}) \in C^1(\mathfrak{U}, \mathcal{O})$ let:

$$\|\xi\|^2_{L^2(\mathfrak{U})} := \sum_{ij} \|f_{ij}\|^2_{L^2(U_i \cap U_j)} .$$

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The set of cochains having finite norm is a vector subspace $C^q_{L^2}(\mathfrak{U}, \mathcal{O}) \subset C^q(\mathfrak{U}, \mathcal{O})$ for $q = 0, 1$. Moreover, they are Hilbert spaces.

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The set of cochains having finite norm is a vector subspace $C^q_{L^2}(\mathfrak{U}, \mathcal{O}) \subset C^q(\mathfrak{U}, \mathcal{O})$ for $q = 0, 1$. Moreover, they are Hilbert spaces. The cocycles in $C^1_{L^2}(\mathfrak{U}, \mathcal{O})$ also form a closed vector subspace, denoted by $Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$.

Proposition

If $\mathfrak{B} \ll \mathfrak{U}$ *(i.e* $V_i \Subset U_i, i = 1, 2, ..., n$*) then for any cochain* $\xi \in C^q(\mathfrak{U}, \mathcal{O})$ *one has* $\|\xi\|_{L^2(\mathfrak{B})} < \infty$*. Moreover, for any* $\varepsilon > 0$ *there exists a closed vector subspace* $A \subset Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$ *of finite codimension such that:*

$$\|\xi\|_{L^2(\mathfrak{B})} \leq \varepsilon \|\xi\|_{L^2(\mathfrak{U})}, \forall \xi \in A.$$

Theorem

Let X be a Riemann surface and $\mathfrak{U}^*$ be a finite family of charts on X such that $z_i(U_i^*)$ are disks. Further suppose that one has $\mathfrak{W} \ll \mathfrak{B} \ll \mathfrak{U} \ll \mathfrak{U}^*$. Then there exists a constant $C > 0$ such that $\forall \xi \in Z^1_{L^2}(\mathfrak{B}, \mathcal{O}), \exists\, \zeta \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O}), \eta \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$ satisfying:

$$\zeta = \xi + \delta\eta \quad \text{on } \mathfrak{W}$$

and

$$\max(\|\zeta\|_{L^2(\mathfrak{U})}, \|\eta\|_{L^2(\mathfrak{W})}) \leq C \|\xi\|_{L^2(\mathfrak{B})}.$$

Proof.

i. Suppose $\xi = (f_{ij}) \in Z^1_{L^2}(\mathfrak{B}, \mathcal{O})$ is given. Since $H^1(|\mathfrak{B}|, \mathcal{E}) = 0$, there exists a cochain $(g_i) \in C^0(\mathfrak{B}, \mathcal{E})$ such that $f_{ij} = g_j - g_i$ on $V_i \cap V_j$.

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Since $\bar{\partial} g_i = \bar{\partial} g_j$ on $V_i \cap V_j$, $\exists\ \omega \in \mathcal{E}^{0,1}(|\mathfrak{B}|)$ with $\omega | V_i = \bar{\partial} g_i$.

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Since $|\mathfrak{W}| \Subset |\mathfrak{B}|$, $\exists\ \psi \in \mathcal{E}(X)$ s.t $Supp(\psi) \subset |\mathfrak{B}|$ and $\psi | \mathfrak{W} = 1$. Hence $\psi\omega$ can be defined on $|\mathfrak{U}^*|$.

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By Dolbeault's lemma, $\exists\ h_i \in \mathcal{E}(U_i^*)$ such that $\bar{\partial} h_i = \psi\omega$ on U_i^*. Let $F_{ij} := h_j - h_i$ on $U_i^* \cap U_j^*$, one has $F_{ij} \in \mathcal{O}(U_i^* \cap U_j^*) \forall i, j$.

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On each $W_i \in \mathfrak{W}_\mathrm{i}$, one has $\bar{\partial} h_i = \psi\omega = \omega = \bar{\partial} g_i$ and thus $h_i - g_i$ is holomorphic on W_i. Since $h_i - g_i$ is bounded on W_i, one has $\eta := (h_i - g_i)|\mathfrak{W} \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$.

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On each $W_i \in \mathfrak{W}_i$, one has $\bar{\partial} h_i = \psi\omega = \omega = \bar{\partial} g_i$ and thus $h_i - g_i$ is holomorphic on W_i. Since $h_i - g_i$ is bounded on W_i, one has $\eta := (h_i - g_i) | \mathfrak{W} \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$.
Note that $F_{ij} - f_{ij} = (h_j - g_j) - (h_i - g_i)$ on $W_i \cap W_j$ and thus

$$\zeta - \xi = \delta\eta \text{ on } \mathfrak{W}.$$

□

Proof.

ii. Consider the Hilbert space:

$$H := Z^1_{L^2}(\mathfrak{U}, \mathcal{O}) \times Z^1_{L^2}(\mathfrak{B}, \mathcal{O}) \times C^0_{L^2}(\mathfrak{W}, \mathcal{O})$$

with the norm

$$\|(\zeta, \xi, \eta)\|_H := \left(\|\zeta\|_{L^2(\mathfrak{U})} + \|\xi\|_{L^2(\mathfrak{B})} + \|\eta\|_{L^2(\mathfrak{W})}\right)^{1/2}$$

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Since L is closed in H, it is also Hilbert. Consider:

$$\pi : L \to Z^1_{L^2}(\mathfrak{B}, \mathcal{O}), \ (\zeta, \xi, \eta) \mapsto \xi$$

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$$\|(\zeta, \xi, \eta)\|_H := \left(\|\zeta\|_{L^2(\mathfrak{U})} + \|\xi\|_{L^2(\mathfrak{B})} + \|\eta\|_{L^2(\mathfrak{W})}\right)^{1/2}$$

Let $L \subset H$ be the subspace

$$L := \{(\zeta, \xi, \eta) \in H : \zeta = \xi + \delta\eta\}$$

Since L is closed in H, it is also Hilbert. Consider:

$$\pi : L \to Z^1_{L^2}(\mathfrak{B}, \mathcal{O}),\ (\zeta, \xi, \eta) \mapsto \xi$$

π is open. Thus exists a constant $C > 0$ s.t $\forall \xi \in Z^1_{L^2}(\mathfrak{B}, \mathcal{O})$, $\exists x = (\zeta, \xi, \eta) \in L$ with $\pi(x) = \xi$ and $\|x\|_H \leq C \|\xi\|_{L^2(\mathfrak{B})}$ $(*)$. The proof is complete.

□

Lemma

Suppose $f : X \to Y$ *is an open linear map between Banach spaces. Then there exists a constant* $C > 0$ *such that for every* $y \in Y$, $\exists x \in X$ *with* $f(x) = y$ *and* $\|x\|_X \leq C\,\|y\|_Y$

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Let $y \in Y$ arbitrarily. One has $\frac{y}{\|y\|_Y} \in f(\{\|x\|_X \leq C\})$, then:

$$\frac{y}{\|y\|_Y} = f(x') \text{ for some } \left\|x'\right\|_X \leq C.$$

Let $x = \|y\|_Y x'$, then $f(x) = y$ and $\|x\|_X = \|y\|_Y \|x'\|_X \leq C \|y\|_Y$. □

Theorem

Let X be a Riemann surface and $\mathfrak{U}^*$ be a finite family of charts on X such that $z_i(U_i^*)$ are disks. Further suppose that one has $\mathfrak{W} \ll \mathfrak{B} \ll \mathfrak{U} \ll \mathfrak{U}^*$. Then there exists a constant $C > 0$ such that $\forall \xi \in Z^1_{L^2}(\mathfrak{B}, \mathcal{O}), \exists\ \zeta \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O}), \eta \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$ satisfying:

$$\zeta = \xi + \delta\eta \quad \textit{on } \mathfrak{W}$$

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$$\mathit{max}(\|\zeta\|_{L^2(\mathfrak{U})}, \|\eta\|_{L^2(\mathfrak{W})}) \leq C \|\xi\|_{L^2(\mathfrak{B})}.$$

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Theorem

Under the same assumptions as in the previous theorem, there exists a finite codimensional vector subspace $S \subset Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$ with the following property: $\forall \xi \in Z^1(\mathfrak{U}, \mathcal{O})$ there exist elements $\sigma \in S$ and $\eta \in C^0(\mathfrak{W}, \mathcal{O})$ s.t:

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Theorem

Let X be a Riemann surface and $\mathfrak{U}^*$ be a finite family of charts on X such that $z_i(U_i^*)$ are disks. Further suppose that one has $\mathfrak{W} \ll \mathfrak{V} \ll \mathfrak{U} \ll \mathfrak{U}^*$. Then there exists a constant $C > 0$ such that $\forall \xi \in Z^1_{L^2}(\mathfrak{V}, \mathcal{O}), \exists\, \zeta \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O}), \eta \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$ satisfying:

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Remark The natural restriction mapping

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Let $\zeta_0 = \xi_0 + \sigma_0$ with $\xi_0 \in A, \sigma_0 \in S$ be the orthogonal decomposition. By Pythagoras one has $\|\xi_0\|_{L^2(\mathfrak{U})} \leq \|\zeta_0\|_{L^2(\mathfrak{U})} \leq CM$. By the definition of A,

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Analogously, $\exists\ \zeta_1 \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$ and $\eta_1 \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$ such that:

$$\zeta_1 = \xi_0 + \delta\eta_1 \text{ on } \mathfrak{W} \text{ and } \|\zeta_1\|_{L^2(\mathfrak{U})}, \|\eta_1\|_{L^2(\mathfrak{W})} \leq \frac{CM}{2}.$$

□

Proof.

By induction, we construct elements $\zeta_k \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O}), \eta_k \in C^0_{L^2}(\mathfrak{W}, \mathcal{O}), \xi_k \in A, \sigma_k \in S$ such that:

$$\begin{cases} \zeta_k = \xi_{k-1} + \delta\eta_k \text{ on } \mathfrak{W} \\ \zeta_k = \xi_k + \sigma_k \\ \|\zeta_k\|_{L^2(\mathfrak{U})}, \|\eta_k\|_{L^2(\mathfrak{W})} \leq \frac{CM}{2^k}. \end{cases}$$

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Then

$$\xi_m + \sum_{k=0}^{m} \sigma_k = \xi + \delta\left(\sum_{k=0}^{m} \eta_k\right) \text{ on } \mathfrak{W}.$$

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Let $m \to \infty$ one has:

$$\sigma = \xi + \delta\eta \text{ on } \mathfrak{W}.$$

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For every open covering $\mathfrak{U} = (U_i)_{i \in I}$ of X, one has $\mathfrak{U} \cap Y := (U_i \cap Y)_{i \in I}$ is an open covering of Y and the natural restriction mapping $Z^1(\mathfrak{U}, \mathcal{F}) \to Z^1(\mathfrak{U} \cap Y, \mathcal{F})$ induces a homomorphism $H^1(\mathfrak{U}, \mathcal{F}) \to H^1(\mathfrak{U} \cap Y, \mathcal{F})$.

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Remark If $Y' \subset Y \subset X$ are open subsets, then the homomorphism $H^1(X, \mathcal{F}) \to H^1(Y', \mathcal{F})$ is the composition of $H^1(X, \mathcal{F}) \to H^1(Y, \mathcal{F})$ and $H^1(Y, \mathcal{F}) \to H^1(Y', \mathcal{F})$.

Theorem

Suppose X is a Riemann surface and $Y_1 \Subset Y_2 \subset X$ are open subsets. Then the restriction homomorphism

$$H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$$

has a finite dimensional image.

Proof.

There exists a finite family of charts $\mathfrak{U}^* = (U_i^*, z_i)_{1 \le i \le n}$ on X and $\mathfrak{W} \ll \mathfrak{V} \ll \mathfrak{U} \ll \mathfrak{U}^*$ such that:

i. $Y_1 \subset \bigcup_{i=1}^n W_i =: Y' \Subset Y'' := \bigcup_{i=1}^n U_i \subset Y_2$,

ii. all $z_i(U_i^*), z_i(U_i)$ and $z_i(W_i)$ are disks in $\mathbb{C}$.

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Note that the restriction mapping $H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$ can be factorized as follows:

$$H^1(Y_2, \mathcal{O}) \to H^1(Y'', \mathcal{O}) \to H^1(Y', \mathcal{O}) \to H^1(Y_1, \mathcal{O}),$$

the proof is complete. □

Theorem

Suppose X is a Riemann surface and $Y_1 \Subset Y_2 \subset X$ are open subsets. Then the restriction homomorphism

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Corollary

If X is a compact Riemann surface then $\dim H^1(X, \mathcal{O}) < \infty$.

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If X is a compact Riemann surface then $g := \dim H^1(X, \mathcal{O})$ is called the **genus** of X.

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Definition

If X is a compact Riemann surface then $g := \dim H^1(X, \mathcal{O})$ is called the **genus** of X.

Remark The Riemann sphere $\mathbb{P}^1$ has genus zero.

Theorem

Suppose X is a Riemann surface and $Y \Subset X$. Then for every point $a \in Y$ there exists a meromorphic function $f \in \mathcal{M}(Y)$ which has a pole at a and holomorphic on $Y \setminus \{a\}$.

Proof.

Let $k := dim\ Im(H^1(X, \mathcal{O}) \to H^1(Y, \mathcal{O})) < \infty$.

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Then $\mathfrak{U} = (U_1, U_2)$ is an open covering of X.

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For each $j = 1, ..., k+1$, we observe that the function z^{-j} is holomorphic on $U_1 \setminus \{a\} = U_1 \cap U_2$ and hence represents a cocycle $\zeta_j \in Z^1(\mathfrak{U}, \mathcal{O})$.

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Note that $dim\ Im(H^1(\mathfrak{U}, \mathcal{O}) \to H^1(\mathfrak{U} \cap Y, \mathcal{O}) \leq k$ then the cocycles

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are linearly dependent modulo the coboundaries. Then there exists complex numbers $c_1, ..., c_{k+1}$ not all zero and a cochain $\eta = (f_1, f_2) \in C^0(\mathfrak{U} \cap Y, \mathcal{O})$ such that:

$$c_1\zeta_1 + ... + c_{k+1}\zeta_{k+1} = \delta\eta \text{ with respect to } \mathfrak{U} \cap Y,$$
$$\Rightarrow \sum_{j=1}^{k+1} c_j z^{-j} = f_2 - f_1 \text{ on } U_1 \cap U_2 \cap Y.$$
$$\Rightarrow f_1 + \sum_{j=1}^{k+1} c_j z^{-j} = f_2 \text{ on } U_1 \cap U_2 \cap Y.$$

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Note that $dim\ Im(H^1(\mathfrak{U}, \mathcal{O}) \to H^1(\mathfrak{U} \cap Y, \mathcal{O}) \leq k$ then the cocycles

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are linearly dependent modulo the coboundaries. Then there exists complex numbers $c_1, ..., c_{k+1}$ not all zero and a cochain $\eta = (f_1, f_2) \in C^0(\mathfrak{U} \cap Y, \mathcal{O})$ such that:

$$c_1\zeta_1 + ... + c_{k+1}\zeta_{k+1} = \delta\eta \text{ with respect to } \mathfrak{U} \cap Y,$$
$$\Rightarrow \sum_{j=1}^{k+1} c_j z^{-j} = f_2 - f_1 \text{ on } U_1 \cap U_2 \cap Y.$$
$$\Rightarrow f_1 + \sum_{j=1}^{k+1} c_j z^{-j} = f_2 \text{ on } U_1 \cap U_2 \cap Y.$$

Hence there exists a function $f \in \mathcal{M}(Y)$ such that $f = f_1 + \sum_{j=1}^{k+1} c_j z^{-j}$ on $U_1 \cap Y$ and $f = f_2$ on $U_2 \cap Y = Y \setminus \{a\}$. This is the desired function. □

Corollary

Suppose X is a compact Riemann surface and $a_1, a_2, ..., a_n$ are distinct points on X. Then there exists a meromorphic function $f \in \mathcal{M}(X)$ such that:

$$f(a_1) = 1, f(a_2) = ... = f(a_n) = 0$$

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For each $k \geq 2$, choose $\lambda_k \in \mathbb{C}^*$ such that $g(a_j) - g(a_k) + \lambda_k \neq 0, \forall j = 1, ..., n$ and set:

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Set

$$f := \prod_{k=2}^{n} g_k \in \mathcal{M}(X),$$

Then $f(a_1) = 1, f(a_2) = ... = f(a_n) = 0$. □

Corollary

Suppose X is a compact Riemann surface and $a_1, ..., a_n$ are distinct points on X. Given any complex numbers $c_1, ..., c_n$. Then there exists a meromorphic function $f \in \mathcal{M}(X)$ such that:

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This is our desire function. □

Some Results For Non-compact Case

Non-compact Case

Theorem

Let X be a non-compact Riemann surface and $Y \Subset X$ be an open subset. Then there exists a holomorphic function $f : Y \to \mathbb{C}$ which is not constant on any connected component of Y.

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Proof.

Choose a domain Y_1 such that $Y \Subset Y_1 \Subset X$ and a point $a \in Y_1 \setminus Y$. Apply the previous theorem to Y_1 and the point a. □

Theorem

Suppose X is a non-compact Riemann surface and $Y \Subset Y' \Subset X$ are open subsets. Then

$$Im(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O})) = 0.$$

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Let $L := Im(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O}))$. We know that L is a finite dimensional vector space.

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Let f be a holomorphic function on Y' which is not constant on any connected component of Y'. Since $H^1(Y', \mathcal{O})$ is a $\mathcal{O}(Y')$-module, then $f\xi_k \in H^1(Y', \mathcal{O})$ are defined.

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By the choice of ξ_k, there exist constants $c_{kp} \in \mathbb{C}$ such that

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$$F := det(f\delta_{kp} - c_{kp})_{1 \le k, p \le n}$$

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Easily see that $F\xi_k|Y = 0$ for $k = 1, ..., n$.

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Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of Y' such that each zero of F is contained in at most one U_i. Thus for $i \neq j$ one has $F|U_i \cap U_j \in \mathcal{O}^*(U_i \cap U_j)$.

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An arbitrarily $\zeta \in H^1(Y', \mathcal{O})$ can be represented by a cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$. By the argument above, there exists $(g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$ such that $f_{ij} = Fg_{ij}$. Let $\xi \in H^1(Y', \mathcal{O})$ be the cohomology class of (g_{ij}). Then $\zeta = F\xi$ and hence $\zeta|Y = F\xi|Y = 0$. □

Corollary

Suppose X is a non-compact Riemann surface and $Y \Subset Y' \subset X$ are open subsets. Then for every differential form $\omega \in \mathcal{E}^{0,1}(Y')$ there exists a function $f \in \mathcal{E}(Y)$ such that $\bar{\partial} f = \omega | Y$.

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Hence there exists a function $f \in \mathcal{E}(Y)$ such that

$$f = f_i - g_i \text{ on } U_i \cap Y, \text{ for every } i \in I.$$

Easily check that $\bar{\partial} f = \omega | Y$. □

THANK YOU FOR LISTENING