Effective Nullstellensatz

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Abstract

Let $\mathbb{K}$ be an algebraically closed field and let $X \subset \mathbb{K}^m$ be an n-dimensional affine variety. Assume that $f_1, \ldots, f_k$ are polynomials which have no common zeros on X. We estimate the degrees of polynomials $A_i \in \mathbb{K}[X]$ such that $1 = \sum_{i=1}^k A_i f_i$ on X. Our estimate is sharp for $k \le n$ and nearly sharp for $k > n$.

Now assume that $f_1, \ldots, f_k$ are polynomials on X. Let $I = (f_1, \ldots, f_k) \subset \mathbb{K}[X]$ be the ideal generated by f_i. It is well-known that there is a number $e(I)$ (the Noether exponent) such that $\sqrt{I}^{e(I)} \subset I$. We give a sharp estimate of $e(I)$ in terms of n, D and deg f_i. We also give similar estimates in the projective case.

Finally we obtain a result from elimination theory: if $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ is a system of polynomials with a finite number of common zeros, then we have the following optimal elimination:

$$\phi_i(x_i) = \sum_{j=1}^{n} f_j g_{ij}, \ i = 1, \ldots, n,$$

where deg $f_j g_{ij} \le \prod_{i=1}^n \deg\ f_i$.