Non-archimedean Hermite's inequality and applications

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Let L be a free $\mathbb{Z}$-module of dimension $n \geq 1$ and $b : L \times L \to \mathbb{R}$ a map which is $\mathbb{Z}$-bilinear and symmetric. Let $(e_1, e_2, \ldots, e_n)$ be a basis of L then the determinant of the matrix $[b(e_i, e_j)]$ is independant of the choice of this basis; it is denoted by $\mathrm{D}(L, b)$. One sets:

$$\mathrm{m}(L,b) \;=\; \inf_{x \in L - \{0\}} |b(x,x)| \qquad .$$

These two invariants are linked by the famous Hermite's inequality:

$$\mathrm{m}(L,b) \;\leq\; (\frac{4}{3})^{\frac{n-1}{2}} \, |\mathrm{D}(L,b)|^{\frac{1}{n}} \qquad .$$

Generalisation

Let K be a field, $|-|$ an absolute value on K and A a subring of K such that the following properties are satisfied:

– One has $|a| \geq 1$ for all non-zero element a in A.

– There exists a real number ρ with $0 < \rho < 1$ such that for all element x in K there exists an element a in A with $|x - a| \leq \rho$.

(These two properties imply that the ring A is a PID. In the classical case K is $\mathbb{R}$, $|-|$ is the usual absolute value, A is $\mathbb{Z}$ and ρ is $\frac{1}{2}$.)

Let L be a free A-module of dimension $n \geq 1$ and $b : L \times L \to \mathbb{R}$ a map which is A-bilinear and symmetric. One has *mutatis mutandis* the inequality:

$$\mathrm{m}(L,b) \;\leq\; (\frac{1}{1-\rho^2})^{\frac{n-1}{2}} \, |\mathrm{D}(L,b)|^{\frac{1}{n}} \qquad .$$

In the case where the absolute value is non-archimedean (*i. e.* if one has $|x+y| \leq \max(|x|, |y|)$), then one has in fact:

$$\mathrm{m}(L,b) \;\leq\; |\mathrm{D}(L,b)|^{\frac{1}{n}}$$

(*non-archimedean Hermite's inequality*).

Application

Let k be a field, let us say with $\mathrm{char}(k) \neq 2$, and $f = \frac{A}{B}$ a rational function with coefficients in k; let $\mathrm{Bez}(f)$ be the non-degenerate symmetric bilinear form on $k^{\max(\deg A, \deg B)}$, associated to f.

The non-archimedean Hermite's inequality is used to show that the "oriented isomorphism class" of $\mathrm{Bez}(f)$ is an invariant of the "algebraic homotopy class" of f. Christophe Cazanave (Algebraic homotopy classes of rational functions, *Ann. Sci. Éc. Norm. Supér.*, **45**, 2012) has shown that it is in fact the only one.