GROUP SCHEMES AND RELATED TOPICS

1. AIM

This is a learning block-seminar that prepares for M. Brion's advanced talk on algebraic groups and inverse Galois problem this November. In addition we shall have three lectures by professor Hashimoto (Osaka Metropolitan University) on Invariant theory.

2. VENUE AND SCHEDULE

Institute of Mathematics, **September 23-27, 2024**.

3. PLAN

Our program tries to cover three topics: reductive groups (4 lectures), abelian varieties (4 lectures), and inverse Galois problem (4 lectures). Throughout this seminar, we will always work over an algebraically closed field.

Each lecture will last for two hours ($\mathbf{2 \times 50}$ **minutes**).

3.1. **Reductive groups.** Our main goal is classifying all connected reductive groups over an algebraically closed field. We assume that the audience knows about linear algebraic groups and also have some basic knowledge of algebraic geometry.
References: [Spr98].

Lecture 1.1: Connected reductive group and root datum:
Abstract: This lecture introduces the concept of connected reductive groups and their basic properties. Our terminology is that a connected reductive group is a smooth linear algebraic group G whose unipotent radical $R_u(G)$ is trivial. Equivalently, since $R_u(G)$ is the unipotent part of $R(G)$ the radical subgroup of G, G is reductive if and only if $R(G)$ is a torus. After giving some examples, the lecturer will introduce the root datum which is our main tool in classifying reductive groups combinatorially. Examples in case of GL_n, or SL_2 are highly recommended.
References: [Tom16, Section 11.1, Section 11.2] and [Spr98, Chapter 7 and 8].

Lecture 1.2: The uniqueness theorem.
Abstract: This lecture starts with constructing a root datum from a connected reductive group. Then the main purpose is to prove the isomorphism theorem roughly saying that two reductive groups are isomorphic if (and only if) their associated root datums are isomorphic. This covers the chapter 9 in [Spr98]. The action of the Weyl group on the character group also needs to be recalled.
References: [Spr98, Chapter 9].

Lecture 1.3: The existence theorem.
Abstract: Given a root datum, this talk will show that each reduced root datum arises from a (unique) connected reductive group.
References: [Spr98, Chapter10].

Lecture 1.4: Classification of almost simple (quasi-simple) group.
Abstract: This talk will give a complete classification of irreducible root systems by using the Dynkin diagram. Consequently, we have a complete classification of quasi-simple groups, and hence, a full classification of reductive groups. Before doing so, maybe we need the structure theorem of reductive groups (see Theorem 11.1.7 in [Tom16]) and the proof of the existence theorem given in the previous lecture (the automorphisms folding technique).
References: [Spr98, Chapter 9, 10].

3.2. **Abelian varieties.** An Algebraic group is a group scheme of finite type over a field. There is Chevalley's structure theorem (see [Bri15]) telling us that every smooth connected algebraic group over a perfect field is an extension of an abelian variety (i.e., a smooth connected proper algebraic group) by a smooth connected algebraic group which is affine, or equivalently linear. Our main purpose in this part is to study the former block, abelian varieties. The main reference is [Mum08].

Lecture 2.1: Complex tori and abelian varieties.

Abstract: This lecture starts with complex tori and their cohomology theory, then explains the equivalent conditions for a complex tori to be an algebraic, that is an abelian variety over the complex numbers. Then the definition an basis properties of abelian variety over any algebraically closed field will be presented, such as the line bundles, definition of abelian varieties and their basic properties such as commutativity as abstract groups, the rigidity lemma and first applications, structure of maps between abelian varieties.

Reference: [Mum08, Sections 1–4], [EGM, Chapter 2].

Lecture 2.2: Line bundles and dual abelian varieties in characteristic 0.

Abstract: Theorem of the cube is at the heart of the theory. Its proof will be mentioned as well as the consequences. Given a line bundle L on an abelian variety X, this talk will construct the natural homomorphism $\phi_L : X \to \text{Pic}(X)$. Moreover, some applications, such as computing the subgroup of $n-$torsion points, and the divisibility of X (([Mum08, Section 4]), the isogeny etc. will be presented. After that we will study the dual abelian variety in characteristic 0.

Reference: [Mum08, Sections 6–8], [EGM, Chapter 6].

Lecture 2.3: Dual abelian varieties in characteristics p.

Abstract: We want to construct the dual abelian variety. In char 0 case, the dual variety is the Jacobian variety $\text{Pic}^0(X)$ whereas in the general case, the dual variety is a quotient of X by a finite subgroup scheme which is defied by choosing an ample line bundle. The lecturer will explain why they are called dual varieties (use Poincare line bundle, the universal property,...). After that we concentrate on the special case of finite commutative group schemes.

References: [Mum08, Section 12–15], [EGM, Chapter 7].

Lecture 2.4: Jacobian variety.

Abstract: In this lecture, we introduce the concept of the Jacobian variety associated with a (non-singular and projective) curve. With some additional conditions on the curve, we prove the existence of the Jacobian variety. Then some basic properties of Jacobian variety will be presented: auto-duality, the canonical map from the symmetric power of the curve to its Jacobian, etc.

References: [Mil86].

3.3. **Inverse Galois problem.** The inverse problem of Galois theory is how to construct finite Galois extensions of the field of rational numbers $\mathbb{Q}$ with prescribed Galois group. To do so, one is led to construct extensions of other fields K with prescribed Galois group, eg. with $K = \mathbb{Q}(t)$.

References: [Ser88].

Lecture 3.1 : Elementary examples in low degrees, and the Scholz-Reichardt theorem.

Abstract: This lecture begins with some examples of low degrees, and then it moves to prove the Scholz-Reichardt theorem saying that any finite nilpotent group of odd order is a Galois group over $\mathbb{Q}$. This is a special case of Shafarevich's theorem for solvable groups (see [ILK90, Chapter 5]).

References: [Ser88, Chapters 1 and 2].

Lecture 3.2 Hilbert property.

Abstract: This lecture introduces an algebraic geometry approach to the inverse Galois problem. Some important concepts need to be discussed: Hilbert property, thin set, and $G-$covering. After some examples, we will prove Hilbert's irreducibility theorem saying that the affine space $\mathbb{A}^n$ has Hilbert property over any number field K.

References:[Ser88, Chapter 3].

Lecture 3.3 Galois extensions of $\mathbb{Q}(T)$.

Abstract: In this lecture, we will try to understand the rigidity method used to obtain extensions of $\mathbb{Q}(T)$ with given Galois groups. We will learn that technique through some examples of Galois groups: finite abelian groups, the quaternion group Q_8, the symmetric groups, and the alternating groups.

References: [Ser88, Chapter 4].

Lecture 3.4 Embedding problems.

Abstract: Let K be a field of characteristic $\neq 2$, and let G_K be the absolute Galois group of K. Then There is a dictionary between etale $K-$algebras of finite ranks and conjugacy classes of homomorphisms $e : G_K \to S_n$. By using the non-degenerate quadratic form $x \mapsto \mathrm{Tr}(x^2)$ on E, we will show that the canonical central extension $\tilde{A}_n$ of A_n is Galois.

References: [Ser88, Chapter 9].

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Three Lectures on Invariant Theory

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Abstract

Invariant theory is a mathematical topic which lies in the midst of algebraic geometry, commutative algebra, and representation theory of algebraic groups. The three talks are on the topic, but from the viewpoint of commutative algebra side. The first is on equivariant sheaves and modules, and the duality. The second is on the representation theory of algebraic groups and its application to the F-regularity of the ring of invariants. The third is on almost principal bundles, which has many applications to the ring-theoretic study in invariant theory, e.g., the study of the canonical modules and the F-representation type.

1 Equivariant Modules, Sheaves and Duality

The study of group actions on schemes and algebras is the heart of invariant theory. When we study rings, we study modules of them, and the category of modules and its derived category are important machinery in (commutative) ring theory. The situation is similar in the study of schemes, and we study (quasi-coherent and coherent) sheaves and the derived category of them. When we consider the group actions, then it is appropriate to enrich the module theory or sheaf theory with the group action. This leads to the notion of equivariant modules. Let k be a field, A a (finitely generated) k-algebra, G an algebraic group (scheme) over k acting on A as k-automorphisms. We say that M is a (G, A)-module if M is a G-module and is an A-module also, the k-vector space coming from the G-module structure and that from the A-module structure are the same, and the product $A \otimes_k M \to M$ is G-linear. The way to generalize this 'enrichment' of modules by the group action to the case of sheaves over schemes is nontrivial, and there are several ways to realize it. In this talk, we use the notion of equivariant sheaves over diagrams of schemes. In particular, equivariant sheaves with respect to a group action is realized using the simplicial scheme or the groupoid associated with the action. We overview the basic operations on equivariant sheaves (and their derived objects), and establish an equivariat version of Grothendieck duality on proper morphisms and the local duality. The main references are as follows. For basics on Hopf algebras, [Swe] and [Mon] are good books. For basics on commutative algebra, [AM] and [Mat] are good textbooks. Bruns–Herzog [BH] is an advanced textbook with various examples. For basics on algebraic geometry, [Har2] is a standard textbook. [Har1] is still a good reading as an introduction to the duality, although the construction of the twisted inverse there is more or less outdated now. For equivariant modules and sheaves as treated in this lecture, please see [Has1], [Has3] and [Has6]. For equivariant local duality, see [HO1] and [HO2]. For introduction to invariant theory, Campbell–Wehlau [CW] on modular invariant theory of finte groups and Dolgachev [Dol] from the algebro-geometric point of view are recommended.

2 Good filtrations, Steinberg modules, and F-regularity

Ring theoretic properties of the ring of invariants are important in commutative algebra direction of invariant theory. Starting from the Cohen–Macaulay property of determinantal rings and the invariant subrings under the action of non-modular finite groups by Hochster and Eagon [HE], purely ring-theoretic or algebro-geometric study of purity was a powerful tool. In particular, the result of Boutot [Boutot] on rational singularities is outstanding. An affine algebraic group (more generally, an affine algebraic group scheme) G over a field is said to be linearly reductive if any G-module is semisimple. In characteristic zero, any reductive group (such as GL_n, SL_n, Sp_{2n}, and SO_n) is linearly reductive. If a linearly reductive algebraic group scheme G over a field k acts on a commutative k-algebra A, then the ring of invariants A^G is a direct summand subring of A. That is, there is an A^G-submodule M of A such that $A = A^G \oplus M$. In particular, A^G is a pure subring of A, and many good properties are inherited by A^G. However, in positive characteristic, a connected affine algebraic group is linearly reductive if and only if it is a torus, although there are many known ring of invariants under the action of reductive groups which are also good from the ring-theoretic point of view. The talk is on the strong F-regularity of the ring of invariants S^G of V, where $S = \mathrm{Sym}\, V^*$, the algebra of polynomial functions on V on which G acts in a natural way. If S has a good filtration as a G-module in the sense of S. Donkin [Don], then S^G is strongly F-regular [Has2]. There are some improvements in [Has4] and [Has5]. The proof heavily depends on the representation-theoretic study of good filtrations due to Donkin and Mathieu [Mathieu]. It also utilize the properties of Steinberg modules, which was also used in the proof of Mumford's conjecture by Haboush, see [Jan, (II.10)]. For F-singularities such as F-regularity and F-rationality, see [BH, Chapter 10].

3 Almost principal bundles

In the invariant theory of finite groups, pseudo-reflections play important roles. Let k be a field, V be a finite-dimensional k-vector space, $G \subset \mathrm{GL}(V)$ be a finite subgroup without pseudo-reflections, and $S = \mathrm{Sym}\, V^*$ the coordinate ring of V. Assume that the order $|G|$ of G is not divisible by the characteristic of k. Kei-ichi Watanabe proved that the ring of invariants S^G is Gorenstein if and only if $G \subset \mathrm{SL}(V)$. This is not true in general if G has a pseudo-reflection. If G is a finite subgroup of $\mathrm{GL}(V)$ without pseudo-reflections, then there is a Zariski closed G-stable subset F of codimension two or more of V such that the action of G on $V \setminus F$ is free. Generalizing this, the almost principal bundle has defined [Has6]. It has applications to the canonical modules, a-invariants, and Gorenstein property [Has7]. It also has some applications to the finite F-representation type problem [HS]. There are some important examples of almost principal bundles where G is not finite or reduced.

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