# PREPARATION SEMINAR FOR M. BRION'S TALK

## 1. AIM

This is a learning block-seminar that prepares for M. Brion's advanced talk on algebraic groups and inverse Galois problem this November. In addition we shall have three lectures by professor Hashimoto (Osaka Metropolitan University) on Invariant theory.

## 2. VENUE AND SCHEDULE

Institute of Mathematics, **September 23-27, 2024**.

## 3. PLAN

Our program tries to cover three topics: reductive groups (4 lectures), abelian varieties (4 lectures), and inverse Galois problem (4 lectures). Throughout this seminar, we will always work over an algebraically closed field.

Each lecture will last for two hours (**2 × 50 minutes**).

3.1. **Reductive groups.** Our main goal is classifying all connected reductive groups over an algebraically closed field. We assume that the audience knows about linear algebraic groups and also have some basic knowledge of algebraic geometry.
**References**: [Spr98].

Lecture 1.1: Connected reductive group and root datum:
**Abstract**: This lecture introduces the concept of connected reductive groups and their basic properties. Our terminology is that a connected reductive group is a smooth linear algebraic group G whose unipotent radical $R_u(G)$ is trivial. Equivalently, since $R_u(G)$ is the unipotent part of $R(G)$ the radical subgroup of $G$, $G$ is reductive if and only if $R(G)$ is a torus. After giving some examples, the lecturer will introduce the root datum which is our main tool in classifying reductive groups combinatorially. Examples in case of $\mathrm{GL}_n$, or $\mathrm{SL}_2$ are highly recommended.
**References**: [Tom16, Section 11.1, Section 11.2] and [Spr98, Chapter 7 and 8].

Lecture 1.2: The uniqueness theorem.
**Abstract**: This lecture starts with constructing a root datum from a connected reductive group. Then the main purpose is to prove the isomorphism theorem roughly saying that two reductive groups are isomorphic if (and only if) their associated root datums are isomorphic. This covers the chapter 9 in [Spr98]. The action of the Weyl group on the character group also needs to be recalled.
**References**: [Spr98, Chapter 9].

Lecture 1.3: The existence theorem.
**Abstract**: Given a root datum, this talk will show that each reduced root datum arises from a (unique) connected reductive group.
**References**: [Spr98, Chapter10].

Lecture 1.4: Classification of almost simple (quasi-simple) group.
**Abstract**: This talk will give a complete classification of irreducible root systems by using the Dynkin diagram. Consequently, we have a complete classification of quasi-simple groups, and hence, a full classification of reductive groups. Before doing so, maybe we need the structure theorem of reductive groups (see Theorem 11.1.7 in [Tom16]) and the proof of the existence theorem given in the previous lecture (the automorphisms folding technique).
**References**: [Spr98, Chapter 9, 10].

3.2. **Abelian varieties.** An Algebraic group is a group scheme of finite type over a field. There is Chevalley's structure theorem (see [Bri15]) telling us that every smooth connected algebraic group over a perfect field is an extension of an abelian variety (i.e., a smooth connected proper algebraic group) by a smooth connected algebraic group which is affine, or equivalently linear. Our main purpose in this part is to study the former block, abelian varieties. The main reference is [Mum08].

Lecture 2.1: Complex tori and abelian varieties.
**Abstract**: This lecture starts with complex tori and their cohomology theory, then explains the equivalent conditions for a complex tori to be an algebraic, that is an abelian variety over the complex numbers. Then the definition an basis properties of abelian variety over any algebraically closed field will be presented, such as the line bundles, definition of abelian varieties and their basic properties such as commutativity as abstract groups, the rigidity lemma and first applications, structure of maps between abelian varieties.
**Reference:** [Mum08, Sections 1–4], [EGM, Chapter 2].

Lecture 2.2: Line bundles and dual abelian varieties in characteristic 0.
**Abstract**: Theorem of the cube is at the heart of the theory. Its proof will be mentioned as well as the consequences. Given a line bundle $L$ on an abelian variety $X$, this talk will construct the natural homomorphism $\phi_L : X \to \mathrm{Pic}(X)$. Moreover, some applications, such as computing the subgroup of $n-$torsion points, and the divisibility of $X$ ( ([Mum08, Section 4]), the isogeny etc. will be presented. After that we will study the dual abelian variety in characteristic 0.
**Reference**: [Mum08, Sections 6–8], [EGM, Chapter 6].

Lecture 2.3: Dual abelian varieties in characteristics $p$.
**Abstract**: We want to construct the dual abelian variety. In char 0 case, the dual variety is the Jacobian variety $\mathrm{Pic}^0(X)$ whereas in the general case, the dual variety is a quotient of $X$ by a finite subgroup scheme which is defied by choosing an ample line bundle. The lecturer will explain why they are called dual varieties (use Poincare line bundle, the universal property,...). After that we concentrate on the special case of finite commutative group schemes.
**References**: [Mum08, Section 12–15], [EGM, Chapter 7].

Lecture 2.4: Jacobian variety.
**Abstract**: In this lecture, we introduce the concept of the Jacobian variety associated with a (non-singular and projective) curve. With some additional conditions on the curve, we prove the existence of the Jacobian variety. Then some basic properties of Jacobian variety will be presented: auto-duality, the canonical map from the symmetric power of the curve to its Jacobian, etc.
**References**: [Mil86].

3.3. **Inverse Galois problem.** The inverse problem of Galois theory is how to construct finite Galois extensions of the field of rational numbers $\mathbb{Q}$ with prescribed Galois group. To do so, one is led to construct extensions of other fields $K$ with prescribed Galois group, eg. with $K = \mathbb{Q}(t)$.
**References**: [Ser88].

Lecture 3.1 : Elementary examples in low degrees, and the Scholz-Reichardt theorem.
**Abstract**: This lecture begins with some examples of low degrees, and then it moves to prove the Scholz-Reichardt theorem saying that any finite nilpotent group of odd order is a Galois group over $\mathbb{Q}$. This is a special case of Shafarevich's theorem for solvable groups (see [ILK90, Chapter 5]).
**References**: [Ser88, Chapters 1 and 2].

Lecture 3.2 Hilbert property.

**Abstract**: This lecture introduces an algebraic geometry approach to the inverse Galois problem. Some important concepts need to be discussed: Hilbert property, thin set, and $G-$covering. After some examples, we will prove Hilbert's irreducibility theorem saying that the affine space $\mathbb{A}^n$ has Hilbert property over any number field $K$.

**References**:[Ser88, Chapter 3].

Lecture 3.3 Galois extensions of $\mathbb{Q}(T)$.

**Abstract**: In this lecture, we will try to understand the rigidity method used to obtain extensions of $\mathbb{Q}(T)$ with given Galois groups. We will learn that technique through some examples of Galois groups: finite abelian groups, the quaternion group $Q_8$, the symmetric groups, and the alternating groups.

**References**: [Ser88, Chapter 4].

Lecture 3.4 Embedding problems.

**Abstract**: Let $K$ be a field of characteristic $\neq 2$, and let $G_K$ be the absolute Galois group of $K$. Then There is a dictionary between etale $K-$algebras of finite ranks and conjugacy classes of homomorphisms $e : G_K \to S_n$. By using the non-degenerate quadratic form $x \mapsto \text{Tr}(x^2)$ on $E$, we will show that the canonical central extension $\tilde{A}_n$ of $A_n$ is Galois.

**References**: [Ser88, Chapter 9].

## REFERENCES


**Algebraic groups**

[Bri15] M. Brion, *Some structure theorems for algebraic groups*, Lecture notes, Tulane University, 2015.

[Hum75] J.E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, 1975.

[Spr98] T. A. Springer, *Linear algebraic groups. Second edition,* Progress in Mathematics 9, Birkhauser Boston, 1998.

**Abelian varieties**

[EGM] B. Edixhoven, G. van der Geer and B. Moonen, *Abelian varieties* (Preliminary version of the first chapters), available online.

[Mil86] J. S. Milne, Jacobian varieties, Chapter VII of Arithmetic geometry (Storrs, Conn., 1984), 167212, Springer, New York, 1986.

[Mum08] D. Mumford, *Abelian varieties*, with appendices by C. P. Ramanujam and Y. Manin, Corrected reprint of the second (1974) edition, Tata Inst. Fundam. Res. Stud. Math. 5, Hindustan Book Agency, New Delhi, 2008.

**Inverse Galois theory**

[Tom16] Tom De Medts, *Linear algebraic groups*, Master course notes, Ghent University, second term 2015-2016, available online.

[ILK90] V.V. Ishanov, B.B. Lure, D.K. Faddeev, *Embedding problems in Galois theory*, Moscow, 1990.

[Ser88] J.-P. Serre, *Topics in Galois theory*, Course at Harvard University, Fall 1988.

[Hel96] H. Völklein, *Groups as Galois groups: an introduction*, Cambridge studies in advanced mathematics 53.