WORKSHOP ON TANNANKIAN CATEGORIES

Tóm Tắt

Phạm trù Tannaka trung tính trên một trường k là một phạm trù tensơ abel $k-$tuyến tính cứng C sao cho vật đơn vị 1 thỏa $\mathrm{End}(1) \cong k$, cùng một hàm tử thớ. Định lý chính của đối ngẫu Tannaka là mọi phạm trù Tannaka đều tương đương với phạm trù các biểu diễn hữu hạn của một lược đồ nhóm trên k sai khác một đẳng cấu.

Trong hội thảo này, bằng cách bắt đầu với những khái niệm cơ bản của lý thuyết phạm trù, chúng ta sẽ tìm hiểu đối ngẫu Tannaka và ứng dụng của lý thuyết Tannaka vào lý thuyết Hodge và lý thuyết Galois vi phân.

Từ Khóa: Phạm trù Tannaka, Biểu diễn của lược đồ nhóm trên k, Đối ngẫu Tannaka, Phạm trù Galois, Lý thuyết Galois vi phân, Nhóm cơ bản Nori, Lý thuyết Hodge.

Abstract

A neutral Tannakian category over a field k is a rigid k-linear abelian tensor category C whose unit 1 satisfies $\mathrm{End}(1) \cong k$, and is moreover equipped with a fibre functor. The central result about Tannakian duality is that every Tannakian category is equivalent to the category of finite dimensional representations of group scheme over k, which is unique up to isomorphism.

In this workshop, by starting from basic notions in category theory, we will study Tannakian duality. Besides, we will study the relationship between Tannakian theory, Hodge theory and Differential Galois theory.

Keywords: Neutral Tannakian category, Representations of group scheme over k; Tannankian Duality, Galois category, Differential Galois theory, Nori's fundamental group, Hodge theory.

1. Programme

Day 1

Goal. To prepare the background on Category, Affine group scheme and Representation theory.

Lecture 1. Tensor Categories.

- Tensor Categories
- Tensor Functor
- Show that if $\mathscr{C}$ and $\mathscr{C}'$ are rigid, then every morphism between monoidal functors $F, G : \mathscr{C} \to \mathscr{C}'$ is an isomorphism (Prop 1.13 in [Deligne-Milne82]).

Lecture 2. Abelian tensor categories.

- Abelian tensor categories
- A criterion to be a rigid tensor category ([Deligne-Milne82], Prop 1.20)
- Example about rigid tensor categories.

Lecture 3. Affine schemes and affine group schemes.

- Affine schemes and the functor of points (Section 1 of [Jantzen87])
- Affine group schemes (Chapter 1 and chapter 2 of [Waterhouse79]) and Hopf algebras (Chapter 4 of [Sweedler69]).

Lecture 4. Representations and Comodules.

- Representation of affine group scheme over field k (Section 2 of [Jantzen87])
- Categories of comodules (Chapter 2 of [Sweedler69])
- Correspondence between reperesentations of affine group scheme and right comodules over the coordinate Hofp algebra (Proposition 6.1.9, [Szamuely09]).

Day 2

Goal. Prove the central result of Tannakian categories: Every neutral Tannakian categories is equivalent to the category of finite dimensional representations of group scheme over k.

Lecture 5. Neutral Tannakian categories.

- Category of finite dimensional representations is Tannakian
- State how properties of G are reflected in category $\mathrm{rep}_k(G)$.

We use section 2 of [Deligne-Milne82] as reference.

Lecture 6. Recovering affine group scheme from its representation

- Every affine $k-$group scheme G is a directed inverse limit of affine algebraic groups over k (Corollary 2.7, [Deligne-Milne82]
- Recovering affine group scheme from its representation (Proposition 2.8 of [Deligne-Milne82])
- Set up the notations for Lemma 2.13 in [Deligne-Milne82] and discuss Hải's injective lemma as a general version in [Hải16].

Lecture 7. Grothendieck-Deligne-Saavedra Theorem

- Every Tannakian category is equivalent to the category of finite dimensional representations of an affine group scheme over k which is unique up to isomorphism (Proposition 2.11 of [Deligne-Milne82]).

We also use [Deligne-Milne82] as reference for Lecture 6 and Lecture 7.

Lecture 8. Examples of Tannakian categories

Examples of neutral tannakian categories such as the category of flat connections over a smooth scheme, the category of essentially finite vector bundles over a proper schemes (after Nori), the category of meromorphic connection in Picard-Vessiot theory, the category of Hodge structures.

We use [Deligne82], [Szamuely09], [Put-Singer03] and [Nori82] as references.

Day 3

Goal. To construct étale fundamental group via Galois category and construct differential Galois group via Tannaka category.

Lecture 9. Galois category.

- Describe the formalism of Galois categories and fundamental groups, as introduced by A. Grothendieck in [SGA1, Chap. V]
- Galois category and fundamental functor
- Main theorem of Galois category: Every Galois category is equivalent to the category of $\pi-$Finsets where π is pro-finite group.

Lecture 10. Étale fundamental groups.

- Finite etale coverings of a scheme with fibre funtor is Galois category
- Define étale fundamental group via main theorem of Galois category and give some examples.

We also use [Lenstra08] and [Murre67] as reference for Lecture 9 and Lecture 10.

Lecture 11 Differential Galois Theory I

- Picard-Vessiot extension (Section 1.3 of [Put-Singer03])
- Differential Galois group (Section 1.4 of [Put-Singer03])
- The fundamental theorem of differential Galois theory (Proposition 1.34 of [Put-Singer03].

We also use [Crespo-Hajito11] as reference.

Lecture 12 Differential Galois Theory II

- Universal Picard-Vessiot extension (Section 10 of [Put-Singer03])
- Picard-Vessiot theory from the Tannaka viewpoint (Apendix B of [Put-Singer03]).

Day 4

Goal. To understand the relation between Tannankian formalism and other theories.

Lecture 13. Nori's fundamental group I

- Principal $G-$bundles or torsors ([Szamuely09] and [Nori82])
- Essentially finite vector bundle and semistable vector bundle ([Nori82] and [Esnault-Hải-Sun08])
- Nori's fundamental group.

Lecture 14. Nori's fundamental group II

- Nori's fundamental group is an inverse limit of finite $k-$group schemes ([Nori82] and ([Esnault-Hải-Sun08])
- Comparison between Nori's fundamental group and étale fundamental group (Prop 6.7.19 of [Szamuely09])

Lecture 15. Mumford-Tate group I

- $\mathbb{Q}-$Hodge structure and Deligne torus [Deligne82]
- Mumford-Tate group [Deligne82]
- Examples of Mumford-Tate group.

Lecture 16. Mumford-Tate group II

- Deligne torus as Tannaka duality of category of $\mathbb{R}-$Hodge structure [Deligne-Milne82]
- Mumford-Tate group from Tannaka viewpoint [Schnell11]
- $\mathbb{Q}-$Hodge structures of $CM-$type form a Tannakian category [Milne20].

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