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Limit theorems for the number of up-crossings of the conjunction set of stationary Gaussian processes

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Abstract In this paper, we investigate the limit theorem for the number of up-crossings of the conjunction set of stationary Gaussian processes. We prove that as the level tends to infinity then the normalized point process of the number of up-crossings converges weakly to a standard Poisson process. Our proof is based on a discretization argument and a Normal comparison lemma for the order statistics.

Keywords Conjunction set, Gaussian processes, Number of up-crossings, Limit theorem, Poisson approximation.

Mathematics Subject Classification (2000) 60G15 · 60G60

1 Introduction

Let $\{X(t), t \in \mathbb{R}\}$ be a centered stationary Gaussian process with $\mathcal{C}^1$ paths almost surely. Without loss of generality, one can assume that the process has unit variance. Hence, its covariance function $r(t)$ is twice differentiable and can be expressed in a neighborhood $[0, \delta]$ for $\delta > 0$ of 0 as

$$r(t) = 1 + \frac{r''(0)}{2}t^2 + \theta(t), \tag{1}$$

with $\theta(t) > 0$, $\frac{\theta(t)}{t^2} \to 0$, $\frac{\theta'(t)}{t} \to 0$, $\theta''(t) \to 0$ as $t \to 0$.

For a given level $u \in \mathbb{R}$, let us define the excursion set of the process $\{X(t)\}$ in the finite interval $[0, T]$

$$\mathcal{E}_u(T) = \{t \in [0, T] : \quad X(t) \geq u,$$

and also the corresponding number of u-crossings

$$N_u(T) = \#\{t \in [0, T] : \quad X(t) = u\},$$

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and also u-upcrossings

$$U_u(T) = \#\{t \in [0,T] : \quad X(t) = u,\ X'(t) \geq 0\}.$$

It is clear that almost surely the excursion set $\mathcal{E}_u(T)$ consists of some finite closed intervals whose the endpoints are u-crossings and the left-endpoint is an u-upcrossing. Hence, the number of upcrossings $U_u(T)$ is just the number of connected components of $\mathcal{E}_u(T)$.

Study the geometry of the excursion set or the number of (up)crossings plays a central role in the theory of stochastic processes with many applications in Telecommunications, Physical oceanography, Neurology, ... see [3]. Therefore, this research direction has attracted a lot of interest and has a long history.

In his pioneering work in 1940s, Rice [23] provided a formula to calculate the expected number of (up)crossings, independent of M. Kac [**?**]. Today, it is usually called the *Kac-Rice formula*. It is stated for general Gaussian process $\{X(t)\}$as follows

$$\mathrm{E}N_u(T) = \int_0^T \mathrm{E}[|X'(t)| \mid X(t) = u] p_{X(t)}(u) dt,$$

and

$$\mathrm{E}U_u(T) = \int_0^T \mathrm{E}[(X'(t))^+ \mid X(t) = u] p_{X(t)}(u) dt,$$

where $p_{X(t)(.)}$ stands for the density function of the random variable $X(t)$. When the process is stationary and with unit variance, it is clear that $X'(t)$ is a centered Gaussian random variable with variance $-r''(0)$ and is independent of $X(t)$. Thus

$$\mathrm{E}N_u(T) = 2\mathrm{E}U_u(T) = T\frac{e^{-u^2/2}\sqrt{-r''(0)}}{\pi}.$$

In the 1960s, Ito [**?**] and Ylvisaker [29] proved that the necessary and sufficient condition for the finiteness of the expectation of the number of (up)crossings is the finiteness of $-r''(0)$.

From this expectation, one can immediately derive an upper bound for the tail distribution of the maximum of the process $\{X(t)\}$ in the interval $[0,T]$, see [3]

$$\begin{aligned} \mathrm{P}(\max_{t\in[0,T]} X(t) \geq u) &\leq \mathrm{P}(X(0) \geq u) + \mathrm{P}(X(0) < u, \max_{t\in[0,T]} X(t) \geq u) \\ &\leq \mathrm{P}(X(0) \geq u) + \mathrm{P}(U_u(T) \geq 1) \leq \mathrm{P}(X(0) \geq u) + \mathrm{E}U_u(T). \end{aligned}$$

Under some more mild conditions, this upper bound is a good approximation for the tail distribution of the maximum with super-exponentially small error.

A formula for the (factorial) second moment of the number of zero-crossings was provided by Cramér and Leadbetter [5] first provided. Then it is generalized to any level (up)crossings and is also called *Kac-Rice formula* today. It is stated as follows, see [3].

$$\mathrm{E}[N_u(T)(N_u(T)-1)] = \int_0^T \int_0^T \mathrm{E}[|X'(t)X'(s)| X(t) = X(s) = u] p_{(X(t),X(s))}(u,u) dt ds,$$

where $p_{(X(t),X(s))}(.,.)$ stands for the joint density function of the random vector $(X(t), X(s))$.

From this formula, Cramér and Leadbetter [?] proved that the second moment of the number of zero crossings is finite if

$$\exists \delta > 0: \quad L(t) = \frac{r''(t) - r''(0)}{t} = \frac{\theta''(t)}{t} \in L^1([0,\delta], dx). \tag{2}$$

Later on, Geman [?] proved that this condition is also sufficient for the finiteness of the second moment $\mathrm{E}N_0^2(T)$. Then, it is now called the *Geman condition.* For general level crossings, Kratz and Leon [17] showed that this condition is necessary and sufficient for the finiteness of second moment. For the moments of higher order (> 2), see Beliaev [4] and Cuzick [?].

Let us now discuss on the limit theorems for the number of upcrossings. The central limit theorem is proved by Kratz and Leon. Here they fixed the level u and let the window size T grow up to infinity. They showed that there exists a positive constant $\sigma^2 > 0$ such that as $T \to \infty$

$$\frac{1}{\sqrt{T}} \left(U_u(T) - \mathrm{E}U_u(T) \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

The constant $\sigma^2 >$ can be expressed through the covariance function r and its derivatives. Their key ingredient is to propose a Wiener-Hermite chaos expansion for the number of upcrossings, and then to derive the central limit theorem for each chaos by approximating the original process by a sequence of m-dependent stationary processes, and finally just to sum up the limit variances of each chaos.

From the extreme value theory point of view, Volkonskii and Rozanov [26] or Leadbetter et al [20] considered the increasing levels and counted the number of exceedances in the windows of suitable scale. They assumed the Geman condition (2) and the Berman condition

$$r(t) \log t \to 0 \quad \text{as} \quad t \to \infty. \tag{3}$$

They defined the point process

$$R_u(t) = U_u(C_u^{-1} t),$$

where $C_u = \mathrm{E}U_u(1)$. Then they proved that as $u \to \infty$, the family of point processes $R_u(t)$ converges weakly to a standard Poisson point process on the nonnegative half axis. As a consequence, they derived a Gumbel limit theorem for the maximum of the process in suitable scale. An intuitive idea for this Poisson approximation result is as follows. The Berman condition represents a weak-dependent property. So for large enough level u, the numbers of upcrossings in disjoint intervals are approximately independent. Hence it deduces the Poisson limit. The rate of this limit theorem was provided by Kratz and Rootzen [18]. See Azais and Mercadier [2] for similar result for non-stationary Gaussian processes with unit variance. Note that, for a special class of Gaussian processes without satisfying the Berman condition, Leadbetter et al [20] or Tan [28] provided the limit process as a Poisson point process with random intensity. For the treatment for high diemnsion random fields, we refer the readers to the monograph by Piterbarg [22].

In this paper, we are interested in the conjunction set of independent Gaussian processes, that is

$$\mathcal{C}_u(T) = \{t \in [0,T]: \ X_i(t) \geq u, \forall 1 \leq i \leq n\}, \tag{4}$$

where X_i's are the copies of a centered stationary Gaussian process $X(t)$ with unit variane and $\mathcal{C}^1$ paths. We can also define the corresponding conjunction probability

$$\mathrm{P}\left(\sup_{t\in[0,T]}\min_{1\leq i\leq n} X_i(t)\geq u\right). \tag{5}$$

This problem has been addressed by Worsley and Friston in the seminal contribution [27]. Their motivation arises from a statistical application in neuroscience to test whether the functional organization of the brain for language differs according to sex, see also Alodat [1]. Here they considered a general setting where X_i's are random fields on $\mathbb{R}^d$. They provided the expectation of the Euler-Poincare characteristic of the conjunction set $\mathcal{C}_u$ and predicted that this expectation is a good approximation for the conjunction probability (5) as $u\to\infty$. In particular, for the considering one-dimensional case, one can summarize their result as follows.

$$\begin{aligned}&\mathrm{P}\left(\sup_{t\in[0,T]}\min_{1\leq i\leq n} X_i(t)\geq u\right) \\ \approx&\mathrm{E}(\chi(C_u(T)))=\overline{\Phi}^n(u)+nT\frac{\overline{\Phi}^{n-1}(u)\varphi(u)}{\sqrt{2\pi}}\sqrt{-r''(0)},\end{aligned} \tag{6}$$

where $\varphi(.)$ and $\overline{\Phi}(.)$ are, respectively, the density function and the tail distribution of a standard normal distribution. However, they could not provide a rigorous proof for this prediction.

The conjunction probability is also considered by other authors as Debicki, Hashorva, et al [8, 9, 10, 11] from another point of view. It is the tail distribution of the minimum of the processes X_i's, that is the first order statistics of these processes. Note that the order statistics are one of the are among the most fundamental tools in nonparametric statistics and inference. Here they provided an asymptotic formula for the conjunction probability and also a Gumbel limit theorem for the order statistics.

Recently, in [21], the author of this paper considered the number of connected components of the conjunction set $\mathcal{C}_u(T)$ by an adapted definition of the number of upcrossings

$$U_u^*(T)=\#\{t\in[0,T]:\ \exists i\in\{1,\dots,n\}\text{ s.t. } X_i(t)=u, X_i'(t)\geq 0, \text{ and } X_j(t)\geq u, \forall j\neq i\}, \tag{7}$$

and then proved that

$$\mathrm{E}U_u^*=nT\frac{\overline{\Phi}^{n-1}(u)\varphi(u)}{\sqrt{2\pi}}\sqrt{-r''(0)}; \tag{8}$$

$$\mathrm{P}\left(\sup_{t\in[0,T]}\min_{1\leq i\leq n} X_i(t)\geq u\right)\leq\overline{\Phi}^n(u)+\mathrm{P}(U_u^*\geq 1)\leq\overline{\Phi}^n(u)+\mathrm{E}U_u^* \tag{9}$$

and provided the validity of the approximation (6) with super-exponentially small error

$$|\mathrm{P}\left(\sup_{t\in[0,T]}\min_{1\leq i\leq n} X_i(t)\geq u\right)-\overline{\Phi}^n(u)+\mathrm{E}U_u^*|\leq O(\varphi(u(n+\delta))), \tag{10}$$

for some $\delta > 0$.

In this paper, we would like to continue this line of research by considering the limit theorems for the number of up-crossings of the conjunction set. We will provide the Poisson approximation. Our results can be viewed as a version Leadbetter et al [20] in the conjunction setting. Our main theorem in this paper is stated as follows.

Theorem 1 *Let $\{X(t),\ t \in \mathbb{R}\}$ be a centered stationary Gaussian process with $\mathcal{C}^2$ paths almost surely. Assume more that $X(t)$ satisfies both the Geman condition (2) and Berman condition (3). Consider $(X_1(t), \dots, X_n(t))$ the independent copies of $X(t)$.*

Let $U_u^*(.)$ be the number of up-crossings as in (7). For $C_u^* = \mathrm{E}U_u^*(1)$, define the point process

$$R_u^*(t) = U_u^*((C_u^*)^{-1}t).$$

Then as $u \to \infty$, the family of point processes $R_u^*(t)$ converges weakly to a standard Poisson point process on $[0, \infty)$.

The detailed proof of main theorem is presented in Section 2. The key ingredients to prove the Poisson approximation are checking the conditions for random measures convergence by Kallenberg [19] by using a discretization argument and a Normal comparison lemma by Debicki et al [11].

Notations. Recall that through out this paper, we will denote $\varphi(.)$ and $\overline{\Phi}(.)$ the density function and the tail distribution of a standard normal distribution. We also use the notations

$$X_{1:n}(t) = \min_{1 \le i \le n} X_i(t),$$

and for any boundend Borel subset $B \subset \mathbb{R}$

$$M(B) = \sup_{t \in B} X_{1:n}(t).$$

2 Proof of the main theorem

This section is devoted to present the proof of Theorem 1. We will use the following result by Kallenberg [19].

Lemma 1 *Let $\{X_n\}$ be a sequence of point processes on $[0, \infty)$ satisfying that as $n \to \infty$*

i. $\mathrm{E}(X_n(B)) \to \lambda(B)$,
ii. $\mathrm{P}(X_n(B) = 0) \to \exp(-\lambda(B))$,

for any subset $B \subset [0, \infty)$ consisting of finite union of bounded and disjoint intervals of $\mathbb{R}$. Then X_n converges weakly to a standard Poisson point process on $[0, \infty)$.

Therefore, we have to check these two conditions for our point processes $R_u^*(t)$.

2.1 Checking the condition Lemma 1 i.

By the linearity of expectation, one can consider $B = [a, b]$. In this case, we have

$$R_u^*([a, b]) = U_u^*([(C_u^*)^{-1}a, (C_u^*)^{-1}b]).$$

By the stationary property and (8), the expectation of the number of upcrossings in the interval $[(C_u^*)^{-1}a, (C_u^*)^{-1}b]$ is proportion to the length of the interval and equal to

$$(C_u^*)^{-1}(b - a)C_u^* = b - a = |B|.$$

2.2 Checking the condition Lemma 1 ii.

Let $B = \bigcup_{j=1}^{p}(c_j, d_j)$. Define the dilated intervals

$$D_{u,j} = ((C_u^*)^{-1}c_j, (C_u^*)^{-1}d_j.)$$

The event $\{R_u^*(B) = 0\}$ means that

$$U_u^*(D_{u,j}) = 0, \quad \forall j = 1, \ldots, p.$$

From (9) and (10), there exists a universal constant C such that for any $j = 1, \ldots, p$,

$$|\mathrm{P}(M(D_{u,j}) \leq u) - \mathrm{P}(U_u^*(D_{u,j}) = 0)| \leq C\overline{\Phi}^n(u).$$

Hence

$$\left| \mathrm{P}\left(\bigcap_{j=1}^{p} \{U_u^*(D_{u,j}) = 0\} \right) - \mathrm{P}\left(\bigcap_{j=1}^{p} \{M(D_{u,j}) \leq u\} \right) \right| \leq pC\overline{\Phi}^n(u).$$

It means that as $u \to \infty$, one can approximate the probability $\mathrm{P}(R_u^*(B) = 0)$ by $\mathrm{P}\left(\bigcap_{j=1}^{p} \{M(D_{u,j}) \leq u\} \right)$. Then, we can finish our checking procedure from the two following lemmas.

Lemma 2 *As* $u \to \infty$,

$$\left| \mathrm{P}\left(\bigcap_{j=1}^{p} \{M(D_{u,j}) \leq u\} \right) - \prod_{j=1}^{p} \mathrm{P}\left(\{M(D_{u,j}) \leq u\} \right) \right| = o(1).$$

Lemma 3 *For any* $j = 1, \ldots, p$,

$$\mathrm{P}\left(\{M(D_{u,j}) \leq u\} \right) = \exp(d_j - c_j) + o(1).$$

Let us now prove the Lemmas 2 and 3. The key idea is a discretization argument. An intuitive observation is that the maximum value of a continuous function can be approximated by the one at the separable points with a suitable step size. To do so, let us first divide a given interval $((C_u^*)^{-1}c, (C_u^*)^{-1}d)$ into N equal sub-intervals $I_1, \ldots, I_N$ of length $O((C_u^*)^{-1/2})$. It is clear $N = O((C_u^*)^{-1/2})$. Once again, we divide each sub-interval I_i into equal smaller sub-intervals of length $q = q_u$. We will choose q later, but at the moment, q satisfies that $qu \to 0$ as $u \to \infty$.

Lemma 4 *For any $A_u \subset D_{u,j}$ such that $A_u = [aq, bq]$ with $a, b \in \mathbb{Z}$,*

$$0 \le \mathrm{P}(X_{1:n} \le u, \forall k, kq \in A_u) - \mathrm{P}(M(A) \le u) \le |A_u| C_u^* o(1).$$

Proof Let us define the number of upcrossings with respect to the dicretization.

$$U_u^q(A_u) = \#\{k \in \mathbb{Z} : \quad kq \in A_u, X_{1:n}((k-1)q) < u, A_u, X_{1:n}(kq) > u\}.$$

It is clear that

$$\begin{aligned} 0 \le &\mathrm{P}(X_{1:n} \le u, \forall k, kq \in A_u) - \mathrm{P}(M(A_u) \le u) \\ = &\mathrm{P}(\{X_{1:n} \le u, \forall k, kq \in A_u\} \cap \{M(A_u) > u\}). \end{aligned}$$

On this intersecting event, there must exist at least one upcrossing point in A_u. Note the, we also have $U_u^q(A_u) = 0$. Hence, this intersecting event is include in the event $\{U_u^*(A_u) - U_u^q(A_u) \ge 1\}$. Therefore

$$\begin{aligned} \mathrm{P}(X_{1:n} \le u, \forall k, kq \in A_u) - \mathrm{P}(M(A_u) \le u) \le &\mathrm{P}(U_u^*(A_u) - U_u^q(A_u) \ge 1) \\ \le &\mathrm{E}[U_u^*(A_u) - U_u^q(A_u)]. \end{aligned}$$

Then the proof is completed from the following claim

$$\mathrm{E}[U_u^*(A_u) - U_u^q(A_u)] = o(|A_u| C_u^*). \tag{11}$$

Indeed, We already have that $\mathrm{E}U_u^*(A_u) = |A_u| C_u^*$. By stationarity, since $A_u = [aq, bq]$,

$$\mathrm{E}U_u^q(A_u) = \frac{|A_u|}{q} \mathrm{P}(X_{1;n}(0) < u < X_{1:n}(q)).$$

Then our aim is to prove that

$$\mathrm{P}(X_{1;n}(0) < u < X_{1:n}(q)) \cong qC_u^*.$$

We have

$$\begin{aligned} &\mathrm{P}(X_{1;n}(0) < u < X_{1:n}(q)) && (12) \\ \cong &\sum_{i=1}^{n} \mathrm{P}\left(X_i(0) < u < X_i(q), \text{ and } u < X_j(0), X_j(q)\ \forall j \ne i\right) && (13) \\ = &n\mathrm{P}(X(0) < u < X(q)) \left(\mathrm{P}(u < X(0), X(q))\right)^{n-1}. && (14) \end{aligned}$$

It is proven in [3][page 265] or [2][Lemma 4.2] that

$$\mathrm{P}(X(0) < u < X(q)) \cong qC_u.$$

Since q is small enough,

$$\mathrm{P}(u < X(0), X(q)) \cong \mathrm{P}(X(0) > u) = \overline{\Phi}(u).$$

Then we finish the proof by the observation

$$C_u^* = nC_u \overline{\Phi}(u).$$

$$\mathrm{P}(X_{1;n}(0) < u < X_{1:n}(q)) \cong qC_u^*.$$

Proof (Proof of Lemma 2) We begin by proving the asymptotic independence of the family of N events $\{X_{1:n}(ql) \le u,\ \forall ql \in I_i\}$ with $i = 1, \dots N$, i.e., as $u \to \infty$

$$\mathrm{P}(X_{1:n}(ql) \le u,\ \forall ql \in \cup_{i=1}^{N} I_i) = \prod_{i=1}^{N} \mathrm{P}(X_{1:n}(ql) \le u,\ \forall ql \in I_i) + o(1). \quad (15)$$

We will use the following Normal Comparision lemma for the order statistics of Gaussian arrays by Debicki et al [11][Theorem 2.4 and Lemma 5.2].

Similar to (15), one can extend from one interval D_u to the union $\cup_{j=1}^{p} D_{u,j}$ as

$$\mathrm{P}(X_{1:n}(ql) \le u,\ \forall ql \in \cup_{j=1}^{p} D_{u,j}) = \prod_{j=1}^{p} \mathrm{P}(X_{1:n}(ql) \le u,\ \forall ql \in D_{u,j}) + o(1). \quad (16)$$

Then, from Lemma 4, the proof of Lemma 2 is completed.

Proof (Proof of Lemma 3) We only need to check for one interval $D_u = ((C_u^*)^{-1}c, (C_u^*)^{-1}d)$. From Lemma 4,

$$\mathrm{P}(M(D_u) \le u) = \prod_{i=1}^{N} \mathrm{P}(X_{1:n}(ql) \le u,\ \forall ql \in I_i) + o(1).$$

Then it si equal to

$$\begin{aligned}
&=(\mathrm{P}(X_{1:n}(ql) \le u,\ \forall ql \in I_i))^N + o(1) \\
&=\Big(\mathrm{P}(M(I_1) \le u) + o((C_u^*)^{1/2})\Big)^N + o(1) \\
&=\Big(1 - |I_1|C_u^* + o((C_u^*)^{1/2})\Big)^N + o(1) \\
&=\exp(|I_1|C_u^* N) + o(1) = \exp(d - c) + o(1).
\end{aligned}$$

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