ON THE LYAPUNOV EXPONENTS OF TRIANGULAR DISCRETE TIME-VARYING SYSTEMS

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ABSTRACT. In this paper, we present upper and lower estimates for the Lyapunov exponents of discrete linear systems with triangular time-varying coefficients. These estimates are expressed by the diagonal elements of the coefficient matrix. As a conclusion from these estimates, we also obtain bounds for the Grobman regularity coefficient.

1. INTRODUCTION

It is well known that every linear system with variable coefficients can be reduced by Lyapunov transformation of coordinates to a triangular system [4], i.e. a system in which all coefficients are triangular matrices. Therefore, the study of dynamic properties invariant with respect to Lyapunov transformations can be reduced, at least from a theoretical point of view, to the study of the properties of triangular systems. It is also known that some of the dynamic properties and the numerical characteristics describing them are uniquely determined for triangular systems by their diagonal, as is the case, for example, with the uniform exponential dichotomy and its spectrum [15]. Such properties or characteristics are called diagonally significant. In this paper, we deal with the Lyapunov exponents and the Grobman regularity coefficient of discrete linear systems with variable coefficients. They belong to the group of numerical characteristics that are invariant with respect to Lyapunov transformations but are not diagonally significant. Therefore, the question arises about the possibility of estimating them, for triangular systems, through the elements of the diagonal. This problem is the main objective of this paper.

Fundamentally speaking, one could raise the question not of an estimate, but of an exact calculation of the Lyapunov exponents of a triangular system, since its solutions are given explicitly. However, apart from the cumbersomeness of calculations using these formulas, there would still be the difficulty that the basis described by them is not necessarily normal, so that its Lyapunov exponents do not always coincide with the Lyapunov exponents of the system, and all known methods of transition from a given basis to a normal one are practically inefficient. Therefore, the purpose of this paper is to construct simple estimates of Lyapunov exponents in terms of diagonal coefficients.

Based on the obtained estimates for the Lyapunov exponents, we will also propose bounds for the so-called Grobman regularity coefficient. While the importance of Lyapunov exponents in the description of dynamic properties is widely known (see e.g. [4] and the references therein), the role of the Grobman coefficient is less known. First of all, the Grobman coefficient determines whether the system

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is Lyapunov regular. A system is regular if and only if the Grobman regularity coefficient is zero ([4]). In [9] it was shown that the perturbation with the Lyapunov exponent less then opposite of the Grobman regularity coefficient, do not change the Lyapunov exponents i.e. the Lyapunov exponents of the perturbed an unperturbed systems are the same. Finally in [3] a condition for stability of non-uniform dichotomy has been established and this condition requires that Grobman regularity coefficient is sufficient small. Thus, having in mind the importance of the stability results in the theory of dynamical systems, it is crucial to obtain sharp estimates for the Grobman regularity coefficient. Let us note that in the literature, apart from the Grobman regularity coefficient, the Lyapunov and Perron regularity coefficients are also considered. A complete description of the relations between Lyapunov, Perron and Grobman regularity coefficients is presented in [12].

It should be noted that one of the upper bounds of Grobman regularity coefficient presented in this paper (right hand side of (18)) has already appeared in paper [3, Theorem 6] and [4, Theorem 3.1.3]. However, its proof is based, in both publications, on a certain bound for elements of the transition matrix ([3, Lemma A.1] and [4, Lemma 3.1.4]), which is not true. The corrected proof has been published in [5] together with a lower bound, however our lower bounds are more accurate. In the last paper an applications of these bounds for a simple proof of the Oseledets ergodic theorem has been also presented.

The work is organized as follows. In the next section we present some facts from abstract theory of Lyapunov exponent and we provide some concepts from theory of linear difference equations. In the third section we formulate the main results. Proofs of the main results are contained in section fourth. The fifth section is devoted to examples that illustrate relations between the obtained bounds.

Notations: We define a sum $\sum_{j=a}^{b}$ to be equal to zero and product $\prod_{j=a}^{b}$ to equal to one if $b < a$. By $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ we will denote the standard scalar product in $\mathbb{R}^d$ and the corresponding norm, respectively. For a d by d matrix M its operator norm will be denoted by $\|M\|$ and the transposition of M by M^T. For any $d \in \mathbb{N}$, a sequence of invertible d by d matrices $A = (A(n))_{n\in\mathbb{N}}$ is called a Lyapunov sequence if it is bounded and its inverse sequence $A^{-1} = (A^{-1}(n))_{n\in\mathbb{N}}$ is bounded. By $\mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ we denote the set of all Lyapunov sequences and by $\mathcal{L}^{\infty}(\mathbb{N}, \mathbb{R}^{d\times d})$ the set of all bounded sequences of d by d matrices. For $A \in \mathcal{L}^{\infty}(\mathbb{N}, \mathbb{R}^{d\times d})$ we define $\|A\|_\infty = \sup_{n\in\mathbb{N}} \|A(n)\|$. If for all $n \in \mathbb{N}$ the matrix $A(n)$ is upper-triangular (lower-triangular), then we will say that the sequence A is upper-triangular (lower-triangular). A sequence A will be called triangular if it is upper-triangular or lower-triangular. We also will use the symbol $\mathbb{R}^d_*$ for the set $\mathbb{R}^d\backslash\{0\}$ and symbol $\mathbb{N}_1$ for the set $\{1, 2, ...\}$.

2. Preliminaries

2.1. **Abstract theory of Lyapunov exponents.** Below we present some facts from the abstract theory of Lyapunov exponents, which we will use later for Lyapunov exponents of discrete time-varying linear systems with coefficients in the set $\mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$. The abstract theory of Lyapunov exponents was initiated by works of Yu. S. Bogdanov ([6] and [7] see also [17]). The modern approach to this theory is presented in monographs [2, 4]. All the results collected below are proven in paragraph 2 of [8, Chapter 1].

Definition 1. *A function* $\lambda : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ *is called a Lyapunov exponent if*

1. $\lambda(vc) = \lambda(v)$ *for each* $v \in \mathbb{R}^d$ *and* $c \in \mathbb{R}_*$,
2. $\lambda(u+v) \le \max\{\lambda(u), \lambda(v)\}$,
3. $\lambda(v) = -\infty$ *if and only if* $v = 0$.

The set $\{\lambda(x_0) : x_0 \in \mathbb{R}^d_*\}$ is finite and contains at most d elements. Denote them as follows

$$-\infty < \lambda'_1 < \lambda'_2 < \ldots < \lambda'_r < \infty.$$

Thus, for each λ'_i, $i = 1, ..., r$, the set

$$E_i := \{v \in \mathbb{R}^d_* : \lambda(v) \le \lambda'_i\} \cup \{0\} \tag{1}$$

is a subspace of $\mathbb{R}^d$. Additionally we set $E_0 = \{0\}$. The multiplicity d_i of λ'_i is defined as

$$\dim E_i - \dim E_{i-1},\ i = 1, ..., r.$$

Define the sequence

$$\Lambda = \left(\underbrace{\lambda'_1, ..., \lambda'_1}_{d_1 \text{ times}}, \underbrace{\lambda'_2, ..., \lambda'_2}_{d_2 \text{ times}}, ... \underbrace{\lambda'_r, ..., \lambda'_r}_{d_r \text{ times}} \right),$$

i.e. Λ is a sequence consisting of numbers $\lambda'_1 < \lambda'_2 < \ldots < \lambda'_r$ and λ'_i appears d_i times, $i = 1, ..., r$.

Definition 2. *A base* $\{v_1, ..., v_d\}$ *of* $\mathbb{R}^d$ *is called normal for the Lyapunov exponent* λ *if for each* $K \subseteq \{1, ..d\}$ *and any* $c_j \in \mathbb{R}_*$, $j \in K$ *we have*

$$\lambda\left(\sum_{j \in K} c_j v_j\right) = \max_{j \in K} \lambda(v_j).$$

Theorem 3. *A base* $\{v_1, ..., v_d\}$ *of* $\mathbb{R}^d$ *is normal for the Lyapunov exponent* λ *if and only if there exists a permutations* π *of the set* $\{1, ..., d\}$ *such that*

$$\Lambda = \left(\lambda\left(v_{\pi(1)}\right), ..., \lambda\left(v_{\pi(d)}\right)\right).$$

Theorem 4. *For any base* $\{x_1, ..., x_d\}$ *of* $\mathbb{R}^d$ *and any Lyapunov exponent* λ *there exists an upper-triangular (lower-triangular) matrix* C *such that the columns* $x'_1, ..., x'_d$ *of* $X' = XC$ *forms a normal base for* λ, *where* X *is a matrix formed from vectors* $x_1, ..., x_d$ *as columns. Moreover, for each such a matrix* C *we have*

$$\lambda(x'_i) \le \lambda(x_i). \tag{2}$$

Definition 5. *Two bases* $\{u_1, ..., u_d\}$ *and* $\{v_1, ..., v_d\}$ *of* $\mathbb{R}^d$ *are called dual if* $\langle u_i, v_j \rangle = 0$ *and* $\langle u_i, v_i \rangle = 1$ *for all* $i, j = 1, ..., d$, $i \ne j$. *Two Lyapunov exponents* λ, μ *are called dual if for any dual bases* $\{u_1, ..., u_d\}$ *and* $\{v_1, ..., v_d\}$ *of* $\mathbb{R}^d$ *we have*

$$\lambda(u_i) + \mu(v_i) \ge 0 \text{ for } i = 1, ..., d.$$

Suppose that $U = \{u_1, ..., u_d\}$ *and* $V = \{v_1, ..., v_d\}$ *are dual bases of* $\mathbb{R}^d$ *and* λ, μ *are dual Lyapunov exponents we define*

$$\gamma(U, V) = \max_{i=1,...,d} (\lambda(u_i) + \mu(v_i)),$$

and then the Grobman regularity coefficient $\sigma_G(\lambda,\mu)$ *of a pair* (λ,μ) *is defined as*

$$\sigma_G(\lambda,\mu) = \inf \gamma(U,V), \tag{3}$$

where the infimum is taken over all pairs (U,V) *of dual bases of* $\mathbb{R}^d$.

Theorem 6. *Suppose that* λ, μ *are dual Lyapunov exponents and* U, V *are dual bases. If* U *is normal for* λ *or* V *is normal for* μ *then* $\sigma_G(\lambda,\mu) = \gamma(U,V)$.

2.2. **Some concepts from theory of difference equations.** Consider a discrete time-varying linear system

$$x(n+1) = A(n)x(n),\ \ n \in \mathbb{N}, \tag{4}$$

where $A = (A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{d\times d})$. The *transition matrix* $\Phi_A = (\Phi_A(n,m))_{n,m\in\mathbb{N}}$ of system (4) is defined by

$$\Phi_A(n,m) = \begin{cases} A(n-1)\dots A(m) & \text{for } n > m, \\ I_d & \text{for } n = m, \\ \Phi_A^{-1}(m,n) & \text{for } n < m. \end{cases}$$

For $x_0 \in \mathbb{R}^d$ we denote by $(x(n,x_0))_{n\in\mathbb{N}}$ the solution of (4) with an initial condition $x(0,x_0) = x_0$. To recall the notion of the Lyapunov exponent of the solution $(x(n,x_0))_{n\in\mathbb{N}}$, we define the *characteristic exponent* $\chi(g)$ (might take the value $+\infty$ or $-\infty$) of a sequence $g = (g(n))_{n\in\mathbb{N}}$ of vectors from $\mathbb{R}^d$ as

$$\chi(g) := \limsup_{n\to\infty} \frac{1}{n} \ln \|g(n)\|.$$

The *Lyapunov exponent* of the solution $(x(n,x_0))_{n\in\mathbb{N}}$ denoted as $\lambda_A(x_0)$ is given by $\lambda_A(x_0) = \chi((x(n,x_0))_{n\in\mathbb{N}})$, i.e.

$$\lambda_A(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \|x(n,x_0)\|.$$

It is well known (see e.g. [3]) that for each $A \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{d\times d})$ the function λ_A is a Lyapunov exponent understood according with Definition 1 and therefore as we know from the previous section the set $\left\{\lambda_A(x_0) : x_0 \in \mathbb{R}^d_*\right\}$ is finite and contains at most d elements. Denote the elements of $\left\{\lambda_A(x_0) : x_0 \in \mathbb{R}^d_*\right\}$ as follows

$$-\infty < \lambda'_1(A) < \lambda'_2(A) < \ldots < \lambda'_r(A) < \infty$$

and denote the multiplicity of $\lambda'_i(A)$ by d_i, $i = 1, ...r$. The sequence

$$\Lambda = \Bigg(\underbrace{\lambda'_1(A),...,\lambda'_1(A)}_{d_1 \text{ times}},\underbrace{\lambda'_2(A),...,\lambda'_2(A)}_{d_2 \text{ times}},\ldots\underbrace{\lambda'_r(A),...,\lambda'_r(A)}_{d_r \text{ times}}\Bigg),$$

i.e. Λ is a sequence consisting of numbers $\lambda'_1(A) < \lambda'_2(A) < \ldots < \lambda'_r(A)$ and λ'_i appears d_i times, $i = 1,...,r$. The sequence Λ will be called the Lyapunov spectrum of (4). We assume that the Lyapunov spectrum is numbered in non-decreasing order. Then, the largest Lyapunov exponent $\lambda_d(A)$ can be computed as follows (see [4]).

Lemma 7. *If* $A \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{d\times d})$, *then*

$$\lambda_d(A) = \limsup_{n\to\infty} \frac{1}{n} \ln \|\Phi_A(n,0)\|.$$

Definition 8. *For any base $\{x_1, ..., x_d\}$ of $\mathbb{R}^d$ the sequence of matrices $X = (X(n))_{n\in\mathbb{N}}$, $X(n) \in \mathbb{R}^{d\times d}$ containing the vectors $x(n, x_1), \ldots, x(n, x_d)$ as columns is called fundamental matrix of system (4). If the base $\{x_1, ..., x_d\}$ is normal for λ_A, then the sequence $X = (X(n))_{n\in\mathbb{N}}$ will be called normal fundamental matrix of system (4).*

We define a so called *adjoint system*

$$y(n+1) = B(n)y(n),\ n \in \mathbb{N} \tag{5}$$

where $B(n) = A^{-\mathrm{T}}(n) := \left(A^{-1}(n)\right)^{\mathrm{T}}$, $n \in \mathbb{N}$. To denote the sequence $\left(\left(A^{-1}(n)\right)^{\mathrm{T}}\right)_{n\in\mathbb{N}}$ we will use the symbol $A^{-\mathrm{T}}$.

Definition 9 (Kinematic equivalence)**.** *System (4) is kinematically equivalent to system*

$$y(n+1) = C(n)y(n), \qquad C \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d}) \tag{6}$$

if there exists a Lyapunov sequence $T = (T(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ such that

$$C(n) = T^{-1}(n+1)A(n)T(n) \qquad \textit{for all } n \in \mathbb{N}. \tag{7}$$

In that case we will also say that sequences $(A(n))_{n\in\mathbb{N}}$ and $(C(n))_{n\in\mathbb{N}}$ are kinematically equivalent and that transformation T establishes this equivalence.

Observe that if (4) and (6) are kinematically equivalent and that transformation T establishes this equivalence, then $(x(n))_{n\in\mathbb{N}}$ is a solution of (4) if and only if $(y(n))_{n\in\mathbb{N}}$, $y(n) = T^{-1}(n)x(n)$ is a solution of (6).

The following result is well known (see e.g. [4]).

Theorem 10. *Suppose that sequences $(A(n))_{n\in\mathbb{N}}$ and $(C(n))_{n\in\mathbb{N}}$ are kinematically equivalent and that transformation T establishes this equivalence. We have*

$$\lambda_A(x_0) = \lambda_C(T(0)x_0),\ x_0 \in \mathbb{R}^d_*.$$

In particular, kinematically equivalent systems have the same Lyapunov spectrum.

Remark 11. *Suppose that $\{u_1, ..., u_d\}$ and $\{v_1, ..., v_d\}$ are dual bases of R^d, then we have*

$$\lambda_A(u_i) + \lambda_{A^{-\mathrm{T}}}(v_i) \geq \limsup_{n\to\infty} \frac{1}{n} \ln \|\Phi_A(n,0)u_i\| \|\Phi_{A^{-\mathrm{T}}}(n,0)v_i\| \geq$$

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\langle \Phi_A(n,0)u_i, \Phi_{A^{-\mathrm{T}}}(n,0)v_i\rangle| = \limsup_{n\to\infty} \frac{1}{n} \ln |\langle u_i, v_i\rangle| = 0$$

i.e. the Lyapunov exponents λ_A and $\lambda_{A^{-\mathrm{T}}}$ are dual. The Grobman coefficients $\sigma_G(A)$ of system (4) is defined as $\sigma_G(A) := \sigma_G(\lambda_A, \lambda_{A^{-\mathrm{T}}})$. This may be defined equivalently as follows

$$\sigma_G(A) = \min_{X\in\Psi(A)} \max_{j=1,\ldots,d} \left(\chi(x_j) + \chi(y_j)\right), \tag{8}$$

where $\Psi(A)$ is the set of all fundamental matrices of system (4) and x_j, y_j is the $j-$th column of X, $X^{-\mathrm{T}}$, respectively.

3. Statements of the main results

3.1. **Symbols and notations.** For a triangular $A = (A(n))_{n\in\mathbb{N}} \in L^{\mathrm{Lya}}(N, R^{d\times d})$, $A(n) = [a_{ij}(n)]_{i,j=1,\ldots,d}$ we define sequences $A_d = (A_d(n))_{n\in\mathbb{N}} \in L^{\mathrm{Lya}}(N, R^{d\times d})$, $A|^l = \left(A|^l(n)\right)_{n\in\mathbb{N}}$ and $A|_l = (A|_l(n))_{n\in\mathbb{N}} \in L^{\mathrm{Lya}}(N, R^{l\times l})$, where $A_d(n)$ is a matrix that has on its diagonal the elements of the diagonal of matrix $A(n)$, and zeros elsewhere, $A|^l(n)$is formed by deleting the last $d-l$ rows and $d-l$ columns in matrix $A(n)$ and $A|_l(n)$ is formed by deleting the first l rows and l columns in matrix $A(n)$ i.e.

$$A_d(n) = \begin{bmatrix} a_{11}(n) & 0 & \cdots & 0 \\ 0 & a_{22}(n) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{dd}(n) \end{bmatrix},$$

$$A|^l(n) = \begin{bmatrix} a_{11}(n) & a_{12}(n) & \cdots & a_{1l}(n) \\ a_{21}(n) & a_{22}(n) & & \vdots \\ \vdots & & \ddots & a_{l-1,l}(n) \\ a_{l1}(n) & \cdots & a_{l,l-1}(n) & a_{ll}(n) \end{bmatrix}$$

and

$$A|_l(n) = \begin{bmatrix} a_{l+1,l+1}(n) & a_{l+1,l+2}(n) & \cdots & a_{l+1,d}(n) \\ 0 & a_{l+2,l+2}(n) & & \vdots \\ \vdots & & \ddots & a_{d-1,d}(n) \\ a_{d1}(n) & \cdots & a_{d,d-1}(n) & a_{dd}(n) \end{bmatrix}.$$

We denote

$$\overline{a}_i := \limsup_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} a_{ii}(n) \right|, \quad \underline{a}_i := \liminf_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} a_{ii}(n) \right|.$$

Let $\Delta a_i := \overline{a}_i - \underline{a}_i$. Moreover, for $n, k \in \mathbb{N}$, $n \geq k$, we define

$$\begin{aligned} \overline{\phi}_{A_d}(n,k) &= \max\left\{ \left| \prod_{i=k}^{n-1} a_{jj}(i) \right| : j = 1, ..., d \right\}, \\ \underline{\phi}_{A_d}(n,k) &= \min\left\{ \left| \prod_{i=k}^{n-1} a_{jj}(i) \right| : j = 1, ..., d \right\}. \end{aligned}$$

Finally, we introduce the following quantities

$$
\begin{aligned}
\overline{\Omega}(A) &= \inf_{N\in\mathbb{N}_1}\frac{1}{N}\left(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln\overline{\phi}_{A_d}((i+1)N,iN)\right), \\
\underline{\Omega}(A) &= \sup_{N\in\mathbb{N}_1}\frac{1}{N}\left(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln\underline{\phi}_{A_d}((i+1)N,iN)\right), \\
\overline{\omega}(A) &= \inf_{N\in\mathbb{N}_1}\frac{1}{N}\left(\liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln\overline{\phi}_{A_d}((i+1)N,iN)\right), \\
\underline{\omega}(A) &= \sup_{N\in\mathbb{N}_1}\frac{1}{N}\left(\liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln\underline{\phi}_{A_d}((i+1)N,iN)\right).
\end{aligned}
\tag{9}
$$

Remark 12. *(i) Observe that the quantities* $\overline{\Omega}(A)$, $\underline{\Omega}(A)$, $\overline{\omega}(A)$ *and* $\underline{\omega}(A)$ *depend only on diagonal elements of matrices* $A(n)$.
(ii) The quantities $\overline{\Omega}(A)$, $\underline{\Omega}(A)$, $\overline{\omega}(A)$ *and* $\underline{\omega}(A)$ *are known in the literature as central exponents and were introduced by R. E. Vinograd in* [16] *(see also* [8, p. 114, 163-167]*) for estimate changes of Lyapunov exponents of linear time-varying differential equations under arbitrary small perturbations. Other properties and applications of central exponents are discussed in the monograph* [14]. *A version of central exponents for discrete linear time-varying systems is analyzed in monograph* [10]. *It should be noted, however, that the central exponents are usually defined based on the entire matrix of coefficients, not just their elements from the main diagonal as we did above. In this approach, our definition becomes a theorem saying that central exponents are diagonally significant (see* [11, Theorem 6]*).*

Remark 13. *If the sequence* $A = (A(n))_{n\in\mathbb{N}}$ *is triangular, then*

$$\overline{\phi}_{A_d^{-\mathrm{T}}\big|_l}(n,k) = \underline{\phi}^{-1}_{A_d|_l}(n,k) \quad \textit{and} \quad \overline{\phi}_{A_d^{-\mathrm{T}}\big|^l}(n,k) = \underline{\phi}^{-1}_{A_d|^l}(n,k)$$

for all $k,\, n\in\mathbb{N}$, $n>k$, $l=1,...,d$. *Therefore, for* $l = 1,...,d$

$$\overline{\Omega}(A|_l) = -\underline{\omega}(A^{-\mathrm{T}}\big|_l),\ \underline{\Omega}(A|_l) = -\overline{\omega}(A^{-\mathrm{T}}\big|_l)$$

and

$$\overline{\Omega}\left(A|^l\right) = -\underline{\omega}(A^{-\mathrm{T}}\big|^l),\ \underline{\Omega}\left(A|^l\right) = -\overline{\omega}(A^{-\mathrm{T}}\big|^l).$$

3.2. **Statements of the main results.** The next three theorems contain the main result of our paper. The first two theorems present the lower and upper bounds for the columns of the fundamental matrix of an upper-triangular system and the adjoint system by diagonal elements. In the first of them, we additionally assume that the fundamental matrix is normal. The third theorem gives upper and lower bounds for the Grobman regularity coefficient. From the first two theorems, the estimations for the Lyapunov exponents of the upper-triangular system, expressed by the diagonal elements, result as a conclusion.

Theorem 14. *Consider system (4) with upper-triangular sequence* $A = (A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{d\times d})$. *Then, the following statements hold:*
(i) For any normal upper-triangular fundamental matrix $X = (X(n))_{n\in\mathbb{N}}$ *of (4) we have*

$$\overline{a}_j \le \chi(x_j) \le \overline{a}_j + \sum_{l=1}^{j-1}\Delta a_l,\ \ j = 1,...d, \tag{10}$$

where $x_j = (x_j(n))_{n\in\mathbb{N}}$ *and* $x_j(n)$ *is the* $j-th$ *column of* $X(n)$, $j = 1, ..., d$.
(ii) For any normal lower-triangular fundamental matrix $Y = (Y(n))_{n\in\mathbb{N}}$ *of system (5) we have*

$$-\underline{a}_j \le \chi(y_j) \le -\underline{a}_j + \sum_{l=j+1}^{d} \Delta a_l, \; j = 1, ..., d, \tag{11}$$

where $y_j = (y_j(n))_{n\in\mathbb{N}}$ *and* $y_j(n)$ *is the* $j-th$ *column of* $Y(n)$, $j = 1, ..., d$.

Theorem 15. *Consider system (4) with upper-triangular sequence* $A = (A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$. *Then, the following statements hold:*
(i) For any upper-triangular fundamental matrix $X = (X(n))_{n\in\mathbb{N}}$ *of (4) we have*

$$\underline{\Omega}(A|_j) \le \chi(x_j) \le \overline{\Omega}(A|_j), \tag{12}$$

where $x_j = (x_j(n))_{n\in\mathbb{N}}$ *and* $x_j(n)$ *is the* $j-th$ *column of* $X(n)$, $j = 1, ..., d$.
(ii) For any lower-triangular fundamental matrix $Y = (Y(n))_{n\in\mathbb{N}}$ *of (5) we have*

$$-\overline{\omega}(A|^j) \le \chi(y_j) \le -\underline{\omega}(A|^j), \tag{13}$$

where $y_j = (y_j(n))_{n\in\mathbb{N}}$ *and* $y_j(n)$ *is the* $j-th$ *column of* $Y(n)$, $j = 1, ..., d$.

Corollary 16. *For any upper-triangular sequence* $A \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ *there exist permutations* π_1, π_2, π_3 *and* π_4 *of the set* $\{1, ..., d\}$ *such that for* $k = 1, ..., d$

$$\overline{a}_k \le \lambda_{\pi_1(k)}(A) \le \overline{a}_k + \sum_{l=1}^{k-1} \Delta a_l, \tag{14}$$

$$\underline{\Omega}(A|_k) \le \lambda_{\pi_2(k)}(A) \le \overline{\Omega}(A|_k), \tag{15}$$

$$-\underline{a}_k \le \lambda_{\pi_3(k)}(A^{-\mathrm{T}}) \le -\underline{a}_k + \sum_{l=k+1}^{d} \Delta a_l \tag{16}$$

and

$$-\overline{\omega}(A|^k) \le \lambda_{\pi_4(k)}(A^{-\mathrm{T}}) \le -\underline{\omega}(A|^k)), \tag{17}$$

Theorem 17. *For any upper-triangular sequence* $A \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ *we have*

$$\max_{l=1,\dots,d} \Delta a_l \le \sigma_G(A) \le \sum_{l=1}^{d} \Delta a_l. \tag{18}$$

and

$$\max_{k=1,\dots,d} \left(\underline{\Omega}(A|_k) - \overline{\omega}(A|^k)\right) \le \sigma_G(A) \le \max_{k=1,\dots,d} \left(\overline{\Omega}(A|_k) - \underline{\omega}(A|^k)\right). \tag{19}$$

Remark 18. *The right side of the inequality (18) was proven in* [5] *(Theorem 4.7) under weaker assumptions that* A *is a tempred sequence, and there (Theorem 3.1), the following lower estimate for* $\sigma_G(A)$ *was shown*

$$\frac{1}{d^2} \sum_{l=1}^{d} \Delta a_l \le \sigma_G(A). \tag{20}$$

Since

$$\frac{1}{d^2} \sum_{l=1}^{d} \Delta a_l \le \frac{1}{d} \sum_{l=1}^{d} \Delta a_l \le \max_{l=1,\dots,d} \Delta a_l,$$

then lower bound given by (18) is more accurate then lower bound from (20).

Remark 19. *For each pair of bounds ((10) and (12), (11) and (13), (18) and (19)) there are systems for which the first estimate gives a better result and those for which the second estimate is more accurate.*

4. Proofs of the main results

Let us denote

$$a = \max\left\{\sup_{n\in\mathbb{N}}|a_{jj}(n)|, \sup_{n\in\mathbb{N}}|a_{jj}(n)|^{-1} : j = 1, ...d\right\}. \tag{21}$$

Lemma 20. *If $A \in L^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ is triangular, then $a \in [1,\infty)$*

$$\left(\frac{1}{a}\right)^{n-k} \leq \left|\prod_{i=k}^{n-1} a_{jj}(i)\right| \leq a^{n-k}, \tag{22}$$

$$\left(\frac{1}{a}\right)^{n-k} \leq \left|\prod_{i=k}^{n-1} a_{jj}(i)\right|^{-1} \leq a^{n-k} \tag{23}$$

for all $n, k \in \mathbb{N}$, $n > k$ and $j = 1, ..., d$. In particular we have

$$-(n-k)\ln a \leq \ln\overline{\phi}_{A_d}(n,k) \leq (n-k)\ln a \tag{24}$$

for all $n, k \in \mathbb{N}$, $n > k$.

Proof. The boundedness of $(A(n))_{n\in\mathbb{N}}$ and $\left(A^{-1}(n)\right)_{n\in\mathbb{N}}$ in particular implies the boundedness of sequences composed of elements from the main diagonal, i.e., sequences $(a_{jj}(n))_{n\in\mathbb{N}}$and $\left(a_{jj}^{-1}(n)\right)_{n\in\mathbb{N}}$,$j = 1, ...d$, which in turn implies that $a \in R$. It is also clear that $a > 0$. The definition of a implies that

$$|a_{jj}(i)| \leq a \text{ and } |a_{jj}(i)|^{-1} \leq a$$

for all $i \in \mathbb{N}$ and $j = 1, ..., d$. These two inequalities imply (22). Considering (22) for $k = n + 1$ we obtain that $1/a \leq a$ i.e. $a \geq 1$.

Next, observe that the constant a is the some for both A and A^{-1}. Applying inequality (22) to A^{-1} we get (23).

Finally, inequality (23) follows from (22) and the definition of $\overline{\phi}_{A_d}$. □

The following result from the work of [5] will play a key role in the proof of Theorem 14. It shows that there exist fundamental systems, not necessarily normal, such that the inequalities (10) and (11) are satisfied. The proof of Theorem 14 will rely on demonstrating how the existence of such fundamental systems implies that the inequalities (10) and (11) are satisfied for any normal fundamental system.

Theorem 21. [5] *(Theorems 4.1 and 4.4) For any upper-triangular sequence $A = (A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ there exists an upper-triangular fundamental matrix $X' = (X'(n))_{n\in\mathbb{N}}$, such that*

$$\chi\left(x'_j\right) \leq \overline{a}_j + \sum_{l=1}^{j-1}\Delta a_l,\ \ j = 1, ...d, \tag{25}$$

where $x'_j = \left(x'_j(n)\right)_{n\in\mathbb{N}}$, $x'_j(n)$ *is the* $j-th$ *column of* $X'(n)$, $j = 1, ..., d$ *and such that for* $Y' = (Y'(n))_{n\in\mathbb{N}}$, $Y'(n) = (X')^{-\mathrm{T}}(n)$ *we have*

$$\chi\left(y'_j\right) \leq -\underline{a}_j + \sum_{l=j+1}^{d} \Delta a_l,\ j = 1, ...d, \tag{26}$$

where $y'_j = \left(y'_j(n)\right)_{n\in\mathbb{N}}$ *and* $y'_j(n)$ *is the* $j-th$ *column of* $Y'(n)$, $j = 1, ..., d$.

Lemma 22. *For any triangular sequence* $A \in L^{\mathrm{Lya}}(N, R^{d\times d})$ *and any triangular fundamental matrices* $X = (X(n))_{n\in\mathbb{N}}$ *and* $Y = (Y(n))_{n\in\mathbb{N}}$ *of system (4) and (5), respectively we have*

$$\overline{a}_j \leq \chi(x_j) \tag{27}$$

$$-\underline{a}_j \leq \chi(y_j) \tag{28}$$

where $x_j = (x_j(n))_{n\in\mathbb{N}}, y_j = (y_j(n))_{n\in\mathbb{N}}$ *and* $x_j(n)$, $y_j(n)$ *is the* $j-th$ *column of* $X(n)$, $Y(n) j = 1, ..., d$, *respectively.*

Proof. We will prove this lemma for upper-triangular system. The proof for lower-triangular is analogical. Consider any upper-triangular fundamental matrix $X = (X(n))_{n\in\mathbb{N}}$ of system (4) with upper-triangular sequence A and denote $x_j(n) = [x_{1j}(n), ..., x_{jj}(n), 0, ..., 0]^{\mathrm{T}}$, $j = 1, ..., d$. Since

$$\|x_j(n)\| \geq |x_{jj}(n)| = \left|\prod_{l=0}^{n-1} a_{jj}(l)\right| |x_{jj}(0)|,$$

which implies that $\chi(x_j) \geq \overline{a}_j$.

Applying the inequality (27) to the adjoint system (5) and having in mind that

$$\limsup_{n\to\infty} \frac{1}{n} \ln\left|\prod_{i=0}^{n-1} a_{ii}^{-1}(n)\right| = -\liminf_{n\to\infty} \frac{1}{n} \ln\left|\prod_{i=0}^{n-1} a_{ii}(n)\right|$$

we get (28). □

Proof of Theorem 14. Let $X' = (X'(n))_{n\in\mathbb{N}}$be a fundamental matrix of system (4) from Theorem 21. According to Theorem 4 there exists an upper-triangular matrix C' such that $X'' = (X''(n))_{n\in\mathbb{N}}$,

$$X^{''}(n) = X'(n)C' \tag{29}$$

is an normal fundamental matrix.

Consider now any normal fundamental matrix $X = (X(n))_{n\in\mathbb{N}}$ of system (4). Since any upper-triangular fundamental matrix$Z = (Z(n))_{n\in\mathbb{N}}$ of system (4) with upper-triangular sequence Ahas the form $Z(n) = \Phi_A(n, 0) D_Z$, $n \in N$ where D_Z is a non-singular upper-triangular matrix, then

$$X''(n) = \Phi_A(n, 0) D_{X''}, n \in \mathbb{N} \tag{30}$$

and

$$X(n) = \Phi_A(n, 0) D_X, n \in \mathbb{N} \tag{31}$$

for certain non-singular upper-triangular matrices $D_{X''}$, D_X. From (29) and (30) we have

$$\Phi_A(n, 0) = X''(n) D_{X''}^{-1} = X'(n) C' D_{X''}^{-1}$$

and therefore by (31) we get $X(n) = X'(n)C$,where $C = C'D^{-1}_{X''}D_X$ is non-singular and upper-triangular.

Using inequality (2) we obtain

$$\chi(x_j) \leq \chi(x'_j), \ j = 1, ...d.$$

Now the right hand side of inequality (10) follows from the last inequality and inequality (25).

The left hand side of inequality (10) follows from Lemma 22.

The proof of inequality (11) is analogical. □

For a fix $N \in \mathbb{N}_1$ let us define sequence $R_{A_d,N} = (R_{A_d,N}(n))_{n\in\mathbb{N}}$ by

$$R_{A_d,N}(n) = \sum_{i=0}^{m-1} \ln \overline{\phi}_{A_d}((i+1)N, iN) + \ln \overline{\phi}_{A_d}(n, mN), \tag{32}$$

where $n \in [mN, (m+1)N - 1]$ and $m \in \mathbb{N}$.

Lemma 23. *Suppose that $A = (A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ is triangular. Then for each $N \in \mathbb{N}_1$*

$$\overline{\phi}_{A_d}(n, k) \leq a^{2N} \exp(R_{A_d,N}(n) - R_{A_d,N}(k)), \tag{33}$$

for all k, $n \in \mathbb{N}$, $n > k$.

Proof. First let us consider the case when n, $k \in [mN, (m+1)N - 1]$ for certain $m \in N$. Then according to Lemma 20 we have $\overline{\phi}_{A_d}(n, k) \leq a^{n-k} \leq a^N$. Hence, to show (33) it is sufficient to show that

$$a^N \leq a^{2N} \exp(R_{A_d,N}(n) - R_{A_d,N}(k)). \tag{34}$$

Indeed, the following equalities are true

$$\exp(R_{A_d,N}(n) - R_{A_d,N}(k)) = \exp\left(\ln \overline{\phi}_{A_d}(n, mN) - \ln \overline{\phi}_{A_d}(k, mN)\right) = \frac{\overline{\phi}_{A_d}(n, mN)}{\overline{\phi}_{A_d}(k, mN)}.$$

and by (23) we have

$$\begin{aligned}
\overline{\phi}_{A_d}(k, mN) &= \max_{j=1,...d} \left| \prod_{i=mN}^{k-1} a_{jj}(i) \right| = \max_{j=1,...d} \left| \prod_{i=mN}^{n-1} a_{jj}(i) \right| \left| \prod_{i=k}^{n-1} a_{jj}(i) \right|^{-1} \\
&\leq a^{n-k} \max_{j=1,...d} \prod_{i=mN}^{n-1} a_{jj}(i) \\
&= a^{n-k} \overline{\phi}_{A_d}(n, mN).
\end{aligned}$$

Therefore,

$$\exp(R_{A_d,N}(n) - R_{A_d,N}(k)) \geq a^{-(n-k)} \geq a^{-N},$$

which verifies (34). Finally, we consider the case that $n \in [m_1 N, (m_1+1)N-1]$, $k \in [m_2 N, (m_2+1)N-1]$, $m_1,\ m_2 \in N$, $m_1 > m_2$.We have the following inequality

$$\begin{aligned}
&\ln \overline{\phi}_{A_d}(n,k)\\
=\ &\ln \max_{j=1,\dots,d}\left|\prod_{i=k}^{n-1} a_{jj}(i)\right|\\
=\ &\ln \max_{j=1,\dots,d}\left|\prod_{i=m_1N}^{n-1} a_{jj}(i)\prod_{i=(m_1-1)N}^{m_1N-1} a_{jj}(i)\dots\prod_{i=(m_2+1)N}^{(m_2+2)N-1} a_{jj}(i)\prod_{i=k}^{(m_2+1)N-1} a_{jj}(i)\right|\\
\le\ &\ln\left(\max_{j=1,\dots,d}\left|\prod_{i=m_1N}^{n-1} a_{jj}(i)\right|\max_{j=1,\dots,d}\left|\prod_{i=(m_1-1)N}^{m_1N-1} a_{jj}(i)\right|\dots\max_{j=1,\dots,d}\left|\prod_{i=(m_2+1)N}^{(m_2+2)N-1} a_{jj}(i)\right|\max_{j=1,\dots,d}\left|\prod_{i=k}^{(m_2+1)N-1} a_{jj}(i)\right|\right)\\
=\ &\ln \overline{\phi}_{A_d}(n,m_1N)+\sum_{i=m_2+1}^{m_1-1}\overline{\phi}_{A_d}((i+1)N,iN)+\ln \overline{\phi}_{A_d}((m_2+1)N,k).
\end{aligned}$$

Then, by (32) we have

$$R_{A_d,N}(n)-R_{A_d,N}(k)=\sum_{i=m_2}^{m_1-1}\ln \overline{\phi}_{A_d}((i+1)N,iN)+\ln \overline{\phi}_{A_d}(n,m_1N)-\ln \overline{\phi}_{A_d}(k,m_2N).$$

Therefore

$$\begin{aligned}
&\ln \overline{\phi}_{A_d}(n,k)-(R_{A_d,N}(n)-R_{A_d,N}(k))\\
\le\ &\ln \overline{\phi}_{A_d}((m_2+1)N,k)-\ln \overline{\phi}_{A_d}((m_2+1)N,m_2N)+\ln \overline{\phi}_{A_d}(k,m_2N)
\end{aligned}$$

Using inequality (24) we may estimate the right hand side of the above equality as follows

$$\begin{aligned}
&\ln \overline{\phi}_{A_d}((m_2+1)N,k)-\ln \overline{\phi}_{A_d}((m_2+1)N,m_2N)+\ln \overline{\phi}_{A_d}(k,m_2N)\\
\le\ &\ln a\,((m_2+1)N-k-((m_2+1)N-m_2N)+k-m_2N)=2N\ln a.
\end{aligned}$$

Consequently,

$$\ln \overline{\phi}_{A_d}(n,k)\le 2N\ln a+R_{A_d,N}(n)-R_{A_d,N}(k)$$

and the proof is completed. □

For a fix $N\in\mathbb{N}_1$ let us define sequence $r_{A_d,N}=(r_{A_d,N}(n))_{n\in\mathbb{N}}$ in the following way

$$r_{A_d,N}(n)=\sum_{i=0}^{m-1}\ln \underline{\phi}_{A_d}((i+1)N,iN)+\ln \underline{\phi}_{A_d}(n,mN)\ \text{ for } n\in[mN,(m+1)N-1],$$

where $m\in\mathbb{N}$.

Lemma 24. *Suppose that $A=(A(n))_{n\in\mathbb{N}}\in\mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{d\times d})$ is triangular. Then for each $N\in\mathbb{N}_1$*

$$\underline{\phi}_{A_d}(n,k)\ge a^{-2N}\exp\left(r_{A_d,N}(n)-r_{A_d,N}(k)\right), \tag{35}$$

for all $k,\ n\in\mathbb{N},\ n>k$.

Proof. Observe that from definitions of $\overline{\phi}_{A_d}$ and $\underline{\phi}_{A_d}$ it follows that $\overline{\phi}_{A_d^{-\mathrm{T}}}(n,k) = \underline{\phi}_{A_d}^{-1}(n,k)$ for all k, $n \in \mathbb{N}$, $n > k$. Therefore

$$R_{A_d^{-\mathrm{T}},N}(n) = -r_{A_d,N}(n) \text{ for } n \in \mathbb{N}.$$

The conclusion of the lemma follows now from Lemma 23 applied to the sequence $A^{-\mathrm{T}}$. □

Lemma 25. *Suppose that* $A = (A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$*. For any* $N \in \mathbb{N}_1$ *and* $x_0 \in \mathbb{R}^d_*$ *we have*

$$\lambda_A(x_0) = \limsup_{n\to\infty} \frac{1}{Nn} \ln \|x(Nn, x_0)\| . \tag{36}$$

Proof. Let us fix $N \in \mathbb{N}_1$ and $x_0 \in \mathbb{R}^d_*$. Observe that from the definition of upper limit it follows that

$$\lambda_A(x_0) \geq \limsup_{n\to\infty} \frac{1}{Nn} \ln \|x(Nn, x_0)\| . \tag{37}$$

Denote by $(n_k)_{k\in\mathbb{N}}$ such a increasing sequences of natural numbers that

$$\lambda_A(x_0) = \lim_{k\to\infty} \frac{1}{n_k} \ln \|x(n_k, x_0)\| \ .$$

For $k \in \mathbb{N}$ denote by p_k such a natural number that $Np_k \leq n_k < N(p_k + 1)$. Let us notice that then

$$0 \leq \frac{n_k - Np_k}{n_k} \leq \frac{1}{n_k}$$

and therefore

$$\lim_{k\to\infty} \frac{n_k - Np_k}{n_k} \ln \|A\|_\infty = 0.$$

Then, we have

$$\begin{aligned}
\lambda_A(x_0) &= \lim_{k\to\infty} \frac{1}{n_k} \ln \|x(n_k, x_0)\| = \lim_{k\to\infty} \frac{1}{n_k} \ln \|\Phi_A(n_k, Np_k)\, x(Np_k, x_0)\| \\
&\leq \limsup_{k\to\infty} \frac{1}{n_k} \ln \|\Phi_A(n_k, Np_k)\| + \limsup_{k\to\infty} \frac{1}{n_k} \ln \|x(Np_k, x_0)\| \\
&\leq \limsup_{k\to\infty} \frac{n_k - Np_k}{n_k} \ln \|A\|_\infty + \limsup_{k\to\infty} \frac{1}{Np_k} \ln \|x(Np_k, x_0)\| \\
&\leq \limsup_{k\to\infty} \frac{1}{Np_k} \ln \|x(Np_k, x_0)\| \leq \limsup_{n\to\infty} \frac{1}{Nn} \ln \|x(Nn, x_0)\| ,
\end{aligned}$$

which together with (37 proves (36). □

In our further consideration we will use the so called $\beta-$transformations which we will define now. Let us fix a $\beta > 0$ and consider a Lyapunov transformation $T_\beta = (T_\beta(n))_{n\in\mathbb{N}}$, $T_\beta(n) \equiv B$, $n \in \mathbb{N}$,where

$$B = \operatorname{diag}\left[1, \beta, ..., \beta^{d-1}\right] . \tag{38}$$

Lemma 26. *Suppose that* A *is triangular and* $\beta \in (0, 1)$*. The transformation* T_β *does not change the diagonal elements of* $A(n)$ *and reduces system (4) to system (7) with* $Q_\beta(n) := C(n) - A_d(n)$ *satisfying that* $\|Q_\beta(n)\| \leq \beta d \|A\|_\infty$ *for* $n \in \mathbb{N}$.

Proof. We will prove this lemma for upper-triangular A. The proof for lower-triangular A is analogical. Observe that

$$C(n)=B^{-1}A(n)B=\begin{bmatrix} a_{11}(n) & \beta a_{12}(n) & \beta^2 a_{13}(n) & \cdots & \beta^{d-1}a_{1d}(n)\\ 0 & a_{22}(n) & \beta a_{23}(n) & \cdots & \beta^{d-2}a_{2d}(n)\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & a_{d-1d-1}(n) & \beta a_{d-1d}(n)\\ 0 & 0 & \cdots & 0 & a_{dd}(n)\end{bmatrix}$$

and therefore

$$Q_\beta(n)=\begin{bmatrix} 0 & \beta a_{12}(n) & \beta^2 a_{13}(n) & \cdots & \beta^{d-1}a_{1d}(n)\\ 0 & 0 & \beta a_{23}(n) & \cdots & \beta^{d-2}a_{2d}(n)\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ & & & & \beta a_{d-1d}(n)\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

It is noted that for any matrix $X=[x_{ij}]_{i,j=1,\dots,d}\in\mathbb{R}^{d\times d}$ we have

$$\|X\|\le d\max_{i,j=1,\dots,d}|x_{ij}|\le d\,\|X\|\,,$$

(see [13, Chapter 5, Section 6, Problem 24, p. 314]). Thus, from $\beta\in(0,1)$ we derive that

$$\|Q_\beta(n)\|\le d\max_{i,j=1,\dots d-1,i<j}\beta^{j-i}\,|a_{ij}(n)|\le\beta d\,\|A\|_\infty\,.$$

The proof is complete. □

Lemma 27. *Suppose that $A=(A(n))_{n\in\mathbb{N}}\in\mathcal{L}^{\rm Lya}(\mathbb{N},\mathbb{R}^{d\times d})$ is triangular. We have*

$$\underline{\Omega}\,(A)\le\lambda_A\,(x_0)\le\overline{\Omega}\,(A)$$

for each $x_0\in\mathbb{R}^d_*$.

Proof. We will prove this lemma for upper-triangular A. The proof for lower-triangular A is analogical. We will first show the most left inequality. Suppose that there exists an upper-triangular sequence $A\in\mathcal{L}^{\rm Lya}(\mathbb{N},\mathbb{R}^{d\times d})$ and $x_0\in\mathbb{R}^d_*$ such that

$$\lambda_A\,(x_0)>\overline{\Omega}\,(A)\,. \tag{39}$$

Let us fix $\varepsilon\in\big(0,\lambda_A\,(x_0)-\overline{\Omega}\,(A)\big)$, $N\in\mathbb{N}$, $N>1$ and define

$$\beta=\frac{e^\varepsilon-1}{a^{4N+1}d\,\|A\|_\infty}. \tag{40}$$

Consider a sequence $C=(C(n))_{n\in\mathbb{N}}\in\mathcal{L}^{\rm Lya}(\mathbb{N},\mathbb{R}^{d\times d})$, where $C(n)=B^{-1}A(n)B$ and B is given by (38) with β defined by (40). Then, $A_d(n)=C_d(n)$ for $n\in\mathbb{N}$ and therefore

$$\overline{\Omega}\,(A)=\overline{\Omega}\,(C)\,, \tag{41}$$

since these quantities depend merely on the diagonal coefficients of $A(n)$ and $C(n)$. Consider matrices $Q_\beta(n)$ from Lemma 26 i.e. $Q_\beta(n)=C(n)-C_d(n)$ for $n\in\mathbb{N}$. Then, by virtue of Lemma 26 and (40) we have

$$q:=\|Q_\beta\|_\infty\le\frac{e^\varepsilon-1}{a^{4N+1}}, \tag{42}$$

where $Q_\beta = (Q_\beta(n))_{n\in\mathbb{N}}$. Since $(T(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\rm Lya}(\mathbb{N},\mathbb{R}^{d\times d})$ where $T(n)=B$, then according to Theorem 10 we have $\lambda_A(x_0)=\lambda_C(Bx_0)$. This together with (39) and (41) implies that

$$\lambda_C(y_0) > \overline{\Omega}(C), \qquad \text{where } y_0 = Bx_0.$$

Consider the system

$$y(n+1) = C(n)y(n),\ n\in\mathbb{N},$$

which can be rewritten as

$$y(n+1) = A_d(n)y(n) + Q_\beta(n)\,y(n),\ n\in\mathbb{N}. \tag{43}$$

Using the variation of constant formula (see [1, Section 2.5]), the solution $y = (y(n,y_0))_{n\in\mathbb{N}}$ of (43) satisfies the following equality

$$y(n,y_0) = \Phi_{A_d}(n,0)y_0 + \sum_{i=0}^{n-1}\Phi_{A_d}(n,i+1)Q_\beta(i)y(i,y_0), \quad n\in\mathbb{N}. \tag{44}$$

Since $\|\Phi_{A_d}(n,j)\| = \overline{\phi}_{A_d}(n,j)$ for $j=0,...,n$ it follows from Lemma 23 that

$$\|y(n,y_0)\| \le a^{2N}\exp R_{A_d,N}(n)\,\|y_0\| + a^{2N}q\sum_{j=0}^{n-1}\exp\left(R_{A_d,N}(n) - R_{A_d,N}(j+1)\right)\|y(j,y_0)\|.$$

Equivalently,

$$\|y(n,y_0)\|\exp\left(-R_{A_d,N}(n)\right) \le a^{2N}\|y_0\| + a^{2N}q\sum_{j=0}^{n-1}\exp\left(-R_{A_d,N}(j+1)\right)\|y(j,y_0)\|. \tag{45}$$

By Lemma 20, we have $\frac{1}{a} \le \overline{\phi}_{A_d}(j+1,j)$ for $j\in\mathbb{N}$. Then, from (33) we derive that

$$\frac{1}{a} \le a^{2N}\exp\left(R_{A_d,N}(j+1) - R_{A_d,N}(j)\right).$$

Therefore,

$$\exp\left(-R_{A_d,N}(j+1)\right) \le a^{2N+1}\exp\left(-R_{A_d,N}(j)\right),$$

this together with (45) implies that

$$u(n) \le a^{2N}\|y_0\| + a^{4N+1}q\sum_{j=0}^{n-1}u(j) \quad \text{for all } n\in\mathbb{N},$$

where $u(n) := \|y(n,y_0)\|\exp\left(-R_{A_d,N}(n)\right)$. By Gronwall's inequality (see e.g. [1, Corollary 4.1.2]) we have

$$u(n) \le a^{2N}\|y_0\|\left(1+a^{4N+1}q\right)^n,$$

consequently

$$\frac{1}{n}\ln\|y(n,y_0)\| \le \frac{1}{n}\ln\left(a^{2N}\|y_0\|\right) + \ln\left(1+a^{4N+1}q\right) + \frac{1}{n}R_{A_d,N}(n).$$

Taking $n=Nm$, $m\in\mathbb{N}$, using the equality

$$R_{A_d,N}(Nm) = \sum_{i=0}^{m-1}\ln\overline{\phi}_{A_d}((i+1)N, iN)$$

and passing to the upper limit when m tends to infinity, we obtain

$$\lambda_C(y_0) \leq \ln\left(1 + a^{4N+1}q\right) + \frac{1}{N}\left(\limsup_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\ln\overline{\phi}_{A_d}((i+1)N, iN)\right), \tag{46}$$

since by Lemma 25 we have

$$\lambda_C(y_0) = \limsup_{m\to\infty}\frac{1}{Nm}\ln\|y(Nm, y_0)\|.$$

Combining (42) in (46), we arrive at

$$\lambda_C(y_0) \leq \varepsilon + \frac{1}{N}\left(\limsup_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\ln\overline{\phi}_{A_d}((i+1)N, iN)\right)$$

and by definition of $\overline{\Omega}(A)$ we have $\lambda_C(y_0) \leq \varepsilon + \overline{\Omega}(A)$. This is contradiction with the choice of ε since $\lambda_C(y_0) = \lambda_A(x_0)$.

To prove the most left inequality suppose that there exists an upper-triangular sequence $A \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ and $x_0 \in \mathbb{R}^d_*$ such that

$$\underline{\Omega}(A) > \lambda_A(x_0). \tag{47}$$

Observe that Lemma 20 implies that

$$|r_{A_d,N}(n+1) - r_{A_d,N}(n)| \leq 2N\ln a$$

and therefore

$$\exp\left(-r_{A_d,N}(n+1)\right) \leq a^{2N}\exp\left(-r_{A_d,N}(n)\right), \ n \in \mathbb{N}. \tag{48}$$

Let us fix $\varepsilon \in (0, \underline{\Omega}(A) - \lambda_A(x_0))$, $N \in N$, $N > 1$ and define

$$\beta = \frac{1 - e^{-\varepsilon}}{a^{4N}d\,\|A\|_\infty}. \tag{49}$$

Again we will consider a sequence $C = (C(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$, where $C(n) = B^{-1}A(n)B$ and B is given by (38) together with system (43), but now we will rewrite (44) in the following form

$$y_0 = \Phi^{-1}_{A_d}(n,0)y(n,y_0) - \sum_{i=0}^{n-1}\Phi^{-1}_{A_d}(i+1,0)Q_\beta(i)y(i,y_0)\ , \tag{50}$$

where $y_0 = Bx_0$, $Q_\beta(n) = A(n) - A_d(n)$. Arguing as previously we have $A_d(n) = C_d(n)$ and therefore $\underline{\Omega}(A) = \underline{\Omega}(C)$. Furthermore,

$$\lambda_A(x_0) = \lambda_C(Bx_0), \quad q := \|Q_\beta\|_\infty \leq \frac{1 - e^{-\varepsilon}}{a^{4N}}. \tag{51}$$

Since $\left\|\Phi^{-1}_{A_d}(i,j)\right\| = \underline{\phi}^{-1}_{A_d}(i,j)$, $i, j \in \mathbb{N}$, $i \geq j$, it follows by Lemma 24 and (50) that

$$\begin{aligned}
\|y_0\| &\leq \underline{\phi}^{-1}_{A_d}(n,0)\|y(n,y_0)\| + \sum_{i=0}^{n-1}\underline{\phi}^{-1}_{A_d}(i+1,0)\|Q_\beta(i)\|\|y(i,y_0)\| \\
&\leq a^{2N}\exp\left(-r_{A_d,N}(n)\right)\|y(n,y_0)\| + a^{2N}q\sum_{i=m}^{n-1}\exp\left(-r_{A_d,N}(i+1)\right)\|y(i,y_0)\|.
\end{aligned}$$

Thus,

$$\mathtt{u}(0) \leq a^{2N} u(n) + a^{2N} q \sum_{i=0}^{n-1} \exp\left(-r_{A_d,N}(i+1) + r_{A_d,N}(i)\right) u(i),$$

where $u(i) = \|y(i, y_0)\| \exp\left(-r_{A_d,N}(i)\right)$. This together with (48) implies that

$$u(0) \leq a^{2N} u(n) + a^{4N} q \sum_{i=0}^{n-1} u(i) \quad \text{for } n \in \mathbb{N}.$$

Observe that (51) implies that $1 - a^{4N} q > 0$ and therefore we may apply Gronwall's backward inequality (see [1, Inequality (4.7.1)]) to get

$$u(0) \leq a^{2N} u(n) \left(1 - a^{4N} q\right)^{-n}$$

and consequently

$$\frac{1}{n} \ln\left(\|y_0\| a^{-2N}\right) + \ln\left(1 - a^{4N} q\right) + \frac{1}{n} r_{A_d,N}(n) \leq \frac{1}{n} \ln \|y(n, y_0)\|.$$

Taking $n = Nk$, $k \in \mathbb{N}$, using the equality

$$r_N(Nk) = \sum_{i=0}^{k-1} \ln \underline{\phi}_{A_d}\left((i+1)N, iN\right)$$

and passing to the upper limit when k tends to infinity, we have

$$\ln\left(1 - a^{4N} q\right) + \frac{1}{N} \left(\limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \ln \underline{\phi}_{A_d}\left((i+1)N, iN\right) \right) \leq \lambda_C(y_0) \tag{52}$$

since by Lemma 25 we have

$$\lambda_C(y_0) = \limsup_{k\to\infty} \frac{1}{Nk} \ln \|y(Nk, y_0)\|.$$

Combining (49) in (52), we obtain

$$-\varepsilon + \frac{1}{N} \left(\limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \ln \underline{\phi}_{A_d}\left((i+1)N, iN\right) \right) \leq \lambda_C(y_0)$$

and by definition of $\underline{\Omega}(A)$ we have $-\varepsilon + \underline{\Omega}(A) \leq \lambda_C(y_0)$. This is contradiction with the choice of ε since $\lambda_C(y_0) = \lambda_A(x_0)$. □

Proof of Theorem 15. Denote by $x_j = (x(n, v_j))_{n\in\mathbb{N}}$ where $x(n, v_j)$ is the $j-$th column of $X(n)$, $j = 1, ..., d$, and v_j is the j-th column of matrix $X(0)$. With this notation x_j is the solution of system (4) satisfying $x(0, v_j) = v_j$, $j = 1, ..., d$. For a vector $\alpha = [\alpha_1, \ldots, \alpha_d]^{\mathrm{T}} \in \mathbb{R}^d$ and $j \in \{1, ..., d\}$ denote by $\alpha|_j$ a vector from $\mathbb{R}^j$ given by $\alpha|_j = [\alpha_1, \ldots, \alpha_j]^{\mathrm{T}}$. Observe that the sequence $z|_j = \left(x(n, v_j)|_j\right)_{n\in\mathbb{N}}$ is a solution of the system

$$z(n+1) = A(n)|_j \, z(n), \ n \in \mathbb{N}, \tag{53}$$

with initial condition $z(0, v_j)|_j = v_j|_j$ and

$$\lambda_{A|_j}\left(v_j|_j\right) = \lambda_A\left(v_j\right),$$

for $j = 1, ..., d$. If we apply Lemma 27, to system (53) and take into account the last equality we get

$$\underline{\Omega}(A|_j) \leq \lambda_A(v_j) \leq \overline{\Omega}(A|_j).$$

Consider any lower-triangular fundamental matrix $Y = (Y(n))_{n\in\mathbb{N}}$ of system (5). Denote by $y_j = (y(n, v_j))_{n\in\mathbb{N}}$ where $y(n, v_j) = [0, ..., 0, y_{j,j}(n, u_j), ..., y_{d,j}(n, u_j)]^{\mathrm{T}}$ the $j-$th column of $Y(n)$, $j = 1, ..., d$, and u_j is the j-th column of matrix $Y(0)$. With this notation y_j is the solution of system (5) satisfying $y(0, u_j) = u_j$, $j = 1, ..., d$. For a vector $\alpha = \begin{bmatrix} \alpha_1 & ... & \alpha_d \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^d$ and $j \in \{1, ..., d\}$ denote by $\alpha|^j$ a vector from $\mathbb{R}^j$ given by $\alpha|^j = [\alpha_{d-j+1}, \ldots, \alpha_d]^{\mathrm{T}}$. Observe that the sequence $t|_j = \left(t(n, u_j)|_j\right)_{n\in\mathbb{N}}$ is a solution of the system

$$t(n+1) = A^{-\mathrm{T}}(n)\big|^j \, t(n), \; n \in \mathbb{N}, \tag{54}$$

with initial condition $t(0, u_j)|_j = u_j|^j$ and

$$\lambda_{A^{-\mathrm{T}}|^j}\left(u_j|^j\right) = \lambda_{A^{-\mathrm{T}}}(u_j) \qquad \text{for } j = 1, ..., d.$$

Applying Lemma 27 to system (54) and taking into account the last equality, we obtain

$$\underline{\Omega}(A^{-\mathrm{T}}\big|^j) \leq \lambda_{A^{-\mathrm{T}}}(u_j) \leq \overline{\Omega}(A^{-\mathrm{T}}\big|^j).$$

Finally by Remark 13 we get (13). □

Proof of Corollary 16. Inequalities (14) and (16) follow from Theorem 14 and 3, whereas inequalities (15) and (17) follow from Theorem 3 and Theorem 15 applied to a normal fundamental matrix X. □

Proof of Theorem 17. Let $X = (X(n))_{n\in\mathbb{N}}$ be any normal upper-triangular fundamental matrix of system (4). By Theorem 4 such a fundamental matrix exists. Consider the lower-triangular fundamental matrix $Y = X^{-\mathrm{T}}$ of the adjoint system (5). Denote by $x_j = (x_j(n))_{n\in\mathbb{N}}$ and $y_j = (y_j(n))_{n\in\mathbb{N}}$ where $x_j(n)$is the $j-$th column of $X(n)$ and $y_j(n)$ is the $j-$th column of $Y(n)$, $j = 1, ..., d$. Applying to X and Y Lemma 22 we get

$$\chi(x_j) \geq \overline{a}_j, \; \chi(y_j) \geq -\underline{a}_j, j = 1, ..., d$$

and applying Theorem 15 we get

$$\chi(x_j) \geq \underline{\Omega}(A|_j), \; \chi(y_j) \geq -\overline{\omega}(A|^j),$$

therefore

$$\max_{j=1,\ldots,d} \left(\chi(x_j) + \chi(y_j)\right) \geq \max_{j=1,\ldots,d} \Delta a_j \tag{55}$$

and

$$\max_{j=1,\ldots,d} \left(\chi(x_j) + \chi(y_j)\right) \geq \max_{j=1,\ldots,d} \left(\underline{\Omega}\left(A|_j\right) - \overline{\omega}\left(A|^j\right)\right). \tag{56}$$

Since X is a normal fundamental matrix and Lyapunov exponents λ_A and $\lambda_{A^{-T}}$ are dual (see Remark 11), then by Theorem 6 we know that

$$\sigma_G(A) = \max_{i=1,\ldots,d} \left(\chi(x_j) + \chi(y_j)\right). \tag{57}$$

From (55) and (57) we obtain the left hand side of inequality (18) and from (56) and (57) we obtain the left hand side of inequality (19).

The right hand side of (18) has been proved in [5, Theorem 4.7]. To prove the right hand side of (19) consider fundamental matrix $X' = (X'(n))_{n\in\mathbb{N}}$ from Theorem 21. Applying the notations of this theorem and using (8) we have

$$\sigma_G(A) \le \max_{j=1,\dots,d} \left(\chi\left(x'_j\right) + \chi\left(y'_j\right)\right). \tag{58}$$

Moreover, since X' is upper-triangular then $Y := (X')^{-T}$ is lower-triangular and we may apply Theorem 15 to get

$$\chi\left(x'_j\right) + \chi\left(y'_j\right) \le \left(\overline{\Omega}(A|_j) - \underline{\omega}(A|^j)\right),\ j = 1, \dots d$$

and consequently

$$\max_{j=1,\dots,d} \left(\chi\left(x'_j\right) + \chi\left(y'_j\right)\right) \le \max_{j=1,\dots,d} \left(\overline{\Omega}(A|_j) - \underline{\omega}(A|^j)\right).$$

The last inequality together with (58) imply the right hand side of (19). □

5. Examples

In this section we will present examples proving that the upper estimates (10) and (12) are independent. Similar examples may by provided for the other pairs of bounds as itis stated in Remark 19.

Example 28. *Consider any sequence $a_1 = (a_1(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R})$ such that $a_1(n) > 0$, $n \in \mathbb{N}$ and $\Delta a_1 > 0$, e.g.*

$$a_1(n) = \begin{cases} 1 & \text{for } n = 0, \\ e & \text{for } n \in \left[2^{2k}, 2^{2k+1}\right), \\ e^{-1} & \text{for } n \in \left[2^{2k+1}, 2^{2k+2}\right), \\ & \end{cases} \qquad k \in \mathbb{N}.$$

and define $a_2 = (a_2(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R})$, $a_2(n) = a_1(n)e$. Consider system (4) with $A = (A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^2)$, where $A(n) = \begin{bmatrix} a_1(n) & 0 \\ 0 & a_2(n) \end{bmatrix}$. Then, we have

$$\lambda_1(A) = \overline{a}_1 \text{ and } \lambda_2(A) = \overline{a}_1 + 1$$

and

$$\overline{\phi}_{A_d}(n,k) = \prod_{j=k}^{n-1} a_2(j), \quad \underline{\phi}_{A_d}(n,k) = \prod_{j=k}^{n-1} a_1(j).$$

Consequently for any $N \in \mathbb{N}_1$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1} \ln \overline{\phi}_{A_d}((i+1)N, iN) = \frac{1}{n}\sum_{i=0}^{n-1} \ln \prod_{j=iN}^{(i+1)N-1} a_2(j) = \frac{1}{n}\sum_{i=0}^{nN-1} \ln a_2(i).$$

By Lemma 25 we know that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{nN-1} \ln a_2(i) = N\overline{a}_2,$$

therefore

$$\overline{\Omega}(A) = \inf_{N\in\mathbb{N}_1} \frac{1}{N}\left(\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln \overline{\phi}_{A_d}((i+1)N, iN)\right) = \overline{a}_2 = \overline{a}_1 + 1 = \lambda_2(A).$$

Similarly we may show that $\underline{\Omega}(A) = \overline{a}_1$. *Applying inequality (10) from Theorem 14 for* $X = (\Phi_A(n,0))_{n\in\mathbb{N}}$ *and* $j = 2$ *we get*

$$\overline{a}_1 + 1 \leq \lambda_2(A) \leq \overline{a}_1 + 1 + \Delta a_1,$$

whereas inequality (12) from Theorem 15 gives

$$\overline{a}_1 + 1 \leq \lambda_2(A) \leq \overline{a}_2 = \overline{a}_1 + 1.$$

Thus, for the considered system, the estimate (10) gives a worse upper value than the estimate (12), and for the lower estimates we have the opposite situation.

In the next example we will use the following result.

Lemma 29. *For any sequence* $(a(n))_{n\in\mathbb{N}} \in \mathcal{L}^\infty(\mathbb{N},\mathbb{R})$ *we have*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} a(i) = \limsup_{n\to\infty} \frac{1}{Nn}\sum_{i=0}^{Nn-1} a(i) \quad \text{for any } N \in \mathbb{N}.$$

Proof. Let K satisfy $|a(i)| < K$ for all $i \in \mathbb{N}$. For any natural number $n \in (Nk, (k+1)N]$ we have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=0}^{n-1} a(i) - \frac{1}{Nk}\sum_{i=0}^{Nk-1} a(i)\right| \\
\leq\ &\left|\frac{1}{n}\sum_{i=0}^{n-1} a(i) - \frac{1}{Nk}\sum_{i=0}^{n-1} a(i)\right| + \left|\frac{1}{Nk}\sum_{i=0}^{n-1} a(i) - \frac{1}{Nk}\sum_{i=0}^{Nk-1} a(i)\right| \\
\leq\ &\left|\frac{Nk-n}{Nk}\right| K + \frac{n-Nk}{Nk}K = 2\frac{n-Nk}{Nk}K \underset{k\to\infty}{\to} 0
\end{aligned}$$

The last conclusion completes the proof of the lemma. □

Example 30. *Fix real numbers* a *and* b *such that* $-b < a < 0 < b$ *and consider system (4) with* $A(n) = \begin{bmatrix} 1 & 0 \\ 0 & a(n) \end{bmatrix}$, *where* $a(n) = \begin{cases} e^a & \text{if } n \in \left[(2k)^2, (2k+1)^2\right), \\ e^b & \text{if } n \in \left[(2k+1)^2, (2k+2)^2\right), \end{cases}$ *for* $k \in \mathbb{N}$. *Denote* $e_1 = [0,1]^T$ *and* $e_2 = [1,0]^T$. *It is clear that* $\lambda_A(e_1) = 0$. *We first show that*

$$\lambda_A(e_2) = \frac{a+b}{2}. \tag{59}$$

Observe that definition of the sequence $(a(n))_{n\in\mathbb{N}}$ *implies*

$$\prod_{i=0}^{n} a(i) = \begin{cases} e^{a\left(n-2k^2-k+1\right)+b\left(2k^2+k\right)} & \text{if } n \in \left[(2k)^2, (2k+1)^2\right), \\ e^{a\left(2k^2+3k+1\right)+b\left(n-2k^2-3k\right)} & \text{if } n \in \left[(2k+1)^2, (2k+2)^2\right), \end{cases}$$

where $k \in \mathbb{N}$. *Since* $a < 0 < b$, *then for* $k \in \mathbb{N}$ *we have*

$$\begin{aligned}
&e^{a\left(2k^2+3k+1\right)} \leq e^{a\left(n-2k^2-k+1\right)} \leq e^{a\left(2k^2-k+1\right)} && \text{if } n \in \left[(2k)^2, (2k+1)^2\right), \\
&e^{b\left(2k^2+k+1\right)} \leq e^{b\left(n-2k^2-3k\right)} \leq e^{b\left(2k^2+5k+3\right)} && \text{if } n \in \left[(2k+1)^2, (2k+2)^2\right),
\end{aligned}$$

and consequently for $n \in \left[(2k)^2, (2k+1)^2\right)$

$$e^{a\left(2k^2+3k+1\right)+b\left(2k^2+k\right)} \leq \prod_{i=0}^{n} a(i) \leq e^{a\left(2k^2-k+1\right)+b\left(2k^2+k\right)} \tag{60}$$

and for $n \in \left[(2k+1)^2, (2k+2)^2\right)$

$$e^{a\left(2k^2+3k+1\right)+b\left(2k^2+k+1\right)} \leq \prod_{i=0}^{n} a(i) \leq e^{a\left(2k^2+3k+1\right)+b\left(2k^2+5k+3\right)}. \tag{61}$$

To prove (59), we consider an increasing sequence of natural number $(n_l)_{l\in\mathbb{N}}$ *such that*

$$\lim_{l\to\infty} \frac{1}{n_l+1} \ln \prod_{i=0}^{n_l} a(i) = \lambda_A(e_2).$$

Two cases are possible. There are infinite many $l \in \mathbb{N}$ *such that there exists* $k_l \in \mathbb{N}$ *with the property* $n_l \in \left[(2k_l)^2, (2k_l+1)^2\right)$ *or there are infinite many* $l \in \mathbb{N}$ *such that there exists* $k_l \in \mathbb{N}$ *with the property* $n_l \in \left[(2k_l+1)^2, (2k_l+2)^2\right)$. *Consider the first case. Let* $(l_m)_{m\in\mathbb{N}}$ *be an increasing sequence of natural numbers such that there exists* $k_{l_m} \in \mathbb{N}$ *with the property* $n_{l_m} \in \left[(2k_{l_m})^2, (2k_{l_m}+1)^2\right)$. *By (60) we have*

$$a\frac{2k_{l_m}^2+3k_{l_m}+1}{n_{l_m}+1}+b\frac{2k_{l_m}^2+k_{l_m}}{n_{l_m}+1} \leq \frac{1}{n_{l_m}+1}\ln\prod_{i=0}^{n_{l_m}} a(i) \leq a\frac{2k_{l_m}^2-k_{l_m}+1}{n_{l_m}+1}+b\frac{2k_{l_m}^2+k_{l_m}}{n_{l_m}+1}. \tag{62}$$

Observe that the condition $n_{l_m} \in \left[(2k_{l_m})^2, (2k_{l_m}+1)^2\right)$, $m \in \mathbb{N}$ *implies that*

$$\lim_{m\to\infty} \frac{2k_{l_m}^2+3k_{l_m}+1}{n_{l_m}+1} = \lim_{m\to\infty} \frac{2k_{l_m}^2+k_{l_m}}{n_{l_m}+1} = \lim_{m\to\infty} \frac{2k_{l_m}^2-k_{l_m}+1}{n_{l_m}+1} = \frac{1}{2}.$$

Passing to the limit when m tends to infinity in (62) we get (59). The consideration in the second case are analogical but we use (61) instead of (60).

Next, we show that

$$\overline{\Omega}(A) \geq \frac{b}{2}. \tag{63}$$

For a fix $N \in \mathbb{N}_1$ *let us consider sequence* $R_{A_d,N} = (R_{A_d,N}(n))_{n\in\mathbb{N}}$ *given by (32). By (33) with* $n = (2l+1)^2$ *and* $k = (2l)^2$, $l \in \mathbb{N}$ *we get*

$$1 \leq a^{2N} \exp\left(R_{A_d,N}\left((2l+1)^2\right) - R_{A_d,N}\left((2l)^2\right)\right)$$

and taking $n = (2l+2)^2$ *and* $k = (2l+1)^2$ *we get*

$$e^{(4l+3)b} \leq a^{2N} \exp\left(R_{A_d,N}\left((2l+2)^2\right) - R_{A_d,N}\left((2l+1)^2\right)\right).$$

Taking logarithm in the last two inequalities, summing them up for $l = 0, ..., m$, *dividing by* $(2m+2)^2$ *and passing to the upper limit when* m *tends to infinity we obtain*

$$\frac{b}{2} \leq \limsup_{m\to\infty} \frac{1}{(2m+2)^2} R_{A_d,N}\left((2m+2)^2\right) \leq \limsup_{m\to\infty} \frac{1}{m} R_{A_d,N}(m).$$

From Lemma 29 and definition of $R_{A_d,N}$ we know that

$$\limsup_{m\to\infty}\frac{1}{m}R_{A_d,N}(m)=\frac{1}{N}\limsup_{m\to\infty}\frac{1}{N}R_{A_d,N}(Nm)$$

and therefore

$$\frac{b}{2}\le\frac{1}{N}\limsup_{m\to\infty}\frac{1}{N}R_{A_d,N}(Nm).$$

Taking infimum over $N\in\mathbb{N}_1$ and having in mind definition of $\overline{\Omega}(A)$ we get (63).

Applying inequality (10) from Theorem 14 for $X=(\Phi_A(n,0))_{n\in\mathbb{N}}$ and $j=2$ we get $\lambda_2(A)\le\frac{a+b}{2}$, whereas inequality (12) from Theorem 15 gives $\lambda_2(A)\le\frac{b}{2}$. Since $\frac{a+b}{2}<\frac{b}{2}$, then for the considered system, the estimate (10) gives a better upper value than the estimate (12).

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