Assignment of spectrum for time-varying linear control systems via kinematic equivalence

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Abstract

In this note, we consider a uniformly completely controllable linear system. For this system, we show the existence of a linear state feedback such that the corresponding closed-loop system is kinematically equivalent to a linear time-invariant system whose spectrum is a prior set of distinct real numbers.

Key words: Linear control systems, Uniform complete controllability, State feedback, Pole placement method, Kinematic equivalence.

1 Introduction

For linear time-invariant systems one of the basic method of control design is the so-called pole placement method. The theoretical basis for this method is the fact that the controllability of the linear time-invariant system is equivalent to the fact that for each set of complex numbers with cardinality equal to the dimension of the state vector and symmetric relative to the real axis, there is a stationary feedback such that the poles of the closed-loop system form this set (see [25]).

Recently, many researchers have attempted to generalize this methodology to systems with variable coefficients. In these works uniformly completely controllable systems were considered, and as the equivalent of poles the Lyapunov spectrum (see [4] for discrete- time systems and [18] for continuous-time systems) or the spectrum of uniform exponential dichotomy (see [6] for discrete-time systems and [5] for continuous-time systems) was adopted. The results of these works show that uniform complete controllability is a necessary and sufficient condition for the assignability of the uniform exponential dichotomy spectrum and sufficient but not necessary for the assignability of the Lyapunov spectrum. The last fact suggests to investigate what, apart from the assignability of the Lyapunov spectrum, can be achieved in control systems which are uniformly completely controllable. In this paper we try to answer the last question partially.

The main result of this work is that uniform complete controllability of a system with bounded coefficients, both for continuous and discrete systems, implies the existence of a bounded feedback that guarantees the kinematic similarity of the closed-loop system to any predetermined stationary diagonal system. It should be emphasized that the concept of kinematic equivalence ensures the preservation of many dynamic properties, including stability and

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asymptotic stability. Furthermore, it is known that reducibility of linear time-varying systems under kinematic equivalent transformations (see e.g. [9, 10, 23]) played an important ingredient in establishing many aspects of qualitative theory such as linearization theory [12, 21], invariant manifold theory [2], normal form theory [24].

The problem of converting a time-varying control system to a given time-invariant system by choosing an appropriate feedback was already discussed in the literature (see [27], [28], [20], [17], [19]). In these works, the authors show that, assuming only controllability for continuous-time case or complete reachability for discrete-time systems, one can find a feedback which guarantees that the closed-loop system can be reduced to a stationary system with any predetermined poles through a linear state transformation. The results of these works, however, do not provide the conditions when this linear state transformation will be bounded together with the inverse and then it will maintain such a dynamic property as stability. If this linear state transformation is not bounded, or the inverse transformation thereto is not bounded, it may be that the closed system is stable and the stationary system will not have this property, or vice versa (See Remark 10 in Section 3 for more details).

The paper is organized as follows: In the first part of Section 2, we state the problem and the result in assigning spectrum of continuous-time time-varying linear control systems via kinematic equivalence. A proof of this result is given in the second part of Section 2. A corresponding result for assignment of spectrum for discrete-time time-varying linear control systems via kinematic equivalence is stated and briefly proved in Section 3.

2 Assignment of spectrum for continuous-time time-varying linear control systems via kinematic equivalence

2.1 Preliminaries and the statement of the main results

For $d, s \in \mathbb{N}$, let $\mathcal{KC}_{d,s}(\mathbb{R})$ denote the set of bounded and piecewise continuous matrix-valued functions $M : \mathbb{R} \to \mathbb{R}^{d\times s}$. Consider a linear time-varying system described by the following equation

$$\dot{x} = A(t)x + B(t)u, \qquad t \in \mathbb{R}, \tag{1}$$

where $A \in \mathcal{KC}_{d,d}(\mathbb{R})$, $B \in \mathcal{KC}_{d,s}(\mathbb{R})$ and $u \in \mathcal{KC}_{s,1}(\mathbb{R})$ is the control. For $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ the solution of system (1) satisfying $x(t_0) = x_0$, will be denoted by $x(\cdot, t_0, x_0, u)$. An important notion relating to (1) called uniform complete controllability is stated below, see [16].

Definition 1 (Uniform complete controllability) *System (1) is called uniformly completely controllable if there exist $\alpha, K > 0$ such that for all $(t_0, \xi) \in \mathbb{R} \times \mathbb{R}^d$ there exists a piecewise continuous control $u : [t_0, t_0 + K] \to \mathbb{R}^s$ such that $x(t_0 + K, t_0, 0, u) = \xi$ and*

$$\|u(t)\| \le \alpha\|\xi\|, \qquad t \in [t_0, t_0 + K]\,.$$

If in system (1) we apply a control of the form

$$u(t) = F(t)x(t),$$

where the feedback $F : \mathbb{R} \to \mathbb{R}^{s\times d}$ is bounded piecewise continuous, we obtain a so called closed loop system

$$\dot{x} = (A(t) + B(t)F(t))\,x. \tag{2}$$

Our interest in this paper is to know the possibility of the dynamics generated by (2) via time-varying bounded linear state feedback. To formulate the main result of this paper, we introduce the kinematic equivalence notion between two continuous time-varying systems. This notion is a well-studied concept and preserves many dynamical properties of linear systems such as Lyapunov spectrum (see e.g. [1, P. 44]) and dichotomy spectrum (see [23, Corollary 2.1]). Let $\mathcal{L}(\mathbb{R}, \mathbb{R}^{d\times d})$ denote the set of bounded, invertible and piecewise C^1 function $T : \mathbb{R} \to \mathbb{R}^{d\times d}$, i.e. there exists $L > 0$ such that

$$\max\left\{\sup_{t\in\mathbb{R}} \|T(t)\|, \sup_{t\in\mathbb{R}} \|T^{-1}(t)\|, \sup_{t\in\mathbb{R}} \|\dot{T}(t)\|\right\} \le L.$$

These functions are also known as Lyapunov transformations, see [1, Definition 3.1.1]. Now, we introduce a notion of equivalence between continuous time-varying linear systems.

Definition 2 *Consider two linear continuous-time time-varying systems*

$$\dot{x} = M(t)x, \qquad \dot{y} = N(t)y, \tag{3}$$

where $M, N \in \mathcal{KC}_{d,d}(\mathbb{R})$. They are said to be kinematically equivalent *if there exists $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^{d\times d})$ such that*

$$\dot{T}(t) = N(t)T(t) - T(t)M(t) \qquad \text{for } t \in \mathbb{R}.$$

Remark 3 *Let $X_M(\cdot,\cdot), X_N(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{d\times d}$ denote the evolution matrices of two systems in (3), i.e. for any $\xi, \eta \in \mathbb{R}^d$ and $s \in \mathbb{R}$, $X_M(\cdot, s)\xi$ and $X_N(\cdot, s)\eta$ solve, respectively, the following equations with initial values*

$$\dot{x} = M(t)x, \quad x(s) = \xi$$

and

$$\dot{y} = N(t)x, \quad y(s) = \eta.$$

If two systems in (3) are kinematically equivalent via transformation $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^{d\times d})$ then

$$T(t)X_M(t, s) = X_N(t, s)T(s) \qquad \text{for all } t, s \in \mathbb{R},$$

see e.g. [23, Lemma 2.1]. It means that the family of linear bounded maps $(T(t))_{t\in\mathbb{R}}$ transforms the solution $X_M(\cdot, s)\xi$ of $\dot{x} = M(t)x$ to the solution $X_N(\cdot, s)T(s)\xi$ of $\dot{y} = N(t)y$.

We are now in a position to state the main result of this paper about a version of pole placement theorem for continuous-time time-varying control systems under kinematically equivalent transformations. In this result, we require a technical condition on the coefficients of (1) that $B : \mathbb{R} \to \mathbb{R}^{d\times s}$ is piecewise uniformly continuous, i.e. there exists a $\Delta_0 > 0$ such that the length of each continuity interval I_j ($j \in J \subset \mathbb{N}$) of the function B satisfies the inequality $|I_j| \geq \Delta_0$, and for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|B(t) - B(s)\| \leq \varepsilon$ for each $j \in J$ and for all $t, s \in I_j$ satisfying the inequality $|t - s| \leq \delta$.

Theorem 4 *Suppose that system (1) is uniformly complete controllable. Assume that $B : \mathbb{R} \to \mathbb{R}^{d\times s}$ is piecewise uniformly continuous and bounded. Let $\sigma = (\lambda_1, \ldots, \lambda_d)$, where $\lambda_d < \lambda_{d-1} < \cdots < \lambda_1$, be an arbitrary collection of d real numbers. Then, there exists a feedback $F \in \mathcal{KC}_{s,d}(\mathbb{R})$ such that the closed-loop system*

$$\dot{x}(t) = (A(t) + B(t)F(t))x(t), \qquad t \in \mathbb{R}$$

is kinematically equivalent to a time-invariant system

$$\dot{x} = \Lambda x, \qquad \text{where } \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_d).$$

2.2 Proof of assignment of spectrum for continuous-time time-varying linear control systems via kinematic equivalence

The first key ingredient in the proof of our main result is taken from [22, Theorem 1] and [18] in which the authors showed that for a uniformly completely controllable time-varying control system there exists a feedback such that the corresponding closed-loop system is kinematically equivalent to an upper-triangular linear system with given diagonal entries.

Theorem 5 *If system (1) is uniformly completely controllable and B is piecewise uniformly continuous, then for arbitrary functions $p_i \in \mathcal{KC}_{1,1}(\mathbb{R})$, $i = 1, ..., d$, there exists a feedback $F \in \mathcal{KC}_{s,d}(\mathbb{R})$ such that the closed system (2) is kinematically equivalent to a system with an upper triangular piecewise continuous bounded matrix function whose diagonal coincides with $(p_1, ..., p_d)$.*

Thanks to the above result, to gain the proof of Theorem 4 it remains to deal with the problem of equivalence between two upper-triangular linear systems. Then, the second key ingredient in the proof of Theorem 4 is stated and proved in the following proposition.

Proposition 6 *Let $\lambda_1 > \cdots > \lambda_d$ be d distinct real numbers. Let $C, D \in \mathcal{KC}_{d,d}(\mathbb{R})$ be of the following forms*

$$C(t) := \begin{pmatrix} \lambda_1 & c_{12}(t) & \dots & c_{1(d-1)}(t) & c_{1d}(t) \\ 0 & \lambda_2 & \dots & c_{2(d-1)}(t) & c_{2d}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{d-1} & c_{(d-1)d}(t) \\ 0 & 0 & \dots & 0 & \lambda_d \end{pmatrix},$$

$$D(t) := \begin{pmatrix} \lambda_1 & c_{12}(t) & \dots & c_{1(d-1)}(t) & 0 \\ 0 & \lambda_2 & \dots & c_{2(d-1)}(t) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{d-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_d \end{pmatrix}.$$

Then, two systems

$$\dot{x} = C(t)x, \qquad \dot{y} = D(t)y \tag{4}$$

are kinematically equivalent.

Proof. To show that two systems in (4) are kinematically equivalent, we need to construct a Lyapunov transformation $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^{d\times d})$ such that

$$\dot{T}(t) = C(t)T(t) - T(t)D(t) \qquad \text{for all } t \in \mathbb{R}. \tag{5}$$

Here, we let $T(t)$ be of the following form

$$T(t) := \begin{pmatrix} 1 & 0 & \dots & 0 & \gamma_1(t) \\ 0 & 1 & \dots & 0 & \gamma_2(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \gamma_{d-1}(t) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \tag{6}$$

where the functions $\gamma_1(\cdot), \dots, \gamma_{d-1}(\cdot)$ are determined later. A direct computation yields that a necessary and sufficient condition for $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^{d\times d})$ is that

$$\max\left\{\sup_{t\in\mathbb{R}} |\gamma_i(t)|, \sup_{t\in\mathbb{R}} |\dot{\gamma}_i(t)|\right\} < \infty \quad \text{for } i = 1, \dots, d-1. \tag{7}$$

Meanwhile, to gain the equality (5) with T of the form as in (6), the functions $\gamma_1(\cdot), \dots, \gamma_{d-1}(\cdot)$ must satisfy for $i = 1, ..., d-1$

$$\dot{\gamma}_i(t) = (\lambda_i - \lambda_d)\,\gamma_i(t) + \sum_{j=i+1}^{d} c_{ij}(t)\gamma_j(t), \tag{8}$$

where we use the convention that $\gamma_d(t) = 1$ for $t \in \mathbb{R}$. To conclude the proof, we will construct step by step $\gamma_{d-1}(\cdot), \dots, \gamma_1(\cdot)$ satisfying (7) and (8).

Initial step: In this step, we construct γ_{d-1}. Replacing $i = d-1$ into (8), we arrive at

$$\dot{\gamma}_{d-1}(t) = (\lambda_{d-1} - \lambda_d)\,\gamma_{d-1}(t) + c_{(d-1)d}(t). \tag{9}$$

A general form of the solution of the above equation is

$$\gamma_{d-1}(t) = e^{(\lambda_{d-1}-\lambda_d)t} \times \left(\gamma_{d-1}(0) + \int_0^t e^{-(\lambda_{d-1}-\lambda_d)s} c_{(d-1)d}(s) ds\right).$$

Choose $\gamma_{d-1}(0) = -\int_0^\infty e^{-(\lambda_{d-1}-\lambda_d)s} c_{(d-1)d}(s)\, ds$, where the existence of the infinite integral comes from the fact that $\lambda_{d-1} > \lambda_d$. Thus, we obtain the following solution of (9)

$$\gamma_{d-1}(t) = -\int_t^\infty e^{(\lambda_{d-1}-\lambda_d)(t-s)} c_{(d-1)d}(s)\, ds.$$

Since $\lambda_{d-1} > \lambda_d$ and $c_{(d-1)d}(\cdot)$ is bounded it follows that

$$\sup_{t\in\mathbb{R}} |\gamma_{d-1}(t)| \le \frac{1}{\lambda_{d-1} - \lambda_d} \sup_{t\in\mathbb{R}} |c_{(d-1)d}(t)|.$$

Thus, by (9) we have

$$\sup_{t\in\mathbb{R}} |\dot{\gamma}_{d-1}(t)| \le 2 \sup_{t\in\mathbb{R}} |c_{(d-1)d}(t)|.$$

Therefore, there exists a function $\gamma_{d-1}(t)$ satisfying (7) and (9).

Induction step: Suppose that $\gamma_{d-1}(\cdot), \ldots, \gamma_{k+1}(\cdot)$ have been constructed satisfying (8) for $i = d-1, \ldots, k+1$ and

$$\max\left\{\sup_{t\in\mathbb{R}} |\gamma_i(t)|, \sup_{t\in\mathbb{R}} |\dot{\gamma}_i(t)|\right\} < \infty \qquad \text{for } i = d-1, \ldots, k+1.$$

We need to construct a function $\gamma_k(t)$ satisfying (7) and (8), i.e.

$$\dot{\gamma}_k(t) = (\lambda_k - \lambda_d)\, \gamma_k(t) + \beta(t), \qquad \text{for } t \in \mathbb{R}, \tag{10}$$

where $\beta(t) := \sum_{j=k+1}^d c_{kj}(t)\gamma_j(t)$ was determined. Analog to the construction in the *initial step* we choose the initial value $\gamma_k(0)$ for (10) as

$$\gamma_k(0) = -\int_0^\infty e^{-(\lambda_k-\lambda_d)s} \beta(s)\, ds.$$

For this choice, we receive the following solution of (10)

$$\gamma_k(t) = -\int_t^\infty e^{(\lambda_k-\lambda_d)(t-s)} \beta(s)\, ds.$$

A direct computation yields that

$$\sup_{t\in\mathbb{R}} |\gamma_k(t)| \le \frac{\sup_{t\in\mathbb{R}} |\beta(t)|}{\lambda_k - \lambda_d}, \sup_{t\in\mathbb{R}} |\dot{\gamma}_k(t)| \le 2 \sup_{t\in\mathbb{R}} |\beta(t)|.$$

Thus, $\gamma_k(t)$ has been constructed and satisfies both (7) and (8). The proof is complete. ■

We are now in a position to prove Theorem 4.

Proof of Theorem 4. From Theorem 5, there exists a feedback $F \in \mathcal{KC}_{s,d}(\mathbb{R})$ and an upper triangular piecewise continuous bounded matrix function C of the form

$$C(t) := \begin{pmatrix} \lambda_1 & c_{12}(t) & \dots & c_{1(d-1)}(t) & c_{1d}(t) \\ 0 & \lambda_2 & \dots & c_{2(d-1)}(t) & c_{2d}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{d-1} & c_{(d-1)d}(t) \\ 0 & 0 & \dots & 0 & \lambda_d \end{pmatrix}$$

such that the closed-loop system

$$\dot{x} = (A(t) + B(t)F(t))x \tag{11}$$

is kinematically equivalent to the system

$$\dot{y} = C(t)y. \tag{12}$$

By Proposition 6, system (12) is kinematically equivalent to

$$\dot{z} = \begin{pmatrix} \lambda_1 & c_{12}(t) & \dots & c_{1(d-1)}(t) & 0 \\ 0 & \lambda_2 & \dots & c_{2(d-1)}(t) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{d-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_d \end{pmatrix} z. \tag{13}$$

Fixing the d-th component of z and applying Proposition 6 to the subsystems corresponding the the first $d-1$ component of z, we see that (13) is kinematically equivalent to

$$\dot{w} = \begin{pmatrix} \lambda_1 & c_{12}(t) & \dots & c_{1(d-2)}(t) & 0 & 0 \\ 0 & \lambda_2 & \dots & c_{2(d-2)}(t) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & \lambda_{d-2} & 0 & 0 \\ 0 & 0 & \dots & 0 & \lambda_{d-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & \lambda_d \end{pmatrix} w.$$

Applying this procedure $d-2$ times more, we arrive that (11) is kinematically equivalent to the time-invariant system

$$\dot{x} = \Lambda x, \qquad \text{where } \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_d).$$

The proof is complete. ■

Remark 7 *In the above proof, the control law $F(t)$ achieving the equivalence in Theorem 4 came from a theoretical result stated in Theorem 5. This control law is constructed implicitly and it is challenging and unclear to the authors how to derive an explicit one.*

3 Corresponding result for assignment of spectrum for discrete-time time-varying linear control systems via kinematic equivalence

This section is devoted to stating a corresponding result for assignment of spectrum for discrete-time time-varying linear control systems. We only provide a sketch of the proof of this result since the structure of the proof is quite similar to the proof of Theorem 4.

For $d, s \in \mathbb{N}$, let $\mathcal{L}^\infty(\mathbb{Z}, \mathbb{R}^{d\times s})$ denote the space of bounded sequences of matrices $M = (M_n)_{n\in\mathbb{Z}}$ with $M_n \in \mathbb{R}^{d\times s}$ satisfying that

$$\|M\|_\infty := \sup_{n\in\mathbb{Z}} \|M_n\| < \infty.$$

For $d \in \mathbb{N}$, let $\mathcal{L}^{\mathrm{Lya}}(\mathbb{Z}, \mathbb{R}^{d\times d})$ denote the set of all Lyapunov sequences $M = (M_n)_{n\in\mathbb{Z}}$ in $\mathbb{R}^{d\times d}$, i.e. $M \in \mathcal{L}^\infty(\mathbb{Z}, \mathbb{R}^{d\times d})$ and its inverse sequence $M^{-1} := (M_n^{-1})_{n\in\mathbb{Z}}$ exists and $M^{-1} \in \mathcal{L}^\infty(\mathbb{Z}, \mathbb{R}^{d\times d})$.

Consider a linear discrete-time time-varying system described by the following equation

$$x(n+1) = A(n)x(n) + B(n)u(n), \qquad n \in \mathbb{Z}, \tag{14}$$

where $A = (A(n))_{n\in\mathbb{Z}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{Z}, \mathbb{R}^{d\times d})$ and $B = (B(n))_{n\in\mathbb{Z}} \in \mathcal{L}^\infty(\mathbb{Z}, \mathbb{R}^{d\times s})$ and the control $u = (u(n))_{n\in\mathbb{Z}} \in \mathcal{L}^\infty(\mathbb{Z}, \mathbb{R}^s)$. Let $x(n, n_0, \xi, u)$ denote the solution of (14) satisfying that $x(n_0, n_0, \xi, u) = \xi$. Now, we recall the notion of uniform complete controllability of (14), see also [3].

Definition 8 (Uniform complete controllability) *System* (14) *is called* uniformly completely controllable *if there exist a positive number α and a natural number K such that for all $\xi \in \mathbb{R}^d$ and $k_0 \in \mathbb{Z}$ there exists a control sequence $u(n), n = k_0, k_0+1, \dots, k_0+K-1$ such that $x(k_0+K, k_0, 0, u) = \xi$ and*

$$\|u(n)\| \le \alpha\|\xi\| \qquad \textit{for all } n = k_0, k_0+1, \dots, k_0+K-1.$$

For a feedback $U = (U(n))_{n\in\mathbb{Z}} \in \mathcal{L}^\infty(\mathbb{Z}, \mathbb{R}^{s\times d})$, the corresponding closed-loop system is

$$x(n+1) = (A(n) + B(n)U(n))x(n). \tag{15}$$

The feedback $U \in \mathcal{L}^\infty(\mathbb{Z}, \mathbb{R}^{s\times d})$ is called *admissible* if $A + BU := (A(n) + B(n)U(n))_{n\in\mathbb{Z}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{Z}, \mathbb{R}^{d\times d})$.

Now, we state and provide a sketch of proof of a corresponding result of Theorem 4 for discrete time-varying linear control systems.

Theorem 9 *Suppose that system* (14) *is uniformly completely controllable. Let $\sigma = (\lambda_1, \dots, \lambda_d)$, where $|\lambda_d| < |\lambda_{d-1}| < \dots < |\lambda_1|$, be an arbitrary collection of d non-zero real numbers. Then, there exists an admissible feedback U such that the closed-loop system*

$$x(n+1) = (A(n) + B(n)U(n))x(n) \quad n \in \mathbb{Z}$$

is kinematically equivalent[1] *to a time-invariant system*

$$y(n+1) = \Lambda y(n), \qquad \textit{where } \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_d).$$

Sketch of proof. Using [3, Theorem 4.6], there exist an admissible feedback U and a upper-triangular matrix $(C(n))_{n\in\mathbb{Z}}$ of the form

$$C(n) := \begin{pmatrix} \lambda_1 & c_{12}(n) & \dots & c_{1(d-1)}(n) & c_{1d}(n) \\ 0 & \lambda_2 & \dots & c_{2(d-1)} & c_{2d}(n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{d-1} & c_{(d-1)d}(n) \\ 0 & 0 & \dots & 0 & \lambda_d \end{pmatrix}$$

[1] Two systems

$$x(n+1) = M(n)x(n), y(n+1) = N(n)y(n), M, N \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{Z}, \mathbb{R}^{d\times d})$$

are called kinematically equivalent if there exists a Lyapunov sequence $(T(n))_{n\in\mathbb{Z}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{Z}, \mathbb{R}^{d\times d})$ such that $C(n) = T^{-1}(n+1)A(n)T(n)$ for $n \in \mathbb{Z}$.

such that the closed-loop system

$$x(n+1) = (A(n) + B(n)U(n))x(n) \tag{16}$$

is kinematically equivalent to the equation

$$y(n+1) = C(n)y(n). \tag{17}$$

Using a corresponding result of Proposition 6 for discrete-time time-varying linear systems, system (16) is kinematically equivalent to the system $z(n+1) = \Lambda z(n)$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$. The proof is complete. ■

Remark 10 *Consider an one dimensional ($d = s = 1$) system*

$$x(n+1) = x(n) + \frac{1}{n^2+1}u(n), \qquad n \in \mathbb{Z}. \tag{18}$$

It is clear that system (18) is completely reachable (see [19, Definition 2.1]). Thus, by [27], [28], [20], [17], [19], there exists a linear state-feedback which guarantees that the closed-loop system can be reduced to a stationary system with any predetermined poles through a linear state transformation (possibly not Lyapunov). In fact, we show below that it is impossible in designing an admissible feedback $u(n) = f(n)x(n)$ such that the corresponding closed loop system

$$x(n+1) = \left(1 + \frac{f(n)}{n^2+1}\right)x(n), \qquad n \in \mathbb{Z} \tag{19}$$

is kinematically equivalent to system

$$y(n+1) = 2y(n), \qquad n \in \mathbb{Z}. \tag{20}$$

Suppose the contrary, i.e. there exists a Lyapunov transformation $t = (t(n))_{n\in\mathbb{Z}}$ which establishes the kinematic equivalence of (19) and (20). Thus,

$$2 = \frac{t(n)}{t\,(n+1)}\left(1 + \frac{f(n)}{n^2+1}\right).$$

Equivalently, we have

$$2\left|\frac{t\,(n+1)}{t(n)}\right| = \left|1 + \frac{f(n)}{n^2+1}\right|. \tag{21}$$

Observe that

$$\lim_{n\to\infty}\left|\frac{t\,(n+1)}{t(n)}\right| = 1.$$

Indeed, if $\limsup_{n\to\infty}\left|\frac{t(n+1)}{t(n)}\right| > 1$, then $(t(n))_{n\in\mathbb{Z}}$ is unbounded and if $\liminf_{n\to\infty}\left|\frac{t(n+1)}{t(n)}\right| < 1$, then $(t^{-1}(n))_{n\in\mathbb{Z}}$ is unbounded. Equality 21 leads to contradiction, since the right hand side tends to 1, when n tends to infinity, whereas the left hand side tends to 2.

4 Example

Example 11 *Consider system (1) with*

$$A(t) = \begin{pmatrix} -\frac{2e^{\sin t}+t^2e^{\sin t}+2t^2+3}{t^2+1} & e^{\sin t}+1 \\ -\frac{\left(t^2+2\right)^2\left(e^{\sin t}+1\right)}{(t^2+1)^2} & \sin t \end{pmatrix}, \; B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for $t \in \mathbb{R}$. It is clear that A is bounded and continuous. Using the definition of uniform complete controllability we can also verify that this system is uniformly complete controllable. Suppose that we want to find an admissible feedback U such that the closed-loop system is kinematically equivalent to diagonal time invariant system with coefficient matrix $\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

Observe that for

$$U(t) = \left(\frac{t^4+3t^2-2t+2}{(t^2+1)^2} \quad -\frac{\sin t - 2e^{\sin t} + t^2 \sin t - t^2 e^{\sin t} + t^2}{t^2+1} \right), \ t \in \mathbb{R}$$

and

$$T_1(t) = \begin{pmatrix} 1 & 0 \\ 1 + \frac{1}{1+t^2} & 1 \end{pmatrix}, \ t \in \mathbb{R}.$$

we have

$$\begin{pmatrix} -1 & e^{\sin t} + 1 \\ 0 & -2 \end{pmatrix} = T_1^{-1}(t)\,(A + BK)\,T_1(t) - T_1^{-1}(t)\dot{T}_1(t).$$

Finally according to proof of Proposition 6 we will find a Lyapunov transformation T_2 such that

$$T_2^{-1}(t) \begin{pmatrix} -1 & e^{\sin t} + 1 \\ 0 & -2 \end{pmatrix} T_2(t) - T_2^{-1}(t)\dot{T}_2(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

This transformation has the form $T_2(t) = \begin{pmatrix} 1 & \tau(t) \\ 0 & 1 \end{pmatrix}$, *where τ is the unique global and bounded solution of the equation*

$$\dot{\tau}(t) = \tau(t) + e^{\sin t} + 1.$$

Example 12 *Consider system (14) with*

$$A(n) = \begin{pmatrix} 1 & \frac{3}{2(1+\sin^2 n)} \\ \sin^2 n & 5 \end{pmatrix}, \qquad B(n) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $n \in \mathbb{Z}$. It is clear that sequence $(A(n))_{n\in\mathbb{Z}}$ is bounded. Calculating $\det A(n)$ *we can show that* $\det A(n) > 3$, *$n \in \mathbb{Z}$, therefore $A = (A(n))_{n\in\mathbb{Z}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{Z}, \mathbb{R}^{2\times 2})$. Using the definition of uniform complete controllability with $K = 2$ we can also verify that this system is uniformly complete controllable. Suppose that we want to find an admissible feedback $U = (U(n))_{n\in\mathbb{Z}}$ such that the closed-loop system is kinematically equivalent to diagonal time invariant system with coefficient matrix* $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$.

Following the proof of [3, Theorem 4.6], we find an admissible feedback U and the Lyapunov transformation $T_1 = (T_1(n))_{n\in\mathbb{Z}}$ such that the closed-loop system is kinematically equivalent to the upper triangular system with elements on the main diagonal equal to $\frac{1}{2}$ and $\frac{1}{3}$. The results are as follows $U(n) = (u_1(n)\ u_2(n))$, where for $n \in \mathbb{Z}$

$$\begin{aligned} u_1(n) &:= \tfrac{1}{6}\cos(2n+2) + \tfrac{4}{9}\cos 2n - \tfrac{5}{6}, \\ u_2(n) &:= -\tfrac{3\cos(2n+2) + 28\cos 2n - 93}{6\cos 2n - 18} \end{aligned}$$

and

$$T_1(n) = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6}\sin^2 n - \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \ n \in \mathbb{Z}.$$

With these U and T_1 we have

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{1+\sin^2 n} \\ 0 & \frac{1}{3} \end{pmatrix} = T_1^{-1}(n+1)\,(A(n) + B(n)U(n))\,T_1(n).$$

Finally according to proof of Proposition 6 we will find a Lyapunov transformation $T_2 = (T_2(n))_{n\in\mathbb{Z}}$ *such that*

$$T_2^{-1}(n+1)\begin{pmatrix} \frac{1}{2} & \frac{1}{1+\sin^2 n} \\ 0 & \frac{1}{3} \end{pmatrix} T_2(n) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \ n \in \mathbb{Z}.$$

This transformation has the form $T_2(n) = \begin{pmatrix} 1 & t(n) \\ 0 & 1 \end{pmatrix}$*, where* $t = (t(n))_{n\in\mathbb{Z}}$ *is the unique global and bounded solution of the equation*

$$t(n+1) = \frac{3}{2}t(n) + \frac{3}{1+\sin^2 n}, \ n \in \mathbb{Z}.$$

Conclusion

In this paper we consider linear systems with variable coefficients in both continuous and discrete time. In the continuous case we assume that the coefficients are bounded, and in the discrete case, additionally, that the coefficients at the state coordinate form a Lyapunov sequence. The main result says uniform complete controllability is a sufficient condition for the existence of a feedback loop guaranteeing the dynamic equivalence of a closed system with a predetermined stationary diagonal system with pairwise different elements on the main diagonal. The main achievement of this work in comparison with the results existing in the literature is that our approach guarantees that the transformation between the closed system and a given diagonal system is a Lyapunov transformation and thus maintains many properties of the dynamical system, including asymptotic stability.

It should be noted that our result is limited to assignment of spectra consisting only distinct eigenvalues and it remains open to know whether assignment of arbitrary spectra which possibly contain complex eigenvalues. Open questions are also how to construct explicitly linear state feedback for the assignment of arbitrary spectra and whether uniform complete controllability is a necessary condition for the considered type of spectrum assignability.

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