A monotonic optimization approach to mixed variational inequality problems

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Abstract

We study a nonconvex mixed variational inequality problem in $\mathbb{R}^{\boldsymbol{n}}$ by using monotonic optimization approach. We show that a class of variational inequality problems can be transformed into a monotonic optimization problem and then propose a branch-reduce-and-bound algorithm as a solution approach. The convergence result of the algorithm is established under monotonic assumptions on the cost and constraint functions. Applications to two equilibrium models are presented.

Keywords: Mixed variational inequality, monotonic optimization, increasing function

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1 Introduction

Throughout this paper, the inequalities between vectors are componentwise. More precisely, for any two vectors $x, y \in \mathbb{R}^n$, we write $x \leq y$ (resp., $x \geq y$) if $x_i \leq y_i$ (resp., $x_i \geq y_i$) for all $i = 1, \ldots, n$. If $x \leq y$, then the box $[x, y]$ is the set of all $z \in \mathbb{R}^n$ satisfying $x \leq z \leq y$. We denote $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \leq 0\}$. Let C be a nonempty closed subset of a given box $[u, v] \subset \mathbb{R}^n_+$, $F : [u, v] \to \mathbb{R}^n_+$, and $g : [u, v] \to \mathbb{R}$. We are concerned with the following mixed variational inequality problem:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle + g(y) - g(x^*) \geq 0 \ \forall y \in C. \quad \text{(MVI)}$$

This problem has various applications in electronics circuits and energy control problems (see [1, 2]). Additionally, some general economic and oligopolistic equilibrium problems can be transformed into (MVI). Note that $g(x)$ may be not convex when the cost function or the benefit function of these models is not convex (see [3–6]). When (MVI) is convex (i.e., when C is convex and $\langle F(x), y-x \rangle + g(y) - g(x)$ is convex with respect to y), many algorithms have been introduced for solving it. Among them, the projection, extragradient, and proximal point algorithms are widely used (see [7–15] and the references therein). However, to get convergent results, most of these algorithms require monotonicity and Lipschitz continuity of the cost mapping F. In case that $\langle F(x), \cdot - x \rangle + g(\cdot) - g(x)$ is quasiconvex, several algorithms for (MVI) can be found in [4, 16–19]. All of these iterative algorithms share a common drawback: at each iteration, they require solving non-convex subproblems that are global optimization problems and computationally expensive.

In case $F \equiv 0$, (MVI) becomes an optimization problem

$$\min\{g(x) : \ x \in C\}. \quad \text{(OP)}$$

The function $g : [u, v] \to \mathbb{R}$ is said to be increasing if $g(x) \leq g(y)$ whenever $u \leq x \leq y \leq v$. Monotonic functions abound in economics, engineering, communications, and information, see [20–25]. When $g(x)$ is increasing on $[u, v]$ and C is defined by

$$C := \{x \in [u, v] \mid r(x) \leq 0 \leq h(x)\}, \quad (1)$$

in which $r : [u, v] \to \mathbb{R}$ is a lower semicontinuous increasing function and $h : [u, v] \to \mathbb{R}$ is an upper semicontinuous increasing function, (OP) becomes a monotonic optimization as defined in [26]. This is a nonconvex optimization problem which can be solved by monotonic optimization algorithms such as polyblock algorithm, rectangular branch-and-bound, and branch-reduce-and-bound ([23, 26–28]). These are global optimization methods that are often computationally expensive. However, they can be applied to (OP) with very lenient assumptions that most iterative algorithms for quasiconvex or convex optimization problems cannot use.

Inspired and motivated by the pioneer research of Tuy [26], in this paper we use the monotonic optimization approach to study (MVI) where $F_i (i = 1, \ldots, n)$ and g are increasing functions on $[u, v]$, while C is defined by (1). Such (MVI) is called monotonic mixed variational inequality. It is worth noting that the monotonic (MVI) differs from the monotone one whose cost mapping F is monotone in the usual sense as Rockafellar [29], i.e., $\langle F(x) - F(y), x - y \rangle \geq 0$ for all $x, y \in C$. We show that this monotonic mixed variational inequality problem can be transformed into a monotonic optimization problem, then we develop a modified branch-reduce-and-bound algorithm for the monotonic (MVI). The convergence theorem of this algorithm is established. It is worth emphasizing that, with the above hypotheses for monotonic (MVI), most of the existing projection, extragradient, point proximal algorithms, and many iterative algorithms for (MVI) cannot be applied since they are designed for monotone F and convex C. We furthermore point out that the monotonic (MVI) appears naturally in some important practical problems including the oligopolistic Nash-Cournot equilibrium model and Bertrand one.

The organization of the paper is as follows. In Section 2 we recall some preliminaries in monotonic optimization theory, then show the equivalence between the monotonic variational inequality problem and a monotonic optimization problem. Section 3 is devoted to describing our algorithm for solving the monotonic (MVI) and proving its convergence results. The selected applications of the monotonic (MVI) are presented in the last section.

2 Monotonic optimization approach

We first recall some preliminaries in monotonic optimization.

Definition 1 (see e.g. [26]) *Given a box* $[a, b]$ *in* $\mathbb{R}^n$. *A subset* $G \subseteq [a, b]$ *is said to be*

(i) *normal if* $x \in G$ *whenever* $a \leq x \leq x'$ *for any* $x' \in G$,
(ii) *conormal if* $x \in G$ *whenever* $x' \leq x \leq b$ *for any* $x' \in G$.

Proposition 2 (Proposition 5, [26]) *Let* f *be an increasing function on* $\mathbb{R}^n_+$. *Then the following assertions hold.*

(i) *The set* $\{x \in \mathbb{R}^n_+ \mid f(x) \leq 1\}$ *is normal and it is closed if* f *is lower semicontinuous.*
(ii) *The set* $\{x \in \mathbb{R}^n_+ \mid f(x) \geq 1\}$ *is conormal and it is closed if* f *is upper semicontinuous.*

Proposition 3 (Proposition 1, [26])

(i) *If* f *and* g *are increasing functions, then so is the function* $\alpha f + \beta g$ *for any* $\alpha, \beta \geq 0$.
(ii) *The pointwise supremum of a bounded above family* $(f_\alpha)_{\alpha \in A}$ *of increasing functions is increasing.*
(iii) *The pointwise infimum of a bounded below family* $(f_\alpha)_{\alpha \in A}$ *of increasing functions is increasing.*

Let us now consider the problem (MVI). We need the following setup for the discussion in sequel.

$$V := \langle F(v), v\rangle + g(v) - \langle F(u), u\rangle - g(u),$$

$$\begin{aligned} a &:= (u,0), \quad b := (v,V), \\ \bar r(x,t) &:= \max\{r(x), t + \langle F(x), x\rangle + g(x) - V\}, \quad \bar h(x,t) := h(x), \\ \overline{R} &:= \{(x,t) \in [a,b] \mid \bar r(x,t) \le 0\}, \quad \overline{H} := \{(x,t) \in [a,b] \mid \bar h(x,t) \ge 0\}, \\ p(x) &:= \inf\{\langle F(x), y\rangle + g(y) \mid y \in C\}, \quad f(x,t) := p(x) + t. \end{aligned}$$

Lemma 4 *Suppose that F_i is increasing upper semicontinuous and non-negative on $[u,v]$ for every $i = 1,\dots,n$ and g is increasing upper semicontinuous on $[u,v]$. Then the following statements hold.*

(i) *$f(x,t)$ is an increasing upper semicontinuous function on $[a,b]$.*

(ii) *$\overline{R}$ is a normal set and $\overline{H}$ is a conormal set.*

Proof (i) Keeping in mind the non-negativity of $F_i (i = 1,\dots,n)$ on $[u,v] \subset \mathbb{R}^n_+$, since these functions and g are increasing on $[u,v]$, so is $\langle F(x), x\rangle + g(x)$ with respect to x. Hence $V = \langle F(v), v\rangle + g(v) - \langle F(u), u\rangle - g(u) \ge 0$ and $a = (u,0) \le b = (v,V)$.

Let $x^1, x^2 \in [u,v]$ with $x^1 \le x^2$. By non-negativity and monotonicity of F_i on $[u,v]$ (for $i = 1,\dots,n$), we have

$$\langle F(x^1), y\rangle \le \langle F(x^2), y\rangle \quad \forall y \in C \subset [u,v],$$

and consequently

$$p(x^1) = \inf\{\langle F(x^1), y\rangle + g(y) \mid y \in C\} \le \inf\{\langle F(x^2), y\rangle + g(y) \mid y \in C\} = p(x^2).$$

It means that $p(x)$ is an increasing function on $[u,v]$. As a consequence, the function $f(x,t) = p(x)+t$ is increasing on $[a,b]$. In the following we prove that $f(x,t)$ is upper semicontinuous on $[a,b]$.

Let $\ell_i(x,y) := F_i(x)y_i$ with $i = 1,\dots,n$, and let $(\bar x, \bar y)$ be an arbitrary point in $[u,v] \times [u,v]$. Let $\{x^k\}$ and $\{y^k\}$ be sequences in $[u,v]$ respectively converging to $\bar x$ and $\bar y$. By increasing monotonicity of F_i, we have $F_i(u) \le F_i(x^k) \le F_i(v)$ for all k. The boundedness of $\{F_i(x^k)\}$ and the convergence of $\{y_i^k\}$ to $\bar y_i$ give us $\limsup F_i(x^k)(y_i^k - \bar y_i) = 0$. In case $\bar y_i = 0$, we obtain

$$\limsup \ell_i(x^k, y^k) = \limsup F_i(x^k) y_i^k = 0 = F_i(\bar x)\bar y_i = \ell_i(\bar x, \bar y). \tag{2}$$

In case $\bar y_i > 0$, since $F_i(x^k)y_i^k = F_i(x^k)\bar y_i + F_i(x^k)(y_i^k - \bar y_i)$, we obtain

$$\begin{aligned} \limsup \ell_i(x^k, y^k) &= \limsup F_i(x^k) y_i^k \\ &\le \limsup F_i(x^k)\bar y_i + \limsup F_i(x^k)(y_i^k - \bar y_i) \\ &= \limsup F_i(x^k)\bar y_i \le F_i(\bar x)\bar y_i = \ell_i(\bar x, \bar y). \end{aligned} \tag{3}$$

The last inequality above is due to the upper semicontinuity of F_i. By (2) and (3) we have the upper semicontinuity of $\ell_i(x,y)$ at $(\bar x, \bar y)$. Consequently, $\langle F(x), y\rangle = \sum_{i=1}^n F_i(x)y_i = \sum_{i=1}^n \ell_i(x,y)$ is also upper semicontinuous at $(\bar x, \bar y)$. Together with the upper semicontinuity of g, it follows that $\langle F(x), y\rangle + g(y)$ is upper semicontinuous at $(\bar x, \bar y)$. Since $(\bar x, \bar y)$ is chosen arbitrarily in $[u,v] \times [u,v]$, we obtain the upper semicontinuity of $\langle F(x), y\rangle + g(y)$ on $\{\bar x\} \times [u,v]$.

We now show that $p(x)$ is upper semicontinuous at $\bar x$. We consider separately two cases where this infimum is finite or infinite at $\bar x$.

In case $p(\bar x) > -\infty$, let $\epsilon > 0$ and a sequence $\{x^j\}$ in $[u,v]$ converging to $\bar x$, then we can choose $y^* \in C$ such that

$$p(\bar x) + \epsilon = \inf\{\langle F(\bar x), y\rangle + g(y) \mid y \in [u,v]\} + \epsilon \ge \langle F(\bar x), y^*\rangle + g(y^*). \tag{4}$$

Taking an arbitrary sequence $\{y^j\}$ in C converging to y^*, we have

$$\begin{aligned}\limsup p(x^j) &= \limsup \inf\{\langle F(x^j), y\rangle + g(y) \mid y \in [u,v]\}\\ &\le \limsup \ \{\langle F(x^j), y^j\rangle + g(y^j) \mid j \in \mathbb{N}\}\\ &\le \langle F(\bar{x}), y^*\rangle + g(y^*). \end{aligned} \tag{5}$$

The last inequality above is due to the upper semicontinuity of $\langle F(x), y\rangle + g(y)$ at $(\bar{x}, y^*) \in \{\bar{x}\} \times [u, v]$. By (4) and (5) we get

$$\limsup p(x^j) \le p(\bar{x}) + \epsilon,$$

and since $\epsilon > 0$ is chosen arbitrarily, we obtain

$$\limsup p(x^j) \le p(\bar{x}).$$

This proves the upper semicontinuity of $p(x)$ at $\bar{x}$ in the considering case.

In case $p(\bar{x}) = -\infty$, by definition of $p(\bar{x})$ there exists a sequence $\{y^j\}$ in C such that

$$\lim_{j\to\infty} [\langle F(\bar{x}), y^j\rangle + g(y^j)] = -\infty. \tag{6}$$

Now, let $\{x^k\}$ be a sequence in $[u, v]$ converging to $\bar{x}$, and for each $j \in \mathbb{N}$ let y^{j^k} be a sequence in C converging to y^j. It follows from the upper semicontinuity of $\langle F(x), y\rangle + g(y)$ on $\{\bar{x}\} \times [u, v]$ that

$$\begin{aligned}\lim_{j\to\infty} [\langle F(\bar{x}), y^j\rangle + g(y^j)] &\ge \lim_{j\to\infty} \limsup_{k\to\infty} [\langle F(x^k), y^{j^k}\rangle + g(y^{j^k})]\\ &\ge \limsup_{k\to\infty} \inf\{\langle F(x^k), y\rangle + g(y) \mid y \in [u,v]\}\\ &\ge \limsup_{k\to\infty} p(x^k),\end{aligned}$$

which, together with (6), implies that $p(x)$ is upper semicontinuous at $\bar{x}$ as desired.

Since $p(x)$ is upper semicontinuous at $\bar{x}$ which is chosen arbitrarily in $[u, v]$, it is upper semicontinuous on $[u, v]$. Consequently, $f(x, t) = p(x) + t$ is upper semicontinuous on $[a, b] = [(u, 0), (v, V)]$.

(ii) Since $r(x)$ is increasing on $[u, v]$ and $t + \langle F(x), x\rangle + g(x) - V$ is increasing with respect to (x, t) on $[a, b]$, we have from Proposition 3 (ii) that $\bar{r}(x, t)$ is an increasing functions on $[a, b]$. By Proposition 2 (i), $\overline{R}$ is a normal set. Similarly, since $h(x)$ is increasing on $[u, v]$, so is $\bar{h}(x, t) = h(x)$ on $[a, b]$, and it follows from Proposition 2 (ii) that $\overline{H}$ is a conormal set. □

The next theorem shows a reduction of the monotonic (MVI) to the following monotonic optimization problem:

$$\max\{f(x,t) \mid (x,t) \in \overline{R} \cap \overline{H}\}. \tag{MOP}$$

This is a monotonic optimization problem of a form that is well-studied in the literature (see [26, 28]). It is known that (MOP) has a solution when $\overline{R} \cap \overline{H}$ is compact and $f(x, t)$ is upper semicontinuous on $\overline{R} \cap \overline{H}$.

Theorem 5 *Suppose that the following conditions are satisfied:*

(A1) *F_i is an increasing, upper semicontinuous, and non-negative function on $[u, v]$ for every $i = 1, \dots, n$;*

(A2) *g is an increasing, continuous, and non-negative function on $[u, v]$;*

(A3) *the solution set of* (MVI) *is nonempty.*

Then the following claims hold true.

(i) *If (x^*, t^*) is an optimal solution of* (MOP)*, then x^* is a solution of* (MVI) *and $t^* = V - \langle F(x^*), x^*\rangle - g(x^*)$.*

(ii) *Conversely, if x^* is a solution of* (MVI)*, then (x^*, t^*) is an optimal solution of* (MOP) *with $t^* = V - \langle F(x^*), x^*\rangle - g(x^*)$, and moreover, the optimal value of* (MOP) *is V.*

Proof Since $g(x)$ is continuous on $[u, v]$ and C is compact, on one hand we have

$$f(x, t) = \min\{\langle F(x), y\rangle + g(y) \mid y \in C\} + t,$$

and on the other hand,

$$\langle F(x), y\rangle + g(y) - (\langle F(x), x\rangle + g(x)) = \langle F(x), y - x\rangle + g(y) - g(x)$$

attains its minimum with respect to $y \in C$ for each fixed $x \in C$. Therefore, for each $x \in C$ we have

$$\begin{aligned} \varphi(x) &:= \min\{\langle F(x), y - x\rangle + g(y) - g(x) \mid y \in C\} \\ &\leq \langle F(x), x - x\rangle + g(x) - g(x) = 0. \end{aligned} \tag{7}$$

Now we prove (i). Let (x^*, t^*) be an optimal solution of (MOP). Then we have $(x^*, t^*) \in \overline{R} \cap \overline{H} \subset [a, b]$, so $x^* \in C \subset [u, v]$ and $t^* \in [0, V]$. Since $(x^*, t^*) \in \overline{R}$, we have $\bar{r}(x^*, t^*) \leq 0$, or equivalently,

$$r(x^*) \leq 0 \tag{8}$$

and

$$t^* + \langle F(x^*), x^*\rangle + g(x^*) - V \leq 0. \tag{9}$$

Let $\bar{t} := V - \langle F(x^*), x^*\rangle - g(x^*)$. By (9) we have $t^* - \bar{t} \leq 0$, so $\bar{t} \geq t^* \geq 0$. By non-negativity of $F(x^*)$, x^*, and $g(x^*)$, we have $\bar{t} \leq V$. Thus $(x^*, \bar{t}) \in [a, b]$. Furthermore, by the definition of $\bar{t}$ we have $\bar{t} + \langle F(x^*), x^*\rangle + g(x^*) - V = 0$, and together with (8) we obtain

$$\bar{r}(x^*, \bar{t}) = \max\{r(x^*), \bar{t} + \langle F(x^*), x^*\rangle + g(x^*) - V\} = 0.$$

In addition, since $(x^*, t^*) \in \overline{H}$, we have $0 \leq \bar{h}(x^*, t^*) = h(x^*) = \bar{h}(x^*, \bar{t})$. To this end, $(x^*, \bar{t})$ is a feasible solution of (MOP). Since $(x^*, t^*) \leq (x^*, \bar{t})$ and $f(x, t)$ is increasing, it follows that $f(x^*, t^*) \leq f(x^*, \bar{t})$. Keeping in mind the optimality of (x^*, t^*), we see that $(x^*, \bar{t})$ is also an optimal solution of (MOP), and since $f(x, t) = p(x) + t$ is separable with respect to t, we must have $\bar{t} = t^*$. So $t^* = V - \langle F(x^*), x^*\rangle - g(x^*)$. This implies that

$$\begin{aligned} &\max\{f(x, t) \mid (x, t) \in \overline{R} \cap \overline{H}\} \\ = \ &\max\{\min\{\langle F(x), y\rangle + g(y) \mid y \in C\} + t \mid x \in C, 0 \leq t \leq V - \langle F(x), x\rangle - g(x)\} \\ = \ &\max\{\min\{\langle F(x), y\rangle + g(y) \mid y \in C\} + t \mid x \in C, 0 \leq t = V - \langle F(x), x\rangle - g(x)\} \\ = \ &\max\{\min\{\langle F(x), y\rangle + g(y) \mid y \in C\} + V - \langle F(x), x\rangle - g(x) \mid x \in C\} \\ = \ &\max\{\min\{\langle F(x), y - x\rangle + g(y) - g(x) \mid y \in C\} \mid x \in C\} + V \\ = \ &\max\{\varphi(x) \mid x \in C\} + V. \end{aligned} \tag{10}$$

It follows that x^* is an optimal solution of the following optimization problem

$$\max\{\varphi(x) \mid x \in C\}. \tag{11}$$

Let $\bar{x}$ be a solution of (MVI), then we have

$$\langle F(\bar{x}), y - \bar{x}\rangle + g(y) - g(\bar{x}) \geq 0 \quad \forall y \in C,$$

which means that

$$\varphi(\bar{x}) = \min\{\langle F(\bar{x}), y - \bar{x}\rangle + g(y) - g(\bar{x}) \mid y \in C\} \geq 0.$$

This implies that the optimal value of (11), which equals $\varphi(x^*)$, is non-negative. Together with (7), it implies that $\varphi(x^*) = 0$, which means

$$\langle F(x^*), y - x^*\rangle + g(y) - g(x^*) \geq 0 \quad \forall y \in C.$$

Hence, x^* is a solution of (MVI).

We now focus on proving (ii). Let x^* be a solution of (MVI). Then, by similar arguments for $\bar{x}$ in the proof of (i), we have $\varphi(x^*) = 0$, and hence by (7), x^* is an optimal solution of problem (11). For every $(x, t) \in \overline{R} \cap \overline{H}$, we have

$$\begin{aligned}
f(x,t) &= \min\{\langle F(x), y\rangle + g(y) \mid y \in C\} + t \\
&= \min\{\langle F(x), y - x\rangle + g(y) - g(x) \mid y \in C\} + t + \langle F(x), x\rangle + g(x) \\
&= \varphi(x) + t + \langle F(x), x\rangle + g(x) \\
&\leq \varphi(x) + V \\
&\leq \varphi(x^*) + V \\
&= \min\{\langle F(x^*), y - x^*\rangle + g(y) - g(x^*) \mid y \in C\} + V \\
&= \min\{\langle F(x^*), y\rangle + g(y) \mid y \in C\} + V - \langle F(x^*), x^*\rangle - g(x^*) \\
&= \min\{\langle F(x^*), y\rangle + g(y) \mid y \in C\} + t^* \\
&= f(x^*, t^*),
\end{aligned}$$

where $t^* = V - \langle F(x^*), x^*\rangle - g(x^*)$. The first inequality above is due to the fact that $(x, t) \in \overline{R}$, and the second inequality above is due to the optimality of x^*. Therefore, (x^*, t^*) is an optimal solution of (MOP). From (10) and the fact that the optimal value of problem (11) is $\varphi(x^*) = 0$, it follows that the optimal value of (MOP) is V. □

Remark 6 (i) The non-negativity of the function g in condition (A2) of Theorem 5 is a technical condition and can be easily attained. In fact, since g is continuous on the compact box $[u, v]$, the value $\alpha = \min\{g(x) \mid x \in [u, v]\}$ exists finitely. Let $g'(x) = g(x) + |\alpha|$, then on one hand we have $g'(x) \geq 0$ for all $x \in [u, v]$, while on the other hand we have $g'(y) - g'(x) = g(y) - g(x)$ for any $x, y \in [u, v]$. The latter means that (MVI) is equivalent to the one in which g is replaced by g'.

(ii) Concerning condition (A3) in Theorem 5, it is known from [30] Theorem 2.3.4 that when C is nonempty compact convex, $\langle F(x), y - x\rangle + g(y) - g(x)$ is quasiconvex with respect to $x \in C$ and upper semicontinuous with respect to $y \in C$, then (MVI) has at least one solution. □

It is worth mentioning the popularity of the constraint set of form (1). Let us first consider a sub-level set D of a d.i. function (i.e. a function that can be represented as the difference of two increasing functions, see e.g. [28]), precisely,

$$D := \{x \in [u, v] \subset \mathbb{R}^n_+ \mid r(x) - h(x) \leq 0\}, \tag{12}$$

where r and h are increasing functions on $[u, v]$. By observing that the inequality $r(x) - h(x) \leq 0$ is equivalent to

$$t = h(v) - h(x) \quad \text{and} \quad r(x) + t - h(v) \leq 0 \leq h(x) + t - h(v),$$

we can convert problem (MVI) into the following problem: Find $(x^*, t^*) \in \overline{C}$ such that

$$\langle F(x^*, t^*), (x, t) - (x^*, t^*)\rangle + \bar{g}(x, t) - \bar{g}(x^*, t^*) \geq 0 \quad \forall (x, t) \in \overline{D},$$

where $\overline{D}$ has the form of (1):

$$\overline{D} := \{(x, t) \in [\bar{u}, \bar{v}] \subset \mathbb{R}^{n+1}_+ \mid r(x) + t - h(v) \leq 0 \leq h(x) + t - h(v)\},$$

with $\bar{u} = (u, 0), \bar{v} = (v, h(v) - h(u))$, $F(x) = (F_1(x), \ldots, F_n(x), 0)$, and $\bar{g}(x, t) = g(x)$. In turn, outstanding examples for d.i. functions are polynomials of the form $P(x) = \sum_\sigma c_\sigma x^\sigma$ with $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}^n$, $c_\sigma \in \mathbb{R}$, and $x^\sigma = x_1^{\sigma_1} \cdots x_n^{\sigma_n}$. To represent such $P(x)$ as a difference of two increasing functions, it suffices to group separately all terms with positive coefficients and all terms with negative coefficients. Another example is that any polyhedral convex set $\{x \in \mathbb{R}^n_+ \mid Ax \leq 0\}$, where $A = (a_{ij})$ is a matrix of order $m \times n$, can be expressed in the form of (12) as follows:

$$\{x \in \mathbb{R}^n_+ \mid Ax \leq 0\} = \{x \in [u, v] \subset \mathbb{R}^n_+ \mid A_1 x - A_2 x \leq 0\},$$

where $A_1 = (a^1_{ij})$ with $a^1_{ij} = \max\{a_{ij}, 0\}$ and $A_2 = (a^2_{ij})$ with $a^2_{ij} = \max\{0, -a_{ij}\}$ are matrices with non-negative elements.

3 Branch-reduce-and-bound algorithm

Thanks to Theorem 5, solving the (MVI) under our consideration consists in finding an optimal solution to (MOP), and it is known that the optimal value of (MOP) equals V. As a monotonic optimization problem, (MOP) can be solved by the branch-reduce-and-bound algorithm proposed in [28]. In this section, we present the algorithm adapted to the setting of our (MOP).

The algorithm starts searching in an initial box containing all feasible points of (MOP), or at least guaranteeing to contain an optimal solution of this problem. An obvious choice of such a box is $[a, b]$. The algorithm proceeds by successively partitioning the initial box according to a branch-reduce-and-bound scheme. In each iteration, for each partition set M, which is a box of form $[p, q] \subset [a, b]$, we compute an upper bound $\mu(M)$ for $f(z)$ with $z = (x, t) \in M$. We then remove a partition box if its corresponding bound is strictly less than the known optimal value V, so only partition sets M with $\mu(M) \geq V$ remain for consideration. We continue by selecting a partition set M with $\mu(M) \geq V$ and further partition it. Then a new collection of boxes is generated for exploration at the next iteration. The algorithm terminates when either no partition set remains for exploration (which means (MOP) is infeasible) or an optimal solution of (MOP) is found. Naturally, for each partition box M we want to have as tight bound as possible, so before computing $\mu(M)$ we should try to replace the box M by a smaller one $M' = [p', q'] \subset M$ without losing any optimal solution of (MOP) if exists in M. In the following we discuss the bounding, reduction, and branching operations in detail.

3.1 Bounding operation

Given a box $M = [p, q] \subset [a, b]$, we need to compute an upper bound $\mu(M)$ for the optimal value of the following sub-problem

$$\max\{f(z) \mid z \in \overline{R} \cap \overline{H} \cap M\}. \tag{13}$$

By Lemma 4, $\overline{R}$ is a normal set, $\overline{H}$ is a conormal set, and f is increasing. Therefore, the optimal value of (13) can be computed (approximately) by the polyblock outer approximation algorithm developed in [26], Section 5. This, however, is very costly to compute. One can simply take $\mu(M) = f(q)$ thanks to the fact that f is increasing on $[p, q]$.

3.2 Reduction

Let $[p, q]$ be a box for exploration. The reduction operation aims to replace the box $[p, q]$ with the smaller box $[p', q']$ without losing any feasible solution $z \in [p, q]$ of (MOP) satisfying $f(z) \geq V$. Since $\bar{r}$, $\bar{h}$, and f are increasing functions, it is easy to see that there exists such z only if $\bar{r}(p) \leq 0$, $\bar{h}(q) \geq 0$, and $f(q) \geq V$. In this case, one can have a reduction of $[p, q]$ as follows.

Proposition 7 *Assume that $\bar{r}(p) \leq 0$, $\bar{h}(q) \geq 0$, and $f(q) \geq V$. Let*

$$p' = q - \sum_{i=1}^{n+1} \alpha_i (q_i - p_i) e^i, \quad q' = p + \sum_{i=1}^{n+1} \beta_i (q_i - p_i) e^i,$$

where

$$\alpha_i = \sup\{\alpha \mid 0 \leq \alpha \leq 1,\ \ell(q - \alpha(q_i - p_i)e^i) \geq 0\} \ \forall i = 1, \ldots, n+1,$$
$$\beta_i = \sup\{\beta \mid 0 \leq \beta \leq 1,\ \bar{r}(p + \beta(q_i - p_i)e^i) \leq 0\} \ \forall i = 1, \ldots, n+1,$$

in which $\ell(z) := \max\{\bar{h}(z), f(z) - V\}$ and e^i is the i-th unit vector in $\mathbb{R}^{n+1}$. Then the box $[p', q'] \subseteq [p, q]$ still contains all feasible solutions of (MOP) *whose objective values are at least equal to V.*

Proof Since $p' = q - \sum_{i=1}^{n+1} \alpha_i(q_i - p_i)e^i$, for all $i = 1, \ldots, n+1$ we have $p'_i = \alpha_i p_i + (1-\alpha_i)q_i$, and since $0 \le \alpha_i \le 1$ it follows that $p'_i \in [p_i, q_i]$. So $p \le p' \le q$. Similarly, we can also prove that $p \le q' \le q$. Hence $[p', q'] \subseteq [p, q]$.

Let z be a feasible solution of (MOP) in $[p, q]$ satisfying $f(z) \ge V$. We first prove that $p' \le z$. Indeed, assume the contrary that $p' \not\le z$, then there exists an index $i \in \{1, \ldots, n+1\}$ such that $z_i < p'_i = q_i - \alpha_i(q_i - p_i)$. It follows that $z_i = q_i - \alpha(q_i - p_i)$ for some $\alpha > \alpha_i$. By virtue of the definition of α_i, we have

$$\ell(q - (q_i - z_i)e^i) = \ell(q - \alpha(q_i - p_i)e^i) < 0. \tag{14}$$

Note that $z \le q - (q_i - z_i)e^i$ and ℓ is an increasing function, we have

$$\ell(z) \le \ell(q - (q_i - z_i)e^i). \tag{15}$$

By (14) and (15) we obtain

$$f(z) - V \le \ell(z) < 0,$$

which contradicts our assumption that $f(z) \ge V$. Therefore, $p' \le z$.

It is left to show that $z \le q'$. Indeed, let i be an arbitrary index in $\{1, \ldots, n+1\}$ and $z^i := p + (z_i - p_i)e^i$. Then $z^i_i = z_i$ and $z^i_j = p_j$ for $j \ne i$, so $z^i \le z$. It follows that $\bar{r}(z^i) \le \bar{r}(z) \le 0$. Note that $z_i \le q_i$, so $z^i \le p + (q_i - p_i)e^i$, and hence $z^i = p + \beta(q_i - p_i)e^i$ for some $\beta \in [0, 1)$. Since $\bar{r}(z^i) \le 0$, we have $\beta \le \beta_i$, i.e. $z^i \le p + \beta_i(q_i - p_i)e^i$. So $z_i = z^i_i \le q'_i$ for all $i = 1, \ldots, n+1$. Hence $z \le q'$. □

The box $[p', q']$ obtained by Proposition 7 is said to be the reduction of the box $[p, q]$, denoted by red$[p, q]$. A similar reduction scheme is proposed in [28], which requires to know the best current objective value γ. When applying to our (MOP), it results in a reduction box $[\bar{p}, \bar{q}]$ with $\bar{q} = q'$ and

$$\bar{p} = q - \sum_{i=1}^{n} \alpha_i(q_i - p_i)e^i,$$

where for each $i = 1, \ldots, n+1$ the parameter α_i is determined by

$$\alpha_i = \sup\{\alpha \mid 0 \le \alpha \le 1, \bar{\ell}(q - \alpha(q_i - p_i)e^i) \ge 0\},$$

in which $\bar{\ell}(z) := \min\{\bar{h}(z), f(z) - \gamma\}$. Since it is already known that V is the optimal value of (MOP), we have $\gamma \le V$, and hence it is clear that $[p', q'] \subseteq [\bar{p}, \bar{q}]$. So the reduction of the box $[p, q]$ obtained by Proposition 7 is better than the one obtained by the reduction in [28].

3.3 Branching process

Starting from the initial box $[a, b]$, the standard branching operation successively bisect it into smaller and smaller boxes using the following simple subdivision rule. Let $M = [p, q]$ be a candidate for subdivision, $i_M = \arg\max_{i=1,\ldots,n+1}\{q_i - p_i\}$, and $w_{i_M} = (p_{i_M} + q_{i_M})/2$, then M is divided into two boxes $M_+ = \{z \in M \mid z_{i_M} \ge w_{i_M}\}$ and $M_- = \{z \in M \mid z_{i_M} \le w_{i_M}\}$.

As introduced in [28], a more efficient branching operation is an adaptive one which takes into account the information at the current step. More precisely, let $M = [p, q]$ be a candidate for further subdivision. In the context of our (MOP), let $z^* \in M$ be a point such that $\bar{h}(z^*) \geq 0$ and $f(z^*)$ is an upper bound of the subproblem

$$\max\{f(z) \mid z \in \overline{R} \cap \overline{H} \cap M\} = \max\{f(z) \mid \bar{r}(z) \leq 0, \bar{h}(z) \geq 0, z \in M\}.$$

Such z^* can be chosen as q due to the increasing property of f. If, furthermore, $\bar{r}(z^*) \leq 0$, then $f(z^*)$ is exactly the optimal value of the subproblem, and we do not need to subdivide this box M. Barring this case, we assume that $\bar{r}(z^*) > 0$. Let y^* be the intersection of the line segment from p to z^* with the surface $\bar{r}(z) = 0$, and then let w^* be the mid point of the line segment connecting y^* and z^*. We then divide the box M into two boxes $M_+ = \{z \in M \mid z_{i_M} \geq w^*_{i_M}\}$ and $M_- = \{z \in M \mid z_{i_M} \leq w^*_{i_M}\}$, in which $i_M = \arg\max_{i=1,\dots,n+1}\{|z^*_i - y^*_i|\}$.

3.4 Algorithm

From the above discussion we come up with the following algorithm to solve (MVI).

Algorithm 1 A branch-reduce-and-bound algorithm to solve (MVI)

Initialization. $M_1 := [a, b]$, $\Xi_1 := \{\text{red}M_1\}$. Determine a feasible solution $s^0 = (x^0, t^0)$ of (MOP). If $f(s^0) = V$, then terminate and x^0 is solution of (MVI). Otherwise, let $k := 1$ and go to **Step 1**.

Step 1. Delete from Ξ_k every box $[p, q]$ such that $\bar{r}(p) > 0$, or $\bar{h}(q) < 0$.

Step 2. For each box $M = [p, q] \in \Xi_k$:

(i) Compute an upper bound $\mu(M)$ for $\max\{f(s) \mid s \in \overline{R} \cap \overline{H} \cap M\}$ such that $\mu(M) \leq f(q)$;

(ii) If $\mu(M) < V$, then delete M from Ξ_k;

(iii) Otherwise, determine a feasible solution $s^M = (x^M, t^M)$ of (MOP) in M. If $f(s^M) = V$, then terminate and x^M is a solution of (MVI). If not, then go to **Step 3**.

Step 3. Let $s^k \in \arg\max\{f(s^M) \mid M \in \Xi_k\}$ and set current best value $CBV = f(s^k)$.

Step 4. Let $M_k \in \arg\max\{\mu(M) \mid M \in \Xi_k\}$. Divide M_k into two boxes M_k^+ and M_k^- according to either the standard subdivision rule or the adaptive one. Let $\Xi_{k+1} := (\Xi_k \backslash \{M_k\}) \cup \{\text{red}M_k^+, \text{red}M_k^-\}$.

Step 5. Set $k := k + 1$ and go to Step 1.

Theorem 8 *Suppose that F_i is increasing, upper semicontinuous, and non-negative on $[u, v]$ for each $i = 1, \dots, n$, g is increasing and continuous on $[u, v]$, and the solution set of* (MVI) *is nonempty. If Algorithm 1 is infinite, then every cluster point $\bar{s} = (\bar{x}, \bar{t})$ of the sequence $\{s^k = (x^k, t^k)\}$ generated in this algorithm belongs to the optimization solution set of* (MOP) *and $\bar{x}$ is a solution of* (MVI).

Proof Assume that Algorithm 1 is infinite. Both standard and adaptive subdivision rules imply that every subsequence of boxes $M_{k_i} := [p_{k_i}, q_{k_i}]$ generated by the algorithm must shrink to a point $\bar{s} = (\bar{x}, \bar{t}) = \lim_{i\to\infty} q_{k_i} = \lim_{i\to\infty} p_{k_i}$. By the deletion in Step 1, we have that $\bar{r}(p_{k_i}) \le 0$ and $\bar{h}(q_{k_i}) \ge 0$ for all $i \in \mathbb{N}$, which, together with the lower semicontinuity of $\bar{r}$ and the upper semicontinuity of $\bar{h}$, implies that $\bar{r}(\bar{s}) \le 0$ and $\bar{h}(\bar{s}) \ge 0$. It follows that $\bar{s} \in \overline{R} \cap \overline{H}$, and so $\bar{x} \in C$. On the other hand, by the selection in Step 4 and the bounding in Step 2(ii) we have

$$f(q_{k_i}) \ge \mu(M_{k_i}) \ge \max\{\mu(M) \mid M \in \Xi_k\} \ge f(x,t) \quad \forall (x,t) \in \overline{R} \cap \overline{H},$$

which, together with the upper semicontinuity of f, implies that

$$f(\bar{s}) \ge f(x,t) \quad \forall (x,t) \in \overline{R} \cap \overline{H}.$$

Hence, $\bar{s}$ is an optimal solution of (MOP). By Theorem 5, $\overline{x}$ is a solution of (MVI). □

Remark 9 Our algorithm follows the general scheme of Algorithm 2 in [28], with some modifications based on the fact that for (MOP) we known V as the optimal value. It is worth emphasizing the importance of this fact in increasing the efficiency of our algorithm in comparison with the general one. Indeed, the known optimal value V helps us to reduce the number of boxes to be subdivided for further investigation. The general scheme in [28] uses the condition $\mu(M) < CBV$ to remove a box from consideration, and since $CBV \le V$, Step 2(ii) in our algorithm removes more boxes than using the former condition. Furthermore, the general scheme in [28] only terminates when $\Xi_k = \emptyset$, which is hard to attain. In contrast, our algorithm uses the condition $CBV = V$, which is easy to check, as a stopping condition. □

A computational issue in our algorithm is to calculate the value of f at $z = (x,t) \in [a,b]$. For that we need to solve the following optimization problem

$$\min\{\langle F(x), y\rangle + g(y) \mid y \in C\}. \tag{16}$$

By the assumptions (A1), (A2) in Theorem 5 and Lemma 4, this is a monotonic optimization problem which can be solved by using algorithms proposed in [26–28]. If C is convex and the function g is concave on C, then (16) is a concave programming problem and can be solved by algorithms proposed in [31]. An interesting case study is when (16) admits separability property in addition. More precisely, in this case, $h(x) \equiv 0$, g is a separable function, i.e., $g(x) = \sum_{i=1}^{n} g_i(x_i)$ with $g_i (i = 1,\dots,n)$ are concave functions on $[u_i, v_i]$, and $r(x) = \max\{r_j(x) \mid k = 1,\dots,m\}$ where $r_j(x) = \sum_{i=1}^{n} r_{ij}(x_i)$ with $r_{ij}(x_i)$ being a finite convex function on $[u_i, v_i]$ for all $j = 1,\dots,m$ and $i = 1,\dots,n$. Then we have (16) as a separable concave programming problem and can be solved on a large scale (see [32–34]). Furthermore, if $r_j(x), i = 1,\dots,m$ are affine functions on $[u,v]$, then (16) can be solved by the finite algorithm proposed in [35]. We note that in some well-known equilibrium models, the separability of functions g, r, and h often appears naturally (see [6, 14, 36]).

4 Applications

In this section we present two practical models that lead to the monotonic mixed variational inequality, including a Nash-Cournot equilibrium model and Bertrand one.

4.1 Nash-Cournot equilibrium model

Let us consider the well-known oligopolistic Nash-Cournot equilibrium model (see e.g. [6, 17, 37]) where it is supposed that n firms are producing a homogeneous commodity. For each $j = 1, \dots, n$ let x_j be the amount of commodity to be produced by firm j, $c_j(x_j)$ the cost of input resource of firm j for producing the quantity x_j of commodity. Generally, in economics, the cost functions are non-negative and increasing. Let $x = (x_1, \dots, x_n)^t$ be the vector of production levels of the firms, then the total quantity of the commodity produced by all firms is $\sigma_x = \sum_{i=1}^n x_i$. Let $p_j(\sigma_x)$ be the price per commodity unit of the firm j, then the profit function of this firm is

$$f_j(x) = x_j p_j(\sigma_x) - c_j(x_j).$$

For each firm $j = 1, \dots, n$, its producing strategy x_j belongs to a given strategy set of form $[u_j, v_j] \subset \mathbb{R}_+$. In addition, it is assumed subject to common constraints with other firms in form $r(x) \le 0 \le h(x)$. Let C be the strategy set of all firms, i.e., $C = \{x \in [u_1, v_1] \times \ldots \times [u_n, v_n] \mid r(x) \le 0 \le h(x)\}$. Each firm aims to achieve maximum its own profit by choosing the corresponding production level under the preassumption that the production levels of the other firms are parametric input. A Nash equilibrium refers to a production pattern where no firm can increase its profit by altering its current production level. Under this equilibrium concept, each firm determines its best response given other firms' actions. Mathematically, a point $x^* = (x_1^*, \dots, x_n^*)^t \in C$ is said to be a Nash-equilibrium point if

$$f_j(x^*) \ge f_j(x^*[y_j]) \quad \forall y_j \in C_j, \forall j = 1, \dots, n,$$

where $x^*[y_j]$ stands for the vector obtained from x^* by replacing the entry x_j^* by y_j and

$$C_j = \{x_j \in [u_j, v_j] \mid \exists x_i \in [u_i, v_i] \ \forall i \ne j \text{ such that } x \in C\}.$$

By setting $\phi(x, y) := -\sum_{j=1}^n f_j(x[y_j])$ and $\psi(x, y) := \phi(x, y) - \phi(x, x)$, the problem of finding such a Nash equilibrium point in this model is equivalent to

$$\text{Find } x^* \in C \text{ such that } \psi(x^*, y) \ge 0 \text{ for all } y \in C. \tag{17}$$

Lemma 10 *Let $p_j(\sigma_x) = \alpha_j - \beta_j \sigma_x$ with $\alpha_j > 0$, $\beta_j > 0$ for all $j = 1, \dots, n$. Assume that $r(x)$ is lower semicontinuous increasing on $[u, v]$, $h(x)$ is upper semicontinuous* *increasing on $[u, v]$, and for each $j = 1, \dots, n$ the cost function $c_j(x_j)$ is continuous increasing function on $[u_j, v_j]$. If (17) is feasible, then it can be transformed into* (MOP)*.*

Proof By simple computations we have

$$\begin{aligned}
\psi(x,y) &= \sum_{j=1}^{n} \left[f_j(x) - f_j(x[y_j]) \right] \\
&= \sum_{j=1}^{n} \left[x_j p_j(\sigma_x) - c_j(x_j) - y_j p_j(\sigma_x - x_j + y_j) + c_j(y_j) \right] \\
&= \sum_{j=1}^{n} \left[x_j(\alpha_j - \beta_j \sigma_x) - y_j(\alpha_j - \beta_j(\sigma_x - x_j + y_j)) \right] + \sum_{j=1}^{n} c_j(x_j) - \sum_{j=1}^{n} c_j(y_j) \\
&= \sum_{j=1}^{n} y_j \beta_j (x_1 + \ldots + x_n - x_j + y_j) - \sum_{j=1}^{n} x_j \beta_j (x_1 + \ldots + x_n) \\
&\quad + \sum_{j=1}^{n} \alpha_j (x_j - y_j) + \sum_{j=1}^{n} c_j(x_j) - \sum_{j=1}^{n} c_j(y_j) \\
&= \sum_{j=1}^{n} \left[\beta_j \sum_{i \neq j} x_i \right] (y_j - x_j) - \sum_{j=1}^{n} \alpha_j (y_j - x_j) \\
&\quad + \sum_{j=1}^{n} \beta_j y_j^2 - \sum_{j=1}^{n} \beta_j x_j^2 + \sum_{j=1}^{n} c_j(x_j) - \sum_{j=1}^{n} c_j(y_j) \\
&= \langle Px, y - x \rangle + \langle Qy - \alpha, y \rangle + \sum_{j=1}^{n} c_j(y_j) - \langle Qx - \alpha, x \rangle - \sum_{j=1}^{n} c_j(x_j),
\end{aligned}$$

in which

$$P = \begin{bmatrix} 0 & \beta_1 & \beta_1 & \ldots & \beta_1 \\ \beta_2 & 0 & \beta_2 & \ldots & \beta_2 \\ \beta_3 & \beta_3 & 0 & \ldots & \beta_3 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \beta_n & \beta_n & \beta_n & \ldots & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \beta_1 & 0 & 0 & \ldots & 0 \\ 0 & \beta_2 & 0 & \ldots & 0 \\ 0 & 0 & \beta_3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & \beta_n \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdot \\ \alpha_n \end{bmatrix}. \tag{18}$$

Hence, (17) can be equivalently restated as the following mixed variational inequality problem

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle + g(x) - g(x^*) \geq 0 \text{ for all } x \in C, \tag{19}$$

where $F(x) = Px$ and $g(x) = \langle Qx - \alpha, x \rangle + \sum_{j=1}^{n} c_j(x_j)$. As P is defined in (18), the component functions $F_i(i = 1, \dots, n)$ are non-negative, continuous, and increasing on $[u, v]$. Let $g_{min} := \min_{x \in [u,v]} g(x)$ and then re-assign $g(x) := g(x) + \xi$ with any $\xi \geq |g_{min}|$. Then, on one hand, due to the definition of Q in (18) and the continuity, increasing property of $c_j(j = 1, \dots, n)$, we have g is non-negative, continuous, and increasing on $[u, v]$. On the other hand, the re-assignment of g does not change the solution set of (19). Hence, by Theorem 5, problem (19) can be transformed equivalently into a monotonic optimization problem of form (MOP). □

Note that g may be not convex when c_j is not convex for some $j = 1, \ldots, n$. In many cases we can further assume that the functions r and h are separable. For example, if it is stipulated that the total quantity of the commodity produced by the firms must be at least m and cannot exceed M, then we have the common constraints with other firms: $r(x) = x_1 + x_2 + \ldots + x_n - M \leq 0$ and $h(x) = x_1 + x_2 + \ldots + x_n - m \geq 0$. Then, the monotonic optimization problem (16) is separable and can be effectively solved by using the branch-bound algorithm given in [26].

4.2 Bertrand equilibrium model

The Bertrand oligopoly model is a different approach to the Cournot model described in Section 4.1. In the Bertrand model, firms produce a common commodity, but instead of setting a production quantity, each firm sets a price. The demand in this model is determined by the price, and the customers tend to buy from firms with lower prices. However, this assumption may not always be applied, as the products of firms are often not entirely interchangeable. Some consumers may prefer one product over the other, even if it costs a bit more. Suppose that each firm's level of production x_j is influenced by the price p and can be expressed as

$$x_j(p) = \gamma_j - \sigma_j p_j + \sum_{j \neq i}^{n} \lambda_{ij} p_j \quad \forall j = 1, \ldots, n.$$

Here, the parameters γ_j and σ_j are both positive, while λ_{ij} is positive when $j \neq i$. The positive value of σ_j implies that the demand for firm j decreases as the price of its products increases, whereas the positive value of λ_{ij} means that the demand for firm j increases when other firms increase their prices. The profit function for firm j can then be expressed as $f_j(p) := p_j x_j - \varphi_j(x_j)$, where $\varphi_j(x_j)$ is the cost of input resource of firm j for producing the quantity x_j of commodity.

Suppose that for each firm $j = 1, \ldots, n$ its strategy p_j belongs to a fixed interval $[u_j, v_j]$. In addition, suppose that the strategy of each firm depends on the others in such a way that the strategy vector $p = (p_1, \ldots, p_n)$ of all firms is subjected to constraints $r(p) \leq 0 \leq h(p)$ for certain functions r, h. Let P be the set of feasible strategies p of all firms. Each firm j attempts to maximize its profit by choosing a corresponding price level on its strategy set C_j by solving the optimization problem

$$f_j(p) = \max_{y_j \in C_j} f_j(p[y_j]) \quad \forall j = 1, \ldots, n,$$

where $p[y_j]$ is the vector obtained from p by replacing p_j with y_j.

Following [3], we assume that $\varphi_j(x_j) = m_j x_j - d_j x_j^2$ with $d_j \geq 0$, so the cost is a concave function of the production level. Then we have following lemma.

Lemma 11 *Assume that $r(x)$ is lower semicontinuous increasing on $[u, v]$, $h(x)$ is upper semicontinuous increasing on $[u, v]$, $1 - 2d_j\sigma_j \geq 0$ and $2\sigma_j(d_j\sigma_j - 1)v_j + \sigma_j m_j + \gamma_j(1 - 2d_j\sigma_j) > 0$ for all $j = 1, \ldots, n$. If the Bertrand equilibrium model has a solution, then it can be transformed into a monotonic optimization problem.*

Proof From $x_j(p) = \gamma_j - \sigma_j p_j + \sum_{j\neq i}^n \lambda_{ij} p_j$ and $\varphi_j(x_j) = m_j x_j - d_j x_j^2$, we obtain

$$\varphi_j(p) = - d_j\sigma_j^2 p_j^2 + \sigma_j \left[2d_j\left(\gamma_j + \sum_{i\neq j}^n \lambda_{ji}p_i\right) - m_j\right] p_j + m_j\left(\gamma_j + \sum_{i\neq j}^n \lambda_{ji}p_i\right) - d_j\left(\gamma_j + \sum_{i\neq j}^n \lambda_{ji}p_i\right)^2.$$

Hence the profit function f_j can be expressed in the following form:

$$f_j(p) = \sigma_j(d_j\sigma_j - 1)p_j^2 + \left[\sigma_j v_j + \left(\gamma_j + \sum_{i\neq j}^n \lambda_{ji}p_i\right)(1 - 2d_j\sigma_j)\right] p_j + d_j\left(\gamma_j + \sum_{i\neq j}^n \lambda_{ji}p_i\right)^2 - m_j\left(\gamma_j + \sum_{i\neq j}^n \lambda_{ji}p_i\right).$$

By the same technique as in the Nash-Cournot model above, the problem of finding a equilibrium point of this Bertrand model can be formulated as the following mixed variational inequality problem of type (MVI) (see [6]):

$$\text{Find } p \in C \text{ such that } \langle G(p), x - p\rangle + g(x) - g(p) \geq 0 \text{ for all } x \in C, \tag{20}$$

where

$$g(x) = \sum_{j=1}^n [\sigma_j(d_j\sigma_j - 1)x_j^2 + [\sigma_j m_j + \gamma_j(1 - 2d_j\sigma_j)]x_j].$$

and $G(x) = Px$ in which $P = (p_{ij})_{n\times n}$ with $p_{ii} = 0$ and $p_{ij} = \lambda_{ij}(1 - 2d_i\sigma_i)$ for all $i, j \in \{1, \ldots, n\}$ with $j \neq i$. For each $j = 1, \ldots, n$, $G_j(x)$ is non-negative continuous increasing on $[u, v]$ since $1 - 2d_j\sigma_j \geq 0$. Let

$$g_{min} := \min\left\{\sum_{j=1}^n \left[\sigma_j(d_j\sigma_j - 1)x_j^2 + [\sigma_j m_j + \gamma_j(1 - 2d_j\sigma_j)]x_j\right] \mid x \in [u, v]\right\}$$

and re-assign $g(x) := g(x) + \xi$ with any $\xi \geq |g_{min}|$. Note that we have $2\sigma_j(d_j\sigma_j - 1)v_j + \sigma_j m_j + \gamma_j(1 - 2d_j\sigma_j) > 0$ by assumption, hence $g(x)$ is non-negative continuous increasing on $[u, v]$. By the Theorem 5, (20) can be transformed equivalently into a monotonic optimization problem. □

It is worth noting that, by the construction in the above proof, G does not satisfy any generalized monotonicity and g is not convex because $d_j\sigma_j - 1 < 0$ for all $j = 1, \ldots, n$.

5 Conclusion

By using monotonic optimization approach, we shown that a class of monotonic mixed variational inequality problems in $\mathbb{R}^n$ can be transformed equivalently into a monotonic optimization problem, for which we proposed a modified branch-reduce-and-bound algorithm to solve and proved the convergence results under very lenient assumptions. Our algorithm is a global optimization method which is often computationally expensive. However, it can be applied to solve (MVI) with very lenient assumptions that most iterative algorithms for convex mixed variational inequality problems cannot use. As an application, we have shown that the solutions of the oligopolistic Nash-Cournot equilibrium model and the Bertrand equilibrium model with very general hypotheses can be solved by the proposed algorithm. We believe that the new results in this paper can provide new light on the implementation of algorithms for solving a class of discrete variational inequality problems that have practical applications, for example, a traffic equilibrium problem (see [38]).

Conflict of interest

No potential conflict of interest was reported by the authors.

Data availability

Data sharing was not applicable to this article as no datasets were generated or analyzed during the current study.

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