

# The oscillatory solutions of multi-order fractional differential equations

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## Abstract

This paper systematically treats the asymptotic behavior of many (linear/nonlinear) classes of higher-order fractional differential equations with multiple terms. To do this, we utilize the characteristics of Caputo fractional differentiable functions, the comparison principle, counterfactual reasoning, and the spectral analysis method (concerning the integral presentations of basic solutions). Some numerical examples are also provided to demonstrate the validity of the proposed results.

**Key works:** Fractional differential equations, multi-order fractional differential equations with Emden–Fowler-type coefficients, oscillatory solutions, non-oscillatory solution, characteristic polynomials, comparison principle, asymptotic behavior of solutions

**AMS subject classifications:** 26A33, 34A08, 34C10, 34D05, 45A05, 45G05

## 1 Introduction

J.C.F. Sturm initiated oscillation theory in his investigations of Sturm–Louville problems in 1836. Due to its great importance in describing many real applications, many papers dealing with non-autonomous ordinary differential equations have appeared, in which various classifications of equations according to the oscillatory properties of their solutions are proposed. Furthermore, the existence or absence of singular, proper, oscillatory, and monotone solutions of various types are shown and the asymptotic properties of such solutions are considered. We refer in particular to the survey paper by Wong [12], to the monographs by Elias [6], Kiguradze and Chanturiya [9], Swanson [11] and the references therein for some representative contributions related to this topic.

Since the Leibnitz rule for derivatives of composite functions is not valid for fractional derivatives, many approaches and methods in ordinary differential equations are often not applicable to fractional differential equations concerning the study of the oscillatory properties of solutions. Hence, only a few studies on this theory have been published, and the development is still in its infancy and requires further investigation. From a personal perspective, we list some typical works below.

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32 Grace [7] reported the first results on oscillatory solutions of fractional differential equations. In  
 33 particular, in this paper, using a counterfactual argument when dealing with the definition of  
 34 non-oscillation, the author obtained some simple sufficient conditions ensuring the existence of  
 35 oscillatory solutions for some basic classes of equations with Riemann–Liouville derivatives of the  
 36 order less than 1. After that in [7], he showed the oscillatory behavior of solutions to nonlinear  
 37 fractional differential equations with the Caputo fractional derivative of the order  $\alpha \in (1, 2)$ .  
 38 Băleanu et al. [1] investigated the eventual sign-changing for the solutions of a linear  $(1 + \alpha)$ -  
 39 order fractional differential equation ( $\alpha \in (0, 1)$ ) by presenting a Kamenev-type theorem in the  
 40 framework of fractional calculus. A survey on one of the mathematical approaches used to solve  
 41 a fractional differential equation whose solution gives the free dynamic response of viscoelastic  
 42 single degree of freedom systems (viscosity is modeled by a fractional displacement derivative  
 43 instead of first-order one) is introduced in [4]. The paper dealt with the Caputo fractional  
 44 derivative and its Laplace transform (on which the resolution method is based).

45 Recently in [2], the authors considered higher-order fractional differential equations with the  
 46 Riemann–Liouville fractional derivatives and Emden–Fowler-type coefficients. They explored  
 47 the effect of different orders of derivatives on the oscillatory and asymptotic properties of the  
 48 studied equations. Moreover, the dissimilarities between integer and non-integer order cases are  
 49 emphasized.

50 Inspired by the works of Grace, Bartušek, and Došlá mentioned above, we focus on multi-term  
 51 fractional differential equations with the Caputo fractional derivatives in the present paper. In  
 52 light of the characteristics of Caputo fractional differentiable functions, the comparison principle,  
 53 counterfactual reasoning, and the spectral analysis method (concerning the integral presentations  
 54 of basic solutions), we systematically treat the asymptotic behavior of many (linear/nonlinear)  
 55 classes of higher-order fractional differential equations with multiple terms. More precisely, in  
 56 Section 3, we discuss the existence of (linear/nonlinear) fractional differential equations. The  
 57 asymptotic behavior of oscillatory solutions is described in Section 4. The asymptotic behavior  
 58 of non-oscillatory solutions of higher-order fractional differential equations with the Emden–  
 59 Fowler-type coefficients is given in Section 5. We also provide numerical examples to illustrate  
 60 the obtained theoretical findings.

## 61 2 Preliminaries

Let  $0 < \alpha \in \mathbb{R}$  with  $[\alpha] = n$  and  $J = [0, T]$  or  $J = [0, \infty)$  we defined Riemann-Liouville  
 fractional integral of a function  $f : J \rightarrow \mathbb{R}$  as below:

$$I_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t \in J,$$

its Riemann-Liouville fractional derivative order  $\alpha$  of  $f$  as

$${}^{RL}D_{0+}^{\alpha} f(t) = D^{(n)} I_{0+}^{n-\alpha} f(t), \quad t \in J \setminus \{0\},$$

and its Caputo fractional derivative order  $\alpha$  of function  $f$  as below:

$${}^C D_{0+}^{\alpha} f(t) := {}^{RL}D_{0+}^{\alpha} (f(t) - T_{n-1}[f; 0](t)), \quad t \in J \setminus \{0\},$$

62 where  $T_{n-1}[f; 0](t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k$  denotes the Taylor polynomial of order  $n - 1$  of  $f$  centered  
 63 at 0 and  $D^{(n)}$  is usually derivative. See [5, Chapter III] and [13] for more details on the Caputo  
 64 fractional derivative.

**Lemma 2.1.** Let  $\alpha, \beta > 0$  and  $x \in L^1[0, T]$ . Then, we have

$$I_{0+}^{\alpha} \left( I_{0+}^{\beta} x(t) \right) = I_{0+}^{\beta} \left( I_{0+}^{\alpha} x(t) \right) = I_{0+}^{\alpha+\beta} x(t), \quad \forall t \in [0, T].$$

65 *Proof.* See [5, Theorem 2.2 and Corollary 2.3 page 14] □

**Lemma 2.2.** Let  $f \in AC[0, T], \alpha \in (0, 1)$ . Then,

$${}^C D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \quad \forall t \in (0, T).$$

66 *Proof.* See [5, Lemma 2.12, p. 27 and Lemma 3.4, p. 53]. □

**Lemma 2.3.** If  $f$  is a continuous and  $\alpha \in (0, 1)$ , then

$${}^C D_{0+}^{\alpha} I_{0+}^{\alpha} f = f.$$

67 *Proof.* See [5, Theorem 3.7, p. 53]. □

68 **Definition 2.4.** [3, Definition 2.2] Let  $0 < \alpha \in \mathbb{R}$  with  $[\alpha] = n$  and  $f \in C^{n-1}[0, T]$ . If  ${}^C D_{0+}^{\alpha} f$   
 69 exists and is in the class  $L_1[0, T]$ , then we say that the function  $f$  is Caputo  $\alpha$ -differentiable on  
 70  $[0, T]$ . If  ${}^C D_{0+}^{\alpha} f$  exists and is in the class  $C[0, T]$ , then we say that the function  $f$  is continuously  
 71 Caputo  $\alpha$ -differentiable on  $[0, T]$ .

**Lemma 2.5.** Let  $\alpha > 0$ . Suppose that  $x$  is continuously Caputo  $\alpha$ -differentiable on  $[0, \infty)$ , then

$$I_{0+}^{\alpha} \left( {}^C D_{0+}^{\alpha} x(t) \right) = x(t) - \sum_{k=0}^{[\alpha]-1} \frac{x^{(k)}(0)}{k!} t^k, \quad \forall t \in [0, \infty).$$

72 *Proof.* See [13, Proposition 5.1]. □

**Lemma 2.6.** Let  $n-1 < \alpha < n, n \in \mathbb{N}$  and assume that  $x : [0, \infty) \rightarrow \mathbb{R}$  is continuously Caputo  
 $\alpha$ -differentiable on  $[0, \infty)$ . Then, for any  $t \in (0, \infty)$ , we have

$${}^C D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{x^{(n-1)}(t) - x^{(n-1)}(0)}{t^{\alpha-n+1}} \right) + (\alpha - n + 1) \int_0^t \frac{x^{(n-1)}(t) - x^{(n-1)}(s)}{(t-s)^{\alpha-n+2}} ds.$$

73 *Proof.* See [13, Theorem 5.2]. □

74 **Lemma 2.7.** [7, Lemma 2.1] For  $X \geq 0$  and  $Y > 0$ , we have

75 (i)  $\lambda XY^{\lambda-1} - X^{\lambda} \leq (\lambda - 1)Y^{\lambda}, \lambda > 1.$

76 (ii)  $\lambda XY^{\lambda-1} - X^{\lambda} \geq (\lambda - 1)Y^{\lambda}, 0 < \lambda < 1.$

77 **Definition 2.8.** Let  $x : [0, \infty) \rightarrow \mathbb{R}$ . It is said to be eventually (non) negative, positive if there  
 78 is a  $T > 0$  such that  $x(t) < 0$  ( $> 0$ ),  $x(t) > 0$  for all  $t \geq T$ , respectively.

Consider the multi-term fractional differential equation

$$\sum_{i=1}^{k-1} a_i {}^C D_{0+}^{\alpha_i} x(t) + {}^C D_{0+}^{\alpha_k} x(t) = f(t, x), \quad t > 0 \quad (1)$$

with the initial conditions

$$x(0) = x_0, x^{(j)}(0) = x_j, j = 1, 2, \dots, k-1, \quad (2)$$

79 where  $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \dots < \alpha_k \leq k$ ,  $a_i, i = 1, 2, \dots, k$  are constants, and  $f : [0, \infty) \times \mathbb{R} \rightarrow$   
80  $\mathbb{R}$  is continuous.

81 **Definition 2.9.** A function  $x : [0, \infty) \rightarrow \mathbb{R}$  is called a solution of the system (1)–(2) if it  
82 is continuously Caputo  $\alpha$ -differentiable on  $[0, \infty)$  and satisfies (1) on  $(0, \infty)$  and the initial  
83 conditions (2).

84 **Definition 2.10.** A solution  $x$  of the system (1)–(2) is oscillatory if it exists globally on  $[0, \infty)$   
85 and there is a sequence  $\{t_n\}_{n=1}^{\infty} \subset [0, \infty)$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $x(t_n) = 0$ ,  $n \in \mathbb{N}$ .  
86 Otherwise, it is called non-oscillatory.

**Theorem 2.11.** Consider the system (1)–(2). Suppose that  $f(\cdot, \cdot)$  is Lipchitz continuous to the  
second variable, that is, there is a continuous function  $L : [0, \infty) \rightarrow [0, \infty)$  such that

$$|f(t, x) - f(t, \hat{x})| \leq L(t)|x - \hat{x}|, \quad \forall t \in [0, \infty), x, \hat{x} \in \mathbb{R}.$$

87 Then, for any  $x_j \in \mathbb{R}$ ,  $j = 0, 1, \dots, k-1$ , the system (1)–(2) has a unique solution on  $[0, \infty)$ .

88 *Proof.* Using the same arguments as in the proof of [3, Theorem 4.5]. □

## 89 3 On the existence of the oscillatory solutions of fractional dif- 90 ferential equations

### 91 3.1 The oscillation of multi-order fractional differential equations

We first focus on the following fractional differential equation

$${}^C D_{0+}^{\alpha} x(t) + a {}^C D_{0+}^{\beta} x(t) = f(t, x(t)) + g(t), \quad t > 0 \quad (3)$$

with the initial conditions

$$x(0) = x_0, x^{(1)}(0) = x_1, \dots, x^{(n-1)}(0) = x_{n-1}, \quad (4)$$

92 where  $0 < \beta < \alpha \leq [\alpha] = n \in \mathbb{N}$ ,  $x_0, x_1, \dots, x_{n-1} \in \mathbb{R}$ ,  $a$  is a non-negative real number,  
93  $f : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g : [0, \infty) \rightarrow \mathbb{R}$  are continuous functions satisfying the assumptions  
94 below.

95 (A) There exists  $T > 0$  such that  $xf(t, x) \leq 0$  for all  $t \geq T$  and  $x \in \mathbb{R}$ .

(B)

$$\limsup_{t \rightarrow +\infty} t^{\beta-\alpha-n+1} \int_0^t (t-s)^{\alpha-1} g(s) ds = +\infty,$$

$$\liminf_{t \rightarrow +\infty} t^{\beta-\alpha-n+1} \int_0^t (t-s)^{\alpha-1} g(s) ds = -\infty.$$

96 Let  $m := \lceil \beta \rceil$ . If  $\beta \notin \mathbb{Z}$ , we set  $\alpha_1 = \beta - \lceil \beta \rceil + 1$ ,  $\alpha_i = \alpha_1 + i - 1, i = 2, 3, \dots, m + 1$ ,  
 97  $\alpha_i = i - 1, i = m + 2, \dots, n - 1$ ,  $\alpha_n = \alpha$ . If  $\beta \in \mathbb{Z}$ , we set  $\alpha_i = i, i = 1, 2, \dots, n - 1$ ,  $\alpha_n = \alpha$ .  
 98 Then the equation (3) is rewritten as

$${}^C D_{0+}^{\alpha_n} x(t) = F(t, x(t), {}^C D_{0+}^{\alpha_1} x(t), {}^C D_{0+}^{\alpha_2} x(t), \dots, {}^C D_{0+}^{\alpha_{n-1}} x(t)), \quad t > 0, \quad (5)$$

here

$$F(t, x(t), {}^C D_{0+}^{\alpha_1} x(t), \dots, {}^C D_{0+}^{\alpha_{n-1}} x(t)) = \begin{cases} f(t, x(t)) + g(t) - a {}^C D_{0+}^{\alpha_{m+1}} x(t) & \text{if } \beta \notin \mathbb{Z}, \\ f(t, x(t)) + g(t) - a {}^C D_{0+}^{\alpha_m} x(t) & \text{if } \beta \in \mathbb{Z}. \end{cases}$$

99 By [3, Theorem 17], if  $f$  is globally Lipschitz continuous concerning the second variable, the  
 100 system (5)–(4) has a unique global solution on  $[0, \infty)$ .

101 **Theorem 3.1.** Assume that the conditions (A) and (B) hold. Then an arbitrary solution of  
 102 (3) (if it exists globally) is oscillatory.

*Proof.* The theorem will be proven by contradiction. Suppose that  $x$  is a non-oscillatory solution of the system (3)–(4). Without loss of generality, assume that  $x(t) > 0$  for  $t \geq t_1$ . According to the assumption (A), we can find  $t_2 > \max\{T, t_1\}$  so that  $x(t)f(t, x(t)) \leq 0$  for all  $t \geq t_2$  and thus  $f(t, x(t)) \leq 0$  for all  $t \geq t_2$ . On the other hand, for all  $t > 0$ , we have

$$\begin{aligned} I_{0+}^{\alpha} \left( {}^C D_{0+}^{\alpha} x(t) + a {}^C D_{0+}^{\beta} x(t) \right) &= x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k + a I_{0+}^{\alpha-\beta} \left( x(t) - \sum_{k=0}^{m-1} \frac{x^{(k)}(0)}{k!} t^k \right) \\ &= x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k - \sum_{k=0}^{m-1} \frac{a x^{(k)}(0)}{\Gamma(\alpha - \beta + 1 + k)} t^{\alpha-\beta+k} + a I_{0+}^{\alpha-\beta} x(t), \end{aligned}$$

where  $m = \lceil \beta \rceil$ . From this,

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k + \sum_{k=0}^{m-1} \frac{a x^{(k)}(0)}{\Gamma(\alpha - \beta + 1 + k)} t^{\alpha-\beta+k} - \frac{a}{\Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad \forall t > 0. \end{aligned}$$

Put  $b = \max\{|x_0|, |x_1|, \dots, |x_{n-1}|\}$ ,  $M_1 = \max_{s \in [0, t_2]} |x(s)|$ ,  $M_2 = \max_{s \in [0, t_2]} |f(s, x(s))|$ . Since  $0 < \beta < \alpha$ ,  $a \geq 0$ ,  $x(s) > 0$ ,  $f(s, x(s)) \leq 0$  for all  $s \geq t_2$ , we obtain

$$\begin{aligned} x(t) &\leq \sum_{k=0}^{n-1} \frac{b}{k!} t^k + \sum_{k=0}^{m-1} \frac{ab}{\Gamma(\alpha - \beta + 1 + k)} t^{\alpha-\beta+k} + \frac{aM_1}{\Gamma(\alpha - \beta + 1)} t^{\alpha-\beta} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} f(s; x(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad \forall t > t_2. \end{aligned}$$

This implies that for all  $t > t_2$ ,

$$\begin{aligned} t^{\beta-\alpha-n+1} x(t) &\leq \sum_{k=0}^{n-1} \frac{b}{k!} t^{k+\beta-\alpha-n+1} + \sum_{k=0}^{m-1} \frac{ab}{\Gamma(\alpha - \beta + 1 + k)} t^{k-n+1} + \frac{aM_1}{\Gamma(\alpha - \beta + 1)} t^{1-n} \\ &\quad + \frac{t^{\beta-n+1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} f(s, x(s)) ds + \frac{t^{\beta-\alpha-n+1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds. \end{aligned} \quad (6)$$

Consider the case when  $\alpha \leq 1$ . Notice that for all  $s \in [0, t_2]$ ,  $t > t_2$ ,  $t(t_2 - s) \leq t_2(t - s)$ . It deduces that  $\frac{t}{t-s} \leq \frac{t_2}{t_2-s}$ . Hence,

$$\begin{aligned} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} f(s, x(s)) ds &\leq \frac{M_2 t^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} ds \\ &= \frac{M_2}{\Gamma(\alpha)} \int_0^{t_2} \left(\frac{t}{t-s}\right)^{1-\alpha} ds \\ &\leq \frac{M_2}{\Gamma(\alpha)} \int_0^{t_2} \left(\frac{t_2}{t_2-s}\right)^{1-\alpha} ds \\ &= \frac{M_2 t_2}{\Gamma(\alpha+1)}, \quad \forall t > t_2. \end{aligned} \tag{7}$$

To deal with  $\alpha > 1$ , we see that

$$\begin{aligned} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} f(s, x(s)) ds &\leq \frac{M_2 t^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} ds \\ &= \frac{M_2}{\Gamma(\alpha)} \int_0^{t_2} \left(\frac{t-s}{t}\right)^{\alpha-1} ds \\ &\leq \frac{M_2}{\Gamma(\alpha)} \int_0^{t_2} ds \\ &= \frac{M_2 t_2}{\Gamma(\alpha)}, \quad \forall t > t_2. \end{aligned} \tag{8}$$

Since  $k + \beta - \alpha - n + 1 < 0$  for all  $k = 0, \dots, n-1$ ,  $k - n + 1 \leq 0$  for all  $k = 0, \dots, m-1$  and  $1 - n \leq 0$ , by combining (6), (7) and (8), there is a positive constant  $c = c(t_2)$  such that

$$t^{\beta-\alpha-n+1} x(t) \leq c + \frac{t^{\beta-\alpha-n+1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad \forall t \geq t_2, \tag{9}$$

which together with (B) leads to

$$\liminf_{t \rightarrow \infty} t^{\beta-\alpha-n+1} x(t) \leq c + \liminf_{t \rightarrow \infty} \frac{t^{\beta-\alpha-n+1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds = -\infty,$$

103 contrary to the counterfactual hypothesis that  $x(t) > 0$  for all  $t > t_2$ . The proof is complete.  $\square$

104 **Corollary 3.2.** Consider the system (3)–(4). Suppose that  $f$  satisfies the assumption (A) in  
105 Theorem 3.1 and  $g$  has the following form.

106 (B)'  $g(t) = {}^C D_{0+}^\alpha (t+1)^\sigma h(t)$ ,  $t \geq 0$ , where  $\sigma > \alpha - \beta + n - 1 > 0$ ,  $h$  is continuously differentiable  
107 on  $[0, \infty)$ ,  $\limsup_{t \rightarrow \infty} h(t) > 0$  and  $\liminf_{t \rightarrow \infty} h(t) < 0$ .

108 Then, its solution (if it exists globally) is oscillatory.

*Proof.* It is sufficient to check that  $g$  verifies the condition (B) in Theorem 3.1. Indeed, due to  $h$  is continuously differentiable on  $[0, \infty)$ , according to Lemma 2.5, we obtain

$$\begin{aligned} t^{\beta-\alpha-n+1} \int_0^t (t-s)^{\alpha-1} g(s) ds &= \Gamma(\alpha) t^{\beta-\alpha-n+1} I_{0+}^\alpha g(t) \\ &= \Gamma(\alpha) t^{\beta-\alpha-n+1} (t+1)^\sigma h(t), \quad t > 0. \end{aligned}$$

Notice that  $\sigma > \alpha - \beta + n - 1$ , therefore

$$\limsup_{t \rightarrow \infty} t^{\beta - \alpha - n + 1} \int_0^t (t - s)^{\alpha - 1} g(s) ds = \lim_{t \rightarrow \infty} t^{\beta - \alpha - n + 1} (t + 1)^\sigma \limsup_{t \rightarrow \infty} h(t) = +\infty.$$

In the same manner, we can see that

$$\liminf_{t \rightarrow \infty} t^{\beta - \alpha - n + 1} \int_0^t (t - s)^{\alpha - 1} g(s) ds = -\infty,$$

109 which completes the proof. □

*Example 3.3.* Consider the fractional differential equation (3) with  $\alpha = 1/2, \beta = 1/3, a = 2$ ,  $f(t, x) = -(t - 1)x - (t^2 - 3)x^3$  and  $g(t) = {}^C D_{0+}^{1/2} (t^{7/6} \sin t)$ . This equation is rewritten as

$${}^C D_{0+}^{1/2} x(t) + 2 {}^C D_{0+}^{1/3} x(t) = -(t - 1)x(t) - (t^2 - 3)x^3(t) + {}^C D_{0+}^{1/2} (t^{7/6} \sin t), \quad t > 0. \quad (10)$$

It is easy to check that  $xf(t, x) \leq 0$  for all  $t \geq 2$ . Thus  $f$  satisfies the condition (A). Moreover, for all  $t > 0$ ,

$$\begin{aligned} t^{-1/6} \int_0^t (t - s)^{-1/2} g(s) ds &= \Gamma(1/2) t^{-1/6} I_{0+}^{1/2} g(t) = \Gamma(1/2) t^{-1/6} I_{0+}^{1/2} \left( {}^C D_{0+}^{1/2} (t^{7/6} \sin t) \right) \\ &= \Gamma(1/2) t^{-1/6} t^{7/6} \sin t = \Gamma(1/2) t \sin t. \end{aligned}$$

This implies that

$$\begin{aligned} \limsup_{t \rightarrow \infty} t^{-1/6} \int_0^t (t - s)^{-1/2} g(s) ds &= +\infty, \\ \liminf_{t \rightarrow \infty} t^{-1/6} \int_0^t (t - s)^{-1/2} g(s) ds &= -\infty. \end{aligned}$$

110 Thus the condition (B) is verified. By Theorem 3.1, all solutions of the equation (10) are  
111 oscillatory. In Figure 1, we simulate the orbit of the solution with the initial condition  $x(0) = 1$   
112 on the interval  $[0, 70]$ .

*Remark 3.4.* Under assumptions (A) and (B), the conclusion of Theorem 3.1 is still true for the following general multi-order fractional system

$${}^C D_{0+}^{\alpha_k} x(t) + \sum_{i=1}^{k-1} a_i {}^C D_{0+}^{\alpha_i} x(t) = f(t, x(t)) + g(t), \quad t > 0$$

with arbitrary initial values

$$x(0) = x_0, x^{(1)}(0) = x_1, \dots, x^{(n-1)}(0) = x_{n-1},$$

113 here  $0 < \alpha_1 < \alpha_2 < \dots < \alpha_k \leq [\alpha_k] = n \in \mathbb{N}$ ,  $a_i, i = 1, \dots, k - 1$ , are nonnegative real numbers.

## 114 3.2 The oscillation of equations with higher-order nonlinearities

We now study the multi-term fractional differential equation with higher-order nonlinearities

$${}^C D_{0+}^{\alpha} x(t) + a {}^C D_{0+}^{\beta} x(t) + p(t)x(t) + \sum_{i=1}^k q_i(t) \operatorname{sgn}(x(t)) |x(t)|^{\lambda_i} = g(t), \quad t > 0 \quad (11)$$

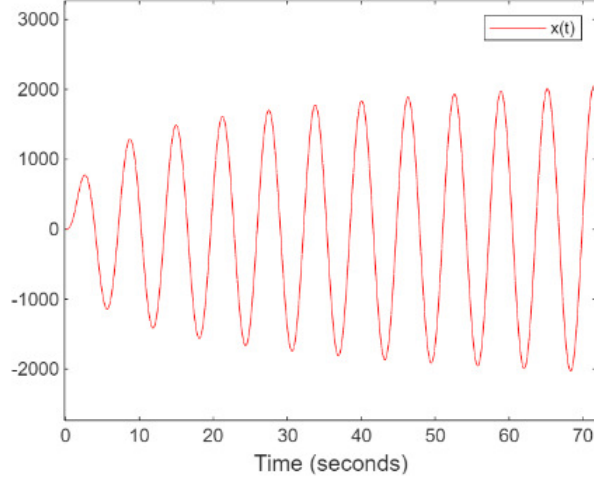


Figure 1: The orbit of the solution to equation (10) with the initial condition  $x_0 = 1$  on the interval  $[0, 70]$ .

with the initial conditions

$$x(0) = x_0, x^{(1)}(0) = x_1, \dots, x^{(n-1)}(0) = x_{n-1}, \quad (12)$$

115 where  $0 < \beta < \alpha \leq \lceil \alpha \rceil = n \in \mathbb{N}$ ,  $x_0, x_1, \dots, x_{n-1} \in \mathbb{R}$ ,  $a$  is a nonnegative real number,  $0 < \lambda_1 <$   
 116  $\lambda_2 < \dots < \lambda_k$ ,  $k \geq 2$  are positive real numbers,  $p : [0, \infty) \rightarrow \mathbb{R}$ ,  $q_i : [0, \infty) \rightarrow \mathbb{R}$ ,  $i = 1, 2, \dots, k$ ,  
 117  $\text{sgn}(\cdot)$  is the sign function, and  $g : [0, \infty) \rightarrow \mathbb{R}$  are continuous functions. Suppose that the  
 118 following conditions are true.

119 (C)  $\lambda_i \neq 1$  for all  $i = 1, 2, \dots, k$  and satisfy one of the following two conditions

120 (C)<sub>1</sub>  $1 < \lambda_1 < \lambda_2 < \dots < \lambda_k$ .

121 (C)<sub>2</sub> There exists  $l \in \{1, \dots, k-1\}$  such that  $\lambda_1 < \dots < \lambda_l < 1 < \lambda_{l+1} < \dots < \lambda_k$ .

122 (D) There exists  $T_1 \geq 0$  such that  $p(t) \neq 0$  for all  $t \geq T_1$ .

123 (E) There exists  $T_2 \geq 0$  such that  $\text{sgn}(\lambda_i - 1)q_i(t) > 0$ ,  $i = 1, 2, \dots, m$  for all  $t \geq T_2$ .

(F) There exists  $T \geq \max\{T_1, T_2\}$  such that

$$\liminf_{t \rightarrow \infty} t^{\beta - \alpha - n + 1} \int_T^t (t-s)^{\alpha-1} \left( \gamma \sum_{i=1}^k |p(s)|^{\frac{\lambda_i}{\lambda_i-1}} |q_i(s)|^{\frac{1}{1-\lambda_i}} + g(s) \right) ds = -\infty,$$

$$\limsup_{t \rightarrow \infty} t^{\beta - \alpha - n + 1} \int_T^t (t-s)^{\alpha-1} \left( -\gamma \sum_{i=1}^k |p(s)|^{\frac{\lambda_i}{\lambda_i-1}} |q_i(s)|^{\frac{1}{1-\lambda_i}} + g(s) \right) ds = +\infty,$$

where

$$\gamma := \left( \frac{1}{k\lambda_1} \right)^{\frac{\lambda_k}{\lambda_k-1}}, \quad \text{if (C)}_1 \text{ holds,}$$

$$\gamma := \max \left\{ \max_{i \in \{1, \dots, l\}} (1 - \lambda_i) \left( \frac{l\lambda_i}{\sum_{j=l+1}^k \lambda_j - 1} \right)^{\frac{\lambda_i}{1-\lambda_i}}, \lambda_k - 1 \right\}, \quad \text{if (C)}_2 \text{ holds.}$$



124 **Theorem 3.5.** Assume that the conditions (C), (D), (E), and (F) hold. Then an arbitrary  
 125 solution of (11) (if it exists globally) is oscillatory.

*Proof.* Suppose to the contrary that there exists a non-oscillatory solution  $x$  of system (11)–(12). Without loss of generality, we can assume that  $x(t) > 0$  for  $t > t_1$ . Due to the assumptions (D) and (E), we can find  $t_2 > \max\{t_1, T\}$  such that  $p(t) \neq 0$  and  $\text{sgn}(\lambda_i - 1)q_i(t) > 0$ ,  $i = 1, 2, \dots, k$ , for  $t \geq t_2$ . Using the same arguments as in the proof of Theorem 3.1, for all  $t > t_2$ , we obtain

$$\begin{aligned}
 x(t) &= \sum_{i=0}^{n-1} \frac{x^{(i)}(0)}{i!} t^i + \sum_{i=0}^{m-1} \frac{ax^{(i)}(0)}{\Gamma(\alpha - \beta + 1 + i)} t^{\alpha - \beta + i} - \frac{a}{\Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha - \beta - 1} x(s) ds \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)x(s) ds - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_0^t (t-s)^{\alpha-1} q_i(s) \text{sgn}(x(s)) |x(s)|^{\lambda_i} ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds \\
 &= \sum_{i=0}^{n-1} \frac{x^{(i)}(0)}{i!} t^i + \sum_{i=0}^{m-1} \frac{ax^{(i)}(0)}{\Gamma(\alpha - \beta + 1 + i)} t^{\alpha - \beta + i} - \frac{a}{\Gamma(\alpha - \beta)} \int_0^{t_2} (t-s)^{\alpha - \beta - 1} x(s) ds \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} p(s)x(s) ds - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_0^{t_2} (t-s)^{\alpha-1} q_i(s) \text{sgn}(x(s)) |x(s)|^{\lambda_i} ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} g(s) ds - \frac{a}{\Gamma(\alpha - \beta)} \int_{t_2}^t (t-s)^{\alpha - \beta - 1} x(s) ds \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} p(s)x(s) ds - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_2}^t (t-s)^{\alpha-1} q_i(s) \text{sgn}(x(s)) |x(s)|^{\lambda_i} ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_T^t (t-s)^{\alpha-1} g(s) ds,
 \end{aligned}$$

where  $m = \lceil \beta \rceil$ . Notice that  $(t-s)^{\alpha - \beta - 1} \leq (t-t_2)^{\alpha - \beta - 1}$  for all  $s \in [0, t_2]$ ,  $t > t_2$  if  $\alpha - \beta - 1 \leq 0$  and  $(t-s)^{\alpha - \beta - 1} \leq t^{\alpha - \beta - 1}$  for all  $s \in [0, t_2]$ ,  $t > t_2$  if  $\alpha - \beta - 1 > 0$ . Thus,  $(t-s)^{\alpha - \beta - 1} \leq (t-t_2)^{\alpha - \beta - 1} + t^{\alpha - \beta - 1}$  for all  $s \in [0, t_2]$ , which implies

$$\begin{aligned}
 -\frac{a}{\Gamma(\alpha - \beta)} \int_0^{t_2} (t-s)^{\alpha - \beta - 1} x(s) ds &\leq \frac{a}{\Gamma(\alpha - \beta)} \left( (t-t_2)^{\alpha - \beta - 1} + t^{\alpha - \beta - 1} \right) \int_0^{t_2} |x(s)| ds \\
 &= M_1 \left( (t-t_2)^{\alpha - \beta - 1} + t^{\alpha - \beta - 1} \right), \quad t > t_2. \tag{13}
 \end{aligned}$$

Furthermore, it is not difficult to check

$$-\frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} p(s)x(s) ds \leq M_2 \left( (t-t_2)^{\alpha-1} + t^{\alpha-1} \right), \tag{14}$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_0^{t_2} (t-s)^{\alpha-1} q_i(s) \text{sgn}(x(s)) |x(s)|^{\lambda_i} ds \leq M_3 \left( (t-t_2)^{\alpha-1} + t^{\alpha-1} \right), \tag{15}$$

and

$$\frac{1}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} g(s) ds \leq M_4 \left( (t-t_2)^{\alpha-1} + t^{\alpha-1} \right) \tag{16}$$

for all  $t > t_2$ . Take  $b := \max\{|x_0|, |x_1|, \dots, |x_{n-1}|\}$ . For  $t \geq t_2$ , it follows from (13), (14), (15) and (16) that

$$\begin{aligned} x(t) &\leq \sum_{k=0}^{n-1} \frac{b}{k!} t^k + \sum_{k=0}^{m-1} \frac{ab}{\Gamma(\alpha - \beta + 1 + k)} t^{\alpha - \beta + k} + M_1 \left( (t - t_2)^{\alpha - \beta - 1} + t^{\alpha - \beta - 1} \right) \\ &\quad + (M_2 + M_3 + M_4) \left( (t - t_2)^{\alpha - 1} + t^{\alpha - 1} \right) + \frac{1}{\Gamma(\alpha)} \int_T^t (t - s)^{\alpha - 1} g(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t - s)^{\alpha - 1} \left( |p(s)|x(s) - \sum_{i=1}^m q_i(s)x(s)^{\lambda_i} \right) ds. \end{aligned} \quad (17)$$

Consider the case when (C)<sub>1</sub> is true, that is,  $1 < \lambda_1 < \lambda_2 < \dots < \lambda_k$ . We get  $q_i(t) > 0$ ,  $i = 1, 2, \dots, k$  and  $t \geq t_2$ . Then,

$$|p(s)|x(s) - \sum_{i=1}^k q_i(s)x(s)^{\lambda_i} = \sum_{i=1}^k \left( \frac{1}{k} |p(s)|x(s) - q_i(s)x(s)^{\lambda_i} \right), \quad s \geq t_2. \quad (18)$$

For each  $i = 1, 2, \dots, k$ , let

$$X_i(s) := q_i(s)^{\frac{1}{\lambda_i}} x(s), Y_i(s) := \left( \frac{1}{m\lambda_i} |p(s)|q_i(s)^{\frac{-1}{\lambda_i}} \right)^{\frac{1}{\lambda_i - 1}}, \quad s \geq t_2.$$

Due to Lemma 2.7(i), we see

$$\lambda_i q_i(s)^{\frac{1}{\lambda_i}} x(s) \frac{1}{k\lambda_i} |p(s)|q_i(s)^{\frac{-1}{\lambda_i}} - q_i(s)x(s)^{\lambda_i} \leq (\lambda_i - 1) \left( \frac{1}{k\lambda_i} |p(s)|q_i(s)^{\frac{-1}{\lambda_i}} \right)^{\frac{\lambda_i}{\lambda_i - 1}},$$

this means that

$$\begin{aligned} \frac{1}{m} |p(s)|x(s) - q_i(s)x(s)^{\lambda_i} &\leq (\lambda_i - 1) \left( \frac{1}{m\lambda_i} |p(s)|q_i(s)^{\frac{-1}{\lambda_i}} \right)^{\frac{\lambda_i}{\lambda_i - 1}} \\ &\leq \gamma |p(s)|^{\frac{\lambda_i}{\lambda_i - 1}} |q_i(s)|^{\frac{1}{1 - \lambda_i}}. \end{aligned} \quad (19)$$

From (18) and (19), it leads to

$$|p(s)|x(s) - \sum_{i=1}^k q_i(s)x(s)^{\lambda_i} \leq \gamma \sum_{i=1}^k |p(s)|^{\frac{\lambda_i}{\lambda_i - 1}} |q_i(s)|^{\frac{1}{1 - \lambda_i}}, \quad s \geq t_2. \quad (20)$$

If the assumption (C)<sub>2</sub> holds, that is, there is an index  $l$  satisfying  $\lambda_1 < \dots < \lambda_l < 1 < \lambda_{l+1} < \dots < \lambda_k$ . It implies that  $q_i(t) < 0, i = 1, \dots, l$  and  $q_i(t) > 0, i = l + 1, \dots, k$  for  $t \geq t_2$ . In this case, we write

$$\begin{aligned} |p(s)|x(s) - \sum_{i=1}^k q_i(s)x(s)^{\lambda_i} &= |p(s)|x(s) + \sum_{i=1}^l |q_i(s)|x(s)^{\lambda_i} - \sum_{i=l+1}^k q_i(s)x(s)^{\lambda_i} \\ &= \sum_{i=1}^l \left( |q_i(s)|x(s)^{\lambda_i} - A|p(s)|x(s) \right) + \sum_{i=l+1}^k \left( \lambda_i |p(s)|x(s) - q_i(s)x(s)^{\lambda_i} \right), \end{aligned} \quad (21)$$

where  $A = \frac{\sum_{j=l+1}^k \lambda_j^{-1}}{l} > 0$ . For each  $i = 1, \dots, l$ , define

$$X_i(s) := |q_i(s)|^{\frac{1}{\lambda_i}} x(s), Y_i(s) := \left( \frac{A}{\lambda_i} |p(s)||q_i(s)|^{\frac{-1}{\lambda_i}} \right)^{\frac{1}{\lambda_i - 1}}, \quad s \geq t_2.$$

By Lemma 2.7(ii),

$$\lambda_i |q_i(s)|^{\frac{1}{\lambda_i}} x(s) \frac{A}{\lambda_i} |p(s)| |q_i(s)|^{\frac{-1}{\lambda_i}} - |q_i(s)| x(s)^{\lambda_i} \geq (\lambda_i - 1) \left( \frac{A}{\lambda_i} |p(s)| |q_i(s)|^{\frac{-1}{\lambda_i}} \right)^{\frac{\lambda_i}{\lambda_i - 1}}.$$

Thus,

$$\begin{aligned} |q_i(s)| x(s)^{\lambda_i} - A |p(s)| x(s) &\leq (1 - \lambda_i) \left( \frac{A}{\lambda_i} |p(s)| |q_i(s)|^{\frac{-1}{\lambda_i}} \right)^{\frac{\lambda_i}{\lambda_i - 1}} \\ &\leq \gamma |p(s)|^{\frac{\lambda_i}{\lambda_i - 1}} |q_i(s)|^{\frac{1}{1 - \lambda_i}}, \quad i = 1, \dots, l, s \geq t_2. \end{aligned} \quad (22)$$

For each  $i = l + 1, \dots, m$ , let

$$X_i(s) := q_i(s)^{\frac{1}{\lambda_i}} x(s), Y_i(s) := \left( |p(s)| |q_i(s)|^{\frac{-1}{\lambda_i}} \right)^{\frac{1}{\lambda_i - 1}}, \quad s \geq t_2.$$

From Lemma 2.7(i),

$$\lambda_i q_i(s)^{\frac{1}{\lambda_i}} x(s) |p(s)| |q_i(s)|^{\frac{-1}{\lambda_i}} - q_i(s) x(s)^{\lambda_i} \leq (\lambda_i - 1) \left( |p(s)| |q_i(s)|^{\frac{-1}{\lambda_i}} \right)^{\frac{\lambda_i}{\lambda_i - 1}},$$

which gives

$$\begin{aligned} \lambda_i |p(s)| x(s) - q_i(s) x(s)^{\lambda_i} &\leq (\lambda_i - 1) \left( |p(s)| |q_i(s)|^{\frac{-1}{\lambda_i}} \right)^{\frac{\lambda_i}{\lambda_i - 1}} \\ &\leq \gamma |p(s)|^{\frac{\lambda_i}{\lambda_i - 1}} |q_i(s)|^{\frac{1}{1 - \lambda_i}}, \quad i = l + 1, \dots, m, s \geq t_2. \end{aligned} \quad (23)$$

Combining (21), (22) and (23) yields

$$|p(s)| x(s) - \sum_{i=1}^k q_i(s) x(s)^{\lambda_i} \leq \gamma \sum_{i=1}^k |p(s)|^{\frac{\lambda_i}{\lambda_i - 1}} |q_i(s)|^{\frac{1}{1 - \lambda_i}}, \quad s \geq t_2. \quad (24)$$

In short, if the condition (C) is true, from (20) and (24), the following estimate is derived

$$|p(s)| x(s) - \sum_{i=1}^k q_i(s) x(s)^{\lambda_i} \leq \gamma \sum_{i=1}^k |p(s)|^{\frac{\lambda_i}{\lambda_i - 1}} |q_i(s)|^{\frac{1}{1 - \lambda_i}}, \quad s \geq t_2, \quad (25)$$

which together with (17) implies

$$\begin{aligned} t^{\beta - \alpha - n + 1} x(t) &\leq \sum_{k=0}^{n-1} \frac{b}{k!} t^{k + \beta - \alpha - n + 1} + \sum_{k=0}^{m-1} \frac{ab}{\Gamma(\alpha - \beta + 1 + k)} t^{k - n + 1} \\ &\quad + M_1 \left( (t - t_2)^{\alpha - \beta - 1} t^{\beta - \alpha - n + 1} + t^{-n} \right) \\ &\quad + (M_2 + M_3 + M_4) \left( (t - t_2)^{\alpha - 1} t^{\beta - \alpha - n + 1} + t^{\beta - n} \right) \\ &\quad + \frac{t^{\beta - \alpha - n + 1}}{\Gamma(\alpha)} \int_T^t (t - s)^{\alpha - 1} \left( \gamma \sum_{i=1}^m |p(s)|^{\frac{\lambda_i}{\lambda_i - 1}} |q_i(s)|^{\frac{1}{1 - \lambda_i}} + g(s) \right) ds, \quad t > t_2. \end{aligned}$$

Therefore, there exist  $c > 0$  and  $t_3 > t_2$  such that

$$t^{\beta - \alpha - n + 1} x(t) \leq c + \frac{t^{\beta - \alpha - n + 1}}{\Gamma(\alpha)} \int_T^t (t - s)^{\alpha - 1} \left( \gamma \sum_{i=1}^m |p(s)|^{\frac{\lambda_i}{\lambda_i - 1}} |q_i(s)|^{\frac{1}{1 - \lambda_i}} + g(s) \right) ds, \quad t > t_3.$$

Under the assumption (F), we conclude

$$\liminf_{t \rightarrow \infty} t^{\beta - \alpha - n + 1} x(t) = -\infty,$$

126 contrary to the counterfactual hypothesis that  $x(t) > 0$  for all  $t \geq t_2$ . The proof is complete.  $\square$

*Example 3.6.* Consider the initial value problem (11)–(12), here  $\alpha = 1/2$ ,  $\beta = 1/3$ ,  $a = 2$ ,  $p(t) = -t$ ,  $\lambda_1 = 1/2$ ,  $\lambda_2 = 2$ ,  $q_1(t) = -t^2$ ,  $q_2(t) = t^3$ ,  $g(t) = {}^C D_{0+}^{1/2}(t^6 \sin t)$  for  $t \geq 0$ . This equation is rewritten as

$${}^C D_{0+}^{1/2} x(t) + 2 {}^C D_{0+}^{1/3} x(t) - tx(t) - t^2 \operatorname{sgn}(x(t)) |x(t)|^{1/2} + t^3 \operatorname{sgn}(x(t)) |x(t)|^2 = {}^C D_{0+}^{1/2}(t^6 \sin t). \quad (26)$$

Then it is easy to check that  $\lambda_1, \lambda_2$  satisfy (C)<sub>2</sub>,  $p$  satisfies (D) and  $q_1, q_2$  satisfy (E) for  $T_1 = T_2 = 1$ , respectively. Next, we see  $\gamma = 1$  and

$$\gamma \sum_{i=1}^m |p(s)|^{\frac{\lambda_i}{\lambda_i-1}} |q_i(s)|^{\frac{1}{1-\lambda_i}} = t^3 + t^5, \quad t \geq 0.$$

Notice that

$$g(t) = \frac{1}{\Gamma(1/2)} \int_0^t \frac{s^5 \sin s + s^6 \cos s}{(t-s)^{1/2}} ds, \quad t \geq 0,$$

which implies

$$|g(t)| \leq \frac{1}{\Gamma(1/2)} \int_0^t \frac{s^5 + s^6}{(t-s)^{1/2}} ds = \frac{5!}{\Gamma(13/2)} t^{11/2} + \frac{6!}{\Gamma(15/2)} t^{13/2}, \quad t \geq 0.$$

Hence, for  $t > 1$ , we obtain

$$\begin{aligned} \int_1^t (t-s)^{-1/2} \left( \gamma \sum_{i=1}^m |p(s)|^{\frac{\lambda_i}{\lambda_i-1}} |q_i(s)|^{\frac{1}{1-\lambda_i}} + g(s) \right) ds &< \Gamma(1/2) I_{0+}^{1/2} (t^3 + t^5 + {}^C D_{0+}^{1/2}(t^6 \sin t)) \\ &+ 2M(t-1)^{1/2} \\ &< \frac{3!}{\Gamma(9/2)} t^{7/2} + \frac{6!}{\Gamma(13/2)} t^{11/2} + \Gamma(1/2) t^6 \sin t + 2Mt^{1/2}, \end{aligned}$$

where  $M = \max_{t \in [0,1]} (t^3 + t^5 + |g(t)|) = 2 + \frac{5!}{\Gamma(13/2)} + \frac{6!}{\Gamma(15/2)}$ . From this,

$$\begin{aligned} \liminf_{t \rightarrow \infty} t^{-1/2} \int_1^t (t-s)^{-1/2} \left( \gamma \sum_{i=1}^m |p(s)|^{\frac{\lambda_i}{\lambda_i-1}} |q_i(s)|^{\frac{1}{1-\lambda_i}} + g(s) \right) ds \\ \leq \liminf_{t \rightarrow \infty} \left( \frac{3!}{\Gamma(9/2)} t^3 + \frac{6!}{\Gamma(13/2)} t^5 + \Gamma(1/2) t^{11/2} \sin t + 2M \right) = -\infty. \end{aligned}$$

By the same arguments, we also see

$$\begin{aligned} \limsup_{t \rightarrow \infty} t^{-1/2} \int_1^t (t-s)^{-1/2} \left( -\gamma \sum_{i=1}^m |p(s)|^{\frac{\lambda_i}{\lambda_i-1}} |q_i(s)|^{\frac{1}{1-\lambda_i}} + g(s) \right) ds \\ \geq \limsup_{t \rightarrow \infty} \left( -\frac{3!}{\Gamma(9/2)} t^3 - \frac{6!}{\Gamma(13/2)} t^5 + \Gamma(1/2) t^{11/2} \sin t + 2M \right) = +\infty, \end{aligned}$$

127 and thus the condition (F) is verified. From Theorem 3.5, we conclude that all solutions (if they  
128 exist globally) are oscillatory. Figure 2 depicts the orbit of the solution with the initial condition  
129  $x(0) = 1$  on the interval  $[0, 80]$ .

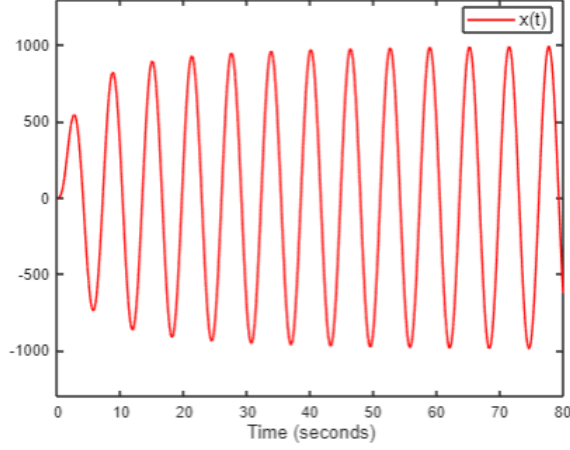


Figure 2: The orbit of the solution to equation (26) with the initial condition  $x_0 = 1$  on the interval  $[0, 80]$ .

## 130 4 On the asymptotic behavior of oscillatory solutions of frac- 131 tional differential equations

132 This section is devoted to the asymptotic behavior of oscillatory solutions of some classes of  
133 fractional differential equations. We first present some comparison results for equations with  
134 two fractional derivatives to do this. Then, we consider linear equations. Finally, we deal  
135 with the nonlinear case by combining Theorem 3.1, the approach as in [10], and the proposed  
136 comparison principles.

### 137 4.1 Comparison results for multi-order fractional differential equations

For given a parameter  $T > 0$ , consider the equation

$${}^C D_{0+}^{\alpha} x(t) + a {}^C D_{0+}^{\beta} x(t) = f(t, x(t)), \quad t \in (0, T], \quad (27)$$

$$x(0) = x_0 \in \mathbb{R}, \quad (28)$$

138 where  $0 < \beta < \alpha \leq 1$ ,  $a$  is a positive constant,  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function  
139 and Lipschitz continuous with respect to the second variable. We will present some comparison  
140 results of the solutions of the system (27)–(28).

**Lemma 4.1.** Let  $x(\cdot)$  be the unique solution of the initial value problem (27)–(28). Suppose that  $y : [0, T] \rightarrow \mathbb{R}$  is continuously Caputo  $\alpha$ -differentiable satisfying

$${}^C D_{0+}^{\alpha} y(t) + a {}^C D_{0+}^{\beta} y(t) \leq f(t, y(t)), \quad t \in (0, T],$$

$$y(0) = y_0 < x_0,$$

141 then  $y(t) < x(t)$  for all  $t \in [0, T]$ .

*Proof.* Suppose by contradiction that there is a  $t \in (0, T]$  such that  $y(t) = x(t)$ . Due to

$y(0) = y_0 < x_0 = x(0)$  and  $x, y \in C[0, T]$ , there exists a  $t_1 \in (0, T]$  such that

$$y(t) < x(t), \quad \forall t \in [0, t_1), \quad (29)$$

$$y(t_1) = x(t_1) \quad (30)$$

Define

$$z(t) := x(t) - y(t), \quad t \in [0, T].$$

Then  $z(t) > 0$  for all  $t \in [0, t_1)$  and  $z(t_1) = 0$ . Notice that for  $0 < \alpha < 1$ , by virtue of Lemma 2.6, we have

$$\begin{aligned} {}^C D_{0+}^\alpha z(t_1) &= \frac{1}{\Gamma(1-\alpha)} \left( \frac{z(t_1) - z(0)}{t_1^\alpha} \right) + \alpha \int_0^{t_1} \frac{z(t_1) - z(s)}{(t_1 - s)^{\alpha+1}} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \left( \frac{-z(0)}{t_1^\alpha} \right) + \alpha \int_0^{t_1} \frac{-z(s)}{(t_1 - s)^{\alpha+1}} ds < 0. \end{aligned} \quad (31)$$

In the case  $\alpha = 1$ , then

$$\begin{aligned} z'(t_1) &= \lim_{h \rightarrow 0^-} \frac{z(t_1 + h) - z(t_1)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{z(t_1 + h)}{h} \\ &\leq 0. \end{aligned} \quad (32)$$

Thus, from (31) and (32), we claim

$${}^C D_{0+}^\alpha z(t_1) \leq 0. \quad (33)$$

Furthermore, due to  $z(\cdot)$  is continuously Caputo  $\alpha$ -differentiable on  $[0, T]$  and  $0 < \beta < \alpha \leq 1$ , it follows from [3, Theorem 3.8 (i)] that this function is also continuously Caputo  $\beta$ -differentiable and thus, by the same arguments as shown above,  ${}^C D_{0+}^\beta z(t_1) < 0$ . This together with (33) implies

$${}^C D_{0+}^\alpha z(t_1) + a {}^C D_{0+}^\beta z(t_1) < 0,$$

142 that is,

$${}^C D_{0+}^\alpha x(t_1) + a {}^C D_{0+}^\beta x(t_1) < {}^C D_{0+}^\alpha y(t_1) + a {}^C D_{0+}^\beta y(t_1). \quad (34)$$

However,

$${}^C D_{0+}^\alpha x(t_1) + a {}^C D_{0+}^\beta x(t_1) = f(t_1, x(t_1)) = f(t_1, y(t_1)) \geq {}^C D_{0+}^\alpha y(t_1) + a {}^C D_{0+}^\beta y(t_1),$$

143 contrary to (34). Therefore,  $y(t) < x(t)$  for all  $t \in [0, T]$ .  $\square$

**Theorem 4.2.** Let  $x(\cdot)$  be the unique solution of the initial value problem (27)–(28). Assume that  $y : [0, T] \rightarrow \mathbb{R}$  is continuously Caputo  $\alpha$ -differentiable satisfying

$$\begin{aligned} {}^C D_{0+}^\alpha y(t) + a {}^C D_{0+}^\beta y(t) &\leq f(t, y(t)), \quad t \in (0, T], \\ y(0) &= y_0 \leq x_0, \end{aligned}$$

144 then  $y(t) \leq x(t)$  for all  $t \in [0, T]$ .

*Proof.* For each  $m \in \mathbb{N}$ , by Theorem 2.11, the following initial value problem

$${}^C D_{0+}^\alpha x(t) + a {}^C D_{0+}^\beta x(t) = f(t, x(t)), \quad t \in (0, T], \quad (35)$$

$$x(0) = x_0 + \frac{1}{m}, \quad (36)$$

has a unique solution  $x_m : [0, T] \rightarrow \mathbb{R}$ . Because  $y(0) = y_0 \leq x_0 < x_0 + \frac{1}{m} = x_m(0)$  for all  $m \in \mathbb{N}$ , according to Lemma 4.1, we see

$$y(t) < x_m(t) \leq x_1(t), \quad \forall t \in [0, T]. \quad (37)$$

In addition, for  $n, m \in \mathbb{N}$  and  $n > m$ , then

$$x_n(t) < x_m(t), \quad \forall t \in [0, T].$$

Thus, the sequence  $\{x_m(\cdot)\}_{m \in \mathbb{N}}$  is uniformly bounded on  $[0, T]$  and for each  $t \in [0, T]$ , the sequence  $\{x_m(t)\}_{m \in \mathbb{N}}$  is strictly decreasing. We next show that this sequence is equicontinuous. Let  $C_1 > 0$  such that  $|x_m(t)| \leq C_1$  for all  $t \in [0, T]$  and  $m \in \mathbb{N}$  and take  $C_2 := \max_{[0, T] \times [-C_1, C_1]} |f(t, x)|$ . For  $0 \leq t_1 < t_2 \leq T$  and  $m \in \mathbb{N}$ , based on [10, Lemma 2.1], we obtain the following estimates

$$\begin{aligned} |x_m(t_2) - x_m(t_1)| &= \left| \frac{a}{\Gamma(\alpha - \beta)} \left( \int_0^{t_1} (t_1 - s)^{\alpha - \beta - 1} x_m(s) ds - \int_0^{t_2} (t_2 - s)^{\alpha - \beta - 1} x_m(s) ds \right) \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \left( \int_0^{t_2} (t_2 - s)^{\alpha - 1} f(s, x_m(s)) ds - \int_0^{t_1} (t_1 - s)^{\alpha - 1} f(s, x_m(s)) ds \right) \right| \\ &\leq \frac{a}{\Gamma(\alpha - \beta)} \int_0^{t_1} \left( (t_1 - s)^{\alpha - \beta - 1} - (t_2 - s)^{\alpha - \beta - 1} \right) |x_m(s)| ds \\ &\quad + \frac{a}{\Gamma(\alpha - \beta)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - \beta - 1} |x_m(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left( (t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1} \right) |f(s, x_m(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} |f(s, x_m(s))| ds \\ &\leq \frac{aC_1}{\Gamma(\alpha - \beta + 1)} (t_1^{\alpha - \beta} + (t_2 - t_1)^{\alpha - \beta} - t_2^{\alpha - \beta}) + \frac{aC_1}{\Gamma(\alpha - \beta + 1)} (t_2 - t_1)^{\alpha - \beta} \\ &\quad + \frac{C_2}{\Gamma(\alpha + 1)} (t_1^\alpha + (t_2 - t_1)^\alpha - t_2^\alpha) + \frac{C_2}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha \\ &\leq \frac{2aC_1}{\Gamma(\alpha - \beta + 1)} (t_2 - t_1)^{\alpha - \beta} + \frac{2C_2}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha. \end{aligned} \quad (38)$$

Let  $\epsilon > 0$  be arbitrarily small. Choosing

$$0 < \delta < \min \left\{ \left( \frac{\epsilon \Gamma(\alpha - \beta + 1)}{2 \cdot 2aC_1} \right)^{\frac{1}{\alpha - \beta}}, \left( \frac{\epsilon \Gamma(\alpha + 1)}{2 \cdot 2C_2} \right)^{\frac{1}{\alpha}} \right\},$$

from (38), we get

$$|x_m(t_2) - x_m(t_1)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all  $t_1, t_2 \in [0, T]$  with  $|t_1 - t_2| < \delta$ . This means that  $\{x_m(\cdot)\}_{m \in \mathbb{N}}$  is equicontinuous on  $[0, T]$ . By Arzelà–Ascoli theorem, there is a subsequence  $\{x_{m_k}(\cdot)\}_{k \in \mathbb{N}}$  of  $\{x_m(\cdot)\}_{m \in \mathbb{N}}$  which uniformly

converges to a function  $x^*(\cdot)$  on  $[0, T]$ . In particular,  $y(t) \leq x^*(t)$  for all  $t \in [0, T]$ . Notice that, for each  $k \in \mathbb{N}$  and  $t \in [0, T]$ , we have (by [10, Lemma 2.1])

$$\begin{aligned} x_{m_k}(t) &= x_0 + \frac{1}{m_k} + \frac{a(x_0 + \frac{1}{m_k})}{\Gamma(1 + \alpha - \beta)} t^{\alpha - \beta} - \frac{a}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} x_{m_k}(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x_{m_k}(s)) ds. \end{aligned}$$

Letting  $k \rightarrow \infty$ , then

$$\begin{aligned} x^*(t) &= x_0 + \frac{ax_0}{\Gamma(1 + \alpha - \beta)} t^{\alpha - \beta} - \frac{a}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} x^*(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x^*(s)) ds, \quad t \in [0, T]. \end{aligned} \quad (39)$$

145 On the other hand, due to Theorem 2.11, the original system (27)–(28) has a unique solution  
146  $x(\cdot)$ , which is also written in the form (39) (by [10, Lemma 2.1]). This implies  $x^* \equiv x$  on  $[0, T]$   
147 and thus  $y(t) \leq x(t)$  for all  $t \in [0, T]$ .  $\square$

**Corollary 4.3.** Let  $x(\cdot)$  be the unique solution of the initial value problem (27)–(28). Assume that  $y : [0, T] \rightarrow \mathbb{R}$  is continuously Caputo  $\alpha$ -differentiable satisfying

$$\begin{aligned} {}^C D_{0+}^\alpha y(t) + a {}^C D_{0+}^\beta y(t) &\geq f(t, y(t)), \quad t \in (0, T], \\ y(0) = y_0 &\geq x_0, \end{aligned}$$

148 then  $y(t) \geq x(t)$  for all  $t \in [0, T]$ .

*Proof.* Put  $u(t) = -x(t)$ ,  $v(t) = -y(t)$ ,  $t \in [0, T]$  and define  $g(t, x) = -f(t, -x)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}$ . Then, the function  $g(\cdot, \cdot)$  is continuous and is Lipschitz continuous to the second variable. It is easy to check that

$$\begin{aligned} {}^C D_{0+}^\alpha u(t) + a {}^C D_{0+}^\beta u(t) &= -{}^C D_{0+}^\alpha x(t) - a {}^C D_{0+}^\beta x(t) \\ &= -f(t, x(t)) = g(t, -x(t)) = g(t, u(t)), \quad t \in (0, T], \\ u(0) &= -x_0. \end{aligned}$$

Similarly,  $v(\cdot)$  is continuously Caputo  $\alpha$ -differentiable and satisfies

$$\begin{aligned} {}^C D_{0+}^\alpha v(t) + a {}^C D_{0+}^\beta v(t) &\leq g(t, v(t)), \quad t \in (0, T], \\ y(0) = -y_0 &\leq -x_0. \end{aligned}$$

149 On account of Theorem 4.2 above, we have  $v(t) \leq u(t)$ ,  $t \in [0, T]$ , that is,  $y(t) \geq x(t)$ ,  $t \in$   
150  $[0, T]$ .  $\square$

## 151 4.2 Asymptotic behavior of oscillatory solutions of fractional-order linear 152 equations

Consider the linear equation

$${}^C D_{0+}^\alpha x(t) + a {}^C D_{0+}^\beta x(t) = -bx(t) + g(t), \quad t > 0 \quad (40)$$

with the initial value

$$x(0) = x_0, \quad (41)$$

153 where  $0 < \beta < \alpha \leq 1$ ,  $a, b$  are positive real constants,  $g : [0, \infty) \rightarrow \mathbb{R}$  is a continuous function.



154 **Theorem 4.4.** Suppose that  $g(\cdot)$  satisfies the following assumption.

155 (B)<sup>''</sup> For  $t \in [0, \infty)$ ,  $g(t) := {}^C D_{0+}^\alpha (t+1)^\sigma h(t)$ , here  $\alpha - \beta < \sigma < \alpha \leq 1$  and  $h(\cdot)$  is continuously  
 156 differentiable and bounded on  $[0, \infty)$  such that  $\limsup_{t \rightarrow \infty} h(t) > 0$ ,  $\liminf_{t \rightarrow \infty} h(t) < 0$ .  
 157 In addition, assume that  $h'(t) = O(t^\eta)$  as  $t \rightarrow \infty$ , here  $-1 < \eta < \alpha - \sigma - 1$ .

158 Then, an arbitrary nontrivial solution  $x(\cdot)$  of the initial value problem (40)–(41) is oscillatory.  
 159 Furthermore, its asymptotic behavior has the form  $O(t^{\max\{-\beta, \eta + \sigma - \alpha + 1\}})$  at infinity.

*Proof.* It is easy to check that for a function  $g(\cdot)$  satisfying the condition (B)<sup>''</sup>, it also satisfies the assumption (B) in Theorem 3.1. Thus, all non-trivial solutions of the system (40)–(41) are oscillatory. Next, we show that  $g(t) = O(t^{\eta + \sigma - \alpha + 1})$  as  $t \rightarrow \infty$ . Indeed, for  $t > 0$ , we have

$$g(t) = {}^C D_{0+}^\alpha (t+1)^\sigma h(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(s+1)^{\sigma-1} h(s) + (s+1)^\sigma h'(s)}{(t-s)^\alpha} ds. \quad (42)$$

Notice that  $h(\cdot)$  is bounded on  $[0, \infty)$ , there is a  $M_1 > 0$  such that

$$\begin{aligned} \left| \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(s+1)^{\sigma-1} h(s)}{(t-s)^\alpha} ds \right| &\leq \frac{M_1}{\Gamma(1-\alpha)} \int_0^t s^{\sigma-1} (t-s)^{-\alpha} ds \\ &\leq \frac{M_1}{\Gamma(1-\alpha)} t^{\sigma-\alpha} B(\sigma, 1-\alpha) \\ &\leq M_2 t^{\sigma-\alpha}, \end{aligned} \quad (43)$$

where  $M_2 := M_1 \frac{B(\sigma, 1-\alpha)}{\Gamma(1-\alpha)}$ . On the other hand, due to  $h'(t) = O(t^\eta)$  as  $t \rightarrow \infty$ , we can find a  $M_3 > 0$  and  $T > 1$  large enough so that  $|h'(t)| \leq M_3 t^\eta$  for all  $t \geq T$ . From this, for any  $t > T$ , we obtain

$$\begin{aligned} \left| \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(s+1)^\sigma h'(s)}{(t-s)^\alpha} ds \right| &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{(s+1)^\sigma |h'(s)|}{(t-s)^\alpha} ds \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_T^t \frac{(s+1)^\sigma |h'(s)|}{(t-s)^\alpha} ds \\ &\leq \frac{1}{\Gamma(1-\alpha)} \frac{(T+1)^\sigma}{(t-T)^\alpha} \int_0^T |h'(s)| ds \\ &\quad + \frac{M_3 (t+1)^\sigma}{\Gamma(1-\alpha)} \int_T^t s^\eta (t-s)^{-\alpha} ds \\ &\leq M_4 (t-T)^{-\alpha} + \frac{M_3 (t+1)^\sigma}{\Gamma(1-\alpha)} \int_0^t s^\eta (t-s)^{-\alpha} ds \\ &= M_4 (t-T)^{-\alpha} + \frac{M_3 (t+1)^\sigma}{\Gamma(1-\alpha)} t^{\eta-\alpha+1} B(1+\eta, 1-\alpha) \\ &\leq M_4 (t-T)^{-\alpha} + M_5 t^{\eta+\sigma-\alpha+1}, \end{aligned} \quad (44)$$

160 where  $M_4 := \frac{1}{\Gamma(1-\alpha)} (T+1)^\sigma \int_0^T |h'(s)| ds$ ,  $M_5 := \frac{2^\sigma M_3 B(\eta+1, 1-\alpha)}{\Gamma(1-\alpha)}$ . Because  $\alpha - \beta < \sigma < \alpha \leq 1$ ,  
 161  $-1 < \eta < \alpha - \sigma - 1$ , from (42), (43), and (44), it shows that  $g(t) = O(t^{\eta + \sigma - \alpha + 1})$  as  $t \rightarrow \infty$ .  
 162 Now, by [10, Theorem 5.2], we conclude that the asymptotic behavior of an arbitrary non-trivial  
 163 solution of the system (40)–(41) has the form  $O(t^{\max\{-\beta, \eta + \sigma - \alpha + 1\}})$  at infinity.  $\square$

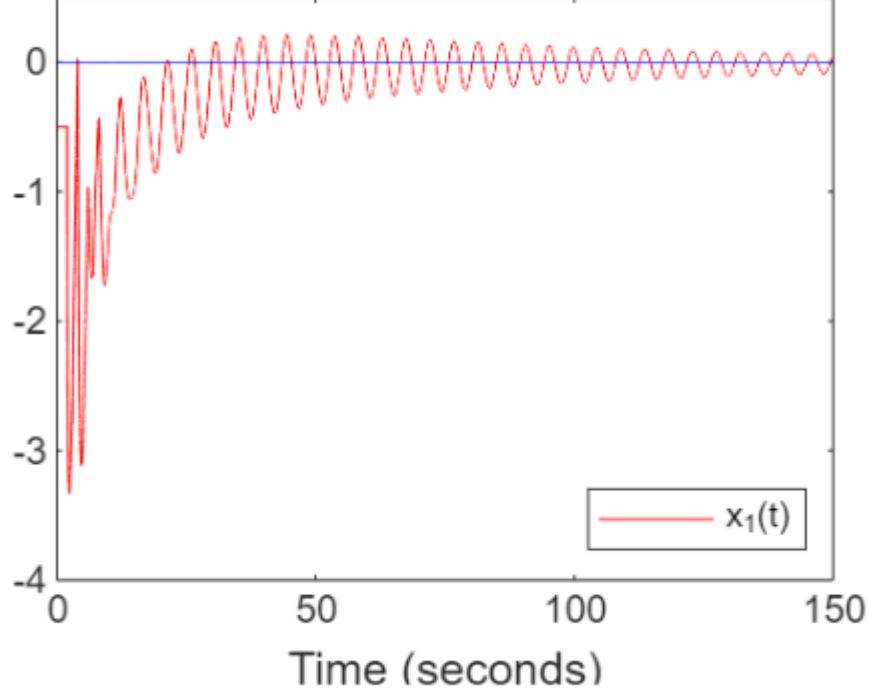


Figure 3: The orbit of the solution to equation (45) with the initial condition  $x(0) = -0.5$  on the interval  $[0, 150]$ .

*Example 4.5.* Consider the system (46)–(47) with  $\alpha = 2/3$ ,  $\beta = 1/2$ ,  $a = 2$ ,  $b = 1$ , and  $g(t) = {}^C D^{2/3}(t+1)^{1/3} \sin((t+1)^{1/4})$ . The system is rewritten as

$${}^C D_{0+}^{2/3} x(t) + 2 {}^C D_{0+}^{1/2} x(t) = -x(t) + {}^C D^{2/3}(t+1)^{1/3} \sin((t+1)^{1/4}), \quad t > 0. \quad (45)$$

164 In this case, we have  $\sigma = 1/3$  and  $h(t) = \sin((t+1)^{1/4})$ ,  $t \geq 0$ . It is obvious to see that  
 165  $\alpha - \beta = 1/6 < 1/3 = \sigma < 2/3 = \alpha$ ,  $h \in C^1[0, \infty)$ ,  $\limsup_{t \rightarrow \infty} h(t) = 1 > 0$ , and  $\liminf_{t \rightarrow \infty} h(t) =$   
 166  $-1 < 0$ . Moreover,  $h'(t) = \frac{1}{4}(t+1)^{-3/4} \cos((t+1)^{1/4})$ , thus  $|h'(t)| < \frac{1}{4}t^{-3/4}$  for all  $t > 0$ . From  
 167 this, we have  $h' = O(t^{-3/4})$  as  $t \rightarrow \infty$ . Notice that  $-1 < -3/4 = \eta < -2/3 = \alpha - \sigma - 1$ , it  
 168 deduces that  $g$  satisfies the assumption (B)” in Theorem 4.4. Therefore, for any  $x_0 \in \mathbb{R}$ , the  
 169 non-trivial solution  $\varphi(\cdot, x_0)$  of (45) is oscillatory and  $\lim_{t \rightarrow \infty} \varphi(t, x_0) = 0$ . Figure 3 shows the  
 170 orbit of the solution with the initial condition  $x(0) = -0.5$  on the interval  $[0, 150]$ .

### 171 4.3 Asymptotic behavior of oscillatory solutions of fractional-order nonlinear 172 equations

Consider the following fractional differential equation

$${}^C D_{0+}^{\alpha} x(t) + a {}^C D_{0+}^{\beta} x(t) = -bx(t) + \sum_{k=1}^m q_k(t) f_k(x(t)) + g(t), \quad t > 0, \quad (46)$$

$$x(0) = x_0 \in \mathbb{R}, \quad (47)$$

173 where  $0 < \beta < \alpha \leq 1$ ,  $a, b$  are positive constants,  $q_k : [0, \infty) \rightarrow \mathbb{R}$ ,  $k = 1, \dots, m$ , are continuous,  
 174 bounded and eventually non-negative. Suppose that  $g : [0, \infty) \rightarrow \mathbb{R}$  is continuous and satisfies  
 175 the condition (B)” in Theorem 4.4 and  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ ,  $k = 1, \dots, m$ , are continuous such that the  
 176 assumptions below are true.

177 (F1)  $f_k(0) = 0$ .

(F2)  $f_k$  is locally Lipschitz continuous at the origin and  $\lim_{r \rightarrow 0} \ell_{f_k}(r) = 0$ , where

$$\ell_{f_k}(r) = \sup_{|x|, |\hat{x}| \leq r, x \neq \hat{x}} \frac{|f_k(x) - f_k(\hat{x})|}{|x - \hat{x}|}.$$

178 (F3)  $xf_k(x) \leq 0$  for all  $x \in \mathbb{R}$ .

179 **Theorem 4.6.** Consider the initial value problem (46)–(47). Then for every  $\epsilon > 0$  small enough,  
 180 there exist positive parameters  $\delta_1 = \delta_1(\epsilon), \delta_2 = \delta_2(\epsilon) > 0$  such that if  $|g(t)| \leq \delta_1$  for all  $t \geq 0$   
 181 and  $|x_0| < \delta_2$ , the solution  $\varphi(\cdot, x_0)$  is oscillatory and  $|\varphi(t, x_0)| \leq \epsilon$  for all  $t \geq 0$ . Moreover,  
 182  $\lim_{t \rightarrow \infty} \varphi(t, x_0) = 0$ .

183 *Proof.* Define  $F(t, x) = -bx + \sum_{k=1}^m q_k(t)f_k(x)$  for all  $t \geq 0$ ,  $x \in \mathbb{R}$ . Since  $q_k$ ,  $k = 1, \dots, m$ ,  
 184 are eventually non-negative, it follows from (F3) that the function  $F$  satisfies the condition (A)  
 185 in Theorem 3.1. Moreover, the condition (B)’ in Corollary 3.2 is verified by the function  $g$ .  
 186 Hence, by Corollary 3.2, for any  $x_0 \in \mathbb{R}$ , the solution  $\varphi(\cdot, x_0)$  of the system (40)–(41) (if it exists  
 187 globally) is oscillatory.

We next show that for the initial condition  $x_0$  small enough and the inhomogeneous term  $g$   
 small uniformly on  $[0, \infty)$ , the solution  $\varphi(\cdot, x_0)$  exists globally and converges to the origin. The  
 approach proposed in [10, Theorem 5.7] will be applied to do that. We first recall some important  
 properties of the function  $G_{\alpha, \beta; a, b}^{\kappa}(t)$  given by

$$G_{\alpha, \beta; a, b}^{\kappa}(t) := \mathcal{L}^{-1} \left( \frac{s^{\kappa-1}}{s^{\alpha} + as^{\beta} + b} \right) (t) \quad t \geq 0, \kappa \in \{\alpha, \beta, 1\}.$$

Taking into account [10, Proposition 4.4 (ii) and (iii)], there is a positive constant  $K > 0$  such  
 that

$$|G_{\alpha, \beta; a, b}^1(t)| \leq \frac{K}{t^{1-\alpha}}, \quad t \in (0, 1), \quad |G_{\alpha, \beta; a, b}^1(t)| \leq \frac{K}{t^{1+\beta}}, \quad t \geq 1. \quad (48)$$

Furthermore, using the same arguments as in the proof of [10, Proposition 4.4 (v)], for any  
 $\gamma \in (0, 1)$ , we can find a constant  $K_{\gamma} > 0$  with

$$\sup_{t \geq 0} t^{\gamma} \int_0^t |G_{\alpha, \beta; a, b}^1(t-s)| s^{-\gamma} ds \leq K_{\gamma}. \quad (49)$$

We conclude that

$$\lim_{t \rightarrow \infty} \int_0^t G_{\alpha, \beta; a, b}^1(t-s)g(s)ds = 0.$$

Indeed, it deduces from the proof of Theorem 4.4 that  $g = O(t^{-\gamma})$  as  $t \rightarrow \infty$ , here  $\gamma =$   
 $-(\eta + \sigma - \alpha + 1) > 0$ . Thus, there is a constant  $M > 0$  and  $t_0 > 0$  large enough satisfying

$$|g(t)| \leq \frac{M}{t^{\gamma}}, \quad t \geq t_0. \quad (50)$$

Then, for  $t > t_0 + 1$ , we have

$$\begin{aligned} \left| \int_0^t G_{\alpha,\beta;a,b}^1(t-s)g(s)ds \right| &\leq \int_0^{t_0} |G_{\alpha,\beta;a,b}^1(t-s)||g(s)|ds + \int_{t_0}^t |G_{\alpha,\beta;a,b}^1(t-s)||g(s)|ds \\ &= I_1(t) + I_2(t). \end{aligned} \quad (51)$$

Let  $s \in [0, t_0]$ . By (48), we estimate

$$|G_{\alpha,\beta;a,b}^1(t-s)| \leq K(t-s)^{-1-\beta} \leq K(t-t_0)^{-1-\beta}.$$

From this,

$$I_1(t) \leq K(t-t_0)^{-1-\beta} \int_0^{t_0} |g(s)|ds = K_1(t-t_0)^{-1-\beta}, \quad (52)$$

where  $K_1 = K \int_0^{t_0} |g(s)|ds$ . To deal with  $I_2(t)$ , by virtue of (49) and (50), we see

$$\begin{aligned} I_2(t) &\leq M \int_{t_0}^t |G_{\alpha,\beta;a,b}^1(t-s)| s^{-\gamma} ds \\ &\leq M \int_0^t |G_{\alpha,\beta;a,b}^1(t-s)| s^{-\gamma} ds \\ &\leq MK_\gamma t^{-\gamma} = K_2 t^{-\gamma}. \end{aligned} \quad (53)$$

By combining (51), (52) and (53), for any  $t > t_0 + 1$ , it implies that

$$\left| \int_0^t G_{\alpha,\beta;a,b}^1(t-s)g(s)ds \right| \leq K_1(t-t_0)^{-1-\beta} + K_2 t^{-\gamma},$$

and thus

$$\lim_{t \rightarrow \infty} \int_0^t G_{\alpha,\beta;a,b}^1(t-s)g(s)ds = 0. \quad (54)$$

Fix  $\hat{\varepsilon} > 0$  small enough such that  $f_k, k = 1, \dots, m$  are Lipchitz continuous on  $B_{\hat{\varepsilon}}(0) := \{x \in \mathbb{R} : |x| \leq \hat{\varepsilon}\}$ . Take  $C > 0$  such that  $|q_k(t)| \leq C, k = 1, \dots, m$  for all  $t \geq 0$  and set

$$h_1(x) := -C \sum_{k=1}^m |f_k(x)| \quad \text{and} \quad h_2(x) := C \sum_{k=1}^m |f_k(x)|.$$

188 It is easy to see that  $h_1$  and  $h_2$  satisfy the assumptions (F1) and (F2).

Consider the initial value problems

$${}^C D_{0+}^\alpha x(t) + a {}^C D_{0+}^\beta x(t) = -bx(t) + h_1(x) + g(t), \quad t > 0, \quad (55)$$

$$x(0) = x_0, \quad (56)$$

and

$${}^C D_{0+}^\alpha x(t) + a {}^C D_{0+}^\beta x(t) = -bx(t) + h_2(x) + g(t), \quad t > 0, \quad (57)$$

$$x(0) = x_0. \quad (58)$$

189 Choosing  $\varepsilon > 0$  be small enough, for example,  $\varepsilon \leq \hat{\varepsilon}$  and

$$\max_{1 \leq k \leq m} \ell_{f_k}(\varepsilon) \int_0^\infty |G_{\alpha,\beta;a,b}^1(t)| dt < \frac{1}{2}.$$

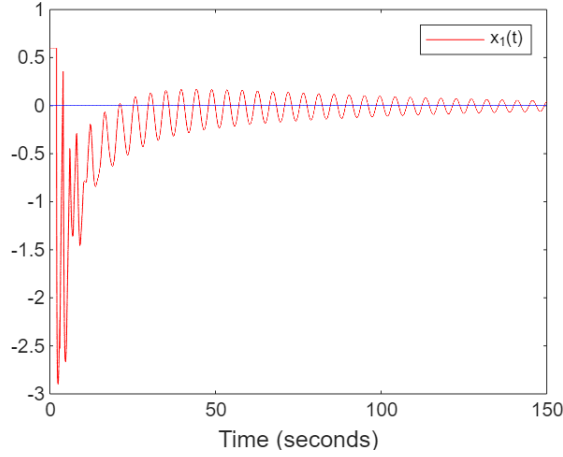


Figure 4: The orbit of the solution to equation (60) with the initial condition  $x(0) = 0.6$  on the interval  $[0, 150]$ .

In light of [10, Theorem 5.7], we can find positive parameters  $\delta_1, \delta_2 > 0$  such that if  $|g(t)| \leq \delta_1$  for all  $t \geq 0$  and  $|x_0| \leq \delta_2$  then  $|\varphi_1(t, x_0)|, |\varphi_2(t, x_0)| \leq \epsilon$  for all  $t \geq 0$  and  $\lim_{t \rightarrow \infty} \varphi_1(t, x_0) = \lim_{t \rightarrow \infty} \varphi_2(t, x_0) = 0$ , here  $\varphi_1(\cdot, x_0), \varphi_2(\cdot, x_0)$  are the solutions of the systems (55)–(56) and (57)–(58), respectively. On the other hand, according to Theorem 4.2 and Corollary 4.3, we claim that

$$\varphi_1(t, x_0) \leq \varphi(t, x_0) \leq \varphi_2(t, x_0), \quad t \geq 0. \quad (59)$$

190 This implies that the solution  $\varphi(\cdot, x_0)$  exists globally and  $\lim_{t \rightarrow \infty} \varphi(t, x_0) = 0$ . The proof is  
 191 complete.  $\square$

*Example 4.7.* Consider the system (46)–(47) with  $\alpha = 2/3$ ,  $\beta = 1/2$ ,  $a = 2$ ,  $b = 1$ ,  $m = 1$ ,  $q_1(t) = 1$ ,  $t \geq 0$ ,  $f_1(x) = -x^3$ ,  $x \in \mathbb{R}$  and  $g(t) = \frac{1}{100} {}^C D^{2/3}(t+1)^{1/3} \sin((t+1)^{1/4})$ . The system is rewritten as

$${}^C D_{0+}^{2/3} x(t) + 2 {}^C D_{0+}^{1/2} x(t) = -x(t) - 2x^2(t) + \frac{1}{100} {}^C D^{2/3}(t+1)^{1/3} \sin((t+1)^{1/4}), \quad t > 0. \quad (60)$$

192 It is obvious to see that the assumptions in Theorem 4.6 are satisfied. Thus, for the initial  
 193 condition  $x_0$  is small enough, the solution  $\varphi(\cdot, x_0)$  is oscillatory and  $\lim_{t \rightarrow \infty} \varphi(t, x_0) = 0$ . Figure  
 194 4 depicts the orbit of the solution to equation (60) with the initial condition  $x(0) = 0.6$  on the  
 195 interval  $[0, 150]$ .

## 196 5 On the asymptotic behavior of non-oscillatory solutions of 197 multi-order fractional differential equations

Consider the equation

$${}^C D_{0+}^\alpha x(t) + a {}^C D_{0+}^\beta x(t) + q(t)f(x) + \operatorname{sgn}(x(t))g(t) = 0, \quad t > 0, \quad (61)$$

$$x(0) = x_0, x^{(i)}(0) = x_i, i = 1, 2, \dots, n-1, \quad (62)$$

198 where  $n - 1 < \beta < \alpha \leq [\alpha] = n$ ,  $n \geq 3$ ,  $a, x_i \in \mathbb{R}$ ,  $i = 0, 1, 2, \dots, n - 1$ ,  $a > 0$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  
 199  $q : [0, \infty) \rightarrow \mathbb{R}$ , and  $g : [0, \infty) \rightarrow \mathbb{R}$  are continuous.

200 In this section, the following restrictions are imposed on the functions  $f(\cdot)$ ,  $q(\cdot)$ , and  $g(\cdot)$ .

201 (G1)  $xf(x) > 0$  for all  $x \neq 0$ .

202 (G2) There are positive constants  $c_1, \eta > 0$  such that  $|f(x)| \geq c_1|x|^\eta$  for all  $|x| \geq 1$ .

203 (G3) There exist positive constants  $c_2 > 0$ ,  $\mu \in (0, \frac{n-2}{n-\beta})$  with  $|f(x)| \geq c_2|x|^\mu$  for all  $|x| < 1$ .

204 (H) There are  $T_1, c_3 > 0$  so that  $q(t) \geq c_3 > 0$  for all  $t \geq T_1$ .

(K) There are  $\sigma \in (n - \beta, 1)$ ,  $c_4 \in (0, \infty)$  and  $T_2 > 0$  such that

$$g(t) \geq c_4 t^{n-\beta-\sigma}, \quad t \geq T_2.$$

205 Under the assumptions mentioned above, we aim to study the asymptotic behavior of the non-  
 206 oscillatory solutions of the higher-order fractional differential equation (61). Our main contri-  
 207 butions are as follows.

**Theorem 5.1.** Assume that the assumptions (G1), (G2), (H), (K) are true and  $n$  is odd. Let  $x(\cdot)$  be a non-oscillatory solution of the initial value problem (61)–(62). Then,

$$x(t) = O(t^{\alpha-n}) \text{ as } t \rightarrow \infty.$$

**Theorem 5.2.** Assume that the assumptions (G1), (G2), (G3), (H), (K) are true and  $n$  is even. Let  $x(\cdot)$  be a non-oscillatory solution of the initial value problem (61)–(62). Then,

$$x(t) = O(t^{\alpha-n}) \text{ as } t \rightarrow \infty.$$

208 The proof of the above results is long and requires much preparation. To make the presentation  
 209 clear and easy to follow, we have divided it into a sequence of lemmas.

Throughout this part, the following notations will be used:

$$\begin{aligned}
 y_1(t) &= I_{0+}^{m-\alpha} x(t) + a I_{0+}^{n-\beta} x(t) \\
 &= I_{0+}^{n-\alpha} \left( x(t) + a I_{0+}^{\alpha-\beta} x(t) \right), \\
 y_2(t) &= {}^C D_{0+}^{\alpha-n+1} x(t) + a {}^C D_{0+}^{\beta-n+1} x(t) + \frac{x_0}{\Gamma(n-\alpha)} t^{n-\alpha-1} + \frac{a x_0}{\Gamma(n-\beta)} t^{n-\beta-1}, \\
 y_i(t) &= {}^C D_{0+}^{\alpha-n-1+i} x(t) + a {}^C D_{0+}^{\beta-n-1+i} x(t) \\
 &\quad + \sum_{k=0}^{i-2} \frac{x_k \prod_{j=0}^{i-2-k} (n-\alpha-i+2+k+j)}{\Gamma(n-\alpha+1)} t^{n-\alpha-i+1+k} \\
 &\quad + \sum_{k=0}^{i-2} \frac{a x_k \prod_{j=0}^{i-2-k} (n-\beta-i+2+k+j)}{\Gamma(n-\beta+1)} t^{n-\beta-i+1+k}, \quad i = 3, \dots, n, \quad t > 0.
 \end{aligned} \tag{63}$$

**Lemma 5.3.** Let  $x(\cdot)$  be a solution of the system (61)–(62) and  $y_i, i = 1, 2, 3, \dots, n$  are defined as in (63). Then,  $y_1 \in AC[0, \infty)$ ,  $y_i \in C(0, \infty)$ ,  $i = 2, 3, \dots, n$ , and

$$\begin{aligned} y'_i(t) &= y_{i+1}(t), \quad t > 0, \quad i = 1, 2, \dots, n-1, \\ y'_n(t) &= -q(t)f(x(t)) - \operatorname{sgn}(x(t))g(t) + \sum_{k=0}^{n-1} \frac{a_k \prod_{j=0}^{n-1-k} (1 - \alpha + k + j)}{\Gamma(n - \alpha + 1)} t^{-\alpha+k} \\ &\quad + \sum_{k=0}^{n-1} \frac{aa_k \prod_{j=0}^{n-1-k} (1 - \beta + k + j)}{\Gamma(n - \beta + 1)} t^{-\beta+k}, \quad t > 0. \end{aligned} \quad (64)$$

210 In addition, if (G1), (H), (K) are satisfied and  $x(\cdot)$  is eventually positive, then  $y_n(\cdot)$  is mono-  
211 tonically decreasing and  $y_i, i = 1, 2, \dots, n-1$ , are monotonic on the interval  $[T, \infty)$  with some  
212  $T > 0$  large enough.

*Proof.* By a simple computation, for  $k \in \mathbb{N} \cup \{0\}$ , we have

$$\begin{aligned} I_{0+}^{n-\alpha} t^k &= \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} s^k ds \\ &= \frac{1}{\Gamma(n - \alpha)} \int_0^1 t^{n-\alpha-1} (1 - u)^{n-\alpha-1} t^k u^k t du \\ &= \frac{1}{\Gamma(n - \alpha)} t^{n-\alpha+k} B(k + 1, n - \alpha) \\ &= \frac{k!}{\Gamma(n - \alpha + 1 + k)} t^{n-\alpha+k}, \quad t > 0, \end{aligned}$$

where  $B(\cdot, \cdot)$  is the Beta function, and thus

$$I_{0+}^{n-\beta} t^k = \frac{k!}{\Gamma(n - \beta + 1 + k)} t^{n-\beta+k}, \quad \forall k \in \mathbb{N} \cup \{0\}, \quad t > 0.$$

To simplify notation, we use the symbol  $D$  instead of  $\frac{d}{dt}$ . First, for any  $t > 0$ , then

$$\begin{aligned} y'_1(t) &= D I_{0+}^{n-\alpha} x(t) + a D I_{0+}^{n-\beta} x(t) \\ &= D I_{0+}^{1-(\alpha-n+1)} (x(t) - x_0) + a D I_{0+}^{1-(\beta-n+1)} (x(t) - x_0) \\ &\quad + D \left( I_{0+}^{n-\alpha} x_0 + a I_{0+}^{n-\beta} x_0 \right) \\ &= {}^C D^{\alpha-n+1}_{0+} x(t) + a {}^C D^{\beta-n+1}_{0+} x(t) + \frac{a_0}{\Gamma(n - \alpha)} t^{n-\alpha-1} + \frac{a a_0}{\Gamma(n - \beta)} t^{n-\beta-1} \\ &= y_2(t). \end{aligned}$$

Next, by direct computation, it is not difficult to check that

$$\begin{aligned}
y_2'(t) &= D \left( {}^C D_{0+}^{\alpha-n+1} x(t) \right) + a D \left( {}^C D_{0+}^{\beta-n+1} x(t) \right) \\
&\quad + \frac{x_0(n-\alpha-1)}{\Gamma(n-\alpha)} t^{n-\alpha-2} + \frac{ax_0(n-\beta-1)}{\Gamma(n-\beta)} t^{n-\beta-2} \\
&= D \left( D I_{0+}^{1-(\alpha-n+1)} (x(t) - x_0) \right) + a D \left( D I_{0+}^{1-(\beta-n+1)} (x(t) - x_0) \right) \\
&\quad + \frac{x_0(n-\alpha-1)}{\Gamma(n-\alpha)} t^{n-\alpha-2} + \frac{ax_0(n-\beta-1)}{\Gamma(n-\beta)} t^{n-\beta-2} \\
&= D^{(2)} \left( I_{0+}^{2-(\alpha-n+2)} (x(t) - x_0 - x_1 t) \right) + a D^{(2)} \left( I_{0+}^{2-(\beta-n+2)} (x(t) - x_0 - x_1 t) \right) \\
&\quad + D^{(2)} \left( I_{0+}^{n-\alpha} a_1 t \right) + a D^{(2)} \left( I_{0+}^{n-\beta} x_1 t \right) + \frac{x_0(n-\alpha-1)}{\Gamma(n-\alpha)} t^{n-\alpha-2} + \frac{aa_0(n-\beta-1)}{\Gamma(n-\beta)} t^{n-\beta-2} \\
&= {}^C D_{0+}^{\alpha-n+2} x(t) + a {}^C D_{0+}^{\beta-n+2} x(t) + \frac{x_1}{\Gamma(n-\alpha)} t^{n-\alpha-1} + \frac{ax_1}{\Gamma(n-\beta)} t^{n-\beta-1} \\
&\quad + \frac{x_0(n-\alpha-1)(n-\alpha)}{\Gamma(n-\alpha+1)} t^{n-\alpha-2} + \frac{ax_0(n-\beta-1)(n-\beta)}{\Gamma(n-\beta+1)} t^{n-\beta-2} \\
&= {}^C D_{0+}^{\alpha-n-1+3} x(t) + a {}^C D_{0+}^{\beta-n-1+3} x(t) \\
&\quad + \sum_{k=0}^{(3-2)} \frac{x_k \prod_{j=0}^{3-2-k} (n-\alpha-3+2+k+j)}{\Gamma(n-\alpha+1)} t^{n-\alpha-3+1+k} \\
&\quad + \sum_{k=0}^{(3-2)} \frac{ax_k \prod_{j=0}^{3-2-k} (n-\beta-3+2+k+j)}{\Gamma(n-\beta+1)} t^{n-\beta-3+1+k} \\
&= x_3(t) \text{ for all } t > 0.
\end{aligned}$$



Using the same arguments, for  $i = 3, \dots, n-1$  and  $t > 0$ , we also obtain

$$\begin{aligned}
y'_i(t) &= D \left( {}^C D_{0+}^{\alpha-n-1+i} x(t) \right) + aD \left( {}^C D_{0+}^{\beta-n-1+i} x(t) \right) \\
&+ \sum_{k=0}^{i-2} \frac{x_k(n-\alpha-i+1+k) \prod_{j=0}^{i-2-k} (n-\alpha-i+2+k+j)}{\Gamma(n-\alpha+1)} t^{n-\alpha-i+k} \\
&+ \sum_{k=0}^{i-2} \frac{ax_k(n-\beta-i+1+k) \prod_{j=0}^{i-2-k} (n-\beta-i+2+k+j)}{\Gamma(n-\beta+1)} t^{n-\beta-i+k} \\
&= D \left( D^{(i-1)} I^{(i-1-(\alpha-n-1+i))} \left( x(t) - \sum_k \frac{x_k t^k}{k!} \right) \right) \\
&\quad + aD \left( D^{(i-1)} I^{(i-1-(\beta-n-1+i))} \left( x(t) - \sum_k \frac{x_k t^k}{k!} \right) \right) \\
&+ \sum_{k=0}^{i-2} \frac{x_k(n-\alpha+k-i+1) \prod_{j=1}^{i-1-k} (n-\alpha-i+1+k+j)}{\Gamma(n-\alpha+1)} t^{n-\alpha-i+k} \\
&+ \sum_{k=0}^{i-2} \frac{ax_k(n-\beta+k-i+1) \prod_{j=1}^{i-1-k} (n-\beta-i+1+k+j)}{\Gamma(n-\beta+1)} t^{n-\beta-i+k} \\
&= D^{(i)} I_{0+}^{i-(\alpha-n+i)} \left( x(t) - \sum_{k=0}^{i-1} \frac{x_k t^k}{k!} \right) + aD^{(i)} I_{0+}^{i-(\beta-n+i)} \left( x(t) - \sum_{k=0}^{i-1} \frac{x_k t^k}{k!} \right) \\
&\quad + D^{(i)} \left( I_{0+}^{n-\alpha} \left( \frac{x_{i-1}}{(i-1)!} t^{i-1} \right) + I_{0+}^{n-\beta} \left( \frac{ax_{i-1}}{(i-1)!} t^{i-1} \right) \right) \\
&+ \sum_{k=0}^{i-2} \frac{x_k \prod_{j=0}^{i-1-k} (n-\alpha-i+1+k+j)}{\Gamma(n-\alpha+1)} t^{n-\alpha-i+k} \\
&\quad + \sum_{k=0}^{i-2} \frac{ax_k \prod_{j=0}^{i-1-k} (n-\beta-i+1+k+j)}{\Gamma(n-\beta+1)} t^{n-\beta-i+k} \\
&= {}^C D_{0+}^{\alpha-n+i} x(t) + a {}^C D_{0+}^{\beta-n+i} x(t) + \frac{x_{i-1}}{\Gamma(n-\alpha)} t^{n-\alpha-1} + \frac{ax_{i-1}}{\Gamma(n-\beta)} t^{n-\beta-1} \\
&+ \sum_{k=0}^{i-2} \frac{x_k \prod_{j=0}^{i-1-k} (n-\alpha-i+1+k+j)}{\Gamma(n-\alpha+1)} t^{n-\alpha-i+k} \\
&+ \sum_{k=0}^{i-2} \frac{ax_k \prod_{j=0}^{i-1-k} (n-\beta-i+1+k+j)}{\Gamma(n-\beta+1)} t^{n-\beta-i+k} \\
&= {}^C D_{0+}^{\alpha-n+i} x(t) + a {}^C D_{0+}^{\beta-n+i} x(t) + \sum_{k=0}^{i-1} \frac{x_k \prod_{j=0}^{i-1-k} (n-\alpha-i+1+k+j)}{\Gamma(n-\alpha+1)} t^{n-\alpha-i+k} \\
&\quad + \sum_{k=0}^{i-1} \frac{ax_k \prod_{j=0}^{i-1-k} (n-\beta-i+1+k+j)}{\Gamma(n-\beta+1)} t^{n-\beta-i+k} \\
&= {}^C D_{0+}^{\alpha-n-1+(i+1)} x(t) + a {}^C D_{0+}^{\beta-n-1+(i+1)} x(t) \\
&+ \sum_{k=0}^{(i+1)-2} \frac{x_k \prod_{j=0}^{(i+1)-2-k} (n-\alpha-(i+1)+2+k+j)}{\Gamma(n-\alpha+1)} t^{n-\alpha-(i+1)+1+k} \\
&+ \sum_{k=0}^{(i+1)-2} \frac{ax_k \prod_{j=0}^{(i+1)-2-k} (n-\beta-(i+1)+2+k+j)}{\Gamma(n-\beta+1)} t^{n-\beta-(i+1)+1+k} \\
&= x_{i+1}(t).
\end{aligned}$$

Finally, for  $t > 0$ , from the definition of  $y_n(\cdot)$  and the observations shown above, then

$$\begin{aligned}
y'_n(t) &= {}^C D_{0+}^\alpha x(t) + a {}^C D_{0+}^\beta x(t) + \sum_{k=0}^{n-1} \frac{x_k \prod_{j=0}^{n-1-k} (1 - \alpha + k + j)}{\Gamma(n - \alpha + 1)} t^{-\alpha+k} \\
&\quad + \sum_{k=0}^{n-1} \frac{ax_k \prod_{j=0}^{n-1-k} (1 - \beta + k + j)}{\Gamma(n - \beta + 1)} t^{-\beta+k} \\
&= -q(t)f(x(t)) - \operatorname{sgn}(x(t))g(t) + \sum_{k=0}^{n-1} \frac{x_k \prod_{j=0}^{n-1-k} (1 - \alpha + k + j)}{\Gamma(n - \alpha + 1)} t^{-\alpha+k} \\
&\quad + \sum_{k=0}^{n-1} \frac{ax_k \prod_{j=0}^{n-1-k} (1 - \beta + k + j)}{\Gamma(n - \beta + 1)} t^{-\beta+k}, \tag{65}
\end{aligned}$$

213 which implies that (64) is verified for all  $t > 0$ . On the other hand, due to  $\beta - n - 1 + i <$   
214  $\alpha - n - 1 + i \leq \alpha - 1 \leq n - 1$  for  $i = 2, 3, \dots, n$  and  $x \in C^{n-1}[0, \infty)$ , it follows from [3,  
215 Theorem 8] that  ${}^C D_{0+}^{\alpha-n-1+i} x, {}^C D_{0+}^{\beta-n-1+i} x \in C[0, \infty)$ ,  $i = 2, 3, \dots, n$ . Therefore,  $y_i \in C(0, \infty)$ ,  
216  $i = 2, 3, \dots, n$ . Notice that by  $y'_1(t) = y_2(t)$  for all  $t > 0$ ,  $y_2(\cdot) \in C(0, \infty)$  and  $y_2(\cdot)$  is integrable  
217 on finite subintervals of  $[0, \infty)$ , we conclude that  $y_1 \in AC[0, \infty)$ .

Now, under the added assumptions (G1), (H), (K) and suppose that  $x(\cdot)$  is eventually positive, we can find a parameter  $t_0 > 0$  such that  $x(t) > 0$ ,  $f(x(t)) > 0$ ,  $q(t) > 0$  and  $g(t) \geq c_4 t^{n-\beta-\sigma}$  for all  $t \geq t_0$ . From this together with the fact  $-\alpha + k < -\beta + k \leq n - \beta - 1 < 0$  for all  $k = 0, 1, 2, \dots, n - 1$ , there are constants  $M > 0$  and  $t_1 > t_0$  so that

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{x_k \prod_{j=0}^{n-1-k} (1 - \alpha + k + j)}{\Gamma(n - \alpha + 1)} t^{-\alpha+k} \\
&\quad + \sum_{k=0}^{n-1} \frac{ax_k \prod_{j=0}^{n-1-k} (1 - \beta + k + j)}{\Gamma(n - \beta + 1)} t^{-\beta+k} \\
&\leq Mt^{n-\beta-1}, \quad \forall t \geq t_1. \tag{66}
\end{aligned}$$

Let  $T = \max \left\{ \left( \frac{M}{c_4} \right)^{\frac{1}{1-\sigma}}, t_1 \right\}$ , then

$$Mt^{n-\beta-1} < c_4 t^{n-\beta-\sigma} < g(t), \quad \forall t > T. \tag{67}$$

218 By combining (65), (66), (67) and the fact  $\operatorname{sgn}(x(t)) = 1$ ,  $q(t)f(x(t)) > 0$  for all  $t > T$  yields  
219  $y'_n(t) < 0$  on  $[T, \infty)$ . This together with the relation  $y'_i(t) = y_{i+1}(t)$ ,  $t > 0$ ,  $i = 1, 2, \dots, n - 1$   
220 leads to that  $y_n(\cdot)$  is monotonically decreasing and  $y_i$ ,  $i = 1, 2, \dots, n - 1$  are monotonic on  
221  $[T, \infty)$ . The proof finishes.  $\square$

222 **Lemma 5.4.** Consider the equation (61). Suppose that (G1), (H), (K) are true. If  $x(\cdot)$  is an  
223 eventually positive solution and  $y_1(\cdot)$  is eventually negative, then  $\lim_{t \rightarrow \infty} y_1(t) = 0$  and  $y_2(\cdot)$  is  
224 eventually positive.

*Proof.* Let  $x(\cdot)$  be a solution of the equation (61) on  $[0, \infty)$ . Assume that it is eventually positive and  $y_1(\cdot)$  is eventually negative. From this, there is a  $t_0 > 0$  large enough such  $x(t) > 0$ ,  $y_1(t) < 0$

for all  $t \geq t_0$ . Taking  $M := \int_0^{t_0} |x(s)| ds$ . For any  $t > t_0$ , it is easy to see

$$\begin{aligned} I_{0^+}^{n-\alpha} x(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x(s) ds \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^{t_0} (t-s)^{n-\alpha-1} x(s) ds + \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t (t-s)^{n-\alpha-1} x(s) ds \\ &\geq \frac{1}{\Gamma(n-\alpha)} \int_0^{t_0} (t-s)^{n-\alpha-1} x(s) ds. \end{aligned} \quad (68)$$

Notice that  $(t-s)^{n-\alpha-1} x(s) \geq -(t-t_0)^{n-\alpha-1} |x(s)|$  for all  $s \in [0, t_0]$  and  $t > t_0$ . This together with (68) lead to

$$I_{0^+}^{n-\alpha} x(t) \geq \frac{-M}{\Gamma(n-\alpha)} (t-t_0)^{n-\alpha-1}, \quad \forall t > t_0. \quad (69)$$

By the same arguments, it is also true that

$$I_{0^+}^{n-\beta} x(t) \geq \frac{-M}{\Gamma(n-\beta)} (t-t_0)^{n-\beta-1}, \quad \forall t > t_0. \quad (70)$$

From (63), (69) and (70), we get

$$\frac{-M}{\Gamma(n-\alpha)} (t-t_0)^{n-\alpha-1} + \frac{-aM}{\Gamma(n-\beta)} (t-t_0)^{n-\beta-1} \leq x_1(t) < 0, \quad \forall t > t_0,$$

225 and thus  $\lim_{t \rightarrow \infty} x_1(t) = 0$ . On the other hand, it follows from Lemma 5.3 that  $y_1(\cdot)$  is strictly  
226 monotonic on  $[T, \infty)$  for some  $T > 0$  large enough. Thus  $y_1(\cdot)$  is strictly increasing on that  
227 interval. Due to  $y_1'(t) = y_2(t)$  for all  $t > 0$ , we conclude that  $y_2(\cdot)$  is eventually positive.  $\square$

**Lemma 5.5.** Suppose that (G1), (G2), (H), (K) hold. Let  $x(\cdot)$  be an eventually positive solution of (61) such that  $y_n(\cdot)$  is also eventually positive, then

$$\limsup_{t \rightarrow \infty} x(t) < \infty.$$

*Proof.* Because (F1), (F2), (Q), (G) are true,  $x(\cdot), y_n(\cdot)$  are eventually positive, there is a  $t_0 > 0$  so that  $x(t) > 0, y_n(t) > 0, f(x(t)) > 0, q(t) \geq c_3 > 0$  and  $g(t) \geq c_4 t^{n-\beta-\sigma}$  for all  $t \geq t_0$ . From this, by (65), (66) and (67), we can find  $t_1 > t_0$  so that

$$y_n'(t) < -q(t)f(x(t)), \quad \forall t > t_1.$$

Moreover,

$$\begin{aligned} y_n(t_1) &> y_n(t_1) - y_n(t) = - \int_{t_1}^t y_n'(s) ds \\ &> \int_{t_1}^t q(s)f(x(s)) ds \\ &\geq c_3 \int_{t_1}^t f(x(s)) ds, \quad \forall t > t_1. \end{aligned}$$

228 Letting  $t \rightarrow \infty$ , then

$$\int_{t_1}^{\infty} f(x(s)) ds < \frac{y_n(t_1)}{c_3} < \infty, \quad (71)$$

which implies that  $f(x(\cdot))$  is bounded on  $[t_1, \infty)$ . This together with (G2) shows that  $x(\cdot)$  is also bounded on  $[t_1, \infty)$  and thus

$$\limsup_{t \rightarrow \infty} x(t) < \infty.$$

229

□

**Lemma 5.6.** Let  $x(\cdot)$  be an eventually positive solution of (61) satisfying

$$y_2(t) \geq c_5 > 0, \forall t \geq T_3 > 0.$$

Then,

$$\limsup_{t \rightarrow \infty} x(t) = \infty.$$

*Proof.* From (63) and Lemma 2.3, we have

$${}^C D_{0+}^{n-\alpha} y_1(t) = x(t) + a I_{0+}^{\alpha-\beta} x(t), \forall t > 0. \quad (72)$$

Define  $x^*(t) = \sup_{s \in [0, t]} x(s)$  for each  $t > 0$ , then

$$\begin{aligned} x(t) + a I_{0+}^{\alpha-\beta} x(t) &= x(t) + \frac{a}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds \\ &\leq x^*(t) + \frac{a x^*(t)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta} \\ &= x^*(t) \left( 1 + \frac{a}{\Gamma(\alpha-\beta)} t^{\alpha-\beta} \right), \forall t > 0. \end{aligned} \quad (73)$$

On the other hand, by the assumption of the lemma, there is a  $t_0 > 0$  such that  $x(t) > 0$ ,  $y_2(t) \geq c_5 > 0$  for all  $t \geq t_0$ . Define  $M_1 := \int_0^{t_0} |y_2(s)| ds$ . It follows from Lemma 2.2 that

$$\begin{aligned} {}^C D_{0+}^{n-\alpha} y_1(t) &= \frac{1}{\Gamma(1+\alpha-n)} \int_0^t \frac{y_2(s)}{(t-s)^{n-\alpha}} ds \\ &= \frac{1}{\Gamma(1+\alpha-n)} \left( \int_0^{t_0} \frac{y_2(s)}{(t-s)^{n-\alpha}} ds + \int_{t_0}^t \frac{y_2(s)}{(t-s)^{n-\alpha}} ds \right) \\ &\geq \frac{1}{\Gamma(1+\alpha-n)} \left( \frac{-1}{(t-t_0)^{n-\alpha}} \int_0^{t_0} |x(s)| ds + \frac{c_5}{\alpha-n+1} (t-t_0)^{\alpha-n+1} \right) \\ &= \frac{1}{\Gamma(1+\alpha-n)} \left( \frac{-M_1}{(t-t_0)^{n-\alpha}} + \frac{c_5}{\alpha-n+1} (t-t_0)^{\alpha-n+1} \right), \forall t > t_0. \end{aligned} \quad (74)$$

Using (72)–(74), for  $t > t_0$ , then

$$x^*(t) \left( 1 + \frac{a}{\Gamma(\alpha-\beta)} t^{\alpha-\beta} \right) \geq \frac{1}{\Gamma(1+\alpha-n)} \left( \frac{-M_1}{(t-t_0)^{n-\alpha}} + \frac{c_5}{\alpha-n+1} (t-t_0)^{\alpha-n+1} \right),$$

which gives

$$x^*(t) \geq \frac{\frac{1}{\Gamma(1+\alpha-n)} \left( \frac{-M_1}{(t-t_0)^{n-\alpha}} + \frac{c_5}{\alpha-n+1} (t-t_0)^{\alpha-n+1} \right)}{1 + \frac{a}{\Gamma(\alpha-\beta)} t^{\alpha-\beta}}, \forall t > t_0,$$

and thus  $x^*(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . In particular, it follows that

$$\limsup_{t \rightarrow \infty} x(t) = \infty.$$

230

□

231 **Lemma 5.7.** Suppose that (G1), (G2), (H), (K) are true. The following statements hold.

(i) If  $n$  is odd, all eventually positive solutions of (61) satisfy

$$(-1)^{i+1}y_i(t) > 0, \quad i = 1, 2, 3, \dots, n, \quad \text{for } t \text{ large enough.} \quad (75)$$

(ii) If  $n$  is even, all eventually positive solutions of (61) satisfy

$$(-1)^i y_i(t) > 0, \quad i = 1, 2, 3, \dots, n, \quad \text{for } t \text{ large enough,} \quad (76)$$

or

$$y_1(t) > 0, \quad y_2(t) > 0 \quad \text{and} \quad (-1)^i y_i(t) > 0, \quad i = 3, \dots, n, \quad \text{for } t \text{ large enough.} \quad (77)$$

*Proof.* Let  $x(\cdot)$  be an eventually positive solution of (61). By Lemma 5.3, there is a  $T > 0$  such that  $y'_n(t) < 0$  and  $y_i, i = 1, 2, 3, \dots, n$ , are strictly monotonic on  $[T, \infty)$ . From this, we can find  $t_0 > T$  with  $y'_n(t) < 0$  and  $y_i(\cdot), i = 1, 2, 3, \dots, n$ , do not change sign on  $[t_0, \infty)$ . We first point out that if there exists  $k \in \{2, 3, \dots, n\}$  satisfying  $y_{k-1}(t)y_k(t) > 0$  for all  $t \geq t_0$ , then

$$y_i(t)y_{i+1}(t) > 0, \quad i = 1, 2, \dots, k-1, \quad t \geq t_0.$$

Indeed, without loss of generality, we assume  $y_{k-1}(t) > 0, y_k(t) > 0$  for all  $t \geq t_0$ . By induction, we only need to prove  $y_{k-2}(t) > 0$  for all  $t \geq t_0$ . Due to  $y'_{k-1}(t) = y_k(t) > 0$  for all  $t \geq t_0$ , the function  $y_{k-1}(\cdot)$  is positive and strictly increasing on  $[t_0, \infty)$ . This implies that

$$y_{k-1}(t) \geq M_3, \quad t \geq t_1,$$

for some  $M_3 > 0$  and  $t_1 > t_0$  and thus

$$y_{k-2}(t) = \int_{t_1}^t y_{k-1}(s)ds + y_{k-2}(t_1) \geq M_3(t - t_1) + y_{k-2}(t_1), \quad \forall t > t_1. \quad (78)$$

232 From (78),  $y_{k-2}(t) > 0$  for all  $t > \frac{M_3 t_1 - y_{k-2}(t_1)}{M_3}$  which together the fact  $x_{k-2}(\cdot)$  does not change  
233 sign on  $[t_0, \infty)$  gives  $y_{k-2}(t) > 0$  for all  $t \geq t_0$ .

234 **Case I:** there exists  $k \in \{2, 3, \dots, n\}$  satisfying  $y_{k-1}(t)y_k(t) > 0$  for all  $t \geq t_0$ . In this case, only  
235 the following possibilities occur.

(i) If  $k = n$ , then

$$y_i(t) > 0, \quad t \geq t_0, \quad i = 1, 2, \dots, n, \quad (79)$$

or

$$y_i(t) < 0, \quad t \geq t_0, \quad i = 1, 2, \dots, n. \quad (80)$$

(ii) If  $2 \leq k \leq n-1$ , then for  $t \geq t_0$ ,

$$y_i(t) > 0, \quad i = 1, 2, \dots, k, \quad y_i(t)y_{i+1}(t) < 0, \quad i = k, \dots, n-1, \quad (81)$$

or

$$y_i(t) < 0, \quad i = 1, 2, \dots, k, \quad y_i(t)y_{i+1}(t) < 0, \quad i = k, \dots, n-1. \quad (82)$$

236 **Case II:** there is not  $k \in \{2, 3, \dots, n\}$  satisfying  $y_{k-1}(t)y_k(t) > 0$  for all  $t \geq t_0$ . It is easy to  
 237 check that then the following statement is true.

(iii)

$$(-1)^i y_i(t) > 0, \quad t \geq t_0, \quad i = 1, 2, \dots, n, \quad (83)$$

or

$$(-1)^{i+1} y_i(t) > 0, \quad t \geq t_0, \quad i = 1, 2, \dots, n. \quad (84)$$

238 Based on Lemma 5.4, the possibilities (80) and (82) do not happen. According to Lemma 5.5  
 239 and Lemma 5.6, the possibility (79) also does not happen.

240 Next, if  $y_n(t) < 0$  for all  $t \geq t_0$ , by the same argument as in the proof of (78), we claim that  
 241  $y_{n-1}(t) < 0$  for all  $t \geq t_0$ . Thus, the possibilities (81), (83) and (84) occur if and only if  $y_n(t) > 0$   
 242 for all  $t \geq 0$ .

243 Consider (81). It follows from Lemma 5.5 and Lemma 5.6 that  $k = 2$ . Hence,  $y_1(t), y_2(t) > 0$ ,  
 244  $(-1)^i y_i(t) > 0$  for all  $t \geq t_0, i = 3, \dots, n$ . This combines with  $y_n(t) > 0, t > t_0$  yields  $(-1)^n > 0$   
 245 and thus  $n$  is even. The statement (77) is checked.

246 Concerning with (83), since  $y_n(t) > 0$  for  $t \geq t_0$ , it implies that  $n$  is even and thus the assertion  
 247 (76) is true.

248 For the case (84), it is obvious that  $n$  is odd. The assertion (75) is verified.  $\square$

249 **Lemma 5.8.** Let  $x(\cdot)$  be an eventually positive solution of (61). Assume that one of the  
 250 following assumptions is true.

(i) For all  $i = 1, 2, 3, \dots, n$ ,

$$(-1)^{i+1} y_i(t) > 0 \text{ for } t \text{ large enough.}$$

(ii) For all  $i = 1, 2, 3, \dots, n$ ,

$$(-1)^i y_i(t) > 0 \text{ for } t \text{ large enough,}$$

(iii)  $y_1(\cdot)$  is bounded. Moreover,

$$y_1(t) > 0, y_2(t) > 0 \text{ and } (-1)^i y_i(t) > 0, \quad i = 3, \dots, n, \text{ for } t \text{ large enough.}$$

Then,

$$x(t) = O(t^{\alpha-n}) \text{ as } t \rightarrow \infty.$$

*Proof.* (a) Suppose that the assumption (i) is true. There exists a  $t_0 > 0$  such that

$$x(t) > 0, \quad y_1(t) > 0, \quad y_2(t) < 0, \quad t \geq t_0.$$

From (63) and Lemma 2.3, it leads to

$${}^C D_{0+}^{n-\alpha} y_1(t) = x(t) + a I_{0+}^{\alpha-\beta} x(t), \quad \forall t > 0. \quad (85)$$

Taking  $M := \int_0^{t_0} |x(s)|ds$ . For  $t > t_0$ , we have

$$\begin{aligned}
x(t) + aI_{0+}^{\alpha-\beta} x(t) &= x(t) + \frac{a}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds \\
&\geq x(t) + \frac{a}{\Gamma(\alpha-\beta)} \int_0^{t_0} (t-s)^{\alpha-\beta-1} x(s) ds \\
&\geq x(t) - \frac{a}{\Gamma(\alpha-\beta)} (t-t_0)^{\alpha-\beta-1} \int_0^{t_0} |x(s)| ds \\
&= x(t) - \frac{aM}{\Gamma(\alpha-\beta)} (t-t_0)^{\alpha-\beta-1}.
\end{aligned} \tag{86}$$

Define  $M_1 := \int_0^{t_0} |y_2(s)|ds$ . Due to  $y_1(\cdot) \in AC[0, \infty)$ , it follows from Lemma 2.2 that, for  $t > t_0$ ,

$$\begin{aligned}
{}^C D_{0+}^{n-\alpha} y_1(t) &= \frac{1}{\Gamma(1+\alpha-n)} \int_0^t \frac{y_1'(s)}{(t-s)^{n-\alpha}} ds \\
&= \frac{1}{\Gamma(1+\alpha-n)} \int_0^{t_0} \frac{y_2(s)}{(t-s)^{n-\alpha}} ds + \frac{1}{\Gamma(1+\alpha-n)} \int_{t_0}^t \frac{y_2(s)}{(t-s)^{n-\alpha}} ds \\
&\leq \frac{1}{\Gamma(1+\alpha-n)} \frac{1}{(t-t_0)^{n-\alpha}} \int_0^{t_0} |y_2(s)| ds \\
&= \frac{M_1}{\Gamma(1+\alpha-n)} \frac{1}{(t-t_0)^{n-\alpha}}.
\end{aligned} \tag{87}$$

Combining (85)–(87), we get

$$x(t) - \frac{aM}{\Gamma(\alpha-\beta)} (t-t_0)^{\alpha-\beta-1} \leq \frac{M_1}{\Gamma(1+\alpha-n)} \frac{1}{(t-t_0)^{n-\alpha}}, \quad \forall t > t_0.$$

251 This means that

$$x(t) \leq \frac{aM}{\Gamma(\alpha-\beta)} (t-t_0)^{\alpha-\beta-1} + \frac{M_1}{\Gamma(1+\alpha-n)} \frac{1}{(t-t_0)^{n-\alpha}}, \quad \forall t > t_0. \tag{88}$$

252 Notice that  $\beta + 1 - \alpha > n - \alpha$ , from (88), we conclude that there are constants  $M_2 > 0$  and  
253  $t_1 > t_0$  such that

$$x(t) \leq \frac{M_2}{t^{n-\alpha}}, \quad \forall t > t_2. \tag{89}$$

(b) Suppose that the assumption (ii) is true. We can find  $t_0 > 0$  so that

$$x(t) > 0, \quad y_1(t) < 0, \quad y_2(t) > 0, \quad y_3(t) < 0, \quad \forall t \geq t_0.$$

Furthermore,  $y_2(\cdot)$  is strictly decreasing on  $[t_0, \infty)$ . Thus,

$$-y_1(t) > y_1(t) - y_1(t_0) = \int_{t_0}^t y_2(s) ds \geq y_2(t)(t-t_0), \quad \forall t > t_0,$$

which implies

$$y_2(t) \leq \frac{-y_1(t_0)}{t-t_0}, \quad \forall t > t_0. \tag{90}$$

Now, for  $t > 2t_0$ , we obtain the following estimates

$$\begin{aligned}
{}^C D_{0+}^{n-\alpha} y_1(t) &= \frac{1}{\Gamma(1+\alpha-n)} \int_0^t \frac{y_2(s)}{(t-s)^{n-\alpha}} ds \\
&= \frac{1}{\Gamma(1+\alpha-n)} \left( \int_0^{t_0} \frac{y_2(s)}{(t-s)^{n-\alpha}} ds + \int_{t_0}^{t/2} \frac{y_2(s)}{(t-s)^{n-\alpha}} ds + \int_{t/2}^t \frac{y_2(s)}{(t-s)^{n-\alpha}} ds \right) \\
&\leq \frac{1}{\Gamma(1+\alpha-n)} \left( \frac{1}{(t-t_0)^{n-\alpha}} \int_0^{t_0} |y_2(s)| ds + \left(\frac{2}{t}\right)^{n-\alpha} \int_{t_0}^{t/2} y_2(s) ds + I(t) \right) \\
&= \frac{1}{\Gamma(1+\alpha-n)} \left( \frac{M_1}{(t-t_0)^{n-\alpha}} + \left(\frac{2}{t}\right)^{n-\alpha} (y_1(t/2) - y_1(t_0)) + I(t) \right) \\
&\leq \frac{1}{\Gamma(1+\alpha-n)} \left( \frac{M_1}{(t-t_0)^{n-\alpha}} - y_1(t_0) \left(\frac{2}{t}\right)^{n-\alpha} + I(t) \right), \tag{91}
\end{aligned}$$

where  $I(t) := \int_{t/2}^t \frac{x_2(s)}{(t-s)^{n-\alpha}} ds$ . By (90),

$$\begin{aligned}
I(t) &= \int_{t/2}^t \frac{y_2(s)}{(t-s)^{n-\alpha}} ds \leq \int_{t/2}^t \frac{-y_1(t_0)}{(s-t_0)(t-s)^{n-\alpha}} ds \\
&\leq \frac{-y_1(t_0)}{t/2-t_0} \frac{1}{\alpha-n+1} \left(\frac{t}{2}\right)^{\alpha-n+1}, \quad \forall t > 2t_0. \tag{92}
\end{aligned}$$

According to (91) and (92), there are  $M_3 > 0$  and  $t_2 > 2t_0$  satisfying

$${}^C D_{0+}^{n-\alpha} y_1(t) \leq \frac{M_3}{t^{n-\alpha}}, \quad \forall t > t_2,$$

which together with (86) leads to

$$x(t) \leq \frac{aM}{\Gamma(\alpha-\beta)} (t-t_0)^{\alpha-\beta-1} + \frac{M_3}{t^{n-\alpha}}, \quad \forall t > t_2,$$

254 and thus

$$x(t) = O(t^{\alpha-n}) \quad \text{as } t \rightarrow \infty. \tag{93}$$

(c) Suppose that the assumption (iii) is true. There exists  $t_0 > 0$  with

$$x(t) > 0, \quad y_1(t) > 0, \quad y_2(t) > 0, \quad y_3(t) < 0, \quad t \geq t_0.$$

Due to  $y_1(\cdot)$  is bounded, there is a  $M_4 > 0$  so that  $y_1(t) < M_4$  for all  $t \geq t_0$ . By the fact  $y_2(\cdot)$  is positive and decreasing on  $[t_0, \infty)$ , we see

$$M_4 > y_1(t) - y_1(t_0) = \int_{t_0}^t y_2(s) ds \geq y_2(t)(t-t_0), \quad \forall t > t_0,$$

which shows that

$$y_2(t) \leq \frac{M_4}{t-t_0}, \quad \forall t > t_0.$$

255 Repeated the arguments as in (b) enables us to claim

$$x(t) = O(t^{\alpha-n}) \quad \text{as } t \rightarrow \infty. \tag{94}$$

In short, based on (89), (93) and (94), we conclude that

$$x(t) = O(t^{\alpha-n}) \quad \text{as } t \rightarrow \infty.$$

256 The proof is complete. □



257 **Lemma 5.9.** Suppose that (G1), (G2), (G3), (H), (K) are true. Let  $n$  be even and  $x(\cdot)$  be an  
 258 eventually positive solution of (61) satisfying (77). Then the limit  $\lim_{t \rightarrow \infty} y_1(t)$  is finite.

*Proof.* It follows from the assumptions of the lemma that there is a  $t_0 > 0$  such that  $x(t) > 0$ ,  $y_1(t) > 0$ ,  $y_2(t) > 0$ ,  $(-1)^i y_i(t) > 0$ ,  $i = 3, \dots, n$ ,  $f(x(t)) > 0$ ,  $q(t) \geq c_3 > 0$  and  $g(t) \geq c_4 t^{n-\beta-\sigma}$  for all  $t \geq t_0$ . From the arguments in the proof of Lemma 5.5, we can find a  $t_1 > t_0$  with

$$y'_n(t) < -q(t)f(x(t)), \quad \forall t > t_1,$$

which implies

$$\begin{aligned} \frac{1}{(n-2)!} \int_{t_1}^t (s-t_1)^{n-2} q(s) f(x(s)) ds &< \frac{-1}{(n-2)!} \int_{t_1}^t (s-t_1)^{n-2} y'_n(s) ds \\ &= \frac{-(t-t_1)^{n-2}}{(n-2)!} y_n(t) + \frac{1}{(n-3)!} \int_{t_1}^t (s-t_1)^{n-3} y'_{n-1}(s) ds \\ &= \frac{-(t-t_1)^{n-2}}{(n-2)!} y_n(t) + \frac{(t-t_1)^{n-3}}{(n-3)!} y_{n-1}(t) - \frac{1}{(n-4)!} \int_{t_1}^t (s-t_1)^{n-4} y'_{n-2}(s) ds. \end{aligned}$$

We continue in this fashion to obtain

$$\begin{aligned} \frac{1}{(n-2)!} \int_{t_1}^t (s-t_1)^{n-2} q(s) f(x(s)) ds &< \frac{-1}{(n-2)!} \int_{t_1}^t (s-t_1)^{n-2} y'_n(s) ds \\ &= \frac{-(t-t_1)^{n-2}}{(n-2)!} y_n(t) + \frac{(t-t_1)^{n-3}}{(n-3)!} y_{n-1}(t) + \dots + \frac{t-t_1}{1!} x_3(t) - \int_{t_1}^t y'_2(s) ds \\ &= \frac{-(t-t_1)^{n-2}}{(n-2)!} y_n(t) + \frac{(t-t_1)^{n-3}}{(n-3)!} y_{n-1}(t) + \dots + \frac{t-t_1}{1!} y_3(t) + y_2(t_1) - y_2(t) \\ &= \sum_{k=0}^{n-2} \frac{(-1)^{k+1} (t-t_1)^k}{k!} y_{k+2}(t) + y_2(t_1), \quad t > t_1. \end{aligned} \tag{95}$$

Because  $(-1)^{k+1} y_{k+2}(t) = -(-1)^{k+2} y_{k+2}(t) < 0$  for  $t \geq t_0$ ,  $k = 0, 1, \dots, n-2$ , we have  $\frac{(-1)^{k+1} (t-t_1)^k}{k!} y_{k+2}(t) \leq 0$  for  $t \geq t_1$ ,  $k = 0, 1, \dots, n-2$ . This combines with (95) leads to

$$y_2(t_1) > \frac{1}{(n-2)!} \int_{t_1}^t (s-t_1)^{n-2} q(s) f(x(s)) ds \geq \frac{c_3}{(n-2)!} \int_{t_1}^t (s-t_1)^{n-2} f(x(s)) ds, \quad \forall t > t_1.$$

Letting  $t \rightarrow \infty$  yields

$$\int_{t_1}^{\infty} (s-t_1)^{n-2} f(x(s)) ds < \frac{(n-2)! y_2(t_1)}{c_3} < \infty.$$

Choosing  $M_1 > 0$  with

$$(t-t_1)^{n-2} f(x(t)) \leq M_1, \quad \forall t \geq t_1.$$

It deduces from (G2) that  $x(t) < 1$  for all  $t > t_2 := \left(\frac{M_1}{c_1}\right)^{\frac{1}{n-2}} + t_1$ . This together with (G3) gives

$$x(t) \leq \left(\frac{M_1}{c_2}\right)^{\frac{1}{\mu}} \frac{1}{(t-t_1)^{\frac{n-2}{\mu}}}, \quad \forall t > t_2,$$

where  $\mu \in \left(0, \frac{n-2}{n-\beta}\right)$ . Hence, there is a  $t_3 > t_2$  with

$$x(t) \leq \frac{1}{t^{n-\beta}}, \quad t \geq t_3.$$

Put  $M_2 := \int_0^{t_3} |x(s)| ds$ . For  $t > t_3$ , then

$$\begin{aligned} x(t) + aI_{0+}^{\alpha-\beta} x(t) &\leq \frac{1}{t^{n-\beta}} + \frac{a}{\Gamma(\alpha-\beta)} \int_0^{t_3} (t-s)^{\alpha-\beta-1} x(s) ds \\ &\quad + \frac{a}{\Gamma(\alpha-\beta)} \int_{t_3}^t (t-s)^{\alpha-\beta-1} x(s) ds \\ &\leq \frac{1}{t^{n-\beta}} + \frac{a}{\Gamma(\alpha-\beta)} (t-t_3)^{\alpha-\beta-1} \int_0^{t_3} |x(s)| ds \\ &\quad + \frac{a}{\Gamma(\alpha-\beta)} \int_{t_3}^t (t-s)^{\alpha-\beta-1} s^{\beta-n} ds \\ &\leq \frac{1}{t^{n-\beta}} + \frac{aM_2}{\Gamma(\alpha-\beta)} (t-t_3)^{\alpha-\beta-1} + \frac{a}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} s^{\beta-n} ds \\ &= \frac{1}{t^{n-\beta}} + \frac{aM_2}{\Gamma(\alpha-\beta)} (t-t_3)^{\alpha-\beta-1} \\ &\quad + \frac{a}{\Gamma(\alpha-\beta)} \int_0^1 t^{\alpha-\beta-1} (1-u)^{\alpha-\beta-1} t^{\beta-n} u^{\beta-n} t du \\ &= t^{\beta-n} + \frac{aM_2}{\Gamma(\alpha-\beta)} (t-t_3)^{\alpha-\beta-1} + \frac{a\Gamma(\beta-n+1)}{\Gamma(\alpha-n+1)} t^{\alpha-n}. \end{aligned}$$

From this there exist  $M_3 > 0$  and  $t_4 > t_3$  satisfying

$${}^C D_{0+}^{n-\alpha} y_1(t) = x(t) + aI_{0+}^{\alpha-\beta} x(t) \leq M_3 t^{\alpha-n}, \quad \forall t \geq t_4. \quad (96)$$

Notice that for  $t > t_4$ ,

$$\begin{aligned} {}^C D_{0+}^{n-\alpha} y_1(t) &= \frac{1}{\Gamma(\alpha-n+1)} \int_0^t \frac{y_1'(s)}{(t-s)^{n-\alpha}} ds \\ &= \frac{1}{\Gamma(\alpha-n+1)} \int_0^{t_4} \frac{y_2(s)}{(t-s)^{n-\alpha}} ds + \frac{1}{\Gamma(\alpha-n+1)} \int_{t_4}^t \frac{y_1'(s)}{(t-s)^{n-\alpha}} ds \\ &\geq \frac{1}{\Gamma(\alpha-n+1)} \frac{-1}{(t-t_4)^{n-\alpha}} \int_0^{t_4} |y_2(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha-n+1)} \frac{1}{(t-t_4)^{n-\alpha}} (y_1(t) - y_1(t_4)) \\ &= \frac{y_1(t) - y_1(t_4) - M_4}{\Gamma(\alpha-n+1)} \frac{1}{(t-t_4)^{n-\alpha}}, \end{aligned} \quad (97)$$

259 where  $M_4 := \int_0^{t_4} |x_2(s)| ds$ . From (96) and (97), it follows

$$M_3 \geq \frac{y_1(t) - y_1(t_4) - M_4}{\Gamma(\alpha-n+1)} \frac{t^{n-\alpha}}{(t-t_4)^{n-\alpha}}, \quad \forall t > t_4. \quad (98)$$

260 Due to  $y_1(\cdot)$  is positive and increasing on  $[t_0, \infty)$ . From (98), we conclude that the limit  
261  $\lim_{t \rightarrow \infty} y_1(t)$  exists and is finite.  $\square$

262 We can now prove the main results of this part.

*Proof of Theorem 5.1.* Let  $x(\cdot)$  be a non-oscillatory solution of the initial value problem (61)–(62). Without loss of generality, we assume that  $x(\cdot)$  is eventually positive. From Lemma 5.7,  $x(\cdot)$  satisfies (75). Applying Lemma 5.8, we conclude

$$x(t) = O(t^{\alpha-n}) \text{ as } t \rightarrow \infty.$$

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□

*Proof of Theorem 5.2.* Let  $x(\cdot)$  be a non-oscillatory solution of the initial value problem (61)–(62). Without loss of generality, we assume that  $x(\cdot)$  is eventually positive. Lemma 5.7 shows that  $x(\cdot)$  satisfies (76) or (77). Applying Lemma 5.8 and Lemma 5.9, we conclude that

$$x(t) = O(t^{\alpha-n}) \text{ as } t \rightarrow \infty.$$

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□

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