Stability criteria for rough systems

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In memory of Björn Schmalfuß

Abstract

We propose a quantitative direct method of proving the local stability for the trivial solution of a rough differential equation and of its regular discretization scheme. Using Doss-Sussmann technique and stopping time analysis, we prove that the trivial solution of the rough system is exponentially stable as long as the noise is small. The same conclusions hold for the regular discretization scheme with small noise and small step size. Our results are significantly stronger than [20, Theorem 14] and [22, Theorem 18] and can be applied to non-flat bounded or linear noises.

Keywords: stochastic differential equations (SDE), Young integral, rough path theory, rough differential equations, exponential stability.

1 Introduction

This paper deals with the local asymptotic stability criteria for rough differential equations of the form

$$dy = f(y)dt + g(y)dx, \tag{1.1}$$

or in the integral form

$$y_t = y_0 + \int_0^t f(y_u)du + \int_0^t g(y_u)dx_u, \qquad t \in [0,T]. \tag{1.2}$$

Equation (1.1) can be viewed as a controlled differential equation driven by rough path $x \in C^\nu([0,T],\mathbb{R}^m)$ in the sense of Lyons [26], [27]. As such, system (1.1) appears as a pathwise approach to solve a stochastic differential equation which is driven by a certain Hölder noise X_t. In this paper, we would like to approach system (1.1) and interpret the second integral as a rough integral in the sense of Gubinelli [23], and consider $\nu \in (\frac{1}{3}, 1]$ for simplicity of presentation (see a detailed description in Section 6).

Under certain assumptions imposed on its coefficient functions, a rough differential equation will have the property of the local existence, uniqueness and continuation of solution given initial conditions, see e.g. [23] or [17] for a version without drift coefficient function, and [32] for a full version using p - variation norms. Moreover, with a stronger assumption on the coefficient functions a rough differential equation will have global solutions which exists on the whole time interval $[0,\infty)$.

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This gives rise to the recent interest in investigation of qualitative properties of rough differential equation with a view to the parallel in the well developed and well known qualitative theory of ordinary differential equations.

The topic of asymptotic stability for path-wise solution of (1.1) is studied in [15, 14, 12] for which the noise is assumed to be fractional Brownian motion [28] (i.e. a family of $B^H = \{B^H_t\}_{t\in\mathbb{R}}$ with continuous sample paths and $E\|B^H_t - B^H_s\| = |t-s|^{2H}$ for all $t,s\in\mathbb{R}$) with small intensity. In addition, the topic of local stability is studied in [20, Theorem 14] and in [22, Theorem 18] for local versions on a small neighborhood $B(0,\rho)$ of the trivial solution, using the cutoff technique and fractional calculus, and under the assumption that $g(x)$ is rather flat, i.e. $g(0) = D_y g(0) = 0$ for the Young differential equations and $g(0) = D_y g(0) = D_{yy} g(0) = 0$ for the rough differential equations. A related topic is random attractors for the random dynamical system generated from (1.1) and its discretization, which is investigated in many works, see e.g. [19], [11], [10], [8] and the references therein.

In this paper we are interested in the Lyapunov asymptotic stability of the rough differential equation (1.1) near an equilibrium point (the origin). Since the stability is a local property of the equation, it is natural to restrict our investigation and impose conditions on the equation only in a neighborhood of the origin. Throughout the paper, we will assume that there exist a positive constant $\epsilon_0 > 0$ such that in the ball $B(0,\epsilon_0) := \{y\in\mathbb{R}^d : \|y\| \le \epsilon_0\} \subset \mathbb{R}^d$ the following conditions are satisfied

($\mathbf{H}_f$) in the ball $B(0,\epsilon_0)$ the coefficient function $f:\mathbb{R}^d\to\mathbb{R}^d$ of (1.1) is locally Lipschitz continuous which can be decomposed into

$$f(y) = Df(0)y + H(y), \qquad \forall y\in B(0,\epsilon_0), \tag{1.3}$$

where $Df(0)\in\mathbb{R}^{d\times d}$ is a matrix which admits all eigenvalues of negative real parts. Meanwhile, $H:\mathbb{R}^d\to\mathbb{R}^d$ is the non-linear part satisfying: there exist positive constants $C_H, L_H, m > 0$ such that

$$\|H(y)\| \le C_H\|y\|^{1+m}, \quad \|H(y)-H(z)\| \le L_H\|y-z\| \quad \forall y,z\in B(0,\epsilon_0), \tag{1.4}$$

($\mathbf{H}_g$) in the ball $B(0,\epsilon_0)$ the coefficient function $g: B(0,\epsilon_0)\to\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)$ of the equation (1.1) belongs to $C^3(B(0,\epsilon_0),\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d))$; we denote

$$C_g := \max\Big\{\|g\|_{\infty,B(0,\epsilon_0)}, \|Dg\|_{\infty,B(0,\epsilon_0)}, \|D^2 g\|_{\infty,B(0,\epsilon_0)}\Big\}, \tag{1.5}$$

$$C_g^* := \max\Big\{C_g, \sqrt{\|g\|_{\infty,B(0,\epsilon_0)}\cdot\|D^3 g\|_{\infty,B(0,\epsilon_0)}}\Big\}. \tag{1.6}$$

($\mathbf{H}_X$) for a given $\nu\in(\frac{1}{3},\frac{1}{2})$, $x\in C^\nu(\mathbb{R},\mathbb{R}^m)$ - the space of all Hölder continuous paths such that x is a realization of a stochastic process $X_t(\omega)$ with stationary increments and that x can be lifted into a realized component $\mathbf{x} = (x,\mathbb{X})$ of a stochastic process $(x_\cdot(\omega),\mathbb{X}_{\cdot,\cdot}(\omega))$ with stationary increments. Moreover the estimate

$$E\Big(\|x_{s,t}\|^p + \|\mathbb{X}_{s,t}\|^q\Big) \le C_{T,\nu}|t-s|^{p\nu}, \qquad \forall s,t\in[0,T], \tag{1.7}$$

holds for any $[0,T]\subset[,\infty)$, with $p\nu\ge 1, q=\frac{p}{2}$ and some constant $C_{T,\nu}$.

Concerning the assumption ($\mathbf{H}_X$), for instance, given $\bar\nu\in(\frac{1}{3},1]$, the path x might be a realization of a $\mathbb{R}^m$-valued centered Gaussian process satisfying: there exists for any $T>0$ a constant C_T such that for all $p\ge\frac{1}{\bar\nu}$

$$E\|X_t - X_s\|^p \le C_T|t-s|^{p\bar\nu}, \quad \forall s,t\in[0,T]. \tag{1.8}$$

By Kolmogorov theorem, for any $\nu \in (0, \bar{\nu})$ and any interval $[0, T]$ almost all realization of X will be in $C^{\nu}([0, T])$. Such a stochastic process, in particular, can be a fractional Brownian motion B^H with Hurst exponent $H \in (\frac{1}{3}, 1)$.

Assumptions $(\mathbf{H}_f)$, $(\mathbf{H}_g)$, $(\mathbf{H}_X)$ are sufficient to prove the existence and uniqueness of the solution of (1.1) defined for any initial value $y_0 \in B(0, \epsilon_0)$ and on a finite or infinite time interval, as well as the continuity of the solution semi-flow and the generation of a continuous random dynamical system, see e.g. [32, Theorem 4.3], [2], [11] and [10].

In this paper, we will always assume that $f(0) = 0$ and $g(0) = 0$. System (1.1) then admits an equilibrium which is the trivial solution. Our aim is to obtain the stability of the system (1.1) in the neighborhood of the origin (an equilibrium). For this we need to show that the solutions starting near to 0 are not exploded, i.e. it can be extended to all the time $t > 0$, and the requirement of the stability are met.

Note that the condition $(\mathbf{H}_f)$ assures that the "unperturbed" ODE $\dot{y} = f(y)$ is exponentially stable (see Perko [31, Theorem 1, p. 130], Chicone [6, Theorem 2.77, p. 183]). We will show that in the rough path setting, considering the rough path integral as a perturbation of this ODE, the system (1.1) remains exponentially stable provided the perturbation is small. Roughly speaking, our main results can be formulated as follows.

Main results. *Assume that $f(0) = 0$ and $g(0) = 0$ so that zero is the trivial solution of* (1.1), *and conditions $(\boldsymbol{H}_f)$, $(\boldsymbol{H}_g)$, $(\boldsymbol{H}_X)$ hold such that the generated Wiener shift is ergodic. Then there exists $C_0 > 0$ depending only on f such that for any $0 < C_g < C_0$, the trivial solution of* (1.1) *is exponentially stable almost surely. The same conclusion holds for the regular discretized system, provided that the step size is sufficiently small.*

Our approach to deal with this problem is first to impose an additional global conditions on f and g to get a global continuous random dynamical system generated by the solutions of (1.1), and then to apply our stopping time technique to get the local stability of the trivial solution for the global system. Finally, we use the extension technique in [16] to prove that such global assumptions on f, g are feasible but do not change the local dynamics of the system around the trivial solution, hence they can all be relaxed.

Our method applies the direct method of Lyapunov, which aims to estimate the norm growth (or a Lyapunov-type function) of the solution in discrete intervals. For the continuous time set, this technique is feasible thanks to the Doss-Sussmann technique to transform the rough differential equation on each stopping time interval to an ordinary differential equation which can be seen as a non-autonomous perturbation of the unperturbed ODE. The tricky part is proving stability, which means one needs to control the norm growth of the solution less than a parameter ϵ on each stopping time interval. The exponential attractivity is then an indirect consequence of the Birkhoff's ergodic theorem (via Lemma 3.1), provided that the generated *Wiener-shift* is ergodic (this holds for fractional Brownian motions, see Lemma 6.2 in Section 6). It is important to note that one can develop a direct argument for the local system without extending it to the global one, by taking care of the time for a solution to exit the local regime of the trivial solution. The results in Theorem 4.2 and Theorem 4.3 are significantly stronger than [20, Theorem 14] and [22, Theorem 18] and can be applied to non-flat bounded or linear noises (see Remark 4.4).

The situation is rather complicated for the discretized system of (1.1), because the Doss-Sussmann transformation fails to control the solution norm to be sufficiently small in the discrete time set, and in general the solution can exit the local regime of the trivial solution right after one discrete time step. Thus similar to the continuous time case, we need first to extend the local system to a global one and then to apply the coupling technique to compare solutions of the rough difference equation and its unperturbed system. A sufficient condition is the integrability of solution, which is straightforward for Young equations but not trivial for the rough case under the

p-variation norm. To overcome this difficulty, we develop further in Section 3 the stopping time technique, which was studied e.g. in [5], [15], [7] for continuous time sets and recently in [8] for discrete time sets, to estimate the norm growth of the solution on each stopping time interval and then to control the initial value y_0 inside a random radius ball, leading to the local exponential stability as in the continuous time case. The stability criteria in Theorem 5.1 and Theorem 5.2 are new, where the choice of C_g and step size Δ are independent and pathwise free.

When the assumption on ergodicity of the generated Wiener shift is relaxed, the ergodic Birkhoff theorem is still applicable in estimating stopping times, but results in random variable limits. In this case, all conclusions in the main theorems still hold almost surely, but all the stability criteria and parameters would be path dependent.

Our method still works for a lower regularity coefficient $\nu \leq \frac{1}{3}$, although the computation would be rather complicated. Moreover, it could also be applied in case f also depends on time, i.e. $f(y)$ is replaced by $f(t, y)$.

We close the introduction part with a counter example that choosing a large C_g might break the stability.

Example 1.1 Consider the Itô stochastic differential equation

$$d\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1(\mu - y_1^2 - y_2^2) \\ y_2(\mu - y_1^2 - y_2^2) \end{pmatrix} dt + \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} dB_t \tag{1.9}$$

or in short $dy = f(y)dt + g(y)dB_t$, where B is a scalar standard Brownian motion will all realization in C^ν for $\nu \in (\frac{1}{3}, \frac{1}{2})$; and μ, σ are real constants. We omit the issue of existence and uniqueness of the solution of equation (1.9), noting that it can be solved either in the Itô sense using polar coordinates $y_1 = r \sin \alpha, y_2 = r \cos \alpha$ or in the pathwise sense as a rough differential equation (see e.g. [11, Theorem 2.1] and [10, Theorem 5.1]). In particular, one can apply Itô formula to check that

$$d\|y\|^2 = \Big[2\langle y, f(y)\rangle + \|g(y)\|^2\Big]dt + 2\langle y, g(y)\rangle dB_t = \Big(2\mu + \sigma^2 - 2\|y\|^2\Big)\|y\|^2 dt.$$

In other words, $\eta = \|y\|^2$ is the solution of the ordinary differential equation

$$\frac{d}{dt}\eta = \eta(2\mu + \sigma^2 - 2\eta). \tag{1.10}$$

Clearly, zero is the trivial solution of (1.9). For $\sigma = 0$ and $\mu < 0$, system (1.9) is an ordinary differential equation, which admits zero as an globally asymptotically stable solution. However, when $\sigma > 0$ large enough such that $\sigma^2 + 2\mu > 0$, then the zero solution of (1.10) becomes unstable. In other words, the zero solution of (1.9) will break its stability when perturbed by the linear noise for σ large enough.

2 Rough differential equations, local solutions and their extension

The existence and uniqueness theorem for system (1.1) is first proved in [32], where the solution is understood in the sense of Friz &Victoir [18]. Using rough path integrals, we interpret the rough differential equation (1.1) by writing it in the integral form

$$y_t = y_a + \int_a^t f(y_s)ds + \int_a^t g(y_s)dx_s, \quad \forall t \in [a, b], \tag{2.1}$$

for any interval $[a,b]$ and an initial value $y_a \in \mathbb{R}^d$. Then we search for a solution in the Gubinelli sense, and solve for a path y which is controlled by x. We refer to [13] and Section 6 for definitions of variation and Hölder norms, Gubinelli rough integrals for controlled rough paths.

The solution of (1.1) with the initial value $y_0 \in B(0,\epsilon_0)$ is understood in the pathwise sense, under the assumptions ($\mathbf{H}_f$)-($\mathbf{H}_g$). To apply the arguments in [13], we provide an indirect argument to first extend the local domain $B(0,\epsilon_0)$ to the whole $\mathbb{R}^d$, and then to apply available results for global solutions. To do that, in the rest of this section (and sometimes later, we will say explicitly if it is a case) we assume the following global condition of f and g.

($\mathbf{H}_f^*$) f satisfies ($\mathbf{H}_f$) and, moreover the decomposition (1.3) holds for all $y \in \mathbb{R}^d$ with H being globally Lipschitz continuous

$$\exists \hat{L}_H > 0 : \|H(y) - H(z)\| \leq \hat{L}_H \|y - z\|, \quad \forall y, z \in \mathbb{R}^d; \tag{2.2}$$

($\mathbf{H}_g^*$) g is in $C_b^3(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ where we define

$$\hat{C}_g := \max\left\{\|g\|_{\infty,\mathbb{R}^d}, \|Dg\|_{\infty,\mathbb{R}^d}, \|D_g^2\|_{\infty,\mathbb{R}^d}, \sqrt{\|g\|_{\infty,\mathbb{R}^d} \cdot \|D^3 g\|_{\infty,\mathbb{R}^d}}\right\} \tag{2.3}$$

Note that from [13, Theorem 3.4], the global solution $\phi_{\cdot}(\mathbf{x},\phi_a)$ of the *pure* rough differential equation

$$d\phi_u = g(\phi_u)dx_u, \quad u \in [a,b], \phi_a \in \mathbb{R}^d \tag{2.4}$$

is C^1 w.r.t. ϕ_a, and $\frac{\partial \phi}{\partial \phi_a}(\cdot,\mathbf{x},\phi_a)$ is the solution of the linearized system

$$d\xi_u = Dg(\phi_u(\mathbf{x},\phi_s))\xi_u dx_u, \quad u \in [a,b], \xi_a = Id, \tag{2.5}$$

where $Id \in \mathbb{R}^{d\times d}$ denotes the identity matrix. The idea is then to prove the existence and uniqueness of the global solution on each small interval $[\tau_k, \tau_{k+1}]$ between two consecutive stopping times, and then concatenate to obtain the conclusion on any interval. The Doss-Sussmann technique used in [13, Theorem 3.7] and [32] ensures that, by a transformation $y_t = \phi_t(\mathbf{x}, z_t)$ there is an one-to-one correspondence between a solution y_t of (1.1) on a certain interval $[0,\tau]$ and a solution z_t of the associate ordinary differential equation

$$\dot{z}_t = \left[\frac{\partial \phi}{\partial z}(t,\mathbf{x},z_t)\right]^{-1} f(\phi_t(\mathbf{x},z_t)), \quad t \in [0,\tau], \ z_0 = y_0. \tag{2.6}$$

The parameter $\tau > 0$ can be chosen such that

$$16 C_p \hat{C}_g \left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|_{p-\mathrm{var},[0,\tau]} \leq \lambda, \quad \text{for some} \quad \lambda \in (0,1), \tag{2.7}$$

where $\left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]}$ is defined in (6.4), $\hat{C}_g$ is defined in (2.3), and C_p is defined in (6.11).

The following result is a direct consequence of [11, Proposition 2.1] and [10, Proposition 2.3], which shows solution norm estimates for equation (2.4). The proof will be omitted here due to similarity.

Proposition 2.1 *Let ϕ_t be the solutions of (2.4). Assume ($\mathbf{H}_g^*$), and $g(0) = 0$. Then for any interval $[a,b]$ such that $16 C_p \hat{C}_g \left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} \leq 1$, the following estimates hold*

$$i) \quad \left|\!\left|\!\left|\phi\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} \leq 8 C_p \hat{C}_g \left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} \min\{\|\phi_a\|, 1\}; \tag{2.8}$$

$$ii) \quad \left\|\frac{\partial \phi}{\partial \phi_a}(t,\mathbf{x},\phi_a) - I\right\|, \left\|\left[\frac{\partial \phi}{\partial \phi_a}(t,\mathbf{x},\phi_a)\right]^{-1} - I\right\| \leq 16 C_p \hat{C}_g \left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]}. \tag{2.9}$$

Note that all the above arguments might fail to be applied if we work directly with the local solution for $y_0 \in B(0,\epsilon_0)$. This is because at stopping time interval $[\tau_k, \tau_{k+1}]$ the Doss-Sussmann transformation $y_t = \phi_t(\mathbf{x}, z_t)$ that starts from $y_{\tau_k} \in B(0,\epsilon_0)$ might exit $B(0,\epsilon_0)$ soon before τ_{k+1}. Hence in general it would be rather technical to estimate the exit time of the local solution from $B(0,\epsilon_0)$.

Next, to apply ($\mathbf{H}_f^*$) and ($\mathbf{H}_g^*$) for a global solution on $\mathbb{R}^d$, we first need to recall a result on extension of differentiable functions on $\mathbb{R}^d$. Namely, we have the following lemma which is a direct corollary of a theorem by C. Fefferman [16, Theorem 1].

Lemma 2.2 *There exists a positive constant $c_3^* \geq 1$ depending only on the dimensions m and d (independent from ϵ_0) such that any function $g : B(0,\epsilon_0) \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$ which is in the class $C^3(B(0,\epsilon_0))$ can be extended to a function $g^* : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$ of the class $C^3(\mathbb{R}^d)$ with bounded derivatives up to order 3, and the following inequality holds*

$$\|g^*\|_{C^3(\mathbb{R}^d)} \leq c_3^* \|g\|_{C^3(B(0,\epsilon_0))}.$$

A similar extension result for the case of locally Lipschitz continuous functions f, H also holds (see [3, Theorem 1.53, p. 54])

Lemma 2.3 *There exists a positive constant $c_1^* \geq 1$ depending only on the dimension d (independent from ϵ_0) such that any function $h : B(0,\epsilon_0) \to \mathbb{R}^d$ which is Lipschitz in the ball $B(0,\epsilon_0)$ with Lipschitz constant $L_h(B(0,\epsilon_0))$ can be extended to a globally Lipschitz function $\bar{h} : \mathbb{R}^d \to \mathbb{R}^d$ with a Lipschitz constant $L_{\bar{h}}(\mathbb{R}^d)$ and the following inequality holds*

$$L_{\bar{h}}(\mathbb{R}^d) \leq c_1^* L_h(B(0,\epsilon_0)).$$

Put

$$c^* := \max\{c_1^*, c_3^*\} \geq 1, \tag{2.10}$$

to be the universal constant that can serve for both estimations of smooth and Lipschitz extensions above. Hence, there are new functions

$$f^* : \mathbb{R}^d \to \mathbb{R}^d, \qquad g^* : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d),$$

such that f^*, g^* are the extensions of f, g from $B(0,\epsilon_0)$ to $\mathbb{R}^d$ provided by Lemma 2.2 and Lemma 2.3, i.e. f^*, g^* coincide with f, g in $B(0,\epsilon_0)$ respectively. Hence, we have

$$C_g \leq C_g^* \leq \hat{C}_{g^*} \leq c^* C_g^*, \quad \hat{L}_H \leq c^* L_H. \tag{2.11}$$

Consider the equation

$$dy_t = f^*(y_t)dt + g^*(y_t)dx_t, \tag{2.12}$$

It is easily seen that the functions f^*, g^* satisfy the strong assumptions ($\mathbf{H}_{f^*}^*$)-($\mathbf{H}_{g^*}^*$)-(2.11) as well as the original assumptions ($\mathbf{H}_{f^*}$)-($\mathbf{H}_{g^*}$). Therefore, there exists a unique global solution for equation (2.12) due to the choice of f^*, g^*. In particular, for any given solution $y_t(\mathbf{x}, y_0)$ with $y_0 \in B(0,\epsilon_0)$ there exists a time

$$\tau(\mathbf{x}, y_0) := \sup\{t > 0 : y_s(\mathbf{x}, y_0) \in B(0,\epsilon_0)\ \forall s \in [0,t]\} > 0, \tag{2.13}$$

such that all solution norm estimates can be computed via f, g (instead of f^*, g^*) during the time interval $[0, \tau(\mathbf{x}, y_0))$.

3 Stopping time analysis

3.1 Stopping times for the continuous time case

The construction of a *greedy sequence of stopping times* in [5] is now used in many recent results, see e.g. [7, 15, 13, 11, 32]. Namely, for any fixed $\gamma \in (0,1)$ the sequence of stopping times $\{\tau_i(\gamma,\mathbf{x})\}_{i\in\mathbb{N}}$ is defined by

$$\tau_0 = 0, \quad \tau_{i+1} := \inf\Big\{t > \tau_i : \|\mathbf{x}\|_{p-\mathrm{var},[\tau_i,t]} = \gamma\Big\}. \tag{3.1}$$

For a fixed closed interval $I \subset [0,\infty)$, we define another sequence of stopping times $\{\tau_i^*(\gamma,\mathbf{x},I)\}_{i\in\mathbb{N}}$ by

$$\tau_0^* = \min I, \quad \tau_{i+1}^* := \inf\Big\{t > \tau_i^* : \|\mathbf{x}\|_{p-\mathrm{var},[\tau_i,t]} = \gamma\Big\} \wedge \max I. \tag{3.2}$$

Define $N^*(\gamma,\mathbf{x},I) := \sup\{i\in\mathbb{N} : \tau_i^* < \max I\} + 1$. It is easy to show a rough estimate

$$N^*(\gamma,\mathbf{x},I) \le 1 + \frac{1}{\gamma^p}\|\mathbf{x}\|^p_{p-\mathrm{var},I}\,. \tag{3.3}$$

In fact, it is proved in [5] that $e^{N^*(\gamma,\mathbf{x},I)}$ is integrable for Gaussian rough paths.

Denote θ the Wiener-type shift in the probability space $\Omega := \mathscr{C}_0^{0,\alpha}(\mathbb{R},T_1^2(\mathbb{R}^m))$ (see (6.6) in Subsection 6.2). Throughout this paper, we will assume that θ is ergodic. In case θ is not ergodic, one can use the ergodic decomposition theorem [33, Theorem 3.2, p. 19] to reduce the problem to the ergodic case.

The following lemma is reformulated from [15, Theorem 14] for the stopping times defined in the p-variation norm. To make the presentation self-contained, we are going to give a short and direct proof here.

Lemma 3.1 *Given the greedy sequence of stopping time* (3.1)*, the following estimate*

$$\liminf_{n\to\infty}\frac{\tau_n}{n} \ge \frac{1}{\mathbb{E}N^*(\gamma,\mathbf{x},[0,1])} \tag{3.4}$$

holds almost surely.

Proof: For each j denote by $N(\gamma,\mathbf{x},[j,j+1))$ the number of stopping times τ_k in $[j,j+1)$. Since the minimal stopping time τ in I is bigger than $\min I = \tau_0^*(\gamma,\mathbf{x},I)$ and the maximal stopping time τ in I is less than $\max I = \tau^*_{N^*(\gamma,\mathbf{x},I)}(\gamma,\mathbf{x},I)$, it follows that

$$N^*(\gamma,\mathbf{x},I) \ge N(\gamma,\mathbf{x},[\min I,\max I)). \tag{3.5}$$

Denote by $\lfloor\tau_k\rfloor$ the integer part of τ_k, then $\tau_k < \lfloor\tau_k\rfloor + 1$. As a result, the number of positive stopping times in the interval $[0,\tau_k]$ (which is k) is less than or equal to the one in the interval $[0,\lfloor\tau_k\rfloor+1)$. As a result,

$$\frac{\tau_k}{k} \ge \frac{\lfloor\tau_k\rfloor}{N\left(\gamma,\mathbf{x},[0,\lfloor\tau_k\rfloor+1)\right)}.$$

On the other hand, by definition of N and N^* and inequality (3.5),

$$\begin{aligned} N(\gamma,\mathbf{x},[0,\lfloor\tau_k\rfloor+1)) &\le \sum_{j=0}^{\lfloor\tau_k\rfloor} N(\gamma,\mathbf{x},[j,j+1)) \\ &\le \sum_{j=0}^{\lfloor\tau_k\rfloor} N^*(\gamma,\mathbf{x},[j,j+1]) = \sum_{j=0}^{\lfloor\tau_k\rfloor} N^*(\gamma,\mathbf{x}(\theta_j\omega),[0,1]), \end{aligned}$$

where the last equality is due to (6.7). Hence,

$$\frac{\tau_k}{k} \geq \frac{\lfloor \tau_k \rfloor}{\sum_{j=0}^{\lfloor \tau_k \rfloor} N^*(\gamma, \mathbf{x}(\theta_j \omega), [0,1])} \geq \frac{\frac{\lfloor \tau_k \rfloor}{\lfloor \tau_k \rfloor + 1}}{\frac{1}{\lfloor \tau_k \rfloor + 1} \sum_{j=0}^{\lfloor \tau_k \rfloor} N^*(\gamma, \mathbf{x}(\theta_j \omega), [0,1])}. \tag{3.6}$$

By applying the Birkhoff ergodic theorem to the last right hand side of (3.6), where $\tau_k \to \infty$ as $k \to \infty$, the numerator tends to one while the denominator converges to $\mathbb{E}N^*(\gamma, \mathbf{x}(\cdot), [0,1])$. This proves (3.4). □

Lemma 3.2 *For every $a, b \in \mathbb{R}, a \leq b$, the following estimate holds*

$$\gamma N^*(\gamma, \mathbf{x}(\cdot), [a,b]) \geq \left\| \mathbf{x}(\cdot) \right\|_{p-\mathrm{var},[a,b]}. \tag{3.7}$$

As a consequence,

$$\gamma \mathbb{E} N^*(\gamma, \mathbf{x}(\cdot), [a,b]) \geq \mathbb{E} \left\| \mathbf{x}(\cdot) \right\|_{p-\mathrm{var},[a,b]}. \tag{3.8}$$

Proof: Indeed, it follows from [7, Lemma 2.1] that

$$\begin{aligned}
\left\| \mathbf{x} \right\|^p_{p-\mathrm{var},[a,b]} &= \left\| \mathbf{x} \right\|^p_{p-\mathrm{var},[\tau_0^*, \tau^*_{N^*(\gamma,\mathbf{x},[a,b])}]} \\
&\leq \Big[N^*(\gamma, \mathbf{x}, [a,b])\Big]^{p-1} \sum_{j=1}^{N^*(\gamma,\mathbf{x},[a,b])} \left\| \mathbf{x} \right\|^p_{p-\mathrm{var},[\tau^*_{j-1}, \tau^*_j]} \\
&\leq \Big[N^*(\gamma, \mathbf{x}, [a,b])\Big]^{p-1} \Big[N^*(\gamma, \mathbf{x}, [a,b])\Big] \gamma^p \\
&\leq \Big[N^*(\gamma, \mathbf{x}, [a,b])\Big]^{p} \gamma^p
\end{aligned}$$

or equivalently

$$\gamma N^*(\gamma, \mathbf{x}, [a,b]) \geq \left\| \mathbf{x} \right\|_{p-\mathrm{var},[a,b]}.$$

Taking the expectation to both sides of the above inequality, we obtain (3.8). □

Lemma 3.3 *The following estimate holds for any $n \geq 1$ and any sequence $t_0 < t_1 < \ldots < t_n$*

$$\sum_{j=0}^{n-1} N^*(\gamma, \mathbf{x}, [t_j, t_{j+1}]) \leq N^*(\gamma, \mathbf{x}, [t_0, t_n]) + n. \tag{3.9}$$

Proof: By definition, on $[t_j, t_{j+1}]$ there are at least $N^*(\gamma, \mathbf{x}, [t_j, t_{j+1}]) - 1$ consecutive stopping time intervals on which the p-variation norm of $\mathbf{x}$ is γ. As a result, on the interval $[t_0, t_n]$ there are at least $\sum_{j=0}^{n-1} \Big(N^*(\gamma, \mathbf{x}, [t_j, t_{j+1}]) - 1 \Big)$ disjoint stopping time intervals on which the p-variation norm of $\mathbf{x}$ is γ. This proves (3.9). □

3.2 Stopping times for discrete time sets

For investigation of discrete time systems arising from Euler scheme applied to the rough equation (1.1) we modify the stopping time technique to the discrete framework. In the discrete time setting we will use more complicated control than the control function $\|x\|^p_{p-\mathrm{var},[s,t]}$ used in the continuous-time setting above. More precisely, given a finite sequence of controls $\omega_{\cdot,\cdot} \in \mathcal{S}$ associated with parameters $\beta_\omega \in (0,1]$, we would like to construct a version of greedy sequence of (discrete-time) stopping times similar to that in [8].

For our rough equations, we consider $\mathcal{S} = \{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$, $\omega^{(1)}_{s,t} = \hat{L}_f(t-s), \beta_1 = 1$, $\omega^{(2)}_{s,t} = \hat{C}_g^p \left\| x \right\|^p_{p,\Pi[s,t]}, \beta_2 = \frac{1}{p}$ and $\omega^{(3)}_{s,t} = \hat{C}_g^p \left\| \mathbb{X} \right\|^q_{q,\Pi[s,t]}, \beta_3 = \frac{1}{q} = \frac{2}{p}$. Here $\hat{L}_f := \|Df(0)\| + \hat{L}_H$ is the global Lipschitz constant of f on $\mathbb{R}^d$.

Let $0 = t_0 < t_1 < t_2 < \cdots$ be an increase sequence which tends to infinity. Given a fixed $\gamma > 0$, assign the starting time $\tau^{\Pi}_0 = 0$. For each $n \in \mathbb{N}$, assume $\tau^{\Pi}_n = t_k$ is determined. Then one can define τ^{Π}_{n+1} by the following rule:

- if $\sum_{\omega \in \mathcal{S}} \omega^{\beta_\omega}_{t_k, t_{k+1}} > \gamma$ then set $\tau^{\Pi}_{n+1} := t_{k+1}$;
- else set $\tau^{\Pi}_{n+1} := \sup\{t_l > \tau_k : \sum_{\omega \in \mathcal{S}} \omega^{\beta_\omega}_{t_k, t_l} \leq \gamma\}$.

Similar to the continuous case, for a given closed interval $[a,b] \subset \mathbb{R}$, and $\Pi[a,b] = \{t_i : 0 \leq i \leq K, a = t^*_0 < t^*_1 < \ldots < t^*_K = b\}$ be an arbitrary finite partition of $[a,b]$. Given a fixed $\gamma > 0$, assign the starting time $\tau^{\Pi}_0 = a$. For each m, assume $\tau^{\Pi*}_n = t^*_k$ is determined. Then $\tau^{\Pi*}_{n+1}$ is determined by the rule:

- if $\sum_{\omega \in \mathcal{S}} \omega^{\beta_\omega}_{t^*_k, t^*_{k+1}} > \gamma$ then set $\tau^{\Pi*}_{n+1} := t^*_{k+1}$;
- else set $\tau^{\Pi*}_{n+1} := \sup\{t^*_l \in (t^*_k, b] : \sum_{\omega \in \mathcal{S}} \omega^{\beta_\omega}_{t^*_k, t^*_l} \leq \gamma\}$.

A detour to the continuous case: Note that for the continuous-time case treated above, based on the set of controls $\mathcal{S}$, one can replace the control $\mathbf{x}$ in (3.1), (3.2) by $\mathcal{S}$ to construct the sequences $\tau^{\mathcal{S}}_k$ from 0 and $\tau^{\mathcal{S}*}_k$ from a on given $[a,b]$ in a similar manner. Denote by $N^*_{\mathcal{S}}(\gamma, \mathbf{x}, [a,b])$ the number of $\tau^{\mathcal{S}*}_k$ on $[a,b]$. Then, due to the choice of the controls $\omega^{(i)}$ $(i = 1,2,3)$, similar to (3.3) we have (see for instance [7, Lemma 2.6])

$$N^*_{\mathcal{S}}(\gamma, \mathbf{x}, [a,b]) < 1 + \frac{1}{\gamma^p} 4^{p-1} \Big[\hat{L}^p_f (b-a)^p + \hat{C}^p_g \left\| x \right\|^p_{p,\Pi[a,b]} + \hat{C}^p_g \left\| \mathbb{X} \right\|^{p/2}_{p/2,\Pi[a,b]} \Big]. \tag{3.10}$$

Furthermore, similar to Lemma 3.1, we have

$$\liminf_{n \to \infty} \frac{\tau^{\mathcal{S}}_n}{n} \geq \frac{1}{\mathbb{E} N^*_{\mathcal{S}}(\gamma, \mathbf{x}, [0,1])} \tag{3.11}$$

almost surely.

Now back to the discrete setting, define $N^*(\gamma, \mathbf{x}, \Pi[a,b])$ to be the number of times $\tau^{\Pi*}_n$ in $[a,b]$. In the following Lemma we relate the sequence of discrete stopping times to the continuous one.

Lemma 3.4

$$\begin{aligned} N^*(\gamma, \mathbf{x}, \Pi[a,b]) &\leq 2 N^*_{\mathcal{S}}(\gamma, \mathbf{x}, [a,b]), & (3.12)\\ \liminf_{n\to\infty} \frac{\tau^{\Pi}_n(\gamma, \mathbf{x})}{n} &\geq \liminf_{n\to\infty} \frac{1}{2} \frac{\tau^{\mathcal{S}}_n(\gamma, \mathbf{x})}{n}. & (3.13) \end{aligned}$$

Proof:

Observe from the definitions of $\tau^{\Pi*}_m(\gamma, \mathbf{x}, \Pi[a,b])$ for the discrete time set $\Pi[a,b]$ and $\tau^{\mathcal{S}}_i(\gamma, \mathbf{x}, [a,b])$ (with the same set $\mathcal{S}$ of controls but for the continuous time set $[a,b]$) that, between two consecutive stopping times $\tau^{\mathcal{S}}_i(\gamma, \mathbf{x}, [a,b]), \tau^{\mathcal{S}}_{i+1}(\gamma, \mathbf{x}, [a,b])$ there are at most two stopping times $\tau^{\Pi}_m(\gamma, \mathbf{x}, \Pi[a,b])$. As a result,

$$N^*(\gamma, \mathbf{x}, \Pi[a,b]) \leq 2 N^*_{\mathcal{S}}(\gamma, \mathbf{x}, [a,b]).$$

Another consequence is that $\tau^{\Pi}_{2m}(\gamma,\mathbf{x}),\tau^{\Pi}_{2m+1}(\gamma,\mathbf{x}) \geq \tau^{\mathcal{S}}_m(\gamma,\mathbf{x})$ (otherwise there exist two consecutive stopping times $\tau^{\mathcal{S}}_i(\gamma,\mathbf{x}),\tau^{\mathcal{S}}_{i+1}(\gamma,\mathbf{x})$ between which there are more than two stopping times $\tau^{\Pi}_m(\gamma,\mathbf{x})$). Therefore, it is easy to check that

$$\liminf_{m\to\infty} \frac{\tau^{\Pi}_m(\gamma,\mathbf{x})}{m} \geq \liminf_{m\to\infty} \frac{1}{2}\frac{\tau^{\mathcal{S}}_m(\gamma,\mathbf{x})}{m}.$$

The lemma is proved. □

4 Local stability for rough differential equations

We give the definition on asymptotic/exponential stability of the trivial solution (cf. [6] and [31] for the ODE case, and [22, Definition 8] for the rough differential equation case).

Definition 4.1 *(A) Stability: The trivial solution of equation* (1.1) *is called stable, if for any* $\varepsilon > 0$ *there exists a positive random variable* $r = r(\mathbf{x}) > 0$ *such that for any initial value* $\|y_0\| < r(\mathbf{x})$ *the solution* y_t *of* (1.1) *starting from* y_0 *exists on the whole half line* $t \in [0,\infty)$ *and the following inequality holds*

$$\sup_{t\geq 0}\|y_t\| < \varepsilon.$$

(B) Attractivity: the trivial solution is called attractive, if there exists a positive random variable $r(\mathbf{x}) > 0$ *such that any solution* y_t *of* (1.1) *with* $\|y_0\| < r(\mathbf{x})$ *exists on the whole half line* $t \in [0,\infty)$ *and satisfies*

$$\lim_{t\to\infty}\|y_t\| = 0. \tag{4.1}$$

(C) Asymptotic stability: The trivial solution of equation (1.1) *is called asymptotically stable, if it is stable and attractive.*
(D) Exponential stability: The trivial solution of equation (1.1) *is called exponentially stable, if it is stable and there exist two positive random variables* $r(\mathbf{x}) > 0$ *and* $\alpha(\mathbf{x}) > 0$ *and a positive constant* $\mu > 0$ *such that for any initial value* y_0 *satisfying* $\|y_0\| < r(\mathbf{x})$ *the solution of equation* (1.1) *starting from* y_0 *exists on the whole half line* $0 \leq t < \infty$ *and the following inequality*

$$\|y_t\| \leq \alpha(\mathbf{x})e^{-\mu t}$$

holds for all $t \geq 0$.

It is easily seen that, like the case of ordinary differential equations, exponential stability implies asymptotic stability, and the asymptotic stability implies stability; but the inverse direction is not true.

Now we show that given the assumptions (1.3) and (1.4) it is enough to prove the conclusions of our theorems under a stronger condition that matrix $Df(0) \in \mathbb{R}^{d\times d}$ satisfies

$$\exists \lambda_f > 0: \quad \langle y, Df(0)y\rangle \leq -\lambda_f\|y\|^2, \quad \forall y \in \mathbb{R}^d. \tag{4.2}$$

Indeed, for any matrix $Df(0)$ with negative real part eigenvalues, there exists a positive definite symmetric matrix $Q = M^2$ (M is positive definite symmetric matrix) which is the solution of the matrix equation

$$Df(0)^{\mathrm{T}}Q + QDf(0) = -I$$

where $I \in \mathbb{R}^{d\times d}$ is the identity matrix [4, Chapter 2 & Chapter 5]. We then introduce the (invertible) transformation $\xi = My$ under which the transformed system has the form

$$\begin{aligned} d\xi = Mdy &= Mf(y)dt + Mg(y)dx \\ &= Mf(M^{-1}\xi)dt + Mg(M^{-1}\xi)dx \\ &= \Big[MDf(0)M^{-1}\xi + MH(M^{-1}\xi)\Big]dt + Mg(M^{-1}\xi)dx \\ &= [DF(0)\xi + K(\xi)]dt + G(\xi)dx. \end{aligned}$$

where $F(\xi) = DF(0)\xi + K(\xi)$ satisfies the same condition of negative real part eigenvalues and condition (1.4) with different constant C_K and the same constant m. Moreover,

$$\begin{aligned} \langle \xi, MDf(0)M^{-1}\xi\rangle &= \langle MM^{-1}\xi, MDf(0)M^{-1}\xi\rangle \\ &= \frac{1}{2}\Big(\langle M^{-1}\xi, M^2Df(0)M^{-1}\xi\rangle + \langle Df(0)^{\mathrm{T}}M^2M^{-1}\xi, M^{-1}\xi\rangle\Big) \\ &= \frac{1}{2}\langle M^{-1}\xi, \Big[QDf(0) + Df(0)^{\mathrm{T}}Q\Big]M^{-1}\xi\rangle \\ &= -\frac{1}{2}\|M^{-1}\xi\|^2 \leq -\frac{1}{2\|M\|^2}\|\xi\|^2 \end{aligned}$$

which has the form of (4.2). Hence (1.3) and (1.4) lead to (4.2) upto a linear transformation.

We emphasize here that Lemma 3.1 holds only almost surely w.r.t. $\omega \in \Omega$ under the assumption that the Wiener-shift θ in Subsection 6.2 is ergodic. From now on, we will only work with a realization $x \in C^\nu$ of the stochastic process X_t satisfying assumption $(\mathbf{H}_X)$, such that x can be lifted into a rough path $\mathbf{x}$. For a little abuse of notation, we only mention the dependence of $\mathbf{x}$ in the proof, without addressing that $\mathbf{x} = \mathbf{x}(\omega) = (x(\omega), \mathbb{X}(\omega))$ for almost surely $\omega \in \Omega$.

Now we are in a position to prove our first main result.

Theorem 4.2 *Assume $(\boldsymbol{H}_X)$ for the noise, $(\boldsymbol{H}_f)$, (4.2) for the drift and $(\boldsymbol{H}_g)$ for the diffusion with $f(0) = 0, g(0) = 0$ so that zero is the trivial solution of (2.4). If there exists a $\hat{\lambda} \in (0,1)$ such that*

$$\lambda_f > \|Df(0)\|\hat{\lambda}(2+\hat{\lambda}) + C_H(1+\hat{\lambda})^{2+m}\epsilon_0^m + \hat{\lambda}(1+\hat{\lambda})\mathbb{E}N^*(\frac{\hat{\lambda}}{16C_pc^*C_g^*}, \mathbf{x}(\cdot), [0,1]) \tag{4.3}$$

then the trivial solution of (1.1) is exponentially stable almost surely.

Proof: First, we assume $(\mathbf{H}_f^*)$ and $(\mathbf{H}_g^*)$ to prove the theorem under additional assumptions on f and g to make the solutions of the equation (1.1) exist on the whole half line $0 \leq t < \infty$, hence (1.1) generate a global random dynamical system, see e.g. [32, Theorem 4.3], [11] and [10]. Moreover, we assume additionally that the constant $\hat{C}_g$ defined by (2.3) satisfies the inequality

$$\hat{C}_g \leq c^*C_g^*. \tag{4.4}$$

We assume that $g \not\equiv 0$, because otherwise we have nothing to prove in Theorem 4.2. By (4.4) we have $0 < \hat{C}_g \leq c^*C_g^* < \infty$. Therefore, for $\lambda := \hat{\lambda}\frac{\hat{C}_g}{c^*C_g^*}$ we have $0 < \lambda \leq \hat{\lambda} \leq 1$ and due to (4.3) the following inequality is satisfied

$$\lambda_f > \|Df(0)\|\lambda(2+\lambda) + C_H(1+\lambda)^{2+m}\epsilon_0^m + \lambda(1+\lambda)\mathbb{E}N^*(\frac{\lambda}{16C_p\hat{C}_g}, \mathbf{x}(\cdot), [0,1]). \tag{4.5}$$

Now, choose and fix a τ satisfying (2.7) (such a τ obviously exists), we follow Section 2, relate the solution y_t of (1.1) to the solution z_t of the ordinary differential equation (2.6) via transformation $y_t = \phi_t(\mathbf{x}, z_t)$, where $\phi_\cdot(\mathbf{x}, \phi_a)$ is solution of the pure rough differential equation (2.4). To estimate the solution norm growth, assign

$$\eta_t := y_t - z_t, \quad \text{and} \quad \psi_t := \Big[\frac{\partial \phi}{\partial z}(t, \mathbf{x}, z_t)\Big]^{-1} - I, \quad \forall t \in [0,\tau],$$

and one rewrites the transformed system (2.6) in the form

$$\dot{z}_t = (I+\psi_t) f(z_t+\eta_t) = (I+\psi_t)\Big[Df(0)(z_t+\eta_t) + H(z_t+\eta_t)\Big], \quad t\in[0,\tau],\ z_0 = y_0. \tag{4.6}$$

Introduce the Lyapunov function $V(z) = \|z\|^2$ and fix an $\epsilon \in (0,\epsilon_0)$. Since λ and τ satisfy (2.7), one deduces from Proposition 2.1 that $\|\psi_t\| \leq \lambda$ and $\|\eta_t\| \leq \lambda \|z_t\|$ for all $t \in [0,\tau]$. It follows from (4.6) that

$$\begin{aligned}
\frac{d}{2dt} V(z_t) &= \langle z_t, Df(0) z_t\rangle + \langle z_t, Df(0)\eta_t\rangle + \langle z_t, \psi_t Df(0)(z_t+\eta_t)\rangle \\
&\quad + \langle z_t, (I+\psi_t) H(z_t+\eta_t)\rangle \\
&\leq -\lambda_f \|z_t\|^2 + \|Df(0)\|\|z_t\|\|\eta_t\| + \|\psi_t\|\|Df(0)\|\|z_t\|\|z_t+\eta_t\| \\
&\quad + (1+\|\psi_t\|)\|z_t\|\|H(z_t+\eta_t)\| \\
&\leq -\lambda_f\|z_t\|^2 + \|Df(0)\|\lambda(2+\lambda)\|z_t\|^2 + C_H(1+\lambda)^{2+m}\|z_t\|^{2+m} \\
&\leq -\Big[\lambda_f - \|Df(0)\|\lambda(2+\lambda) - C_H(1+\lambda)^{2+m}\|z_t\|^m\Big]\|z_t\|^2,
\end{aligned} \tag{4.7}$$

where the last two inequalities of (4.7) hold as long as $(1+\lambda)\|z_t\| \leq \epsilon < \epsilon_0$ in order to apply (1.4). Due to (4.3), λ and ϵ satisfy

$$\kappa = \kappa(\lambda,\epsilon) := \lambda_f - \|Df(0)\|\lambda(2+\lambda) - C_H(1+\lambda)^{2+m}\epsilon^m > 0. \tag{4.8}$$

As a result, given $\|z_0\| \leq \frac{\epsilon}{1+\lambda}$, define $\tau^* := \sup\{t \in [0,\tau] : \|z_s\| \leq \frac{\epsilon}{1+\lambda}\ \forall 0 \leq s \leq t\}$, then $\|z_{\tau^*}\| \leq \frac{\epsilon}{1+\lambda}$ and it follows from (4.7) and (4.8) that

$$\frac{d}{dt} V(z) \leq -2\kappa V(z), \quad t \in [0,\tau^*].$$

In particular, $V(z_{\tau^*}) \leq V(z_0)\exp\{-2\kappa\tau^*\} = V(y_0)\exp\{-2\kappa\tau^*\} < V(z_0)$ or $\|z_{\tau^*}\| < \|z_0\| \leq \frac{\epsilon}{1+\lambda}$, thus z_t can still be extended into the right hand side of τ^* until its norm hits $\frac{\epsilon}{1+\lambda}$. Due to its definition, $\tau^* = \tau$. In other words, provided that $\|y_0\| \leq \frac{\epsilon}{1+\lambda}$, we have just proved that

$$\begin{aligned}
\|y_t\| = \|z_t + \eta_t\| &\leq (1+\lambda)\|z_t\| \\
&\leq \|y_0\|(1+\lambda)\exp\{-\kappa t\} \\
&\leq \|y_0\|\exp\{-\kappa t + \lambda\}, \quad \forall t \in [0,\tau].
\end{aligned} \tag{4.9}$$

Next, construct the greedy sequence of stopping times (3.1) on $[0,\infty)$ where $\gamma = \frac{\lambda}{16 C_p C_g^*}$. One can then prove similarly that: whenever $\|y_{\tau_n}\| \leq \frac{\epsilon}{1+\lambda}$ then

$$\|y_t\| \leq \|y_{\tau_n}\|(1+\lambda)\exp\{-\kappa(t-\tau_n)\} \leq \|y_{\tau_n}\|\exp\{-\kappa(t-\tau_n)+\lambda\} \tag{4.10}$$

for all $t \in [\tau_n, \tau_{n+1}], n \in \mathbb{N}$. By induction,

$$\|y_{\tau_n}\| \leq \|y_0\|\exp\{-\kappa\tau_n + n\lambda\} \wedge \frac{\epsilon}{1+\lambda}, \quad \forall n \in \mathbb{N}.$$

Thus, in order for y_{τ_n} to satisfies $y_{\tau_n} \leq \frac{\epsilon}{1+\lambda}$, it is sufficient that

$$\|y_0\| \exp\{-\kappa\tau_n + n\lambda\} \leq \frac{\epsilon}{1+\lambda}, \quad \forall n \in \mathbb{N},$$

or

$$\|y_0\| \leq R(\mathbf{x}) := \frac{\epsilon}{1+\lambda} \inf\Big\{ \exp\{\kappa\tau_n(\mathbf{x}) - n\lambda\} : n \in \mathbb{N}\Big\}. \tag{4.11}$$

We show that $R(\mathbf{x})$ defined in (4.11) is a positive random variable. Indeed, by Lemma 3.1, there exists a positive random variable $M(\mathbf{x})$ such that

$$\inf\{\frac{\tau_n}{n} : n \geq M(\mathbf{x})\} > \frac{1}{(1+\lambda)C_1}, \text{ where } C_1 := \mathbb{E}N^*(\frac{\lambda}{16C_p\hat{C}_g}, \mathbf{x}(\cdot), [0,1]). \tag{4.12}$$

As a result,

$$\epsilon \geq R(\mathbf{x}) \geq \frac{\epsilon}{1+\lambda} \exp\Big\{ \min_{0\leq n < M(\mathbf{x})} \Big(\kappa\tau_n - n\lambda\Big) \bigwedge \inf_{n\geq M(\mathbf{x})} n\Big(\frac{\kappa}{(1+\lambda)C_1} - \lambda\Big)\Big\}.$$

It follows from (4.5) and (4.8) that

$$\kappa > \lambda(1+\lambda)C_1.$$

Hence, taking into account that $\kappa\tau_0 - 0\lambda = 0$, it follows that

$$\epsilon \geq R(\mathbf{x}) = \frac{\epsilon}{1+\lambda} \exp\Big\{ \min_{0\leq n < M(\mathbf{x})} \Big(\kappa\tau_n - n\lambda\Big)\Big\}. \tag{4.13}$$

As such, the right hand side of (4.13) is positive and thus $R(\mathbf{x})$ defined in (4.11) is a positive random variable. All in all, there exists a random neighborhood $B(0, R(\mathbf{x}))$ of zero such that whenever $\|y_0\| \leq R(\mathbf{x})$ then $\|y_{\tau_n}\| \leq \frac{\epsilon}{1+\lambda}$ for all $n \in \mathbb{N}$ and then $\|y_t\| \leq \epsilon$ for all $t \geq 0$ due to (4.10). Since the conclusion holds for any fixed $\epsilon \leq \epsilon_0$, this proves stability. Moreover, since $\sup\{\frac{n}{\tau_n} : n \geq M(\mathbf{x})\} < (1+\lambda)C_1$ for $n \geq M(\mathbf{x})$,

$$\|y_{\tau_n}\| \leq \|y_0\| \exp\Big\{ -\tau_n\Big(\kappa - \lambda\frac{n}{\tau_n}\Big)\Big\} \leq \|y_0\| \exp\Big\{ -\Big(\kappa - \lambda(1+\lambda)C_1\Big)\tau_n\Big\}.$$

Put

$$\mu := \kappa - \lambda(1+\lambda)C_1,$$

we have $0 < \mu < \kappa$. Using (4.10), and taking into account that $\|y_0\| \leq \frac{\epsilon}{1+\lambda}$, we obtain that

$$\|y_t\| \leq \|y_0\|(1+\lambda)e^{-\kappa(t-\tau_n)}e^{-\mu\tau_n} \leq \epsilon e^{-\mu t} \quad \text{for all} \quad t \in [\tau_n, \tau_{n+1}], n \geq M(\mathbf{x}).$$

Therefore, since $\|y_t\| \leq \epsilon$ for all $t \geq 0$, provided the initial value $\|y_0\| \leq R(\mathbf{x})$, we have

$$\|y_t\| \leq \alpha(\mathbf{x}) \exp(-\mu t) \quad \text{for all} \quad t \geq 0,$$

where

$$\alpha(\mathbf{x}) := \epsilon \exp\Big\{\Big(\kappa - \lambda(1+\lambda)C_1\Big)\tau_{M(\mathbf{x})}\Big\} = \epsilon \exp\{\mu\tau_{M(\mathbf{x})}\} > 0$$

is a positive random variable. This proves the exponential stability of the zero solution of (1.1) under the additional assumptions ($\mathbf{H}_f^*$)-($\mathbf{H}_g^*$)-(4.4).

Finally we show that we can relax our arguments from the additional assumptions ($\mathbf{H}_f^*$)-($\mathbf{H}_g^*$)-(4.4). Given the equation (1.1) on $B(0,\epsilon_0)$, we follow the extension process in Section 2 to extend f, g on $B(0,\epsilon_0)$ to f^*, g^* on $\mathbb{R}^d$. Then consider equation (2.12)

$$dy_t = f^*(y_t)dt + g^*(y_t)dx_t,$$

It is easily seen that the functions f^*, g^* satisfy the strong assumptions $(\mathbf{H}^*_{f^*})$-$(\mathbf{H}^*_{g^*})$-(2.11) as well as the original assumptions $(\mathbf{H}_{f^*})$-$(\mathbf{H}_{g^*})$. Therefore, the foregoing arguments are applicable to this equation (2.12) due to the choice of f^*, g^*, implying that the trivial solution of equation (2.12) is exponentially stable. That is, there exist two positive random variables $\epsilon_0 > \epsilon \geq R^*(\mathbf{x}) > 0$, $\alpha^*(\mathbf{x}) > 0$ and a positive constant μ such that if $\|y_0\| \leq R^*(\mathbf{x})$ then the solution y_t of (2.12), starting from y_0, satisfies $\|y_t\| \leq \epsilon < \epsilon_0$ and

$$\|y_t\| \leq \alpha^*(\mathbf{x}) \exp(-\mu t), \quad \forall t \geq 0. \tag{4.14}$$

We notice here that since $f^* = f$ and $g^* = g$ on $B(0, \epsilon_0)$ the quantities $\|Df(0)\|$, C_H, λ_f, C^*_g defined for the equation (1.1) coincide with their counterparts defined for (2.12).

For any initial value $\|y_0\| \leq R^*(\mathbf{x}) \leq \epsilon < \epsilon_0$ the solution y_t of the equation (2.12), starting from y_0, satisfies $\|y_t\| \leq \epsilon$ for all $t \geq 0$, hence it is the solution of (1.1) starting from y_0 because (1.1) coincides with (2.12) for all those solutions. Therefore, for any initial value y_0 satisfying $\|y_0\| \leq R^*(\mathbf{x})$ the solution y_t of (1.1) starting from y_0 satisfies $\|y_t\| \leq \epsilon$ and (4.14) for all $t \geq 0$. This implies that the equation (1.1) is exponential stable. The proof is complete. □

Theorem 4.3 *Assume that $f(0) = 0$ and $g(0) = 0$ so that zero is the trivial solution of* (1.1)*, and the conditions ($\boldsymbol{H}_f$), ($\boldsymbol{H}_g$), ($\boldsymbol{H}_X$) hold. Then there exists $C_0 > 0$ depending only on f such that if $0 < C_g < C_0$ the trivial solution of* (1.1) *is exponentially stable almost surely.*

Proof: We apply the Theorem 4.2 to prove this theorem. Note that from the condition $(\mathbf{H}_f)$ we have (1.3), (1.4) and as shown above we can also have (4.2) by a linear transformation. Therefore, for simplicity we assume that (4.2) holds.

To prove our theorem we will use the criterion of stability provided by Theorem 4.2.

First observe that there exists a small positive number $C_0 \in (0, 1)$ (dependent on f but not on g) such that

$$\frac{\lambda_f}{4} \geq \|Df(0)\| C_0 (2 + C_0) + (1 + C_0) C_0 \mathbb{E} N^*(\frac{1}{16 C_p c^*}, \mathbf{x}(\cdot), [0, 1]), \tag{4.15}$$

(because the right hand side of the above inequality is a continuous increasing function of non-negative C_0 and vanishes at 0). We will show that this constant C_0 furnishes the conclusion of the theorem. To this end, let g is such that $0 < C_g < C_0$ we will show below by an application of Theorem 4.2 that (1.1) is exponentially stable.

It is easily seen that there exists a small positive number $\epsilon_1 \in (0, 1)$ such that

$$C_H (1 + C_0)^{2+m} \epsilon_1^m \leq \frac{\lambda_f}{4}, \tag{4.16}$$

and

$$\epsilon_1 \leq \min\{\frac{C_0}{1 + \|D^3 g\|_{\infty, B(0, \epsilon_0)}}, 1, \epsilon_0\}. \tag{4.17}$$

Then $B(0, \epsilon_1) \subset B(0, \epsilon_0)$ and we have

$$\|D^i g\|_{\infty, B(0, \epsilon_1)} \leq \|D^i g\|_{\infty, B(0, \epsilon_0)} \quad \text{for} \quad i = 0, 1, 2, 3. \tag{4.18}$$

Since $\|Dg\|_{\infty, B(0, \epsilon_1)} \leq C_0$ and $g(0) = 0$ we have $\|g\|_{\infty, B(0, \epsilon_1)} \leq C_0 \epsilon_1$. Hence

$$\|g\|_{\infty, B(0, \epsilon_1)} \|D^3 g\|_{\infty, B(0, \epsilon_1)} \leq C_0 \epsilon_1 \|D^3 g\|_{\infty, B(0, \epsilon_1)} < C_0^2. \tag{4.19}$$

We notice that if we restrict the consideration of (1.1) on the ball $B(0,\epsilon_1)$ instead of the ball $B(0,\epsilon_0)$ then the stability criterion (4.3) provided by Theorem 4.2 now read

$$\lambda_f > \|Df(0)\|\hat{\lambda}(2+\hat{\lambda}) + C_H(1+\hat{\lambda})^{2+m}\epsilon_1^m + \hat{\lambda}(1+\hat{\lambda})\mathbb{E}N^*(\frac{\hat{\lambda}}{16C_pc^*\tilde{C}_g^*}, \mathbf{x}(\cdot), [0,1]), \tag{4.20}$$

where $\tilde{C}_g^*$ is the counterpart of C_g^* computed in the ball $B(0,\epsilon_1)$. By virtue of (4.18), (4.19), (1.5) and (1.6) we have

$$\tilde{C}_g^* \le \max\{C_g, C_0\} \le C_0. \tag{4.21}$$

Now we show that if we choose $\hat{\lambda} = \tilde{C}_g^*$ then the inequality (4.20) is satisfied. Indeed, by the choice of $\hat{\lambda}$ and (4.21), (4.15) we have

$$\frac{\lambda_f}{4} \ge \|Df(0)\|\hat{\lambda}(2+\hat{\lambda}) + \hat{\lambda}(1+\hat{\lambda})\mathbb{E}N^*(\frac{\hat{\lambda}}{16C_pc^*\tilde{C}_g^*}, \mathbf{x}(\cdot), [0,1]).$$

This together with (4.16) imply (4.20). Therefore, Theorem 4.2 is applicable and leads to the exponential stability for system (1.1). □

Remark 4.4 (i) Theorem 4.3 implies the result in [22, Theorem 17]. Indeed, if $g \in C^3$ and $g(0) = 0, Dg(0) = 0, D^2g(0) = 0$ then by restricting consideration to a smaller ball $B(0,\epsilon_1) \subset B(0,\epsilon_0)$ we may make C_g small enough to satisfy the assumptions of Theorem 4.3 implying that the equation is exponentially stable. On the other hand, in the case g is a linear map, $g = Cy$, $C \in \mathbb{R}^{d\times m}$ with $\|C\| < C_0$, (1.1) is exponentially stable by Theorem 4.3. However, condition $Dg(0) = 0$ in [22, Theorem 17] is not satisfied so that [22, Theorem 17] is not applicable. Thus, Theorem 4.3 is significantly stronger than [22, Theorem 17].

(ii) We notice that the derivative of order 3 of g does not affect the local exponential stability of (1.1). An interesting example is the one-dimensional case with $g = C_1y + C_2y^2 + C_3y^3$, where C_1, C_2, C_3 are constants. By Theorem 4.3 if C_1, C_2 are small enough then (1.1) is exponentially stable for any C_3 (provided other assumptions on f, g are satisfied).

(iii) Criterion (4.3) can be relaxed to

$$\lambda_f > \|Df(0)\|\hat{\lambda}(2+\hat{\lambda}) + C_H(1+\hat{\lambda})^{2+m}\epsilon_0^m + \hat{\lambda}(1+\hat{\lambda})\mathbb{E}N^*(\frac{\hat{\lambda}}{16C_pC_g^*}, \mathbf{x}(\cdot), [0,1]) \tag{4.22}$$

without constant c^*. The reason is that all the arguments in the proof of Theorem 4.2 are applied in the local regime $B(0,\epsilon_0)$ where f^*, g^* coincide with f, g. Hence condition (4.15) in Theorem 4.3 can also be relaxed without c^*. However, the difference between the criteria are small and only upto constant c^* that happens in controlling the number of stopping time N^*, hence is negligible. We will see in the next section that the direct arguments for the local system fails to be applied for the discretization scheme of system (1.1), because at each time step there is a possibility that the solution can exit $B(0,\epsilon_0)$, making it difficult to construct a sequence of stopping time τ_n as seen in Theorem 4.2. For that reason, we prefer to maintain the strategy to study the local stability of system (1.1) and its discretization scheme via the extended system (2.12), that helps us simplify the arguments of the proofs.

5 Local stability for discrete rough systems

In this section, we consider the discretization scheme of system (1.1), i.e. the explicit Euler scheme for the regular grid with step size $\Delta > 0$, i.e. $\Pi = \{t_k := k\Delta\}_{k\in\mathbb{N}}$ and

$$\begin{aligned} &y_0^\Delta \in \mathbb{R}^d, \\ &y_{t_{k+1}}^\Delta = y_{t_k}^\Delta + f(y_{t_k}^\Delta)\Delta + g(y_{t_k}^\Delta)x_{t_k,t_{k+1}} + Dg(y_{t_k}^\Delta)g(y_{t_k}^\Delta)\mathbb{X}_{t_k,t_{k+1}}, \quad k \in \mathbb{N}. \end{aligned} \tag{5.1}$$

The global dynamics of the discrete system (5.1) has been studied recently in [10], [8], which show that there is a similarity in asymptotic behavior of the continuous system (1.1) and its discretization (5.1). Note that we can not apply the Doss-Sussmann technique for the rough difference equation, simply because it is difficult to control the solution growth in a smooth way for the discrete time set. Instead, we will couple the discrete system (5.1) with its unperturbed discrete system and control the difference of the two trajectories.

Here in our local setting, the global arguments in [8] can not be applied because one needs to control at each discrete time step the solution to stay inside the ball $B(0,\epsilon_0)$ to prove the local stability (the notion of stability of discrete systems is an obvious modification of the one for the continuous-time case). By extending the local system from $B(0,\epsilon_0)$ to $\mathbb{R}^d$, we can prove the following results.

Theorem 5.1 *Assume ($\boldsymbol{H}_f$) for the noise, ($\boldsymbol{H}_f$), (4.2) for the drift and ($\boldsymbol{H}_g$) with $f(0) = 0, g(0) = 0$. Assume further that there exist $\Delta > 0$ and $\lambda \in (0, \frac{1}{2}), \gamma^* \in (0,1)$ such that*

$$\begin{aligned} \frac{1}{2\Delta} > \qquad \lambda_f &> \frac{1}{2}C_H\epsilon_0^m + \frac{1}{2}(\|Df(0)\| + L_H c^*)^2\Delta \\ &+ (3C_p + 2e^{\frac{1}{2}})c^* C_g \gamma^* \Big[\mathbb{E}N^*(\gamma^*, \mathbf{x}(\cdot), [0,1]) + \mathbb{E}N^*_{\mathcal{S}}(\lambda, \mathbf{x}(\cdot), [0,1])\Big]. \end{aligned} \tag{5.2}$$

Then the trivial solution of system (5.1) *is exponentially stable almost surely.*

Proof: We will do in a similar manner as in the proof of Theorem 4.2. At first, we assume global conditions of f, g as in ($\mathbf{H}_f^*$)-($\mathbf{H}_g^*$). Moreover, we assume additionally that the constants $\hat{C}_g$ defined by (2.3) and $\hat{L}_H$ defined by (2.2) satisfy the inequality

$$\hat{C}_g \le c^* C_g, \quad \hat{L}_H \le c^* L_H. \tag{5.3}$$

Due to the assumption ($\mathbf{H}_f^*$) the function f is global Lipschitz continuous with constant $\hat{L}_f = \|Df(0)\| + \hat{L}_H$. From (5.2), by virtue of (5.3) we have

$$\begin{aligned} \frac{1}{2\Delta} > \qquad \lambda_f &> \frac{1}{2}C_H\epsilon_0^m + \frac{1}{2}\hat{L}_f^2\Delta \\ &+ (3C_p + 2e^{\frac{1}{2}})\hat{C}_g \gamma^* \Big[\mathbb{E}N^*(\gamma^*, \mathbf{x}(\cdot), [0,1]) + \mathbb{E}N^*_{\mathcal{S}}(\lambda, \mathbf{x}(\cdot), [0,1])\Big]. \end{aligned} \tag{5.4}$$

For each $\tau \in \Pi$ fixed, we would like to compare the solution of this difference equation (5.1) with the unperturbed difference equation

$$z_0 = y_\tau^\Delta, \quad z_{t_{k+1}} = z_{t_k} + f(z_{t_k})\Delta, \quad \forall t_k \ge \tau. \tag{5.5}$$

We would like to study the dynamics of this difference equation (5.5). Due to the global Lipschitz continuity of f, for any $s, t \in \Pi, t \ge s \ge \tau$,

$$\|z_{s,t}\| \le \sum_{s \le t_k < t} \|f(z_{t_k})\|\Delta \le \hat{L}_f \|z\|_{\infty,\Pi[s,t]}(t-s);$$

which implies

$$\|z\|_{p-\mathrm{var},\Pi[s,t]} \leq \|z_s\| + \hat{L}_f \|z\|_{\infty,\Pi[s,t]}(t-s) \leq \Big[1+\hat{L}_f(t-s)\Big]\|z\|_{\infty,\Pi[s,t]}. \tag{5.6}$$

On the other hand, by using Lyapunov function $V(z) = \|z\|^2$ and applying the same arguments as the ones in Theorem 4.2, we obtain for any $\|z_{t_k}\| \leq \epsilon \leq \epsilon_0$

$$\begin{aligned}
\|z_{t_{k+1}}\|^2 &= \|z_{t_k}\|^2 + 2\langle z_{t_k}, f(z_{t_k})\rangle \Delta + \|f(z_{t_k})\|^2\Delta^2 \\
&\leq \|z_{t_k}\|^2 - 2\lambda_f \Delta \|z_{t_k}\|^2 + C_H\epsilon^m \Delta \|z_{t_k}\|^2 + \hat{L}_f^2 \Delta^2 \|z_{t_k}\|^2 \\
&\leq \|z_{t_k}\|^2 \Big[1 - \Big(2\lambda_f - C_H\epsilon^m - \hat{L}_f^2\Delta\Big)\Delta\Big] \\
&\leq \|z_{t_k}\|^2(1-2\kappa\Delta) \leq \|z_{t_k}\|^2 \exp\{-2\kappa\Delta\},
\end{aligned}$$

where one deduces from (5.4) that

$$\kappa := \lambda_f - \frac{1}{2}C_H\epsilon^m - \frac{1}{2}\hat{L}_f^2\Delta \in (0, \frac{1}{2\Delta}). \tag{5.7}$$

As a result, we can prove by induction that,

$$\text{if} \quad \|z_0\| \leq \epsilon \quad \text{then} \quad \|z_t\| \leq \|z_0\|(1-\kappa t) \leq \|z_0\|\exp\{-\kappa t\} < \epsilon, \quad \forall t \in \Pi. \tag{5.8}$$

Next, define $h_{t_k} = y^\Delta_{t_k} - z_{t_k}$ for all $t_k \geq \tau$, then by definition $h_\tau = 0$ and for all $t_k \geq \tau$

$$\begin{aligned}
h_{t_{k+1}} &= h_{t_k} + [f(h_{t_k}+z_{t_k}) - f(z_{t_k})]\Delta \\
&\quad + g(h_{t_k}+z_{t_k})x_{t_k,t_{k+1}} + Dg(h_{t_k}+z_{t_k})g(h_{t_k}+z_{t_k})\mathbb{X}_{t_k,t_{k+1}}.
\end{aligned}$$

Define $R^h_{s,t} := h_{s,t} - g(h_s+z_s)x_{s,t}$ for all $s \leq t$. We then apply the discrete sewing lemma [23, 8] for

$$F_{s,t} = g(h_s+z_s)x_{s,t} + Dg(h_s+z_s)g(h_s+z_s)\mathbb{X}_{s,t}$$

for all $s,t \in \Pi, t \geq s \geq \tau$ and similar estimates to [8, Theorem 3.3], to obtain

$$\begin{aligned}
&\|h,R^h\|_{p-\mathrm{var},\Pi[s,t]} \\
\leq\ & \|h_s\| + C_p\Big[\hat{C}_g \left\|x\right\|_{p-\mathrm{var},\Pi[s,t]} + \hat{C}_g^2 \left\|\mathbb{X}\right\|_{p-\mathrm{var},\Pi[s,t]}\Big]\|z\|_{p-\mathrm{var},\Pi[s,t]} \\
&+ C_p\Big[\hat{L}_f(t-s) + \hat{C}_g \left\|x\right\|_{p-\mathrm{var},\Pi[s,t]} + \hat{C}_g^2 \left\|\mathbb{X}\right\|_{p-\mathrm{var},\Pi[s,t]}\Big]\|h,R^h\|_{p-\mathrm{var},\Pi[s,t]}.
\end{aligned} \tag{5.9}$$

We now apply (5.9) for $s = \tau = \tau^\Delta_n, t = \tau^\Delta_{n+1}$ where $\tau^\Delta_n, \tau^\Delta_{n+1}$ are the stopping times defined in Section 3.2 for $\lambda \in (0, \frac{1}{2})$. Then $h_{\tau^\Delta_n} = 0$. Consider two cases:

Case 1: If $C_p\Big[\hat{L}_f(\tau^\Delta_{n+1} - \tau^\Delta_n) + \hat{C}_g \left\|x\right\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} + \hat{C}_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]}\Big] \leq \lambda$, then (5.9) yields

$$\begin{aligned}
&\|h\|_{\infty,\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} \\
\leq\ & \|h,R^h\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} \\
\leq\ & \lambda\|h,R^h\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} + C_p\hat{C}_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} \|z\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} \\
\leq\ & \lambda\|h,R^h\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} \\
&+ C_p\hat{C}_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} \left[1+\hat{L}_f(\tau^\Delta_{n+1}-\tau^\Delta_n)\right]\|z\|_{\infty,\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} \\
\leq\ & \lambda\|h,R^h\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} + (1+\lambda)C_p\hat{C}_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]} \|z\|_{\infty,\Pi[\tau^\Delta_n,\tau^\Delta_{n+1}]}
\end{aligned}$$

which yields

$$\begin{aligned}\|h\|_{\infty,\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]} &\leq \|h,R^h\|_{p-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\\ &\leq \frac{C_p\hat{C}_g \left\lVert\!\left\lvert \mathbf{x}\right\rvert\!\right\rVert_{p-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}(1+\lambda)}{1-\lambda}\|z\|_{\infty,\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\\ &\leq 3C_p\hat{C}_g \left\lVert\!\left\lvert \mathbf{x}\right\rvert\!\right\rVert_{p-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\|z\|_{\infty,\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\end{aligned}$$

where the last inequality holds due to $\lambda\in(0,\frac{1}{2})$. As a result, whenever $\|y^\Delta_{\tau_n^\Delta}\|=\|z_{\tau_n^\Delta}\|\leq\frac{\epsilon}{1+3\lambda}$, we obtain from (5.8)

$$\begin{aligned}\|y^\Delta_t\| &\leq \|z_t\|+3C_p\hat{C}_g \left\lVert\!\left\lvert \mathbf{x}\right\rvert\!\right\rVert_{p-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\|z\|_{\infty,\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]} \qquad (5.10)\\ &\leq \|z_{\tau_n^\Delta}\|\Big[\exp\{-\kappa(t-\tau_n^\Delta)\}\Big(1+3C_p\hat{C}_g \left\lVert\!\left\lvert \mathbf{x}\right\rvert\!\right\rVert_{p-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\Big)\Big]\\ &\leq \|y^\Delta_{\tau_n^\Delta}\|(1+3\lambda)\leq\epsilon,\quad \forall t\in\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta].\end{aligned}$$

In particular

$$\begin{aligned}\|y^\Delta_{\tau_{n+1}^\Delta}\| &\leq \|y^\Delta_{\tau_n^\Delta}\|\Big[1-\kappa(\tau_{n+1}^\Delta-\tau_n^\Delta)+3C_p\hat{C}_g \left\lVert\!\left\lvert \mathbf{x}\right\rvert\!\right\rVert_{p-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\Big]\\ &\leq \|y^\Delta_{\tau_n^\Delta}\|\exp\{-\kappa(\tau_{n+1}^\Delta-\tau_n^\Delta)+3C_p\hat{C}_g \left\lVert\!\left\lvert \mathbf{x}\right\rvert\!\right\rVert_{p-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\}.\end{aligned}$$

By applying Lemma 3.2 for a certain $\gamma^*>0$ fixed, one obtains

$$\|y^\Delta_{\tau_{n+1}^\Delta}\|\leq\|y^\Delta_{\tau_n^\Delta}\|\exp\{-\kappa(\tau_{n+1}^\Delta-\tau_n^\Delta)+3C_p\hat{C}_g\gamma^*N^*(\gamma^*,\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta])\}. \qquad (5.11)$$

Case 2: If $C_p\Big[\hat{L}_f(\tau_{n+1}^\Delta-\tau_n^\Delta)+\hat{C}_g \left\lVert\!\left\lvert x\right\rvert\!\right\rVert_{p-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}+\hat{C}_g^2 \left\lVert\!\left\lvert \mathbb{X}\right\rvert\!\right\rVert_{q-\mathrm{var},\Pi[\tau_n^\Delta,\tau_{n+1}^\Delta]}\Big]\geq\lambda$ so that $\tau_n^\Delta,\tau_{n+1}^\Delta$ are two consecutive times in Π. Construct the sequence of stopping times $\tau_i(\frac{\lambda}{C_p\hat{C}_g},\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta])$ for continuous time interval $[\tau_n^\Delta,\tau_{n+1}^\Delta]$, thus

$$\|h_{\tau_{n+1}^\Delta}\| \leq \|z_{\tau_n^\Delta}\|\Big(\hat{C}_g\|x_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|+\hat{C}_g^2\|\mathbb{X}_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|\Big).$$

Then it follows from (5.8) and the fact that $\tau_{n+1}^\Delta=\tau_n^\Delta+\Delta$ in this case, that

$$\begin{aligned}\|y^\Delta_{\tau_{n+1}^\Delta}\| &\leq \|z_{\tau_{n+1}^\Delta}\|+\|z_{\tau_n^\Delta}\|\Big[\hat{C}_g\|x_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|+\hat{C}_g^2\|\mathbb{X}_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|\Big]\\ &\leq \|y^\Delta_{\tau_n^\Delta}\|\Big[1-\kappa(\tau_{n+1}^\Delta-\tau_n^\Delta)+\hat{C}_g\|x_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|+\hat{C}_g^2\|\mathbb{X}_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|\Big] \qquad (5.12)\end{aligned}$$

Note that $\kappa(\tau_{n+1}^\Delta-\tau_n^\Delta)=\kappa\Delta<\frac{1}{2}$, we now apply inequality $1-u+v\leq e^{-u}(1+e^uv)$ to (5.12) to obtain

$$\begin{aligned}\|y^\Delta_{\tau_{n+1}^\Delta}\| &\leq \|y^\Delta_{\tau_n^\Delta}\|\exp\Big\{-\kappa(\tau_{n+1}^\Delta-\tau_n^\Delta)\Big\}\\ &\quad\Big[1+\exp\Big\{\kappa(\tau_{n+1}^\Delta-\tau_n^\Delta)\Big\}\Big(\hat{C}_g\|x_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|+\hat{C}_g^2\|\mathbb{X}_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|\Big)\Big]\\ &\leq \|y^\Delta_{\tau_n^\Delta}\|\exp\Big\{-\kappa(\tau_{n+1}^\Delta-\tau_n^\Delta)\Big\}\\ &\quad\Big[1+e^{\kappa\Delta}\Big(\hat{C}_g\|x_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|+\hat{C}_g^2\|\mathbb{X}_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|\Big)\Big]. \qquad (5.13)\end{aligned}$$

Again we apply Lemma 3.2 so that for a certain $\gamma^* > 0$,

$$\begin{aligned}
& 1+e^{\kappa\Delta}\Big(\hat{C}_g\|x_{\tau_n^\Delta,\tau_{n+1}^\Delta}\| + \hat{C}_g^2\|\mathbb{X}_{\tau_n^\Delta,\tau_{n+1}^\Delta}\|\Big) \\
\leq\ & 1+e^{\kappa\Delta}\hat{C}_g\,\|\!|\mathbf{x}|\!\|_{p-\mathrm{var},[\tau_n^\Delta,\tau_{n+1}^\Delta]} + e^{\kappa\Delta}\hat{C}_g^2\,\|\!|\mathbf{x}|\!\|^2_{p-\mathrm{var},[\tau_n^\Delta,\tau_{n+1}^\Delta]} \\
\leq\ & 1+e^{\kappa\Delta}\hat{C}_g\gamma^* N^*(\gamma^*,\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta]) \\
& + e^{\kappa\Delta}\hat{C}_g^2(\gamma^*)^2\Big[N^*(\gamma^*,\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta])\Big]^2 \\
\leq\ & 1+e^{\kappa\Delta}\hat{C}_g\gamma^* N^*(\gamma^*,\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta]) + [e^{\kappa\Delta}\hat{C}_g\gamma^*]^2\Big[N^*(\gamma^*,\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta])\Big]^2 \\
\leq\ & \exp\Big\{2e^{\frac{1}{2}}\hat{C}_g\gamma^* N^*(\gamma^*,\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta])\Big\},
\end{aligned}$$

where the last inequality is due to the fact that $1+u+u^2 \leq e^{2u}, \forall u \geq 0$. Hence (5.13) has the form

$$\|y^\Delta_{\tau^\Delta_{n+1}}\| \leq \|y^\Delta_{\tau^\Delta_n}\| \exp\Big\{-\kappa(\tau^\Delta_{n+1}-\tau^\Delta_n) + 2e^{\frac{1}{2}}\hat{C}_g\gamma^* N^*(\gamma^*,\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta])\Big\}. \tag{5.14}$$

Combining (5.11) and (5.14) and setting

$$C = 3C_p + 2e^{\frac{1}{2}}$$

yields for any $n \in \mathbb{N}$

$$\|y^\Delta_{\tau^\Delta_{n+1}}\| \leq \|y^\Delta_{\tau^\Delta_n}\| \exp\Big\{-\kappa(\tau^\Delta_{n+1}-\tau^\Delta_n) + C\hat{C}_g\gamma^* N^*(\gamma^*,\mathbf{x},[\tau_n^\Delta,\tau_{n+1}^\Delta])\Big\}$$

provided that $\|y^\Delta_{\tau^\Delta_n}\| \leq \frac{\epsilon}{1+3\lambda}$. One can then prove by induction that for any $n \geq 1$

$$\|y^\Delta_{\tau^\Delta_n}\| \quad\leq\quad \|y^\Delta_0\| \exp\Big\{-\kappa\tau^\Delta_n + C\hat{C}_g\gamma^* \sum_{j=0}^{n-1} N^*(\gamma^*,\mathbf{x},[\tau_j^\Delta,\tau_{j+1}^\Delta])\Big\}. \tag{5.15}$$

By applying (3.9) in Lemma 3.3 to (5.15), we finally obtain

$$\|y^\Delta_{\tau^\Delta_n}\| \quad\leq\quad \|y^\Delta_0\| \exp\Big\{-\kappa\tau^\Delta_n + C\hat{C}_g\gamma^* \Big[N^*(\gamma^*,\mathbf{x},[\tau_0^\Delta,\tau_n^\Delta]) + n\Big]\Big\}. \tag{5.16}$$

Hence, in order for $\|y^\Delta_{\tau^\Delta_n}\| \leq \frac{\epsilon}{1+3\lambda}$ for any $n \in \mathbb{N}$ it suffices if one chooses y^Δ_0 so that

$$\begin{aligned}
\|y^\Delta_0\| \quad&\leq\quad R^\Delta(\mathbf{x}) \\
&:=\quad \frac{\epsilon}{1+3\lambda}\inf_{n\geq 1}\exp\Big\{\kappa\tau^\Delta_n - C\hat{C}_g\gamma^* N^*(\gamma^*,\mathbf{x},[0,\tau^\Delta_n]) - C\hat{C}_g\gamma^* n\Big\} \\
&=\quad \frac{\epsilon}{1+3\lambda}\inf_{n\geq 1}\exp\tau^\Delta_n\Big\{\kappa - C\hat{C}_g\gamma^*\frac{n}{\tau^\Delta_n} - C\hat{C}_g\gamma^*\frac{N^*(\gamma^*,\mathbf{x},[0,\tau^\Delta_n])}{\tau^\Delta_n}\Big\}.
\end{aligned} \tag{5.17}$$

Denote $\tau_n(\gamma^*,\mathbf{x})$ the maximal stopping time that is less than τ^Δ_n. By (3.4) in Lemma 3.1

$$\limsup_{n\to\infty}\frac{N^*(\gamma^*,\mathbf{x},[0,\tau^\Delta_n])}{\tau^\Delta_n} \leq \limsup_{n\to\infty}\frac{n+1}{\tau_n} \leq \mathbb{E}N^*(\gamma^*,\mathbf{x}(\cdot),[0,1]) \tag{5.18}$$

Mean while, applying (3.13) and (3.11) for $\gamma := \lambda$ yields

$$\limsup_{n\to\infty} C\hat{C}_g\gamma^*\frac{n}{\tau^\Delta_n} \leq \frac{C\hat{C}_g\gamma^*}{\liminf\limits_{n\to\infty}\frac{\tau^{\mathcal{S}}_n}{2n}} \leq 2C\hat{C}_g\gamma^*\mathbb{E}N^*_{\mathcal{S}}(\lambda,\mathbf{x}(\cdot),[0,1]) \quad \text{a.s.}$$

Since $\lambda \in (0, \frac{1}{2})$ and $\gamma^* \in (0,1)$ satisfy (5.4), and by assumption $\hat{C}_g \leq c^* C_g$, we can use the same arguments as in the proof of Theorem 4.2 to prove that $R^\Delta(\mathbf{x})$ defined in (5.17) is a random variable which is positive almost surely. We has proved that if $\|y_0\| \leq R^\Delta(\mathbf{x})$ then $\|y^\Delta_{\tau^\Delta_n}\| \leq \frac{\epsilon}{1+3\lambda}$ for all n, and using (5.10) we have $\|y_t\| \leq \epsilon$ for all $t \in \Pi$. Moreover (5.15), (5.10) and (5.4) yield the exponential convergence of $y^\Delta_{t_k}$ to zero as $k \to \infty$. Thus, exponential stability of the trivial solution of (5.1) is proved.

Finally, following the same way as in the proof of Theorem 4.2 we use extension techniques to relax the proof from the additionally required strong global conditions $(\mathbf{H}^*_f)$-$(\mathbf{H}^*_g)$-(5.3). Namely, by virtue of Lemma 2.2 and Lemma 2.3 we can find extensions g^* of g and H^* of H respectively from $B(0,\epsilon_0)$ to the whole space $\mathbb{R}^d$. Put

$$f^*(y) := Df(0)y + H^*(y), \qquad \forall y \in \mathbb{R}^d,$$

according to (1.3). Then for the difference equation of type (5.1) in which f, g are replaced by f^*, g^* defined in Theorem 4.2 respectively, i.e.

$$\begin{aligned} & y^\Delta_0 \in \mathbb{R}^d, \\ & y^\Delta_{t_{k+1}} = y^\Delta_{t_k} + f^*(y^\Delta_{t_k})\Delta + g^*(y^\Delta_{t_k})x_{t_k,t_{k+1}} + Dg^*(y^\Delta_{t_k})g^*(y^\Delta_{t_k})\mathbb{X}_{t_k,t_{k+1}}, \quad k \in \mathbb{N}. \end{aligned} \tag{5.19}$$

the criterion (5.2) for the pair f^*, g^* as well as the strong additional global conditions $(\mathbf{H}^*_{f^*})$-$(\mathbf{H}^*_{g^*})$-(5.3) are satisfied. Therefore, the foregoing arguments are applicable to (5.19), yielding exponential stability of the trivial solution of (5.19). Moreover, since inside the ball $B(0,\epsilon)$ the equation (5.19) coincides with the equation (5.1), the same arguments as in the proof of Theorem 4.2 show that the trivial solution of (5.1) is exponentially stable. The proof is completed. □

Theorem 5.2 *Assume ($\boldsymbol{H}_X$) for the noise, ($\boldsymbol{H}_f$) for the drift and ($\boldsymbol{H}_g$) for the diffusion with $f(0) = 0, g(0) = 0$. Then there exists $\Delta_0 > 0$, $C_0 > 0$ depending only on f such that for any $0 < \Delta < \Delta_0$, $0 < C_g < C_0$, the trivial solution of* (5.1) *is exponentially stable almost surely.*

Proof: We apply the criterion of exponential stability given in Theorem 5.1 to prove this theorem. Similar to the continuous case in Section 4 we have (4.2) by a linear transformation. Therefore, for simplicity we assume that (4.2) holds.

At first, we note that once given (5.1) on $B(0,\epsilon_0)$, we may consider (5.1) on a smaller ball $B(0,\epsilon_1)$ with $0 < \epsilon_1 \leq \epsilon_0$, and on $B(0,\epsilon_1)$ the corresponding parameters C_H, L_H, C_g are not bigger than their counterparts defined on $B(0,\epsilon_0)$. Therefore, we may apply Theorem 5.1 to (5.1) with the change of ϵ_0 to ϵ_1 in the criterion (5.2), that means the criterion (5.2) now becomes a new (and better) criterion

$$\begin{aligned} \frac{1}{2\Delta} > \qquad \lambda_f > & \frac{1}{2}C_H\epsilon_1^m + \frac{1}{2}(\|Df(0)\| + L_H c^*)^2\Delta \\ & + (3C_p + 2e^{\frac{1}{2}})c^* C_g \gamma^* \Big[\mathbb{E}N^*(\gamma^*, \mathbf{x}(\cdot), [0,1]) + \mathbb{E}N^*_{\mathcal{S}}(\lambda, \mathbf{x}(\cdot), [0,1])\Big]. \end{aligned} \tag{5.20}$$

Choose and fix $0 < \epsilon_1 < \epsilon_0$ such that

$$C_H \epsilon_1^m < \frac{1}{2}\lambda_f. \tag{5.21}$$

We fix $\Delta_0 = \min\{\frac{1}{2\lambda_f}, \frac{\lambda_f}{(\|Df(0)\| + L_H c^*)^2}\}$. At this position, one can assign $\gamma^* = \frac{1}{2}$, $\lambda := \frac{1}{3}$ and choose $C_0 \in (0, \frac{1}{3})$ from criterion (5.20) and inequality (5.21) such that

$$\frac{\lambda_f}{4} > \frac{1}{2}(3C_p + 2e^{\frac{1}{2}})c^* C_0 \Big[\mathbb{E}N^*(\frac{1}{2}, \mathbf{x}(\cdot), [0,1]) + \mathbb{E}N^*_{\mathcal{S}}(\frac{1}{3}, \mathbf{x}(\cdot), [0,1])\Big].$$

This is possible, because we can estimate the two expectations

$$\begin{aligned}
\mathbb{E}N^*(\frac{1}{2},\mathbf{x}(\cdot),[0,1]) &\leq 1+2^p\mathbb{E}\,\|\!|\mathbf{x}(\cdot)|\!\|^p_{p,[0,1]} =: C_1;\\
\mathbb{E}N^*_{\mathcal{S}}(\frac{1}{3},\mathbf{x}(\cdot),[0,1]) &\leq 1+\big(\frac{\|Df(0)\|+L_H}{\lambda}\big)^p+\big(\frac{C_g}{\lambda}\big)^p\mathbb{E}\,\|\!|x(\cdot)|\!\|^p_{p,[0,1]}+\frac{C_g^p}{\lambda^p}\mathbb{E}\,\|\!|\mathbb{X}(\cdot)|\!\|^q_{q,[0,1]}\\
&\leq 1+3^p(\|Df(0)\|+L_H)^p+\mathbb{E}\,\|\!|\mathbf{x}(\cdot)|\!\|^p_{p,[0,1]} =: C_2
\end{aligned}$$

by the quantities which are independent of C_g. We then determine C_0 as follows

$$C_0 := \min\Big\{\frac{1}{4},\frac{\lambda_f}{2c^*(3C_p+2e^{\frac{1}{2}})(C_1+C_2)}\Big\}.$$

It is easily seen that with such choice of Δ_0 and C_0 if $0<\Delta\leq\Delta_0$ and $0<C_g\leq C_0$ then the criterion (5.20) is satisfied, hence Theorem 5.1 is applicable to (5.1) implying that (5.1) is exponentially stable. The proof is complete. □

Remark 5.3 (i). We notice that the arguments in the proof of Theorems 4.2, 4.3, 5.1, 5.2 are conducted for one fixed specific path $\mathbf{x}=\mathbf{x}(\omega)$ for which inequality (3.4) in Lemma 3.1 holds. Those ω form a set of full measure $\Omega'\subset\Omega$.

(ii). If one deals with non ergodic cases, one can use the ergodic decomposition theorem [33, Theorem 3.2, p. 19] and have the conclusion of local stability for each ergodic component of Ω. Otherwise, the ergodic Birkhoff theorem is still applicable, but the right hand side of (3.4) and (3.11) would be random variables. As a result, the right hand side of criteria (4.3) and (5.2) are path dependent, which implies that $\hat{\lambda}$ and λ are also chosen to be path dependent. In this case, C_0 in Theorem 4.3 and 5.2 is also path dependent.

6 Appendix

6.1 Rough paths

Let us briefly present the concept of rough paths in the simplest form, following [17] and [26].

For any finite dimensional vector space W, denote by $C([a,b],W)$ the space of all continuous paths $y:[a,b]\to W$ equipped with the sup norm $\|\cdot\|_{\infty,[a,b]}$ given by $\|y\|_{\infty,[a,b]}=\sup_{t\in[a,b]}\|y_t\|$, where $\|\cdot\|$ is the norm in W. We write $y_{s,t}:=y_t-y_s$. For $p\geq 1$, denote by $C^{p-\rm var}([a,b],W)\subset C([a,b],W)$ the space of all continuous paths $y:[a,b]\to W$ of finite p-variation $\|\!|y|\!\|_{p-{\rm var},[a,b]} := \Big(\sup_{\Pi([a,b])}\sum_{i=1}^n\|y_{t_i,t_{i+1}}\|^p\Big)^{1/p}<\infty$, where the supremum is taken over the whole class of finite partitions of $[a,b]$.

Also for each $0<\alpha<1$, we denote by $C^\alpha([a,b],W)$ the space of Hölder continuous functions with exponent α on $[a,b]$ equipped with the norm

$$\|y\|_{\alpha,[a,b]} := \|y_a\|+\|\!|y|\!\|_{\alpha,[a,b]}, \quad \text{where} \quad \|\!|y|\!\|_{\alpha,[a,b]} := \sup_{s,t\in[a,b],\ s<t}\frac{\|y_{s,t}\|}{(t-s)^\alpha}<\infty. \tag{6.1}$$

For $\alpha\in(\frac{1}{3},\frac{1}{2})$, a couple $\mathbf{x}=(x,\mathbb{X})\in\mathbb{R}^m\oplus(\mathbb{R}^m\otimes\mathbb{R}^m)$, where $x\in C^\alpha([a,b],\mathbb{R}^m)$ and

$$\begin{aligned}
&\mathbb{X}\in C^{2\alpha}([a,b]^2,\mathbb{R}^m\otimes\mathbb{R}^m)\\
:=\ &\Big\{\mathbb{X}\in C([a,b]^2,\mathbb{R}^m\otimes\mathbb{R}^m):\sup_{s,t\in[a,b],\ s<t}\frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\alpha}}<\infty\Big\},
\end{aligned}$$

is called a *rough path* if it satisfies Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = x_{s,u} \otimes x_{u,t}, \qquad \forall a \leq s \leq u \leq t \leq b. \tag{6.2}$$

We introduce the rough path semi-norm

$$\begin{aligned} \|\mathbf{x}\|_{\alpha,[a,b]} &:= \|x\|_{\alpha,[a,b]} + \|\mathbb{X}\|^{\frac{1}{2}}_{2\alpha,[a,b]^2}, \\ \text{where} \quad \|\mathbb{X}\|_{2\alpha,[a,b]^2} &:= \sup_{s,t\in[a,b];s<t} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\alpha}} < \infty. \end{aligned} \tag{6.3}$$

Throughout this paper, we will fix parameters $\frac{1}{3} < \alpha < \nu < \frac{1}{2}$ and $p = \frac{1}{\alpha}$ so that $C^{\alpha}([a,b],W) \subset C^{p-\text{var}}([a,b],W)$. We also set $q = \frac{p}{2}$ and consider the $p-$var semi-norm

$$\begin{aligned} \|\mathbf{x}\|_{p-\text{var},[a,b]} &:= \Big(\|x\|^p_{p-\text{var},[a,b]} + \|\mathbb{X}\|^q_{q-\text{var},[a,b]^2} \Big)^{\frac{1}{p}}, \\ \|\mathbb{X}\|_{q-\text{var},[a,b]^2} &:= \left(\sup_{\Pi([a,b])} \sum_{i=1}^n \|\mathbb{X}_{t_i,t_{i+1}}\|^q \right)^{1/q}, \end{aligned} \tag{6.4}$$

where the supremum is taken over the whole class of finite partitions $\Pi([a,b])$ of $[a,b]$.

Example 6.1 (i), Let $B = \{B_t : t \in \mathbb{R}^m\}$ be a Brownian motion. Define

$$\mathbb{B}^{\text{Itô}}_{0,t} := \int_0^t B_{0,r} \otimes dB_r \in \mathbb{R}^m \otimes \mathbb{R}^m = \mathbb{R}^{m\times m},$$

where the stochastic integral is understood in the Itô sense. We consider continuous versions of B_t and $\mathbb{B}^{\text{Itô}}_{0,t}$ and define

$$\mathbb{B}^{\text{Itô}}_{s,t} := \mathbb{B}^{\text{Itô}}_{0,t} - \mathbb{B}^{\text{Itô}}_{0,s} - B_s \otimes B_{s,t},$$

which is continuous in (s,t) and satisfies Chen's relation (6.2). By the additivity of the Itô integral

$$\mathbb{B}^{\text{Itô}}_{s,t} = \int_s^t B_{s,t} \otimes dB_r.$$

By the Kolmogorov criterion, there is a version of $(B, \mathbb{B}^{\text{Itô}})$ which is in $\mathscr{C}^{\alpha}(I,\mathbb{R}^m)$ for all $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Note that by Itô formula

$$\text{Sym}(\mathbb{B}^{\text{Itô}}_{s,t}) = \frac{1}{2} B_{s,t} \otimes B_{s,t} - \frac{1}{2} I_m (t-s).$$

Define

$$\mathbb{B}^{\text{Strat}}_{s,t} := \mathbb{B}^{\text{Itô}}_{s,t} + \frac{1}{2} I_m(t-s), \quad \text{then} \quad \text{Sym}(\mathbb{B}^{\text{Strat}}_{s,t}) = \frac{1}{2} B_{s,t} \otimes B_{s,t},$$

which implies that $(B, \mathbb{B}^{\text{Strat}}) \in \mathscr{C}^{\alpha}_g(I,\mathbb{R}^m)$ for all $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Moreover

$$\mathbb{B}^{\text{Strat}}_{s,t} = \int_s^t B_{s,r} \otimes dB_r.$$

where the integral is in the Stratonovich sense. It is then easy to check that both $\mathbb{B}^{\text{Itô}}$ and $\mathbb{B}^{\text{Strat}}$ satisfy Chen's relation (6.2).

(ii) When $X = B^H$ is a scalar fractional Brownian motion, we define the stochastic integral $\int y \delta B^H$ in the sense of Skorohod-Wick-Itô by using the Wick product as in [30, Chapter 5]. Then by using the Wick-Itô formula [30] for the Skorohod-Wick-Itô integral

$$f(B^H_t) - f(B^H_s) = \int_s^t H u^{2H-1} f''(B^H_u) du + \int_s^t f'(B^H_u) \delta B^H_u \tag{6.5}$$

for any function $f \in C^2$, we can compute explicitly

$$\mathbb{X}_{s,t} := \int_s^t B^H_{s,u} \delta B^H_u = \frac{1}{2}(B^H_{s,t})^2 - \frac{1}{2}\Big(t^{2H} - s^{2H}\Big),$$

In general, $\mathbb{X}$ can also be defined for a scalar centered Gaussian process of the form $X_t = \int_0^t K(t,s) dB_s$ where B is a standard Brownian motion, and $K(t,s)$ is a square integrable kernel. In particular, the stochastic integral $\int \delta X$ can be computed as the limit of Riemann sums defined w.r.t. the Wick product [1]. As such $\mathbb{X}_{s,t} := \int_s^t X_{s,u} \delta X_u$ can be computed explicitly and satisfies Chen's relation (6.2).

The reader is also referred to [17, Chapter 10] for a detailed construction of $\mathbb{X}$ of a multi-dimensional Gaussian process $X = (X_i)_{i=1}^m$ with mutually independent components.

6.2 Probabilistic settings

Following [10], denote by $T_1^2(\mathbb{R}^m) = 1 \oplus \mathbb{R}^m \oplus (\mathbb{R}^m \otimes \mathbb{R}^m)$ the set with the tensor product

$$(1, g^1, g^2) \otimes (1, h^1, h^2) = (1, g^1 + h^1, g^1 \otimes h^1 + g^2 + h^2),$$

for all $\mathbf{g} = (1, g^1, g^2), \mathbf{h} = (1, h^1, h^2) \in T_1^2(\mathbb{R}^m)$. Then $(T_1^2(\mathbb{R}^m), \otimes)$ is a topological group with unit element $\mathbf{1} = (1, 0, 0)$ and $\mathbf{g}^{-1} = (1, -g^1, g^1 \otimes g^1 - g^2)$.

Given $\alpha \in (\frac{1}{3}, \nu)$, denote by $\mathscr{C}^{0,\alpha}(I, T_1^2(\mathbb{R}^m))$ the closure of $\mathscr{C}^{\infty}(I, T_1^2(\mathbb{R}^m))$ in the Hölder space $\mathscr{C}^{\alpha}(I, T_1^2(\mathbb{R}^m))$, and by $\mathscr{C}^{0,\alpha}_0(\mathbb{R}, T_1^2(\mathbb{R}^m))$ the space of all paths $\mathbf{g} : \mathbb{R} \to T_1^2(\mathbb{R}^m))$ such that $\mathbf{g}|_I \in \mathscr{C}^{0,\alpha}(I, T_1^2(\mathbb{R}^m))$ for each compact interval $I \subset \mathbb{R}$ containing 0. Then $\mathscr{C}^{0,\alpha}_0(\mathbb{R}, T_1^2(\mathbb{R}^m))$ is equipped with the compact open topology given by the α-Hölder norm (6.1), i.e the topology generated by the metric

$$d_\alpha(\mathbf{g}, \mathbf{h}) := \sum_{k \geq 1} \frac{1}{2^k} (\|\mathbf{g} - \mathbf{h}\|_{\alpha, [-k,k]} \wedge 1).$$

Let us consider a stochastic process $\bar{\mathbf{X}}$ defined on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with realizations in $(\mathscr{C}^{0,\alpha}_0(\mathbb{R}, T_1^2(\mathbb{R}^m)), \mathcal{F})$. Assume further that $\bar{\mathbf{X}}$ has stationary increments. Assign $\Omega := \mathscr{C}^{0,\alpha}_0(\mathbb{R}, T_1^2(\mathbb{R}^m))$ and equip it with the Borel σ-algebra $\mathcal{F}$ and let $\mathbb{P}$ be the law of $\bar{\mathbf{X}}$. Denote by θ the *Wiener-type shift*

$$(\theta_t \omega)_\cdot = \omega_t^{-1} \otimes \omega_{t+\cdot}, \forall t \in \mathbb{R}, \omega \in \mathscr{C}^{0,\alpha}_0(\mathbb{R}, T_1^2(\mathbb{R}^m)), \tag{6.6}$$

and define the so-called *diagonal process* $\mathbf{X} : \mathbb{R} \times \Omega \to T_1^2(\mathbb{R}^m), \mathbf{X}_t(\omega) = \omega_t$ for all $t \in \mathbb{R}, \omega \in \Omega$. Due to the stationarity of $\bar{\mathbf{X}}$, it can be proved that θ is invariant under $\mathbb{P}$, then forming a continuous (and thus measurable) dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$ [2, Theorem 5]. Moreover, $\mathbf{X}$ forms an *α-rough path cocycle*, namely, $\mathbf{X}_\cdot(\omega) \in \mathscr{C}^{0,\alpha}_0(\mathbb{R}, T_1^2(\mathbb{R}^m))$ for every $\omega \in \Omega$, which satisfies the *cocycle relation*:

$$\mathbf{X}_{t+s}(\omega) = \mathbf{X}_s(\omega) \otimes \mathbf{X}_t(\theta_s \omega), \forall \omega \in \Omega, t, s \in \mathbb{R},$$

in the sense that $\mathbf{X}_{s,s+t} = \mathbf{X}_t(\theta_s\omega)$ with the increment notation $\mathbf{X}_{s,s+t} := \mathbf{X}_s^{-1} \otimes \mathbf{X}_{s+t}$. It is important to note that the two-parameter flow property

$$\mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} = \mathbf{X}_{s,t}, \forall s,t \in \mathbb{R}$$

is equivalent to the fact that $\mathbf{X}_t(\omega) = (1, \mathbf{x}_t(\omega)) = (1, x_t(\omega), \mathbb{X}_{0,t}(\omega))$, where $x.(\omega) : \mathbb{R} \to \mathbb{R}^m$ and $\mathbb{X}_{\cdot,\cdot}(\omega) : I^2 \to \mathbb{R}^m \otimes \mathbb{R}^m$ are random functions satisfying Chen's relation relation (6.2).

As pointed out in [11, Remark 1] and due to [2, Corollary 9], the above construction is possible for X_t to be a continuous, centered Gaussian process with stationary increments and independent components, satisfying: there exists for any $T > 0$ a constant C_T such that for all $p \geq \frac{1}{\nu}$, $\mathbb{E}\|X_t - X_s\|^p \leq C_T|t-s|^{p\nu}$ for all $s,t \in [0,T]$. Then $\mathbf{X}$ can be chosen to be the natural lift of X in the sense of Friz-Victoir [18, Chapter 15] with sample paths in the space $C_0^{0,\alpha}(\mathbb{R}, T_1^2(\mathbb{R}^m))$, for a certain $\alpha \in (0,\nu)$. In particular, the Wiener shift (6.6) implies that

$$\begin{aligned} \|\mathbf{x}(\theta_h\omega)\|_{p-\mathrm{var},[s,t]} &= \|\mathbf{x}(\omega)\|_{p-\mathrm{var},[s+h,t+h]}\,; \\ N_{[s,t]}(\mathbf{x}(\theta_h\omega)) &= N_{[s+h,t+h]}(\mathbf{x}(\omega)). \end{aligned} \tag{6.7}$$

As said above, in this paper, we need an assumption on ergodicity of θ. It is known (see [21, Lemma 3]) that if X is a m-dimensional fractional Brownian motion with mutual independent components, we have the ergodicity of θ.

Lemma 6.2 *Assume that $X = B^H$, then θ is ergodic.*

Proof: We sketch out a short proof here. For $H = \frac{1}{2}$, the canonical process w.r.t. the Wiener measure $\mathbb{P}_{\frac{1}{2}}$ and Wiener shift $\theta_t^*\omega. = \omega_{t+\cdot} - \omega.$ on $\Omega^* = C_0^0(\mathbb{R},\mathbb{R})$ is ergodic. By [21], the Wiener shift $\eta_t x. = x_{t+\cdot} - x_t$ is ergodic on $\Omega' = C^{0,\alpha}(\mathbb{R},\mathbb{R}^m)$ w.r.t. $\mathbb{P}_H = B^H\mathbb{P}_{\frac{1}{2}}$. Because of [17, Theorem 10.4], there exists a full measure subset $\Omega_1 \subset \Omega'$ such that $\omega = (1, x, \mathbb{X}) \in \Omega$ for any $x \in \Omega_1$. Moreover, by [18, Theorem 15.42, 15.45], one can choose this full measure subset Ω_1 such that it satisfies the piece-wise linear approximations (mollifier approximation). Then consider the natural lift $\mathcal{S}$ on smooth paths

$$\mathcal{S}(x)_{s,t} = (1, x_{s,t}, \int_s^t x_{s,r}dx_r), \tag{6.8}$$

which can be extended to Ω_1 such that $\mathcal{S} : \Omega_1 \to \Omega$. One can now apply the arguments in [2] to conclude that there exists a metric dynamical system $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\theta})$ such that $\hat{\Omega} \subset \Omega$ and $\hat{\mathbb{P}} = \mathbb{P}^H \circ \mathcal{S}^{-1}$. Furthermore,

$$\theta_t\mathcal{S}(x) = \lim_{n\to\infty} \hat{\theta}_t\mathcal{S}(x^{(n)}) = \lim_{n\to\infty} \mathcal{S}(\eta_t x^{(n)}) = \mathcal{S}(\eta_t x). \tag{6.9}$$

Since η is ergodic, it follows from [21, Lemma 3] that θ is also ergodic. □

6.3 Gubinelli's rough path integrals

Following Gubinelli [23], a rough path integral can be defined for a continuous path $y \in C^\alpha([a,b],W)$ which is *controlled by* $x \in C^\alpha([a,b],\mathbb{R}^m)$ in the sense that, there exists a couple (y', R^y) with $y' \in C^\alpha([a,b],\mathcal{L}(\mathbb{R}^m,W)), R^y \in C^{2\alpha}([a,b]^2,W)$ such that

$$y_{s,t} = y'_s x_{s,t} + R^y_{s,t}, \qquad \forall a \leq s \leq t \leq b. \tag{6.10}$$

y' is called the *Gubinelli derivative* of y.

Denote by $\mathscr{D}_x^{2\alpha}([a,b])$ the space of all the couples (y,y') controlled by x. Then for a fixed rough path $\mathbf{x} = (x,\mathbb{X})$ and any controlled rough path $(y,y') \in \mathscr{D}_x^{2\alpha}([a,b])$, the integral $\int_s^t y_u dx_u$ can be defined as the limit of the Darboux sum

$$\int_s^t y_u dx_u := \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} \Big(y_u \otimes x_{u,v} + y'_u \mathbb{X}_{u,v} \Big)$$

where the limit is taken on all finite partitions Π of $[a,b]$ with $|\Pi| := \max\limits_{[u,v]\in\Pi} |v-u|$. Moreover, there exists a constant $C_p > 1$ independent of $\mathbf{x}$ and (y,y') such that

$$\begin{aligned} &\Big\| \int_s^t y_u dx_u - y_s \otimes x_{s,t} - y'_s \mathbb{X}_{s,t} \Big\| \\ \leq & C_p \Big(\left\| x \right\|_{p-\mathrm{var},[s,t]} \left\| R^y \right\|_{q-\mathrm{var},[s,t]^2} + \left\| y' \right\|_{p-\mathrm{var},[s,t]} \left\| \mathbb{X} \right\|_{q-\mathrm{var},[s,t]^2} \Big). \end{aligned} \tag{6.11}$$

6.4 Stochastic integrals as rough integrals

In general, a Gubinell rough integral $\int y dx$ is defined in the pathwise sense with respect to a driving path x, yet we can compare it to classical stochastic integrals in some special cases. Namely, let B be a m-dimensional Brownian motion which is enhanced to an Itô rough path $(B(\omega), \mathbb{B}^{\text{Itô}}(\omega)) \in \mathscr{C}^\alpha$ for any $\alpha \in (\frac{1}{3}, \frac{1}{2})$ a.s. Assume $(y(\omega), y'(\omega)) \in \mathscr{D}^{2\alpha}_{B(\omega)}$ a.s. Then the rough integral

$$\int_I y_r dB_r^{\text{Itô}} = \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} \Big(y_u B_{u,v} + y'_u \mathbb{B}^{\text{Itô}}_{u,v} \Big)$$

exists a.s. If y, y' are adapted, then a.s. $\int_I y_r d_r^{\text{Itô}} = \int_I y_r dB_r$ where the latter is the Itô integral. The same conclusions also hold for $(B, \mathbb{B}^{\text{Strat}})$ and the corresponding Stratonovich integral (see e.g. [17]).

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