EUCLIDEAN DISTANCE DEGREE OF CURVE IN $\mathbb{C}^3$

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ABSTRACT. Many models in data science or mechanical engineering can be represented as a real algebraic set, leading to the need to solve the problem of finding the nearest point. For example, in the field of computer vision, the problem of determining a point in space when knowing its image from two cameras with the positions of the two cameras and a given shooting angle. This problem would be easy if the information obtained was absolutely accurate but in reality it is not. Therefore, the problem is to find a point in space that is most compatible with the information obtained from the cameras. This is the nearest problem mentioned above. Here, the Euclidean distance degree (EDD) is a measure to determine the computational complexity of this problem (read more[7]). The author of the [1] paper gave a meaningful result for the EDD of the hyperplane $f = 0$ in 2022. The main purpose of this note is to study the case that the manifold is defined by two polynomials $f_1(x) = f_2(x) = 0$. We show that the Euclidean distance degree of this variety is not greater than the mixed volume of Newton polytopes of the associated Lagrange multiplier equations.

1. Introduction

The nearest point problem for points in the Euclidean plane was among the first geometric problems that were treated at the origins of the systematic study of the computational complexity of geometric algorithms.

Nearest point problem: In $\mathbb{R}^n$ given the algebraic set X and a point u, find a point x of X that minimizes the squared Euclidean distance function $d_u(x) = \Sigma\,(x_i - u_i)^2$ from the given point u.

Algebraic varieties are the central objects of study in algebraic geometry, a sub-field of mathematics. Classically, an algebraic variety is defined as the set of solutions of a system of polynomial equations over the real or complex numbers.

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Specifically, an algebraic varieties is a set defined by:

$$M = \{x \in \mathbb{R}^n : P_1(x) = ... = P_n(x) = 0\},$$

where $P_1, .., P_n$ are polynomials.

Example 1.1. Here are some examples of algebraic manifolds:

$$M = \left\{m \in \mathbb{R} : 5m^3 + 6m^3 + 12m + 8 = 0\right\}.$$

$$N = \left\{(n_1, n_2, n_3) : n_1^2 - 2n_2^2 + 5 = 0, n_1^2 + n_3 = 0\right\} \subset \mathbb{R}^3.$$

Below is the definition of a submanifold in $\mathbb{R}^n$.

Definition 1.2. A subset $M \subset \mathbb{R}^n$ is called a k-dimensional submanifold if for every $p \in M$ there exists a diffeomorphism $\Phi : U \to V$ between open subsets of $\mathbb{R}^n$ such that $p \in U$ and

$$\Phi(M \cap U) = \left(\mathbb{R}^k \times \{0\}\right) \cap V$$

Example 1.3. Example of a sub-manifold of $\mathbb{R}^n$:

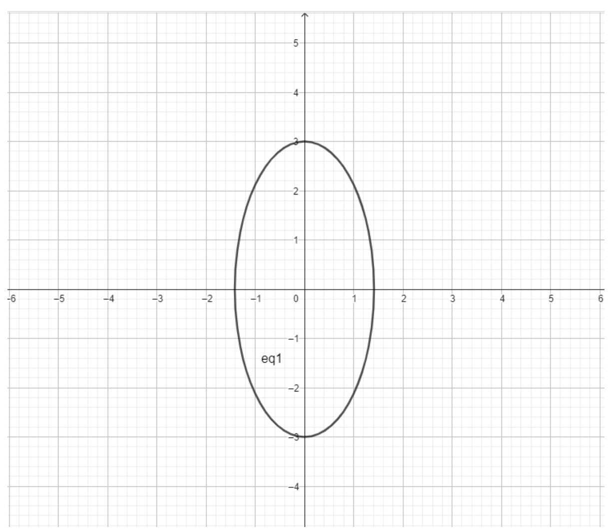

FIGURE 1. One-dimensional manifold $\dfrac{x^2}{2} + \dfrac{y^2}{9} = 1$.

Definition 1.4. Let $M \subset \mathbb{R}^n$ be a k-dimensional submanifold and $p \in M$. A vector $v \in \mathbb{R}^n$ is called a **tangent vector** to M at p if there exist an $\epsilon > 0$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that

$$\gamma(0) = p, \quad \gamma'(0) = v$$

where γ' is the calculus derivative of γ (or more precisely, the derivative of $\iota \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$, where $\iota : M \to \mathbb{R}^n$ is the inclusion map).
The set T_pM of all tangent vectors to M at p is called the tangent space to M at p.

Definition 1.5. The differential of the map f at point x_0 is the map

$$\begin{aligned} df(p) : \mathrm{T}_p M &\to \mathrm{T}_{f(p)}\mathrm{N} \\ v &\mapsto df(p)(v) \end{aligned}$$

is defined as follows: if v is a tangent vector to the curve $\gamma(t)$ at $\gamma(t_0) = p$ then $df(p)(v)$ is a tangent vector to the curve $f(\gamma(t))$ at $f(p) = f(\gamma(t_0))$.

Definition 1.6. Given an open set $M \subset \mathbb{R}^n$ and a smooth map $f : M \to \mathbb{R}^m$, a point $p \in M$ is called a critical point for f if the derivative $df_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. The image of a critical point under f is called the critical value.

Sard's theorem says that the set of critical values of f has measure zero.

Therefore, one approach to solving the distance function optimization problem mentioned in the introduction is to find and check all critical points of f_p.

Initially, we construct the Lagrange equation for the case of a manifold defined by two polynomials and show that the EDD corresponds to the number of solutions of the Lagrange equation. The final task is to determine the number of solutions of this Lagrange equation, using Bernstein's theorems to prove that the EDD of this manifold approximates the mixing volume (MV) of Newton polytopes.

The Newton polytope and the mixing volume are defined as follows:

Definition 1.7. Let $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be polynomials with the support $A \subset \mathbb{N}^{\mathrm{n}}$, such that

$$f(x) = \sum_{a \in A} c_a x^a, \quad (c_a \in \mathbb{C}).$$

Then, the Newton polytope of f is defined as the convex hull of the set $\{a \in \mathbb{N}^n : c_a \neq 0\}$ in $\mathbb{R}^n$.

Definition 1.8. Let $K_1, K_2, \ldots, K_r$ be convex sets in $\mathbb{R}^n$ and consider the function $f(\lambda_1, \ldots, \lambda_r) = \mathrm{Vol}_n(\lambda_1 K_1 + \cdots + \lambda_r K_r), \lambda_i \geq 0$ where Vol_n stands for the n-dimensional volume with its argument being the Minkowski sum of the convex sets K_i. Then f is a homogeneous polynomial of degree n, so it can be written as

$$f(\lambda_1, \ldots, \lambda_r) = \sum_{j_1,\cdots,j_n=1}^{r} V(K_{j_1}, \ldots, K_{j_n}) \lambda_{j_1} \cdots \lambda_{j_n},$$

where the functions V are symmetric. For a function with a particular index $j \in \{1, \ldots, r\}^n$, the coefficient $V(K_{j_1}, \ldots, K_{j_n})$ is called the mixing volume of $K_{j_1}, \ldots, K_{j_n}$.

Below, we study the distance function from a given point to an algebraic varieties and the problem of counting the number of critical points of that function.

2. Definition of Euclidean distance degree

Definition 2.1. Let $u = (u_1, u_2, \ldots, u_n)$ be a point in the Euclidean space $\mathbb{R}^n$. Let X be an algebraic variety in $\mathbb{R}^n$, with $x = (x_1, x_2, \ldots, x_n) \in X$, consider the function $f_u : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_u(x) = \sum (x_i - u_i)^2 .$$

Then, for a general point u , the distance function $f_{u|X} : X \to \mathbb{R}$ (the function f_u on X) has a finite number of critical points. The number of complex critical points does not depend on the general point u and is called the Euclidean distance degree of the set X, denoted by $\mathrm{EDD}(X)$.

Example 2.2. Consider

$$X = \mathcal{V}_{\mathbb{R}} \left(x_1^2 x_2^2 - 3x_1^2 - 3x_2^2 + 5 \right) \subset \mathbb{R}^2,$$

and the point $p(0.025, 0.2)$ has 12 critical points of the distance function $f_{p|X}$. Therefore, the Euclidean distance degree of X is 12.

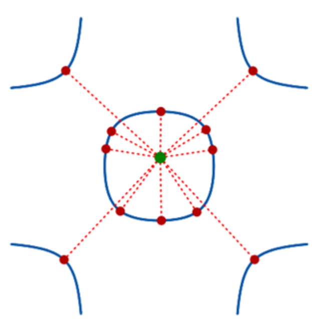

Figure 2. $\mathrm{EDD}(X) = 12$.

The relationship between the number of roots of a polynomial system and the mixing volume is given by Bernstein's theorem below.

Theorem 2.3 (see [2, 3]). *Let* $h_1, \ldots, h_m \in \mathbb{C}[x_1, \ldots, x_m]$ *be* m *polynomials with Newton polytopes* $H_1, \ldots, H_m$. *Let* $\#\mathcal{V}_{\mathbb{C}^\times}(h_1, \ldots, h_m)$ *be the number of solutions of* $h_1 = \cdots = h_m = 0$ *in* $(\mathbb{C}^\times)^m$, *computed by their algebraic multiples. Bernstein's theorem states that*

$$\#\mathcal{V}_{\mathbb{C}^\times}(h_1, \ldots, h_m) \leq \mathrm{MV}(H_1, \ldots, H_m)$$

and equality occurs when h_i *are general polynomials.*

Note that a general polynomial means that the coefficients of the polynomial are general to fix its Newton polytope.

We also have another theorem of Bernstein.

Theorem 2.4 (xem [2, 3]). *Let* $H = (h_1, \ldots, h_m)$ *be a system of Laurent polynomials with variables* $x_1, \ldots, x_c$. *For each* $1 \leq i \leq m$, *let* $\mathscr{H}_i$ *be the support of* h_i *and* $H_i = \mathrm{conv}(\mathscr{H}_i)$ *be its Newton polytope. Then,*

$$\#\mathcal{V}_{\mathbb{C}^\times}(h_1, \ldots, h_m) < \mathrm{MV}(H_1, \ldots, H_m)$$

if there exists $0 \neq w \in \mathbb{Z}^m$ *such that the facial system* $H_w := ((h_1)_w, \ldots, (h_m)_w)$ *has a solution in* $(\mathbb{C}^\times)^m$. *On the other hand,* $\#\mathcal{V}_{\mathbb{C}^\times}(h_1, \ldots, h_m)$ *is equal to* $\mathrm{MV}(H_1, \ldots, H_m)$.

3. Euclidean distance degree of the curve in $\mathbb{C}^3$

In this paper, differentiable varieties are considered as sub-manifolds of $\mathbb{R}^N$.

Let $f_1, f_2 \in \mathbb{R}[x_1, x_2, x_3]$ be two polynomials such that $M = \{x \in \mathbb{R}^3 : f_1(x) = f_2(x) = 0\}$ is a differentiable variety. We consider:

$$\varphi : M \to \mathbb{R}$$
$$x \mapsto \|x - u\|^2,$$

with $\|x - u\|^2$ being the Euclidean norm.

Since the set of nearest points lies in the set of critical points of the function φ, the number of critical points is considered to be the complexity of the nearest point problem. This implies that $EDD(M)$ is equal to the number of complex solutions of the following system of equations:

$$\begin{cases} f_1(x) = f_2(x) = 0, \\ d_x\varphi = 0 \end{cases}$$

where $d_x\varphi : \mathrm{T}_x M \to \mathrm{T}_{\varphi(x)}\mathbb{R}$ is a tangent map (or a differential of the map φ) and x is a critical point.

Assume that $x = (x_1, x_2, x_3)$ is a critical point if

$$\langle \nabla f_i(x), v\rangle = 0 (i = 1, 2) \Leftrightarrow \langle \nabla\varphi(x), v\rangle = 0.$$

This implies that there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$\nabla\varphi(x) = \lambda_1 \cdot \nabla f_1(x) + \lambda_2 \cdot \nabla f_2(x).$$

Thus, the critical points x must satisfy

$$u - x = \lambda_1 \cdot \nabla f_1(x) + \lambda_2 \cdot \nabla f_2(x).$$

The Lagrange multiplier below is a system of five polynomial equations with five variables $(\lambda_1, \lambda_2, x_1, x_2, x_3)$:

$$\mathscr{L}_{f_1,f_2,u}(\lambda, x) := \{f_1(x) = f_2(x) = 0 \text{ and } u - x = \lambda_1 \cdot \nabla f_1(x) + \lambda_2 \cdot \nabla f_2(x)\},$$

where λ_1, λ_2 are auxiliary variables.

Next, consider the number of solutions to $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$. For u general, this number is called the Euclidean distance degree of the manifold

$$M = \left\{x \in \mathbb{C}^3 : f_1(x) = f_2(x) = 0\right\}.$$

Thus, $\mathrm{EDD}(M) :=$ the number of solutions to $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ in $\mathbb{C}^5$.
Here, "general" means for all u outside some algebraic set, that is, outside the set of measure zero.

Theorem 3.1. *Let* $\sum := \left\{(a_{\alpha_1}, \ldots, a_{\alpha_N}, x_1, \ldots, x_n) \in \mathbb{C}^N \times (\mathbb{C}^\times)^n\right\}$. *Let* Ω *be the set of critical points of the function* $p : \Sigma \to \mathbb{C}^N$. *If* $f\left(a_{\alpha_1}, \ldots, a_{\alpha_N}\right) \in \Omega$ *then* $p^{(-1)}(0)$ *has no critical points in* $(\mathbb{C}^\times)^n$.

Proof. Suppose $p\left(x_1, x_2, \ldots, x_n\right) = \sum_{i=1,\ldots,N} a_{\alpha_i} x^{\alpha_i}$ is a polynomial.

If $\{a_{\alpha_i}\}$ is general, then the roots of the polynomial $p(x)$ lie in $(\mathbb{C}^\times)^n$:

$$\left\{x \in \left(\mathbb{C}^\times\right)^n : \partial_1 p(x) = \ldots = \partial_n p(x) = p(x) = 0\right\}.$$

Consider the equation

$$\sum_{i=1,\ldots,N} a_{\alpha_i} x^{\alpha_i} = 0,$$

the equation has a solution because clearly $\sum_{i=1,\ldots,N} a_{\alpha_i} x^{\alpha_i}$ is an exponential function, $x_i \in (\mathbb{C}^\times)^n$ and $\sum_{i=1,\ldots,N} a_{\alpha_i} x^{\alpha_i}$ has partial derivatives so $\sum_{i=1,\ldots,N} a_{\alpha_i} x^{\alpha_i}$ is

smooth.
Let π be a map:

$$\pi : \sum \to \mathbb{C}^N$$
$$(a_{\alpha_1}, \ldots, a_{\alpha_N}, x_1, \ldots, x_n) \mapsto (a_{\alpha_1}, \ldots, a_{\alpha_N})$$

then

$$d_\pi : \mathrm{T}_x\Sigma \to \mathbb{C}$$
$$v = \varphi'(t)\Big|_{t=0} \mapsto \pi\left(\varphi'(t)\right) = 0.$$

We write

$$\sum := \left\{(a_{\alpha_1}, \ldots, a_{\alpha_N}, x_1, \ldots, x_n) \in \mathbb{C}^N \times \left(\mathbb{C}^\times\right)^n\right\}.$$

We have

$$\sum(x) = \sum(\varphi(t)),$$

then

$$d\sum(x) = d\sum(\varphi(t)) = 0.$$

Let x be the critical point, then

$$\langle \nabla\Sigma(x), v\rangle = 0 \text{ and } \langle \nabla\pi(x), v\rangle = 0.$$

Therefore,

$$\sum = \left\{(a, x) \in \mathbb{C}^N \times \left(\mathbb{C}^\times\right)^n : \operatorname{rank}\begin{pmatrix} x^{\alpha_1} & \ldots & x^{\alpha_N} & \frac{\partial p}{\partial x_1} & \ldots & \frac{\partial p}{\partial x_n} \\ 1 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1\ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix} = 1\right\}$$
$$= \left\{(a, x) \in \mathbb{C}^N \times \left(\mathbb{C}^\times\right)^n : \frac{\partial p}{\partial x_i} = 0\right\}.$$

Thus, $(a_{\alpha_1}, \ldots, a_{\alpha_N})$ is a critical value if and only if $p^{(-1)}(0)$ has no critical points in $(\mathbb{C}^\times)^n$. □

Let $f_1, f_2 \in \mathbb{C}[x_1, \ldots, x_m]$ be polynomials with support $\mathscr{H} \subset \mathbb{N}^n$, which is the set of exponents of the monomials f_1, f_2. Suppose that $0 \in \mathscr{H}$. We write $\partial_i\mathscr{H} \subset \mathbb{N}^n$ as the support of the partial derivatives $\partial_i f_1$ and $\partial_i f_2$. For $w \in \mathbb{Z}^n$, the linear function $x \mapsto w \cdot x$ takes on minimum values on $\mathscr{H}$ and on $\partial_i\mathscr{H}$,

$$h^* = h_w(\mathscr{H}) := \min_{a\in\mathscr{H}} w \cdot a \quad \text{ and } \quad h_i^* = h_w\left(\partial_i\mathscr{H}\right) := \min_{a\in\partial_i\mathscr{H}} w \cdot a.$$

Since $0 \in \mathscr{H}$, we have $h^* \leq 0$. Furthermore, if $h^* = 0$ and if there exists $a \in \mathscr{H}$ with $a_i > 0$, then $w_i \geq 0$. The subsets of $\mathscr{H}$ and $\partial_i \mathscr{H}$ for which the linear function $x \mapsto w \cdot x$ is minimal are their faces tangent to w,

$$\mathscr{H}_w := \{a \in \mathscr{H} \mid w \cdot a = h^*\} \text{ and } (\partial_i \mathscr{H})_w := \{a \in \partial_i \mathscr{H} \mid w \cdot a = h_i^*\}.$$

Lemma 3.2. *For each $1 \leq i \leq n$ and $w \in \mathbb{Z}^n$, we have $h_i^* \geq h^* - w_i$. If $\partial_i (f_w) = 0$, then $\partial_i (f_w) = (\partial_i f)_w$ and $h_i^* = h^* - w_i$.*

Proof. See [1]. □

Lemma 3.3. *Let $w \in \mathbb{Z}^n$, we have*

$$h^* \cdot f_w = \sum_{i=1}^{n} w_i x_i \partial_i (f_w).$$

Proof. See [1]. □

Theorem 3.4. *Let $f_1, f_2 \in \mathbb{R}[x_1, x_2, x_3]$ be two polynomials. If $\mathscr{H}$ is the support of the polynomials f_1, f_2 containing 0 , then*

$$\operatorname{EDD}(f_1, f_2) \leq MV(P_1, P_2, P_{1'}, P_{2'}, P_{3'}),$$

where P_1, P_2 are Newton polytopes of f_1, f_2 and $P_{i'}$ are Newton polytopes of $u - x_i - \lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2$ with $i = 1, 2, 3$.

Proof. Suppose that $u \in \mathbb{C}^n \backslash \mathrm{N}(f_1, f_2)$ is general, with

$$\mathrm{N}(f_1, f_2) := \left\{x \in \mathbb{C}^3 \mid f_1 = f_2 = 0\right\}.$$

By Bernstein's Theorem, the Lagrange equation system $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ has $MV(P_1, P_2, P_{1'}, P_{2'}, P_{3'})$ solutions in $(\mathbb{C}^\times)^5$. We need to prove that the Lagrange system has no solutions outside $(\mathbb{C}^\times)^5$, which means that all solutions of $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ must lie inside $(\mathbb{C}^\times)^5$.

Consider

$$S := \left\{(u, \lambda, x) \in \mathbb{C}_u^3 \times \mathbb{C}_\lambda^2 \times \mathbb{C}_x^3 \mid \mathscr{L}_{f_1,f_2,u} = 0\right\}$$

is an affine manifold.

Recall that $f_1 = f_2 = 0$ are equations in $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$, let $X_{\mathrm{C}} = \mathrm{N}(f_1, f_2)$ be a complex curve.

Let $x \in X_{\mathrm{C}}$ and denote h the projection from S to X_{C}. The fiber $h^{-1}(x)$ on x is

$$\left\{(u, \lambda) \in \mathbb{C}_u^3 \times \mathbb{C}_\lambda^2 \mid u - x = \lambda_1 \cdot \nabla f_1(x) + \lambda_2 \cdot \nabla f_2(x)\right\}.$$

It is easy to see that the fiber $h^{-1}(x)$ is homomorphic to $\mathbb{C}_h^2$, showing that $S \xrightarrow{h} \mathbb{C}_u^3$ is $\mathbb{C}^2$-bundle and $\dim S = 3$.

Consider the projection from S to $\mathbb{C}_u^3$ as dominant. By Sard's theorem, the general fiber has dimension $3-3=0$ and is smooth. It means that when $u \in \mathbb{C}_u^3$ is general, the system $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ has finite solutions, i.e. is $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ has finite critical points and the number of critical points does not depend on u.

Since $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ has a finite number of critical points in $(\mathbb{C}^\times)^5$, on the other hand, according to Theorem 2.4 when $m = 5$, the number of solutions of $f_1 = f_2 = 0$ in $(\mathbb{C}^\times)^5$ is always less than or equal to $MV(Q_1, Q_2, Q_3, Q_4, Q_5)$. So the number of critical points of the Lagrange system $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ is less than or equal to $MV(P_1, P_2, P_{1'}, P_{2'}, P_{3'})$. □

Theorem 3.5. *If f_1, f_2 are general with support $\mathscr{H}$, such that $0 \in \mathscr{H}$ and $u \in \mathbb{C}^3$ are general. For any nonzero $w \in \mathbb{Z}^5$, the facial system $(\mathscr{L}_{f_1,f_2,u})_w$ has no solution in $(\mathbb{C}^\times)^5$.*

Proof. Fix

$$0 \neq w = (v_1, v_2, w_1, w_2, w_3) \in \mathbb{Z}^5.$$

The initial coordinates of w are $v \in \mathbb{Z}^2$. It has index 0 and corresponds to the variable $\lambda \in \mathbb{Z}^2$. The first two functions of the facial system $(\mathscr{L}_{f_1,f_2,u})_w$ are f_{1w} and f_{2w}.
The remaining functions depend on w as follows.
Let

$$\left(h^j\right)^* := \min\{w \cdot a \mid a \in \mathscr{H}, j = 1, 2\}$$

and

$$\left(h_i^j\right)^* := \min\left\{w \cdot a \mid a \in \partial_i \mathscr{H}; i = 1, 2, 3; j = 1, 2\right\}.$$

There are fifteen cases for these remaining functions, which are
$(u_i - x_i - \lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2)_w =$

$$\begin{cases} -x_i & \text{if } w_i < 0, v_1 + (h_i^1)^* < w_i, v_2 + (h_i^2)^* < w_i, & (3.1) \\ u_i & \text{if } w_i > 0, v_1 + (h_i^1)^* > 0, v_2 + (h_i^2)^* > 0, & (3.2) \\ (-\lambda_1 \partial_i f_1)_w & \text{if } v_1 + (h_i^1)^* < 0 \text{ and } v_2 + (h_i^2)^* \text{ and } w_i, & (3.3) \\ (-\lambda_2 \partial_i f_2)_w & \text{if } v_2 + (h_i^2)^* < 0 \text{ and } v_1 + (h_i^1)^* \text{ and } w_i, & (3.4) \\ u_i - x_i & \text{if } w_i = 0 < v_1 + (h_i^1)^* \text{ and } v_2 + (h_i^2)^*, & (3.5) \\ -x_i - (\lambda_1 \partial_i f_1)_w & \text{if } w_i = v_1 + (h_i^1)^* < 0 \text{ and } v_2 + (h_i^2)^*, & (3.6) \\ (-\lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2)_w & \text{if } v_1 + (h_i^1)^* = v_2 + (h_i^2)^* < 0 \text{ and } w_i, & (3.7) \\ u_i - (\lambda_1 \partial_i f_1)_w & \text{if } v_1 + (h_i^1)^* = 0 < w_i \text{ and } v_2 + (h_i^2)^*, & (3.8) \\ u_i - (\lambda_2 \partial_i f_2)_w & \text{if } v_2 + (h_i^2)^* = 0 < w_i \text{ and } v_1 + (h_i^1)^*, & (3.9) \\ -x_i - (\lambda_2 \partial_i f_2)_w & \text{if } v_2 + (h_i^2)^* = w_i < 0 \text{ and } v_1 + (h_i^1)^*, & (3.10) \\ u_i - x_i - (\lambda_1 \partial_i f_1)_w & \text{if } w_i = v_1 + (h_i^1)^* = 0 < v_2 + (h_i^2)^*, & (3.11) \\ u_i - x_i - (\lambda_2 \partial_i f_2)_w & \text{if } w_i = v_2 + (h_i^2)^* = 0 < v_1 + (h_i^1)^*, & (3.12) \\ -x_i - (\lambda_1 \partial_i f_1 + \lambda_2 \partial_i f_2)_w & \text{if } w_i = v_2 + (h_i^2)^* = v_1 + (h_i^1)^* < 0, & (3.13) \\ u_i - (\lambda_1 \partial_i f_1 + \lambda_2 \partial_i f_2)_w & \text{if } v_2 + (h_i^2)^* = v_1 + (h_i^1)^* = 0 < w_i, & (3.14) \\ u_i - x_i - (\lambda_1 \partial_i f_1 + \lambda_2 \partial_i f_2)_w & \text{if } w_i = v_2 + (h_i^2)^* = v_1 + (h_i^1)^* = 0. & (3.15) \end{cases}$$

Note that if one of the polynomials $(u_i - x_i - \lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2)_w$ is a monomial then $(\mathscr{L}_{f_1, f_2, u})_w$ has no solution in $(\mathbb{C}^\times)^3$.

Given the subset $I \subset \{1, 2, 3\}$ and the vector $u \in \mathbb{C}^3$, let $u_I := \{u_i \mid i \in I\}$. We denote w_I for $w \in \mathbb{Z}^3$ and x_I for the variables $x \in \mathbb{C}^3$ and denote $\mathbb{C}^I$ for the corresponding subspace of $\mathbb{C}^3$.

Case 1.
Suppose that $\partial_i f_1 w = 0$ and $\partial_i f_2 w = 0$ for all $1 \le i \le 3$. Since $0 \in \mathscr{H}$, f_{1w}, f_{2w} are constant terms of general f_1, f_2 and the facial system $(\mathscr{L}_{f_1, f_2, u})_w$ has no solutions. Suppose that $I \sqcup J = \{1, 2, 3\}$ with $I \neq \varnothing$ such that $\partial_i f_{1w} \neq 0, \partial_i f_{2w} \neq 0$ for $i \in I$ and $\partial_j f_1 w = \partial_j f_2 w = 0$ for $j \in J$.
Since $j \in J$, $\partial_j f_{1w} = \partial_j f_{2w} = 0$. If $a \in \mathscr{H}_w$, then $a_J = 0$. Hence f_{1w}, f_{2w} are polynomials with only the variables x_I, i.e. $f_{1w}, f_{2w} \in \mathbb{C}[x_I]$ or

$$(\mathscr{L}_{f_1, f_2, u})_w = \begin{cases} -x_i = 0 \\ u_i = 0 \end{cases},$$

has no solution in $(\mathbb{C}^\times)^5$.

Case 2.

Suppose that for $i \in I, w_i \geq 0$ means that $w_I \geq 0$. This implies that $w_I = 0$. To prove this, suppose $a \in \mathscr{H}_w$. We see that $a_J = 0$. We have

$$0 \geq h^* = w \cdot a = w_I \cdot a_I \geqslant 0.$$

Thus $h^* = w_I \cdot a_I = 0$ means that $0 \in \mathscr{H}_w$. Let $i \in I$. Since $\partial_i f_1 w \neq 0$ and $\partial_i f_2 w \neq 0$, there exists some $a \in \mathscr{H}_w$ with $a_i > 0$. Since $w_I \cdot a_n = 0$ for all $a \in \mathscr{H}_w$, we have $w_i = 0$.

Since $w_i = 0$, $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$ includes the equations (3.3), (3.4), (3.5), (3.8), (3.9), (3.11), (3.12), (3.14), (3.15). We consider the three cases $v_k < 0, v_k > 0, v_k = 0$ with $k = 1, 2$.

Case 2.1.

Suppose that $v_k < 0$ and $(\lambda_1, \lambda_2, x_1, x_2, x_3) \in (\mathbb{C}^\times)^5$ is a solution of $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$. We have $u_i - x_i = 0$ for all $i \in I$, we conclude that $x_I = u_I$. Since $f_{1w}, f_{2w} \in \mathbb{C}[x_I]$ are general with support $\mathscr{H}_w$ and u_1, u_2 are also general, we do not have $f_1(u_I) = f_2(u_I) = 0$. Therefore $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$ has no solution when $v_k < 0$.

Case 2.2.

Assume that $v_k > 0$. Then the subsystem of $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$ includes f_{1w}, f_{2w} and the equations with indices in I are

$$\begin{cases} f_{1w} = -\lambda_1 \partial_i (f_{1w}) = 0 \\ f_{2w} = -\lambda_2 \partial_i (f_{2w}) = 0 \end{cases}, \text{ with } i \in I. \tag{3.16}$$

Since $f_{1w}, f_{2w} \in \mathbb{C}[x_I]$, the system (3.16) implies the hypersurface $\mathcal{V}_{(\mathbb{C}^\times)^I} f_{1,2w} \subset (\mathbb{C}^\times)^I$ is singular. However, since $f_1 w, f_2 w$ is general, $\mathcal{V}_{(\mathbb{C}^\times)^I}(f_{1,2w})$ must be smooth. Therefore $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$ has no solution when $v_k > 0$.

Case 2.3.

When $v_k = 0$, the subsystem of $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$ includes f_{1w}, f_{2w} and the equations with index I:

$$u_i - x_i - \lambda_1 \partial_i (f_{1w}) - \lambda_2 \partial_i (f_{2w}) = 0 \text{ with } i \in I.$$

This is the system $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$ in $\mathbb{C}_\lambda \times \mathbb{C}^I$ for the critical points of the Euclidean distance from $u_I \in \mathbb{C}^I$ to $\mathcal{V}_{\mathbb{C}^I}(f_{1,2w}) \subset \mathbb{C}^I$. Therefore $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$ is triangular, indeed:

Since $\partial_j f_{1w} = \partial_j f_{2w} = 0$ with $j \in J$, the remaining equations do not depend on u_I and f_{1w}, f_{2w}.

Since $(h^1)^* = (h^2)^* = 0$, if $a \in \mathscr{H} \backslash \mathscr{H}_w$, then $w \cdot a > 0$. If $a \in g_w$, then $a_j = 0$ for $j \in J$ we have defined $h_j^* = \min\{w \cdot a \mid a \in \partial_j \mathscr{H}\}$. Furthermore, if $a \in \partial_j \mathscr{H}$ then $a + e_j \in \mathscr{H}$, it follows that $a + e_j \in \mathscr{H} \backslash \mathscr{H}_w$.

We consider

$$\begin{aligned} & w \cdot (a + e_j) > 0 \\ \Rightarrow & w \cdot a + w \cdot e_j > 0 \\ \Rightarrow & w \cdot a > -w_j, \end{aligned}$$

it follows that $h_j^* > -w_j$.
When $w_j \geqslant 0$ for all $j \in J$, we obtain $h_j^* > 0$ for all $j \in J$.

Therefore, the equations (3.8), (3.9), (3.11), (3.12), (3.14), (3.15) do not occur because of the contradiction with $v_k + \left(h_j^k\right)^* = 0$ and $v_k = 0$ with $k = 1, 2$.

Case 3.
Let $i \in I$ be an index with $w_i < 0$. Suppose that the facial system $\left(\mathscr{L}_{f_1, f_2, u}\right)_w$ has a solution; so the equation (3.1) of $(u_1 - x_i - \lambda_1 \partial_i f_1 - \lambda_2 \partial_1 f_2)_w$ does not occur. Thus, one of the following four equations is possible

$$\begin{aligned} & (u_1 - x_i - \lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2)_w\,, \\ = & \begin{cases} -x_i - (\lambda_1 \partial_i f_1)_w & \text{if } w_i = v_1 + (h_i^1)^* < 0, \\ -x_i - (\lambda_2 \partial_i f_2)_w & \text{if } w_i = v_2 + (h_i^2)^* < 0, \\ (-\lambda_1 \partial_i f_1 - \lambda_2 f_2)_w & \text{if } v_1 + (h_i^1)^* = v_2 + (h_i^2)^* < 0 \text{ and } w_i, \\ -x_i - (\lambda_1 \partial_1 f_1 + \lambda_2 \partial_2 f_2)_w & \text{if } w_i = v_1 + (h_i^1)^* = v_2 + (h_i^2)^* < 0. \end{cases} \end{aligned}$$

Since $w_i > 0$,

$$\left(h_i^1\right)^* \leq w_i + v_1 < v_1 \text{ and } \left(h_i^2\right)^* \leq w_i + v_2 < v_2.$$

By Lemma 3.2, we have

$$\left(h^1\right)^* = \left(h_i^1\right)^* + w_i \leq 2w_i + v_1 < v_1$$

and

$$\left(h^2\right)^* = \left(h_i^2\right)^* + w_i \leq 2w_i + v_2 < v_2.$$

For each $i \in I$, we have

$$\begin{cases} (h_i^1)^* = (h^1)^* - w_i < v_1 - w_i, \\ (h_i^2)^* = (h^2)^* - w_i v_2 - w_i. \end{cases}$$

So

$$\left(h_i^2\right)^* = \left(h^2\right)^* - w_i < v_2 - w_i.$$

If $w_i \geq 0$, then

$$\left(h_i^1\right)^* < v_1, \left(h_i^2\right)^* < v_2.$$

Therefore, only one of the four equations is possible for $i \in I$.
That is

$$(\mathscr{L}_{f_1,f_2,u})_w = \begin{cases} -x_i - (\lambda_1 \partial_i f_1)_w & \text{if } (h^1)^* = 2w_i + v_1 \text{ and } w_i < 0, \\ -x_i - (\lambda_2 \partial_i f_2)_w & \text{if } (h^2)^* = 2w_i + v_2 \text{ and } w_i < 0, \\ (-\lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2)_w & \text{if } \begin{cases} (h^1)^* - w_i < \min\{v_1, w_i + v_1\}, \\ (h^2)^* - w_i < \min\{v_2, w_i + v_2\}, \end{cases} \\ -x_1 - (\lambda_1 \partial_i f_1 + \lambda_2 \partial_i \partial_2)_w & \text{if } (h^1)^* = (h^2)^* \text{ and } w_i < 0. \end{cases} \tag{3.17}$$

This case further divides I into sets L and M, where

$$L := \left\{ l \in I \mid \left(h^1\right)^* - w \leq \min\{v_1, w + v_1\}, \left(h^2\right)^* - w < \min\{v_2, w_l + v_2\} \right\}$$

and

$$M := \left\{ m \in I \mid \left(h^1\right)^* = 2w_m + v_1, \left(h^2\right)^* = 2w_m + v_2 \text{ and } w_m < 0 \right\}.$$

For $l \in L$, we have

$$\lambda_1 \partial_l f_{1w} + \lambda_2 \partial_l f_{2w} = 0.$$

It follows that for $m \in M$, we have

$$\begin{cases} \lambda_1 \partial_m f_1 + \lambda_2 \partial_m f_{2w} = -x_m, \partial_m f_{1w} = \dfrac{-x_m}{\lambda_1}, \\ \partial_m f_{2w} = \dfrac{-x_m}{\lambda_2}. \end{cases}$$

For $M = \varnothing$, then $L = I$ and the subsystem of $(\mathscr{L}_{f_1,f_2,u})_w$ includes f_{1w}, f_{2w} and the equation (3.13) has no solution as we have seen.
For $M \neq \varnothing$, let $w' := \min\{w_i \mid i \in I\}$ then $w' < 0$. Furthermore, from the system (3.17), if $m \in M$ then we have

$$w_m = \frac{1}{2}\left(h^1\right)^* - v_1 = \frac{1}{2}\left(h^2\right)^* - v_2.$$

Thus, $w_m = w'$, for each $m \in M$.

Suppose that $(\lambda_1, \lambda_2, x_1, x_2, x_3)$ is a solution of $(\mathscr{L}_{f_1,f_2,u})_w$.
By Lemma 3.3, we obtain:

$$\left(h^1\right)^* \cdot f_{1w}(x) = \sum_{i \in I} w_i x_i \partial_i \left(f_1 w\right)(x) = \frac{-1}{\lambda_1} w' \sum_{m \in M} x_m^2,$$
$$\left(h^2\right)^* \cdot f_{2w}(x) = \sum_{i \in I} w_i x_i \partial_i \left(f_2 w\right) x = \frac{-1}{\lambda_2} w' \sum x_m^2.$$

Because

$$\left(h^1\right)^* f_{1w}(x) = 0 = \left(h^2\right)^* f_2 w(x)$$

so we have

$$-\frac{1}{\lambda_1} w' \sum_{m \in M} x_m^2 = -\frac{1}{\lambda_2} w' \sum_{m \in M} x_m^2 = 0.$$

Since

$$\lambda_1 \neq 0, \lambda_2 \neq 0 \text{ and } w' \neq 0$$

we have

$$\sum_{m \in M} x_m^2 = 0.$$

Let Q be this quadratic form. Then point x_I lies on both (f_{1w}, f_{2w}) and $\mathcal{V}(Q)$.
Since

$$\partial_l f_{1w}(x_I) = \partial_l f_{2w}(x_I) = \partial_l Q = 0,$$

with $l \in L$ and

$$\begin{cases} 2\partial_m f_{1w}(x_I) = \lambda_1 \partial_m Q \\ 2\partial_m f_{2w}(x_I) = \lambda_2 \partial_m Q, \text{ for } m \in M \end{cases}$$

so we see that the hypersurfaces do not intersect at x_I. But this contradicts f_{1w}, f_{2w} which is general. Therefore, there is no solution for the facial system $(\mathscr{L}_{f_1,f_2,u})_w = 0$. □

Theorem 3.6. *Let $f_1, f_2 \in \mathbb{R}[x_1, x_2, x_3]$ be two polynomials . If the support $\mathscr{H}$ of the polynomials f_1, f_2 contain 0 and f_1, f_2 are general and $u \in \mathbb{C}^3$ is also general, then*

$$EDD(f_1, f_2) = MV(P_1, P_2, P_{1'}, P_{2'}, P_{3'}),$$

where P_1, P_2 are the Newton polytopes of f_1, f_2 and $P_{i'}$ are the Newton polytopes of

$$u_i - x_i - \lambda_1 \partial_i f_1 - \lambda_2 \partial_i f_2,$$

with $i = 1, 2, 3$.

Proof. Since

$$\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$$

has finite critical points and lies in $(\mathbb{C}^{\times})^5$, by Theorem 2.3 when $m = 5$, the number of critical points of $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ is less than or equal to $MV\left(P_1, P_2, P_{1'}, P_{2'}, P_{3'}\right)$.

We use Theorem 2.4 when

$$G = \mathscr{L}_{f_1,f_2,u}$$

and

$$m = 5.$$

This shows that for two general polynomials f_1, f_2 all the solutions of $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ lies in $(\mathbb{C}^{\times})^5$.

However, by Theorem 3.5, the facial system $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ has no solution in $(\mathbb{C}^{\times})^5$. In this case, the number of critical points of the system $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ is $MV\left(P_1, P_2, P_{1'}, P_{2'}, P_{3'}\right)$.

While the number of solutions of the system $\mathscr{L}_{f_1,f_2,u}(\lambda, x) = 0$ is equal to the Euclidean distance degree, we obtain

$$EDD\left(f_1, f_2\right) = MV\left(P_1, P_2, P_1', P_2', P_3'\right).$$

□

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