Modified Mikhailov stability criterion for non-commensurate fractional-order neutral differential systems with delays

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Abstract

This paper studies the asymptotic stability of non-commensurate fractional-order neutral differential systems with constant delays. To do this, we propose a modified Mikhailov stability criterion. Our work not only generalizes the existing results in the literature but also provides a rigorous mathematical basis for the frequency domain analysis method concerning fractional-order systems with delays. Specific examples and numerical illustrations are also provided to demonstrate the validity of the obtained result.

Key works: Fractional differential equations with delays, non-commensurate fractional order neutral differential systems, modified Mikhailov stability criterion, asymptotic stability

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1 Introduction

Consider the system

$$\frac{d}{dt}x(t) = Ax(t),\ t > 0, \tag{1}$$

$$x(0) = x_0 \in \mathbb{R}^d, \tag{2}$$

where A is a real matrix of size $d \times d$. This system is called asymptotically stable if for any $x_0 \in \mathbb{R}^d$, the solution $\Phi(\cdot; 0, x_0)$ of the initial value problem (1)–(2) satisfies

$$\lim_{t\to\infty} \|\Phi(t; 0, x_0)\| = 0,$$

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here $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$. By the final value theorem for Laplace transforms, it is known that the system (1) is asymptotically stable if and only if its characteristic polynomial has only zeros with negative real parts. From this observation, an important task arose in the study of the asymptotic behavior of solutions of continuous-time dynamical systems: set a criterion to check whether a polynomial with real coefficients

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

has roots only in the left open half of the complex plane or not. E.J. Routh and A. Hurwitz independently derived an equivalence criterion for stability using an algebraic procedure. They provided necessary and sufficient conditions for all roots of the polynomial p to lie in the left half plane without needing to determine them (see [10]). However, when the degree of p is large, applying the Routh–Hurwitz criterion will be difficult (one has to calculate a lot of determinants of large matrices). Therefore, geometric methods were developed. H. Nyquist [19] and A.V. Mikhailov [18] given graphical solutions in the frequency domain. It is worth noting that Nyquist and Mikhailov-style graphical techniques can also be applied to time-delay systems, see, e.g., [16, 6].

In the past three decades, fractional calculus has become an active research area. One of the main reasons is that it provides an excellent instrument for describing memory and hereditary properties of real-world processes. This is an advantage over classical differential models in which such effects are neglected. The nterested reader can find updated applications of fractional calculus in the monographs [4, 5, 20, 23, 24].

Many tools have been developed to investigate the asymptotic behavior of solutions to fractional dynamical systems up to now: Lyapunov-type first and second methods, generalized comparison principle, and modified frequency domain analysis. Depending on the specific situation, each approach has different strengths and weaknesses. Within the scope of the current paper, we limit our attention to the fourth topic mentioned above. Below, we briefly list notable papers based on frequency domain analysis.

In [25], the authors prove that the fractional transfer function

$$H(s) = \frac{1}{s^{\nu_n} + a_{n-1}s^{\nu_{n-1}} + \cdots + a_1 s^{\nu_1} + a_0}$$

has the same poles as a closed-loop system $\tilde{H}(s)$ that the open-loop is

$$H_{OL} = \frac{a_{n-1}}{s^{\nu_n-\nu_{n-1}}} + \frac{a_{n-2}}{s^{\nu_n-\nu_{n-2}}} + \cdots + \frac{a_0}{s^{\nu_n}}.$$

Then, under the method based on Nyquist's theorem, they have given Routh-like stability conditions for fractional order systems involving a maximum of two fractional derivatives. Unfortunately, for higher numbers of differential operators, this method is unsuitable for its numerical implementation. J. Sabatier et al. [21] have presented another realization of the fractional system recursively defined and involves nested closed loops. By exploiting Cauchy's argument principle on a frequency range, the numerical limitation in [25] is removed (however, no formal proof is shown in [21]). In [14], E. Ivanova et al. studied a second-order fractional transfer function in the form

$$G(s) = \frac{1}{(\frac{s}{\omega_0})^2 + 2\xi(\frac{s}{\omega_0})^\nu + 1},$$

here $\nu \in [0,2]$, $\xi \in \mathbb{R}$, and ω_0 is a parameter related to the physical properties of the system. Using a simplified Nyquist criterion applied on a Nichols chart of the corresponding open-loop transfer function, they establish several stability and resonance conditions in the form of a pseudo-damping factor and a fractional differentiation order. After that, the approach in [14] has been successfully extended in [29] for a non-commensurate elementary fractional-order system without delay and in [30] for a non-commensurate elementary fractional-order delay system.

Although there have been some works on frequency domain analysis criteria have appeared. Until [7, 8] (on commensurate fractional systems with and without delays) and then [22] (on non-commensurate fractional systems without delays), it seems that no fundamental and systematic contributions to this research direction have been announced.

The fractional neutral delay differential equations (FNDDEs) have received considerable attention in recent years. In [1], the authors proved the existence of at least one solution of FNDDEs. In [28], a new Halanay-type inequality was derived to describe the behavior of solutions of FNDDEs. In [2], the robust stability of a class of FNDDEs with uncertainty and input saturation is discussed. After that, in [26], an analysis of the asymptotical stability for some scalar linear FNDDEs has been introduced.

As a continuation of the studies on FNDDEs mentioned above, inspired by [7, 22, 8], we focus on the following non-commensurate fractional-order neutral differential system with constant delays:

$$^{C}D^{\hat{\alpha}}_{0+}\left(x(t) + Ax(t-\tau)\right) = B_0 x(t) + B_1 x(t-\gamma),\ t > 0, \tag{3}$$

where $\hat{\alpha} = (\alpha_1, \dots, \alpha_d) \in (0,1]^d$ is a multi-index,

$$^{C}D^{\hat{\alpha}}_{0+}x(t) = \left(^{C}D^{\alpha_1}_{0+}x_1(t), \dots, {}^{C}D^{\alpha_i}_{0+}x_i(t), \dots, {}^{C}D^{\alpha_d}_{0+}x_d(t)\right)^T$$

with $^{C}D^{\alpha_i}_{0+}x_i(t)$ is the Caputo fractional derivative of the order α_i, A, B_0, B_1 are real matrices of size $d \times d$, τ, γ are positive constant delays.

Our aim in this paper is to build a rigorous mathematical basis for the modified Mikhailov curve method to study the asymptotic stability of the system (3). It is a development of previous results on frequency domain analysis approaches for continuous-time dynamical systems.

The organization of the paper is the following. In section 2, we introduce the necessary preparatory knowledge for further analysis in the following section. The main contribution is the modified Mikhailov stability criterion for fractional semi-polynomials stated in Section 3. Then, a detailed comparison of our result with those published in the literature is mentioned in Remarks 3.9, 3.10, 3.11. As a consequence of the main result, a three-step scheme for checking the asymptotic stability of the system (3) is established in Subsection 3.3. Finally, specific examples and numerical illustrations have been provided to demonstrate the correctness of the obtained theoretical results.

To conclude this part, we present some notations used throughout the rest of the paper. Let $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_+$ be the set of integers, non-negative integers, real, non-negative real, and positive real numbers, respectively. For a vector $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we define the norm $\|x\| := \max\{|x_1|, |x_2|, \cdots, |x_d|\}$ and x^{T} is its transpose. Denote $\mathbb{C}$ as the set of complex numbers. For any $z \in \mathbb{C}$, let $\Re z$, $\Im z$ be its real and imaginary part. Set $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $\mathbb{C}_{\geq 0} := \{z \in \mathbb{C} : \Re z \geq 0\}$, and $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$. For any $a, b \in \mathbb{R}, a < b$, the space of all continuous functions (continuously differentiable functions) $\xi : [a, b] \to \mathbb{R}^d$ is denoted by $C([a, b]; \mathbb{R}^d)$ $(C^1([a, b]; \mathbb{R}^d))$.

2 Preliminaries

For $\alpha \in (0, 1]$ and $J = [0, T]$ or $J = [0, \infty)$, the Riemann-Liouville fractional integral of a function $f : J \to \mathbb{R}$ is defined by

$$I^{\alpha}_{0+}x(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} x(s)\, \mathrm{d}s, \qquad t \in J,$$

and its Caputo fractional derivative of the order $\alpha \in (0, 1)$ as

$${}^{C}D^{\alpha}_{0+}x(t) := \frac{d}{dt} I^{1-\alpha}_{0+}(f(t) - f(0)), \quad t \in J \setminus \{0\},$$

where $\Gamma(\cdot)$ is the Gamma function and $\frac{d}{dt}$ is the first derivative, see. e.g., [15, Chapters 2 and 3] or [27].

Let $d \in \mathbb{N}$, $\hat{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in (0, 1]^d$ be a multi-index and $x = (x_1, \ldots, x_d)^{\mathrm{T}}$ with $x_i : J \to \mathbb{R}$, $i = 1, \ldots, d$, be a vector valued function. Then, we denote

$${}^{C}D^{\hat{\alpha}}_{0+}x(t) := \left({}^{C}D^{\alpha_1}_{0+}x_1(t), \ldots, {}^{C}D^{\alpha_n}_{0+}x_d(t)\right)^T.$$

We consider the following non-commensurate fractional neutral differential system with constant delays:

$${}^{C}D^{\hat{\alpha}}_{0+}\left(x(t) + Ax(t - \tau)\right) = f(t, x(t), x(t - \gamma)),\ t \in (0, T], \tag{4}$$

where $\hat{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0, 1]^d$ is a multi-index, $A = (a_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$, τ, γ are positive constant delays, $f = (f_1, \cdots, f_d)^{\mathrm{T}}$ with $f_i : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a continuous function. For each $i = 1, \ldots, d$, assume that f_i satisfies the Lipschitz condition

$$|f_i(t, x, y) - f_i(t, \tilde{x}, y)| \leq L_i(t, y)\|x - \tilde{x}\|, \tag{5}$$

for all $t \in [0, T]$, $x, \tilde{x}, y \in \mathbb{R}^d$. Here, $L_i : [0, T] \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ is a continuous function.

For $\nu := \max\{\tau, \gamma\}$, we take the initial condition of the system (4) as below

$$x = \phi \in C^1([-\nu, 0]; \mathbb{R}^d). \tag{6}$$

Definition 2.1. A function $x \in C([-\nu, T]; \mathbb{R}^d)$ is said to be a solution of the system (4) with the initial condition (6) if for any $t \in (0, T]$, we have

$${}^{C}D^{\hat{\alpha}}_{0+}\left(x(t) + Ax(t - \tau)\right) = f(t, x(t), x(t - \gamma)).$$

Furthermore, $x(t) = \phi(t)$, $\forall t \in [-\nu, 0]$.

Using the same arguments as in the proof of [15, Lemma 6.2], we obtain the following lemma.

Lemma 2.2. For an initial condition $\phi \in C^1([-\nu, 0]; \mathbb{R}^d)$, a function $x \in C([-\nu, T]; \mathbb{R}^d)$ is a solution of the system (4) with the initial condition (6) if and only if it is a solution of the following delay integral system

$$\begin{aligned} x_i(t) &= \phi_i(0) + \sum_{j=1}^{d} a_{ij}\phi_j(-\tau) - \sum_{j=1}^{d} a_{ij}x_j(t-\tau) \\ &\quad + \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1} f_i(s, x(s), x(s-\gamma))ds,\ t \in (0, T],\ i = 1, \dots, d, \end{aligned} \tag{7}$$

and satisfies $x(t) = \phi(t)$ on $[-\nu, 0]$.

With the help of Lemma 2.2 and a slight modification of the arguments in the proof of [26, Theorem 3.1], we receive a result on the existence of a unique solution of the system (4).

Theorem 2.3. Assume that the condition (5) holds. Then, for each initial condition $\phi \in C^1([-\nu, 0]; \mathbb{R}^d)$, the system (4) has a unique solution on $[-\nu, T]$.

Corollary 2.4. Consider the system (4) on $[0, \infty)$. Assume the condition (5) holds. Then, the system (4) with the initial condition (6) has a unique global solution on $[0, \infty)$.

Proof. The proof of this corollary is similar to [26, Corollary 3.2] and thus we omit it. □

We now discuss the exponential boundedness of solutions to the following system:

$$^{C}D^{\hat{\alpha}}_{0+}\left(x(t) + Ax(t-\tau)\right) = f(t, x(t), x(t-\gamma)),\ t > 0, \tag{8}$$

$$x(t) = \phi(t),\ t \in [-\nu, 0], \tag{9}$$

here $f = (f_1, \dots, f_d)^{\mathrm{T}}$ and for each $i = 1, \dots, d$, $f_i : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a continuous function such that the following conditions are true.

(F1) There exists a positive constant L such that

$$|f_i(t, x, y) - f_i(t, \tilde{x}, \tilde{y})| \le L\left(\|x - \tilde{x}\| + \|y - \tilde{y}\|\right), \tag{10}$$

for all $t \in [0, \infty)$, $i = 1, \dots, d$, $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d$.

(F2) There exists a positive constant $\lambda > 1$ such that

$$\max_{i \in \{1,\dots,d\}} \sup_{t \ge 0} \frac{\int_0^t (t-s)^{\alpha_i - 1}|f_i(s, 0, 0)|ds}{e^{\lambda t}} < +\infty. \tag{11}$$

Theorem 2.5. Assume that (F1) and (F2) are true. Then, the unique global solution $\Phi(\cdot, \phi)$ of the initial value problem (8)–(9) is exponentially bounded. More precisely, there is a positive constant M such that

$$\|\Phi(\cdot, \phi)\| \le M \exp(\lambda t),\ t \ge 0.$$

Proof. The proof of this theorem is easily obtained by modifying the arguments in the proof of [26, Theorem 4.1]. □

Denote by $\mathcal{L}$ the Laplace transform, it is well known that

$$\mathcal{L}\left({}^{C}D^{\alpha}_{0+}x(\cdot)\right)(s) = s^{\alpha}X(s) - s^{\alpha-1}x(0), \tag{12}$$

and

$$\mathcal{L}\left(x(\cdot-\tau)\right)(s) = e^{-s\tau}X(s) + e^{-s\tau}\int_{-\tau}^{0} e^{-su}x(u)du, \tag{13}$$

where $\alpha \in (0,1]$, $\tau > 0$ and $X(\cdot)$ is the Laplace transform of $x(\cdot)$.

Theorem 2.6. ([15, Theorem D.13, p. 232]) Assume that $\mathcal{L}(f)$ does not have any singularities in the closed right half-plane $\mathbb{C}_{\geq 0}$ except for possibly a simple pole at the origin. Then, $\lim_{x\to\infty} f(x) = \lim_{s\to 0+} s\mathcal{L}(f)(s)$.

Theorem 2.7. (Rouché's Theorem, see, e.g., [9, Theorem 8.18]) Let U be a bounded open subset of $\mathbb{C}$, f, g continuous on $U \cup \partial U$ and holomorphic in U. Suppose that $|g(s)| < |f(s)|$ on ∂U. Then, counting multiplicities, the functions f and $f+g$ have the same number [which is finite] of zeros in U.

3 The asymptotic behavior of solutions to noncommensurate fractional neutral differential system with delays

For any initial condition $\phi \in C^1([-\nu, 0]; \mathbb{R}^d)$, we consider the following non-commensurate fractional neutral differential system with constant delays

$${}^{C}D^{\hat{\alpha}}_{0+}\left(x(t) + Ax(t-\tau)\right) = B_0x(t) + B_1x(t-\gamma),\ t > 0, \tag{14}$$

with the initial condition

$$x(t) = \phi(t),\ t \in [-\nu, 0], \tag{15}$$

where $\hat{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0,1]^d$ is a multi-index, $A = (a_{ij})_{d\times d}$ and $B_l = (b^{(l)}_{ij})_{d\times d}$, $l \in \{0,1\}$ are real matrices.

In this section, we will present a theoretical basis of the modified Mikhailov curve method to study the asymptotic stability of the system (14).

3.1 The characteristic polynomial of the system (14)

Due to Theorem 2.3 and Theorem 2.5, the system (14)–(15) has a unique global solution $\Phi(\cdot, \phi)$ on $[-\nu, \infty)$. Moreover, this solution is exponentially bounded. Taking the Laplace transform on both the sides of (14) and paying attention to the facts (12), (13), we see that

$$(s^{\hat{\alpha}}I)X(s) - (s^{\hat{\alpha}-1}I)x(0) + (s^{\hat{\alpha}}I)\left(A\left(e^{-\tau s}X(s) + e^{-\tau s}\int_{-\tau}^{0} e^{-su}\phi(u)du\right)\right)$$
$$- \left(s^{\hat{\alpha}-1}I\right)A\phi(-\tau) = B_0X(s) + B_1\left(e^{-\gamma s}X(s) + e^{-\gamma s}\int_{-\gamma}^{0} e^{-su}\phi(u)du\right),$$

where I is the identity matrix of size d, $s^{\hat{\alpha}}I = diag(s^{\alpha_1}, \dots, s^{\alpha_d})$,

$$s^{\hat{\alpha}-1}I = diag(s^{\alpha_1 - 1}, \dots, s^{\alpha_d - 1}),$$

$X(s) = (X_1(s), \dots, X_d(s))^{\mathrm{T}}$ with $X_i(s) = \mathcal{L}\{x_i(\cdot)\}(s)$ and

$$\int_{-t}^{0} e^{-su}\phi(u)du = \left(\int_{-t}^{0} e^{-su}\phi_1(u)du, \dots, \int_{-t}^{0} e^{-su}\phi_d(u)du\right)^{\mathrm{T}}, \ t \in [-\nu, 0].$$

Hence, the characteristic polynomial of (14) is

$$Q(s) := \det\left(s^{\hat{\alpha}}I + e^{-\tau s}(s^{\hat{\alpha}}I)A - B_0 - e^{-\gamma s}B_1\right). \tag{16}$$

Our first task in this section is to expand Q in formal monomials of the forms $s^{\alpha_{i_1}} \cdots s^{\alpha_{i_r}}$, where $1 \le r \le d$, $1 \le i_1 < \cdots < i_r \le d$. Put

$$C(s) := \left(s^{\hat{\alpha}}I + e^{-\tau s}(s^{\hat{\alpha}}I)A - B_0 - e^{-\gamma s}B_1\right). \tag{17}$$

The element in the i-th row and j-th column of the matrix $C(s)$ is

$$c_{ij}(s) := \begin{cases} s^{\alpha_i} - s^{\alpha_i}a_{ii}e^{-\tau s} - b_{ii}^{(0)} - b_{ii}^{(1)}e^{-\gamma s} & \text{if} \quad j = i, \\ -s^{\alpha_i}a_{ij}e^{-\tau s} - b_{ij}^{(0)} - b_{ij}^{(1)}e^{-\gamma s} & \text{if} \quad j \neq j. \end{cases} \tag{18}$$

Define

$$p_{ij}(s) := \begin{cases} 1 - a_{ii}e^{-\tau s} & \text{if} \quad j = i, \\ -a_{ij}e^{-\tau s} & \text{if} \quad j \neq i, \end{cases}$$
$$q_{ij}(s) := -b_{ij}^{(0)} - b_{ij}^{(1)}e^{-\gamma s}, \tag{19}$$

for $1 \le i, j \le d$. Then, the matrix $C(s)$ is rewritten as

$$C(s) = \begin{pmatrix} s^{\alpha_1}p_{11}(s) + q_{11}(s) & s^{\alpha_1}p_{12}(s) + q_{12}(s) & \cdots & s^{\alpha_1}p_{1d}(s) + q_{1d}(s) \\ s^{\alpha_2}p_{21}(s) + q_{21}(s) & s^{\alpha_2}p_{22}(s) + q_{22}(s) & \cdots & s^{\alpha_2}p_{2d}(s) + q_{2d}(s) \\ \cdots & \cdots & \ddots & \cdots \\ s^{\alpha_d}p_{d1}(s) + q_{d1}(s) & s^{\alpha_2}p_{d2}(s) + q_{d2}(s) & \cdots & s^{\alpha_d}p_{dd}(s) + q_{dd}(s) \end{pmatrix}. \tag{20}$$

Since $p_{ij}(s)$ and $q_{ij}(s)$ do not contain the components $s^{\alpha_1}, \dots, s^{\alpha_d}$, $1 \le i, j \le d$, $Q(s) = \det C(s)$ is equal to the sum of the forms $s^{\alpha_{i_1} + \cdots + \alpha_{i_r}} h_{i_1, i_2, \cdots, i_r}(s)$, where $1 \le i_1 < \cdots < i_r \le d, 0 \le r \le d$, $h_{i_1, \dots, i_r}(s)$ do not contain the monomials $s^{\alpha_1}, \dots, s^{\alpha_d}$ and depend only on the functions p_{ij}, q_{ij}, $1 \le i, j \le d$. For simplicity we take the convention $\alpha_{i_1} + \cdots + \alpha_{i_r} = 0$ when $r = 0$. Let $0 = \beta_0 < \beta_1 < \cdots < \beta_N = \alpha_1 + \cdots + \alpha_d$ be distinct elements of the set

$\{\alpha_{i_1} + \cdots + \alpha_{i_r} : 0 \le r \le d, 1 \le i_1 < \cdots < i_r \le d\}$. Here, for convenience, we call an index tuple of length β_j as a tuple $\{\alpha_{i_l}\}_{l=1}^{k}$, $1 \le i_1 < \cdots < i_k \le d$ with $\sum_{l=1}^{k} \alpha_{i_l} = \beta_j$. It is easy to check that Q has the form

$$Q(s) = \sum_{j=0}^{N} h_j(s) z^{\beta_j}, \tag{21}$$

where $h_j(s)$ are functions that do not contain the components $s^{\alpha_1}, \ldots, s^{\alpha_d}$ and only depend on the functions $p_{mn}(s), q_{mn}(s)$, $1 \le m, n \le d$.

From a definition of the determinant, we have

$$Q(s) = \sum_{\sigma \in S_d} sgn(\sigma) \prod_{i=1}^{d} (s^{\alpha_i} p_{i\sigma(i)}(s) + q_{i\sigma(i)}(s)), \tag{22}$$

here S_d is the symmetric group on the set $\{1, \ldots, d\}$ and $sgn(\sigma)$ is the signature of the permutation σ.

First, we express $h_0(s)$ (the term that does not contain any components of the forms $s^{\alpha_i}, i = 1, \ldots, d$). From (22),

$$\begin{aligned} h_0(s) &= \sum_{\sigma \in S_d} sgn(\sigma) \prod_{i=1}^{d} q_{i\sigma(i)} \\ &= \det(-B_0 - e^{-\gamma s} B_1) \\ &= \sum_{k=0}^{d} a_k e^{-k\gamma s}, \end{aligned} \tag{23}$$

where $a_k \in \mathbb{R}, k = 0, 1, \cdots, d$, are coefficients that depend only on the elements of the matrices B_0, B_1.

Next, for $1 \le j \le N-1$, we will determine $h_j(s)$. Suppose $\beta_j = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r}$ with $1 \le i_1 < \cdots < i_r \le d$ and $1 \le r < d$. We first describe the components of Q that contain only elements of the form $s^{\alpha_{i_1}} \cdots s^{\alpha_{i_r}}$ (we consider s^{α_i} as formal variables and therefore $s^{\alpha_{i_1}} s^{\alpha_{i_2}} \ne s^{\alpha_{i_2}} s^{\alpha_{i_1}}$ for $i_1, i_2 \in \{1, \ldots, d\}$). By (22), the component containing only the monomial $s^{\alpha_{i_1}} \cdots s^{\alpha_{i_r}}$ in Q is

$$\sum_{\sigma \in S_d} sgn(\sigma) \prod_{k=1}^{r} \prod_{l=1, l \ne i_1, \ldots, i_r}^{d} p_{i_k \sigma(i_k)} q_{l\sigma(l)}.$$

It is worth noting that $\prod_{k=1}^{r} p_{i_k\sigma(i_k)} = \sum_{m=0}^{r} c_m e^{-m\tau s}$, where c_m, $m = 0, 1, \ldots, r$, are constants that depend only on the elements of the matrix A. Meanwhile, $\prod_{l=1, l\ne i_1, \ldots, i_r}^{d} q_{l\sigma(l)} = \sum_{n=0}^{d-r} d_n e^{-n\gamma s}$, where d_n, $n = 0, 1, \ldots, d-r$, are constants that depend only on the elements of the matrices B_0, B_1. Combining the above observations, we obtain the representation of h_j as

$$h_j(s) = \sum_{0 \le m \le r, 0 \le n \le d-r} g_{m,n} e^{-(m\tau + n\gamma)s}, \tag{24}$$

where $g_{m,n}$, $0 \le m \le r, 0 \le n \le d-r$, are constants that depend on the elements of the matrices A, B_0, B_1, $r \in \{1, \dots, d-1\}$ is the number of elements in an index tuple of length β_j. Let p_j be the number of distinct elements of the set $\{m\tau + n\gamma : 0 \le m \le r, 0 \le n \le d-r\}$ such that there exists an index tuple of length β_j with r elements (we call this set as M_j). Then, h_j has another representation as follows

$$h_j(s) = \sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s}, \tag{25}$$

where $\tau_{kj} \in M_j$. Finally, consider $j = N$. It is not difficult to see that the coefficient of the highest-order term of Q is

$$\sum_{\sigma \in S_d} sgn(\sigma) \prod_{k=1}^{d} p_{k\sigma(k)}(s) = \sum_{k=0}^{d} b_k e^{-k\tau s}, \tag{26}$$

here b_i, $0 < i \le d$, depends only on the elements of the matrix A and $b_0 = 1$. It implies from (23), (25) and (26) that

$$Q(s) = \sum_{j=0}^{N} \left(\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s} \right) s^{\beta_j}, \tag{27}$$

here $\tau_{kj} \ge 0$, $0 \le j \le N$, $0 \le k \le p_j$. In particular, $p_0 = p_N = d$, $\beta_N = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $\tau_{k0} = k\gamma$, $k = 0, 1, \dots, d$, $\tau_{kN} = k\tau$, $k = 0, 1, \dots, d$, and $c_{0N} = 1$.

Define $i_0 := \max\{i \in \{1, \dots, d\} : c_{iN} \neq 0\}$. We obtain a simple relation between the positions of the zeros of the characteristic polynomial Q and its coefficients.

Proposition 3.1. Consider the characteristic polynomial Q as in (27). If $\sum_{k=0}^{d} c_{k0} < 0$ or $|c_{i_0 N}| > 1$, then Q has at least one zero point in the open right half of the complex plane.

Proof. (i) Suppose that $\sum_{k=0}^{d} c_{k0} < 0$. Due to $Q(0) = \sum_{k=0}^{d} c_{k0}$, hence $Q(0) < 0$. On the other hand, from (27), we derive

$$Q(s) = \sum_{j=0}^{N-1} \left(\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s} \right) s^{\beta_j} + \left(\sum_{k=0}^{d} c_{kN} e^{-k\tau s} \right) s^{\beta_N}, \tag{28}$$

where $c_{0N} = 1$. Notice that $\lim_{|s| \to \infty, s \in \mathbb{R}_+} \sum_{k=0}^{d} c_{kN} e^{-k\tau s} = c_{0N} = 1$, and $\lim_{|s| \to \infty, s \in \mathbb{R}_+} s^{\beta_N} = +\infty$, we conclude

$$\lim_{|s| \to \infty, s \in \mathbb{R}_+} Q(s) = +\infty.$$

Hence, Q has at least one positive root.

(ii) Suppose that $c_{i_0 N} > 1$. Using (27), we write

$$Q(s) = (1 + \sum_{k=1}^{d} c_{kN} e^{-k\tau s}) s^{\beta_N} + \sum_{j=0}^{N-1} \left(\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s} \right) s^{\beta_j}. \tag{29}$$

From the representation of Q above, we see that $s \neq 0$ is a solution of Q if and only if it is also a solution of the following polynomial:

$$P(s) := 1 + \sum_{k=1}^{d} c_{kN} e^{-k\tau s} + \sum_{j=0}^{n-1} \left(\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s} \right) s^{\beta_j - \beta_n}. \tag{30}$$

Take

$$\begin{aligned} f(s) :&= 1 + \sum_{k=1}^{d} c_{kN} e^{-k\tau s} \\ &= 1 + \sum_{k=1}^{i_0} c_{kN} e^{-k\tau s}, \end{aligned} \tag{31}$$

and

$$g(s) := \sum_{j=0}^{N-1} \left(\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s} \right) s^{\beta_j - \beta_N}. \tag{32}$$

by the change of variables $u = e^{-\tau s}$, f is a polynomial of degree i_0 concerning the variable u. Therefore, the equation $f(u) = 0$ has i_0 solutions as $u_1, \dots, u_{i_0}$. Moreover, these solutions satisfy $\prod_{i=1}^{i_0} |u_i| = \frac{1}{|c_{i_0 N}|} < 1$. It implies that there is a solution u_i with $|u_i| < 1$, and thus there is at least one solution s of the equation $f(s) = 0$ with $\Re s > 0$. Let s^0 be a zero point of f satisfying $\Re s^0 > 0$. It is easy to check that $\{s_k^0\}_{k \in \mathbb{Z}_{\geq 0}}$ with $z_k^0 = s^0 \pm i\frac{2k\pi}{\tau}$ are also the solutions of this polynomial. Due to the nature of f, we can find $\delta > 0$ small enough such that for every $z \in S_\delta(s_k^0)$, $k \in \mathbb{Z}_{\geq 0}$, the distance from z to all the zero points of f is larger or equal to δ, here $S_\delta(s_k^0)$ is the circle with the center at s_k^0 and the radius δ (in this way, s_k^0 is the only zero point of f in $S_\delta(s_k^0)$ and $S_\delta(s_k^0)$ is completely in the open right half plane). By [17, Lemma 1, p. 268], there exists $m(\delta) > 0$ such that

$$|f(z)| \geq m(\delta),\ \forall z \in S_\delta(s_k^0),\ \forall k \in \mathbb{Z}_{\geq 0}.$$

By virtue of the facts that $\lim_{|s| \to \infty, s \in \mathbb{R}_+} |s^{\beta_j - \beta_N}| = 0$ for all $j = 0, 1, \dots, N-1$,

$$|\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s}| \leq \sum_{k=0}^{p_j} |c_{kj}|$$

for all $s \in \mathbb{C}_+$, $j = 0, 1, \dots, N-1$, there is an index k_0 large enough so that

$$|g(z)| \leq m(\delta)/2,\ \forall z \in S_\delta(s_{k_0}^0).$$

Thus, $|f(z)| > |g(z)|$ for all $z \in S_\delta(s_{k_0}^0)$. Following from Rouché's theorem (Theorem 2.7), Q has exactly one zero point in $S_\delta(s_{k_0}^0)$. The proof is complete. □

Remark 3.2. Consider the system (14). Suppose that $\hat{\alpha} = (1, \dots, 1)^{\mathrm{T}}$, $B_0 = B_1 = 0$. By [3, Theorem 5.2], the system is asymptotically stable if and only if

$$\sup\{\Re s : \det(I + e^{-\tau s} A) = 0\} < 0.$$

This implies that a necessary condition for the stability of (14) is

$$\sum_{k=1}^{d} |c_{kN}| < 1.$$

Remark 3.3. Consider the case $d = 1$, $A = -1$, and $B = B_1 = 0$. The characteristic function of (14) becomes $Q(s) = (1 + e^{-\tau s})\, s^\alpha$. Excluding the origin point, this characteristic polynomial has only purely imaginary roots. Choosing the initial condition $\phi = \lambda \neq 0$, then the equation (14) has the solution $x(t) = \lambda$ for all $t \geq 0$. This means that the trivial solution is not asymptotically stable.

Remark 3.4. Consider the system (14) in the case $d = 1$, $\tau = \gamma$, $|A| > 1$, $B_0 < 0$, $|B_1| < |B_0|$. Then, we have $Q(s) = s^\alpha + As^\alpha e^{-\tau s} - B_0 - B_1 e^{-\tau s}$ and thus $c_{11} = A$, $c_{00} = -B_0$, $c_{10} = -B_1$. In [26], the authors have proven if $|c_{11}| > 1$, $c_{00} > 0$, and $|c_{10}| < |c_{00}|$ then Q has at least one root in the open right half plane.

3.2 Modified Mikhailov stability criterion for the characteristic function Q

We begin this subsection by recalling some basic knowledge of complex analysis.

Proposition 3.5. ([13, Proposion A.2.3] Given an arbitrary interval $I \subset \mathbb{R}$ and a continuous function $\gamma : I \to \mathbb{C}^*$, there exists a continuous function $\theta : I \to \mathbb{R}$ such that

$$\gamma(t) = |\gamma(t)|e^{i\theta(t)} = e^{\ln|\gamma(t)|+i\theta(t)}, \quad t \in I. \tag{33}$$

Moreover, the function θ is differentiable at each point $t \in I$ where γ is differentiable.

Definition 3.6. ([13, Definition A.2.4]) Given an arbitrary interval $I \subset \mathbb{R}$ and a continuous function $\gamma : I \to \mathbb{C}^*$, any continuous function $\theta : I \to \mathbb{R}$ satisfying (33) is called an argument function of the complex curve γ. In this case, we write $\arg\gamma(\cdot) = \theta(\cdot)$. If $I = [a, b]$, the net change of the argument of $\gamma(t)$ as t moves from a to b is given by

$$\Delta \arg \gamma(t)\big|_I = \Delta \arg \gamma(t)\big|_a^b = \theta(b) - \theta(a). \tag{34}$$

If $I = [0, \infty)$, then the change of the argument of γ as t move from 0 to ∞ is defined by

$$\Delta \arg \gamma(t)\big|_I = \Delta \arg \gamma(t)\big|_0^\infty = \lim_{k\to\infty} \Delta \arg \gamma(t)\big|_0^k = \lim_{k\to\infty} \theta(k) - \theta(0). \tag{35}$$

If $\gamma : [a, b] \to \mathbb{C}$ is a closed curve and $c \in \mathbb{C} \setminus \gamma([a, b])$, then the winding number of the point c with respect to the closed curve γ is defined by

$$w(\gamma, c) = (2\pi)^{-1}(\psi(b) - \psi(a)) = (2\pi)^{-1}\Delta \arg(\gamma(t) - c)\big|_a^b, \tag{36}$$

where ψ is any argument function of the closed curve $t \mapsto \gamma(t) - c$.

Theorem 3.7. (Argument principle, see, e.g., [9, Corollary 9.15]) Let C be a simple closed curve, oriented in c counterclockwise direction, f is analytic on and inside C, except for (possibly) some finite poles inside (not on) C and some zeros inside (not on) C. Then, $w(f(C), 0) = Z - P$, where $w(f(C), 0)$ is the winding number of $f(C)$ around 0, i.e., the total number of times that the curve $f(C)$ encircles the point 0 in the positive direction and Z, P are the number of zeros and poles of f inside C, respectively.

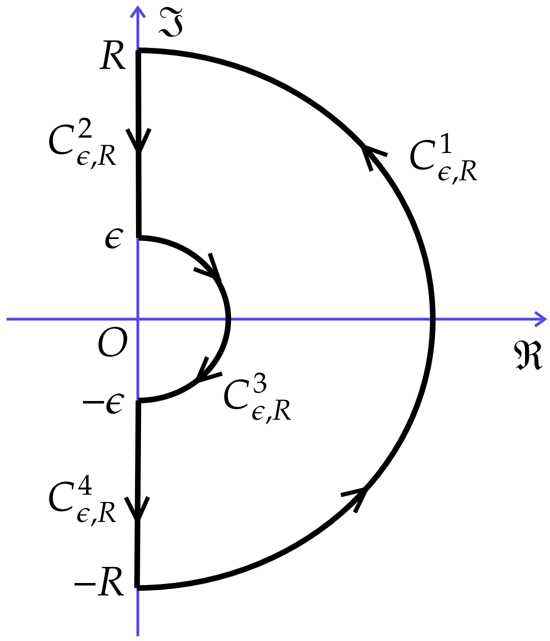


Figure 1: The modified Nyquist contour.

Let Ω be an open subset of $\mathbb{C}$ and a given holomorphic function $f : \Omega \to \mathbb{C}$. Suppose that $\alpha : I = [a, b] \subseteq \mathbb{R} \to \Omega$ is an oriented complex curve that does not pass through any zero point of the function f. From the definition above, we see that the change of the argument of f along the curve α equals the change of the function $\gamma : I \to \mathbb{C}^*$ given by $\gamma(t) = f(\alpha(t))$ as t moves from a to b. More precisely, we have

$$\Delta \arg f(s)\Big|_\alpha = \Delta \arg \gamma(t)\Big|_I. \tag{37}$$

In particular, for the case when $\alpha : [0, \infty) \to \mathbb{C}$ given by $\alpha(t) = it$, $t \geq 0$ and $f(i\omega) \neq 0$ for all $\omega \in [0, \infty)$, we obtain

$$\Delta \arg f(s)\Big|_\alpha = \Delta \arg f(i\omega)\Big|_0^\infty. \tag{38}$$

As shown in (27), the characteristic function Q of the system (14) has the form

$$Q(s) = \sum_{j=0}^{N} \left(\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s} \right) s^{\beta_j}.$$

From Proposition 3.1 and Remarks 3.2, 3.3, 3.3, to establish an asymptotic stability criterion for the system (14), it is natural and necessary to add the following assumptions:

$$\sum_{k=1}^{d} |c_{kN}| < 1, \; \sum_{k=0}^{d} c_{k0} > 0. \tag{39}$$

For any $\epsilon \in (0, R)$, we define the modified Nyquist curve $C_{\epsilon,R} = C^1_{\epsilon,R} \cup C^2_{\epsilon,R} \cup C^3_{\epsilon,R} \cup C^4_{\epsilon,R}$

with

$$
\begin{aligned}
C^1_{\epsilon,R} &:= \left\{ s = Re^{i\varphi} : \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \right\}, \\
C^2_{\epsilon,R} &:= \left\{ s = i\omega : \omega \in [R, \epsilon] \right\}, \\
C^3_{\epsilon,R} &:= \left\{ s = \epsilon e^{i\varphi} : \varphi \in [\frac{\pi}{2}, -\frac{\pi}{2}] \right\}, \\
C^4_{\epsilon,R} &:= \left\{ s = i\omega : \omega \in [-\epsilon, -R] \right\},
\end{aligned}
\tag{40}
$$

Let $\Omega_{\epsilon,R}$ be the bounded domain surrounded by the curve $C_{\epsilon,R}$.

Our main contribution to the current work is the result below.

Theorem 3.8. (Modified Mikhailov stability criterion) Consider the characteristic function Q as in (27). Assume that the condition (39) holds and $Q(i\omega) \neq 0$ for all $\omega \in (0, \infty)$. Then, all zero points of Q lie in the open left half of the complex plane if and only if

$$
\beta_N \frac{\pi}{2} - \Theta \leq \Delta \arg Q(i\omega)\big|_0^\infty \leq \beta_N \frac{\pi}{2} + \Theta, \tag{41}
$$

where $\beta_N = \alpha_1 + \cdots + \alpha_d$, $\Theta = \arcsin \left(\sum_{k=1}^d |c_{kN}| \right)$.

Proof. First, we see that

$$
\begin{aligned}
Q(s) &= \left(c_{0N} + \sum_{k=1}^d c_{kN} e^{-k\tau s} + \sum_{j=0}^{N-1} \left(\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s} \right) s^{\beta_j - \beta_N} \right) s^{\beta_N} \\
&= (1 + H(s) + K(s)) s^{\beta_N},
\end{aligned}
\tag{42}
$$

where $H(s) = \sum_{k=1}^d c_{kN} e^{-k\tau s}$, and $K(s) = \sum_{j=0}^{N-1} \left(\sum_{k=0}^{p_j} c_{kj} e^{-\tau_{kj} s} \right) s^{\beta_j - \beta_N}$. Since $\beta_j < \beta_N$ for all $j = 1, \ldots, N-1$, and $\tau_{kj} \geq 0$ for all $k = 0, 1, \ldots, p_j$, $j = 0, 1, \ldots, N-1$, we have

$$
\lim_{|s| \to \infty,\, s \in \mathbb{C}_+} |K(s)| = 0. \tag{43}
$$

On the other hand,

$$
|H(s)| \leq \sum_{k=1}^d |c_{kN}| e^{-k\tau \Re s} \leq \sum_{k=1}^d |c_{kN}| < 1, \ \ \forall s \in \mathbb{C}_+. \tag{44}
$$

By the condition (39), we see that $Q(0) > 0$. Moreover, we can find $\epsilon_1 > 0$ such that $Q(s) \neq 0$ for all $s \in \mathbb{C}$, $|s| \leq \epsilon_1$. From the fact that $\lim_{|s| \to \infty, s \in \mathbb{C}_+} |Q(s)| = \infty$, there is an $R_1 > \epsilon_1$ so that its every zero point located in the open right half of the complex plane is in the domain $\{z \in \mathbb{C}_{\geq 0} : |z| \leq R_1\}$. Since $M = \{z \in \mathbb{C}_{\geq 0} : \epsilon_1 \leq |s| \leq R_1\}$ is a compact set in $\mathbb{C}$ and Q is analytic in M, Q has only at most r zero points in M. It follows that Q only has a finite number of zero points on the open right half of the complex plane, and these points must belong to the domain M.

Due to $\lim_{|s| \to \infty,\, s \in \mathbb{C}_+} |K(s)| = 0$, we can choose ϵ small enough and R_ϵ large enough so that the contour C_{ϵ,R_ϵ} defined as (40) does not hit the zero points of Q. Furthermore, the following facts are verified:

- $0 < \epsilon < \min\{\epsilon_1, 1 - \sum_{k=1}^{d} |c_{kN}|\}$;
- $|K(s)| < \epsilon,\ \forall s \in \mathbb{C}_+,\ |s| \geq R_\epsilon$;
- $R_\epsilon > R_1$;
- $R_\epsilon \to \infty$ as $\epsilon \to 0$.

Since Q is analytic on $\bar{\Omega}_{\epsilon,R_\epsilon}$ and there is no zero point on C_{ϵ,R_ϵ}, according to (36), (37) and Theorem 3.7, we have

$$\Delta \arg Q(s)\big|_{C_{\epsilon,R_\epsilon}} = 2\pi(Z - P) = 2r\pi, \tag{45}$$

which implies that

$$\Delta \arg Q(s)\big|_{C^1_{\epsilon,R_\epsilon}} + \Delta \arg Q(s)\big|_{C^2_{\epsilon,R_\epsilon}} + \Delta \arg Q(s)\big|_{C^3_{\epsilon,R_\epsilon}} + \Delta \arg Q(s)\big|_{C^4_{\epsilon,R_\epsilon}} = 2r\pi. \tag{46}$$

Due to the fact that $\overline{Q(s)} = Q(\overline{s})$ for all $s \in \mathbb{C}$, we see that

$$\Delta \arg Q(s)\big|_{C^2_{\epsilon,R_\epsilon}} = \Delta \arg Q(s)\big|_{C^4_{\epsilon,R_\epsilon}}, \tag{47}$$

this means that

$$\Delta \arg Q(s)\big|_{C^2_{\epsilon,R_\epsilon}} + \Delta \arg Q(s)\big|_{C^4_{\epsilon,R_\epsilon}} = 2\Delta \arg Q(s)\big|_{C^2_{\epsilon,R_\epsilon}} = -2\Delta \arg Q(i\omega)\big|_{\epsilon}^{R_\epsilon}. \tag{48}$$

Notice that, for ϵ chosen as above, $(1 + H(s) + K(s)$ lies entirely within the circle with center 1 and radius $\sum_{k=1}^{d} |c_{kN}| + \epsilon$ for all $s \in C^1_{\epsilon,R_\epsilon}$. On the other hand, this circle does not surround the origin, so the curve $\{1 + H(s) + K(s) : s \in C^1_{\epsilon,R_\epsilon}\}$ also does not surround the origin. Thus,

$$\begin{aligned}\Delta \arg(1{+}H(s) + G(s))\big|_{C^1_{\epsilon,R_\epsilon}} &= \arg(1 + H(Re^{i\frac{\pi}{2}}) + G(Re^{i\frac{\pi}{2}})) \\ &- \arg(1 + H(Re^{-i\frac{\pi}{2}}) + G(Re^{-i\frac{\pi}{2}})) = 2\arg(1 + H(Re^{i\frac{\pi}{2}}) + G(Re^{i\frac{\pi}{2}})).\end{aligned} \tag{49}$$

In addition, for all $s \in C^1_{\epsilon,R_\epsilon}$, $1 + H(s) + K(s)$ is in the cone with the vertex being the origin and two edges being tangents going from the origin to the circle with center 1 of radius $\sum_{k=1}^{d} |c_{kN}| + \epsilon$, the following estimate is true

$$-\arcsin\left(\sum_{k=1}^{d} |c_{kN}| + \epsilon\right) \leq \arg\left(1 + H(s) + K(s)\right) \leq \arcsin\left(\sum_{k=1}^{d} |c_{kN}| + \epsilon\right). \tag{50}$$

By combining (42), (49) and (50), it deduces that

$$\begin{aligned}\Delta \arg s^{\beta_N}\big|_{C^1_{\epsilon,R_\epsilon}} - 2\arcsin\left(\sum_{k=1}^{d} |c_{kN}| + \epsilon\right) &\leq \Delta \arg Q(s)\big|_{C^1_{\epsilon,R_\epsilon}} \\ &\leq \Delta \arg s^{\beta_N}\big|_{C^1_{\epsilon,R_\epsilon}} + 2\arcsin\left(\sum_{k=1}^{d} |c_{kN}| + \epsilon\right).\end{aligned} \tag{51}$$

From this, we obtain

$$\beta_N\pi - 2\arcsin\left(\sum_{k=1}^{d}|c_{kN}| + \epsilon\right) \le \Delta \arg Q(s)\big|_{C^1_{\epsilon,R_\epsilon}}$$
$$\le \beta_N\pi + 2\arcsin\left(\sum_{k=1}^{d}|c_{kN}| + \epsilon\right), \quad (52)$$

where $0 < \epsilon < 1 - \sum_{k=1}^{d}|a_{kN}|$. Now, using the fact that $Q(0) = \sum_{k=0}^{d} c_{k0} > 0$ and that Q, $\Re Q$ are continuous at the origin, there exists a constant $\epsilon_0 \in (0, 1 - \sum_{k=1}^{d}|c_{kN}|)$ small enough to satisfy the estimates below.

(a) $|Q(s) - Q(0)| \le \frac{1}{2}\sum_{k=0}^{d} c_{k0}$ for all $|s| \le \epsilon_0$.

(b) $\Re Q(s) \ge \frac{1}{2}\sum_{k=0}^{d} c_{k0} > 0$ for all $|s| \le \epsilon_0$.

Following from (a), for all $\epsilon \in (0, \epsilon_0)$, $Q(C^3_{\epsilon,R_\epsilon})$ lies completely within the circle with center $\sum_{k=0}^{d} c_{0k}$, radius $\frac{1}{2}\sum_{k=0}^{d} c_{0k}$. Therefore, $Q(C^3_{\epsilon,R})$ does not surround the origin for all $\epsilon \in (0, \epsilon_0)$. Hence,

$$\Delta \arg Q(s)\big|_{C^3_{\epsilon,R}} = \arg Q(\epsilon e^{i\frac{-\pi}{2}}) - \arg Q(\epsilon e^{i\frac{\pi}{2}}) = -2\arg Q(\epsilon e^{i\frac{\pi}{2}}),\ \forall \epsilon \in (0, \epsilon_0). \quad (53)$$

By (b), we have $\Re Q(\epsilon e^{i\frac{\pi}{2}}) > 0$ for all $\epsilon \in (0, \epsilon_0)$. Thus,

$$\arg Q(\epsilon e^{i\frac{\pi}{2}}) = \arctan\left(\frac{\Im Q(\epsilon e^{i\frac{\pi}{2}})}{\Re Q(\epsilon e^{i\frac{\pi}{2}})}\right),\ \forall \epsilon \in (0, \epsilon_0). \quad (54)$$

From (53) and (54), then

$$\Delta \arg Q(s)\big|_{C^3_{\epsilon,R_\epsilon}} = -2\arctan\left(\frac{\Im Q(\epsilon e^{i\frac{\pi}{2}})}{\Re Q(\epsilon e^{i\frac{\pi}{2}})}\right),\ \forall \epsilon \in (0, \epsilon_0),$$

which together with (46), (48), (52) leads to

$$\beta_N\pi - 2\arcsin\left(\sum_{k=1}^{d}|c_{kN}| + \epsilon\right) \le 2\Delta \arg Q(i\omega)\big|_{\epsilon}^{R_\epsilon} + 2\arctan\left(\frac{\Im Q(\epsilon e^{i\frac{\pi}{2}})}{\Re Q(\epsilon e^{i\frac{\pi}{2}})}\right) + 2r\pi$$
$$\le \beta_N\pi + 2\arcsin\left(\sum_{k=1}^{d}|c_{kN}| + \epsilon\right).$$

Let $\epsilon \to 0$ and note that $R_\epsilon \to \infty$ as $\epsilon \to 0$, we get the inequalities

$$\beta_N\frac{\pi}{2} - \arcsin\left(\sum_{k=1}^{d}|c_{kN}|\right) \le \Delta \arg Q(i\omega)\big|_0^\infty + r\pi \le \beta_N\frac{\pi}{2} + \arcsin\left(\sum_{k=1}^{d}|c_{kN}|\right). \quad (55)$$

It is worth noting that $0 \le \sum_{k=1}^{d}|c_{kN}| < 1$ and thus $0 \le \arcsin\left(\sum_{k=1}^{d}|c_{kN}|\right) < \frac{\pi}{2}$. From (55), the desired assertion (41) is satisfied if and only if $r = 0$. The proof is complete. □

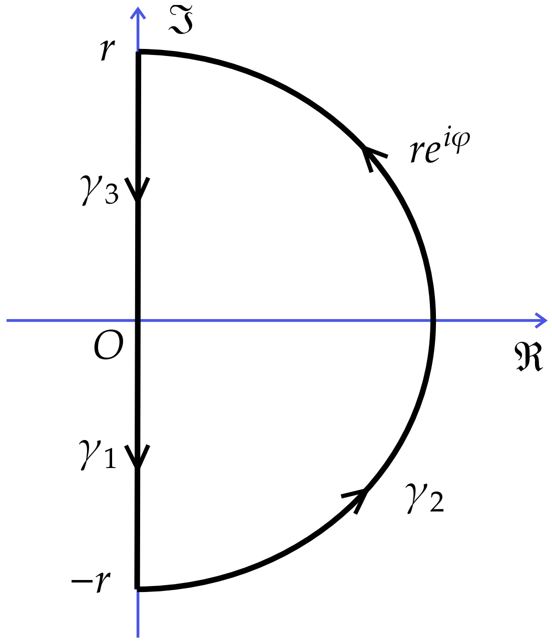


Figure 2: The contour γ.

Remark 3.9. Consider the system (14) when $A = 0$. From the observations above, we have $c_{kN} = 0$ for $k = 1, \ldots, d$. Therefore, $\Theta = 0$ and the condition (41) in Theorem 3.8 becomes

$$\Delta \arg Q(i\omega)\big|_0^\infty = \beta_N \frac{\pi}{2}.$$

Thus, with the added assumption $B_1 = 0$, we get again Theorem 3 in the paper [22].

Remark 3.10. Although the statement of [22, Theorem 3] is correct, the proof of this result seems incomplete. Indeed, because the contour γ (see Figure 2) passes through the origin, the characteristic polynomial p is not analytic on this curve. Therefore, using Cauchy's argument principle as the author did is not legal. To fill the gap, we suggest replacing γ by the contour $C_{\epsilon,R}$ defined in Theorem 3.8 above (see Figure 1).

Remark 3.11. In [7, 8], the authors developed modified Mikhailov criteria to study the asymptotic stability property for fractional differential systems both in the case of delays and without delays. To prove the proposed main results, they applied the transformation $\lambda = s^\alpha$ to the characteristic polynomial f, here α is the order of the basic fractional-order derivative of the system (the orders of other fractional order derivatives appearing in the system are multiples of α). In our opinion, this is probably the reason the approach in the mentioned articles does not apply to non-commensurate fractional differential systems.

3.3 An approach to analysis the stability of the system (14) and simulation examples

We propose a 3-step scheme to check the stability of the system (14) as follows:
Step 1: Calculating the characteristic polynomial Q;
Step 2: Using Theorem 3.8 to check the position of zero points of Q;
Step 3: Based on the Final Value Theorem 2.6 to conclude the stability of the system.

In the above approach, Step 2 is the most difficult to implement. Therefore, to help the reader easily visualize the role and validity of the proposed theoretical results, we give some specific examples and numerical simulations in which we calculate the argument of Mikhailov curves.

Example 3.12. Consider the equation

$$ {}^{C}D^{\hat{\alpha}}_{0+}\left(x(t)+Ax(t-1)\right)=B_0x(t)+B_1x(t-2),\ t>0, \tag{56}$$

where $\hat{\alpha}=(1/2,\frac{\pi}{6},\frac{1}{\sqrt{3}})$,

$$A=\begin{pmatrix}-0.3 & -0.3 & -0.2\\ 0.3 & -0.2 & 0.3\\ -0.2 & -0.1 & 0.2\end{pmatrix},\ B_0=\begin{pmatrix}-5 & -1 & -2\\ -3 & -5 & -4\\ -1 & -2 & -5\end{pmatrix},\ B_1=\begin{pmatrix}1 & 1 & -3\\ 1.5 & -2 & -1\\ -1 & 1.5 & 1\end{pmatrix}.$$

The characteristic polynomial $Q(s)$ of (56) is described explicitly as

$$\begin{aligned}Q(s)=&s^{1.6009}(1-0.3e^{-s}+0.04e^{-2s}+0.053e^{-3s})+s^{1.1009}(5+0.1e^{-s}-1.31e^{-2s}\\ &+0.9e^{-3s}-0.08e^{-4s})+s^{1.0774}(5+0.8e^{-s}+1.86e^{-2s}-0.55e^{-3s}-0.29e^{-4s})\\ &+s^{1.0236}(5-2.9e^{-s}-0.32e^{-2s}+1.15e^{-3s}-0.325e^{-4s})+s^{0.5774}(22+7e^{-s}\\ &+9.5e^{-2s}+5.8e^{-3s}-3.5e^{-4s}+1.05e^{-5s})+s^{0.5236}(23-7.6e^{-s}-15e^{-2s}\\ &+8.55e^{-3s}-2e^{-4s}-2e^{-5s})+s^{0.5}(17-2e^{-s}+9e^{-2s}-4.45e^{-3s}-0.5e^{-4s}\\ &+0.25e^{-5s})+(76+28.5e^{-2s}-70.5e^{-4s}+1.75e^{-6s}).\end{aligned} \tag{57}$$

From (57), we have $N=7$, $\sum_{k=1}^{3}|c_{kN}|=0.393<1$ and $\sum_{k=0}^{3}c_{k0}=76+28.5-70+1.75=$

$36.35 > 0$. Furthermore, $\Theta = \arcsin(\sum_{k=1}^{3} |c_{kN}|) \approx \frac{\pi}{7.7783}$. Let $s = i\omega$, $0 \le \omega < \infty$, then

$$\begin{aligned}Q(i\omega) &= \omega^{1.6009}(-0.8098 + i0.5866)[1 - 0.3(\cos\omega - i\sin\omega) + 0.04(\cos(2\omega) - i\sin(2\omega))\\ &\quad + 0.053(\cos(3\omega) - i\sin(3\omega))] + \omega^{1.1009}(-0.1578 + i0.9875)[5 + 0.1(\cos\omega - i\sin\omega)\\ &\quad - 1.31(\cos(2\omega) - i\sin(2\omega)) + 0.9(\cos(3\omega) - i\sin(3\omega)) - 0.08(\cos(4\omega) - i\sin(4\omega))]\\ &\quad + \omega^{1.0774}(-0.1213 + i0.9926)[5 + 0.8(\cos\omega - i\sin\omega) + 1.86(\cos(2\omega) - i\sin(2\omega))\\ &\quad - 0.55(\cos(3\omega) - i\sin(3\omega)) - 0.29(\cos(4\omega) - i\sin(4\omega))] + \omega^{1.0236}(-0.0371\\ &\quad + i0.9993)[5 - 2.9(\cos\omega - i\sin\omega) - 0.32(\cos(2\omega) - i\sin(2\omega)) + 1.15(\cos(3\omega)\\ &\quad - i\sin(3\omega)) - 0.325(\cos(4\omega) - i\sin(4\omega))] + \omega^{0.5774}(0.6161 + i0.7876)[22 + 7(\cos\omega\\ &\quad - i\sin\omega) + 9.5(\cos(2\omega) - i\sin(2\omega)) + 5.8(\cos(3\omega) - i\sin(3\omega)) - 3.5(\cos(4\omega)\\ &\quad - i\sin(4\omega)) + 1.05(\cos(5\omega) - i\sin(5\omega))] + \omega^{0.5236}(0.6804 + i0.7328)[23 - 7.6(\cos\omega\\ &\quad - i\sin\omega) - 15(\cos(2\omega) - i\sin(2\omega)) + 8.55(\cos(3\omega) - i\sin(3\omega)) - 2(\cos(4\omega)\\ &\quad - i\sin(4\omega)) - 2(\cos(5\omega) - i\sin(5\omega))] + \omega^{0.5}(0.7071 + i0.7071)[17 - 2(\cos\omega - i\sin\omega)\\ &\quad + 9(\cos(2\omega) - i\sin(2\omega)) - 4.45(\cos(3\omega) - i\sin(3\omega)) - 0.5(\cos(4\omega) - i\sin(4\omega))\\ &\quad + 0.25(\cos(5\omega) - i\sin(5\omega))] + [76 + 28.5(\cos(2\omega) - i\sin(2\omega)) - 70.5(\cos(4\omega)\\ &\quad - i\sin(4\omega)) + 1.75(\cos(6\omega) - i\sin(6\omega))].\end{aligned}$$

Set

$$\begin{aligned}h_1(\omega) :&= \Re(Q(i\omega)) = \omega^{1.6009}(-0.8098 + 0.2429\cos\omega - 0.176\sin\omega - 0.0324\cos(2\omega)\\ &\quad + 0.0235\sin(2\omega) - 0.0429\cos(3\omega) + 0.0311\sin(3\omega)) + \omega^{1.1009}(-0.789 - 0.0158\cos\omega\\ &\quad + 0.0988\sin\omega + 0.2067\cos(2\omega) - 1.2936\sin(2\omega) - 0.142\cos(3\omega) + 0.8888\sin(3\omega)\\ &\quad + 0.0126\cos(4\omega) - 0.0790\sin(4\omega)) + \omega^{1.0774}(-0.6065 - 0.097\cos\omega + 0.7941\sin\omega\\ &\quad - 0.2256\cos(2\omega) + 1.8462\sin(2\omega) + 0.0667\cos(3\omega) - 0.5459\sin(3\omega)\\ &\quad + 0.0352\cos(4\omega) - 0.2879\sin(4\omega)) + \omega^{1.0236}(-0.1855 + 0.1076\cos\omega - 2.898\sin\omega\\ &\quad + 0.0119\cos(2\omega) - 0.3198\sin(2\omega) - 0.0427\cos(3\omega) + 1.1492\sin(3\omega)\\ &\quad + 0.0121\cos(4\omega) - 0.3248\sin(4\omega)) + \omega^{0.5774}(13.5542 + 4.3127\cos\omega + 5.5132\sin\omega\\ &\quad + 5.8529\cos(2\omega) + 7.4822\sin(2\omega) + 3.5734\cos(3\omega) + 4.5681\sin(3\omega)\\ &\quad - 2.1564\cos(4\omega) - 2.7566\sin(4\omega) + 0.6469\cos(5\omega) + 0.827\sin(5\omega)) + \omega^{0.5236}\times\\ &\quad \times(15.6492 - 5.171\cos\omega - 5.5693\sin\omega - 10.206\cos(2\omega) - 10.992\sin(2\omega)\\ &\quad + 5.8174\cos(3\omega) + 6.2654\sin(3\omega) - 1.3608\cos(4\omega) - 1.4656\sin(4\omega)\\ &\quad - 1.3608\cos(5\omega) - 1.4656\sin(5\omega)) + \omega^{0.5}(12.0207 - 1.4142\cos\omega - 1.4142\sin\omega\\ &\quad + 6.3639\cos(2\omega) + 6.3639\sin(2\omega) - 3.1466\cos(3\omega) - 3.1466\sin(3\omega)\\ &\quad - 0.3535\cos(4\omega) - 0.3535\sin(4\omega) + 0.1768\cos(5\omega) + 0.1768\sin(5\omega)) + (76\\ &\quad + 28.5\cos(2\omega) - 70.5\cos(4\omega) + 1.75\cos(6\omega)),\end{aligned}$$

and

$$
\begin{aligned}
h_2(\omega) : = \Im(Q(i\omega)) = \omega^{1.6009}&(0.5866 - 0.176\cos\omega - 0.2429\sin\omega + 0.0235\cos(2\omega)\\
&+ 0.0324\sin(2\omega) + 0.0311\cos(3\omega) + 0.0429\sin(3\omega)) + \omega^{1.1009}(4.9375 + 0.0988\cos\omega\\
&+ 0.0158\sin\omega - 1.2936\cos(2\omega) - 0.2067\sin(2\omega) + 0.8888\cos(3\omega) + 0.1420\sin(3\omega)\\
&- 0.0790\cos(4\omega) - 0.0126\sin(4\omega)) + \omega^{1.0774}(4.9630 + 0.7941\cos\omega + 0.097\sin\omega\\
&+ 1.8462\cos(2\omega) + 0.2256\sin(2\omega) - 0.5459\cos(3\omega) - 0.0667\sin(3\omega)\\
&- 0.2879\cos(4\omega) - 0.0352\sin(4\omega)) + \omega^{1.0236}(4.9965 - 2.898\cos\omega - 0.1076\sin\omega\\
&- 0.3198\cos(2\omega) - 0.0119\sin(2\omega) + 1.1492\cos(3\omega) + 0.0427\sin(3\omega)\\
&- 0.3248\cos(4\omega) - 0.0121\sin(4\omega)) + \omega^{0.5774}(17.3272 + 5.5132\cos\omega - 4.3127\sin\omega\\
&+ 7.4822\cos(2\omega) - 5.8529\sin(2\omega) + 4.5681\cos(3\omega) - 3.5734\sin(3\omega)\\
&- 2.7566\cos(4\omega) + 2.1564\sin(4\omega) + 0.8270\cos(5\omega) - 0.6469\sin(5\omega)) + \omega^{0.5236}\times\\
&\times(16.8544 - 5.5693\cos\omega + 5.1710\sin\omega - 10.992\cos(2\omega) + 10.206\sin(2\omega)\\
&+ 6.2654\cos(3\omega) - 5.8174\sin(3\omega) - 1.4656\cos(4\omega) + 1.3608\sin(4\omega)\\
&- 1.4656\cos(5\omega) + 1.3608\sin(5\omega)) + \omega^{0.5}(12.0207 - 1.4142\cos\omega + 1.4142\sin\omega\\
&+ 6.3639\cos(2\omega) - 6.3639\sin(2\omega) - 3.1466\cos(3\omega) + 3.1466\sin(3\omega)\\
&- 0.3535\cos(4\omega) + 0.3535\sin(4\omega) + 0.1768\cos(5\omega) - 0.1768\sin(5\omega))\\
&+ (-28.5\sin(2\omega) + 70.5\sin(4\omega) - 1.75\sin(6\omega)).
\end{aligned}
$$

Using the bisection method, we find the approximating solutions of the equation $h_2(\omega) = 0$ in the interval $(0, \infty)$ within the accuracy of 10^{-4} as

$$\omega_1 \approx 0.8552, \qquad \omega_2 \approx 1.3653.$$

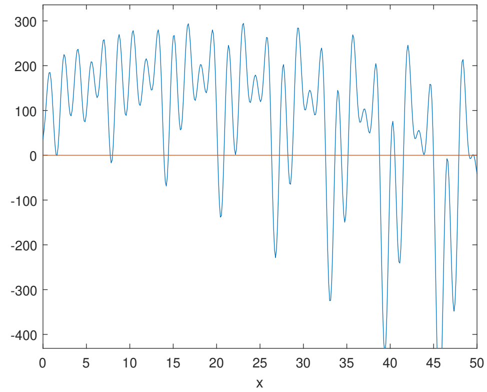
The approximating solutions of the equation $h_1(\omega) = 0$ in the interval $(0, \infty)$ within the accuracy of 10^{-4} are

$\omega_3 \approx 1.5205$,	$\omega_4 \approx 1.6258$,	$\omega_5 \approx 7.7438$,	$\omega_6 \approx 8.0205$,
$\omega_7 \approx 13.9262$,	$\omega_8 \approx 14.4442$,	$\omega_9 \approx 20.1282$,	$\omega_{10} \approx 20.8463$,
$\omega_{11} \approx 26.3392$,	$\omega_{12} \approx 27.2404$,	$\omega_{13} \approx 28.2071$,	$\omega_{14} \approx 28.7303$,
$\omega_{15} \approx 32.5544$,	$\omega_{16} \approx 33.6355$,	$\omega_{17} \approx 34.3693$,	$\omega_{18} \approx 35.144$,
$\omega_{19} \approx 38.7704$,	$\omega_{20} \approx 40.0454$,	$\omega_{21} \approx 40.5307$,	$\omega_{22} \approx 41.5325$,
$\omega_{23} \approx 44.9837$,	$\omega_{24} \approx 47.9089$,	$\omega_{25} \approx 49.083$,	$\omega_{26} \approx 49.4637$,
$\omega_{27} \approx 50.489$,	$\omega_{28} \approx 51.1893$,	$\omega_{29} \approx 54.2795$,	$\omega_{30} \approx 55.1944$,
$\omega_{31} \approx 56.9355$,	$\omega_{32} \approx 57.3742$,	$\omega_{33} \approx 60.649$,	$\omega_{34} \approx 61.3807$,
$\omega_{35} \approx 67.023$,	$\omega_{36} \approx 67.5768$,	$\omega_{37} \approx 73.4152$,	$\omega_{38} \approx 73.76$.

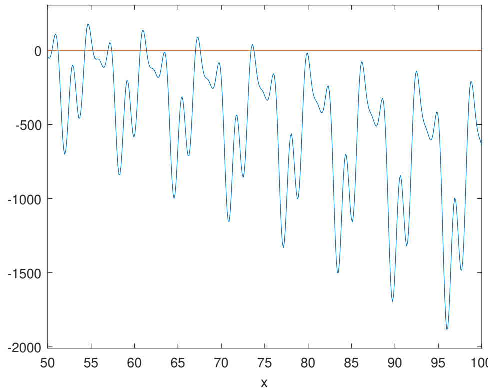
From this, we write

$$\Delta argQ(i\omega)\Big|_0^\infty = \Delta argQ(i\omega)\Big|_0^{\omega_1} + \sum_{j=1}^{37} \Delta argQ(i\omega)\Big|_{\omega_j}^{\omega_{j+1}} + \Delta argQ(i\omega)\Big|_{\omega_{38}}^\infty. \qquad (58)$$

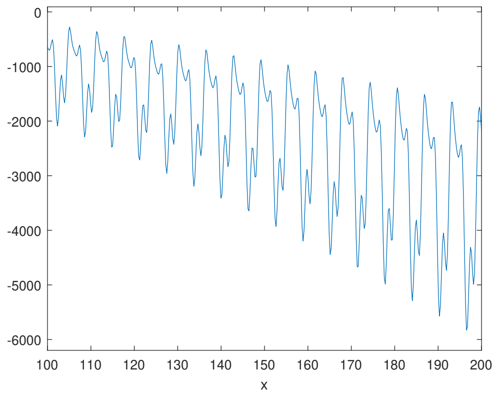
On the interval $(0, \omega_1)$, it is easy to check that $h_1(\omega) > 0$ and $h_2(\omega) > 0$. Hence, $Q(i\omega)$ starts from the point $(35.75, 0)$, moves in the open part of the first quadrant, and returns



(a) The graph on the interval from 0 to 50.

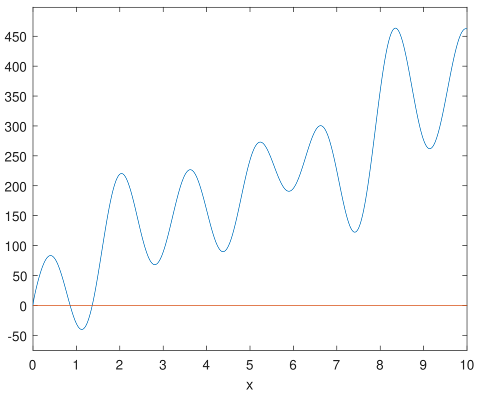


(b) The graph on the interval from 50 to 100.

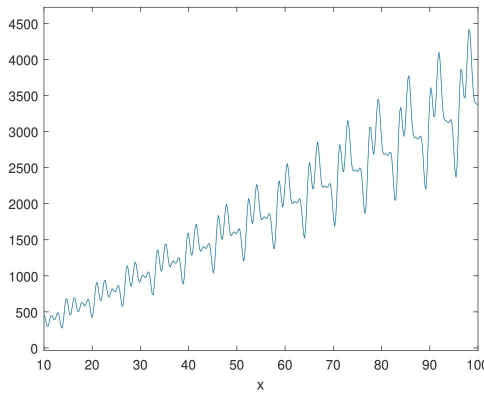


(c) The graph on the interval from 100 to 200.

Figure 3: The graph of $h_1(\omega)$ on the interval $[0, 200]$.



(a) The graph on the interval from 0 to 10.



(b) The graph on the interval from 10 to 100.

Figure 4: The graph of $h_2(\omega)$ on the interval $[0, 100]$.

to intersect the real axis at $(180.4078, 0)$ as the variable ω increases from 0 to ω_1. This implies that $\Delta argQ(i\omega)\big|_0^{\omega_1} = 0$. Similarly, we have $\Delta argQ(i\omega)\big|_{\omega_1}^{\omega_2} = 0$. On (ω_2, ω_3), then $h_1(\omega) > 0$, $h_2(\omega) > 0$, and thus $Q(i\omega)$ starts from $(28.2422, 0)$ then moves in the open part of the first quadrant and intersect the imaginary axis at $(0, 61.0713)$ as ω increases from ω_2 to ω_3. Hence, $\Delta argQ(i\omega)\big|_{\omega_2}^{\omega_3} = \frac{\pi}{2}$. On (ω_3, ω_4), due to $h_1(\omega) < 0$ and $h_2(\omega) > 0$, $Q(i\omega)$ moves in the open part of the second quadrant and returns to intersect the imaginary axis at $(0, 108.7798)$ as the variable ω increases from ω_3 to ω_4. This leads to that $\Delta argQ(i\omega)\big|_{\omega_3}^{\omega_4} = 0$. Using the same arguments, the assertion $\Delta argQ(i\omega)\big|_{\omega_j}^{\omega_{j+1}} = 0$, $j = 4, \cdots, 37$, is also true. From these facts, we receive

$$\Delta argQ(i\omega)\big|_0^{\omega_1} + \sum_{j=1}^{37} \Delta argQ(i\omega)\big|_{\omega_j}^{\omega_{j+1}} = \frac{\pi}{2}. \tag{59}$$

We now focus on the case $\omega \in (\omega_{38}, \infty)$. Noting that $h_1(\omega) < 0$ and $h_2(\omega) > 0$ for all $\omega > \omega_{38}$. Thus, $Q(i\omega)$ moves from $(0, 2.5815 \times 10^3)$ to the open part of the second quadrant as ω increases from ω_{38} to $+\infty$. On the other hand,

$$\begin{aligned}-1.1399 < -0.8098 &+ 0.2429\cos\omega - 0.176\sin\omega - 0.0324\cos(2\omega)\\ &+ 0.0235\sin(2\omega) - 0.0429\cos(3\omega) + 0.0311\sin(3\omega) < -0.5861,\end{aligned}$$

and

$$\begin{aligned}0.2701 < 0.5866 &- 0.176\cos\omega - 0.2429\sin\omega + 0.0235\cos(2\omega)\\ &+ 0.0324\sin(2\omega) + 0.0311\cos(3\omega) + 0.0429\sin(3\omega) < 0.8315.\end{aligned}$$

for all $\omega > \omega_{38}$. Thus, for $\omega > \omega_{38}$, then

$$-1.4186 < \frac{h_4(\omega)}{h_3(\omega)} < -0.2369,$$

where $h_3(\omega) = -0.8098 + 0.2429\cos\omega - 0.176\sin\omega - 0.0324\cos(2\omega) + 0.0235\sin(2\omega) - 0.0429\cos(3\omega) + 0.0311\sin(3\omega)$, $h_4(\omega) = 0.5866 - 0.176\cos\omega - 0.2429\sin\omega + 0.0235\cos(2\omega) + 0.0324\sin(2\omega) + 0.0311\cos(3\omega) + 0.0429\sin(3\omega)$. Let $\omega > 0$ be large enough, with the help of the obtained calculations, the following estimate holds

$$\pi - \frac{\pi}{3.2835} < \arg Q(i\omega) < \pi - \frac{\pi}{13.5058}.$$

This reduces that

$$\frac{\pi}{5.1165} < \Delta \arg Q(i\omega)\big|_{\omega_{38}}^{\infty} < \frac{\pi}{2.3477},$$

which together with (58) shows that

$$\frac{\pi}{1.438} < \Delta \arg Q(i\omega)\big|_0^{\infty} < \frac{\pi}{1.08}.$$

As shown above,

$$(\alpha_1 + \alpha_2 + \alpha_3)\frac{\pi}{2} - \Theta \approx \frac{\pi}{1.4883},$$

and

$$(\alpha_1 + \alpha_2 + \alpha_3)\frac{\pi}{2} + \Theta \approx \frac{\pi}{1.0764},$$

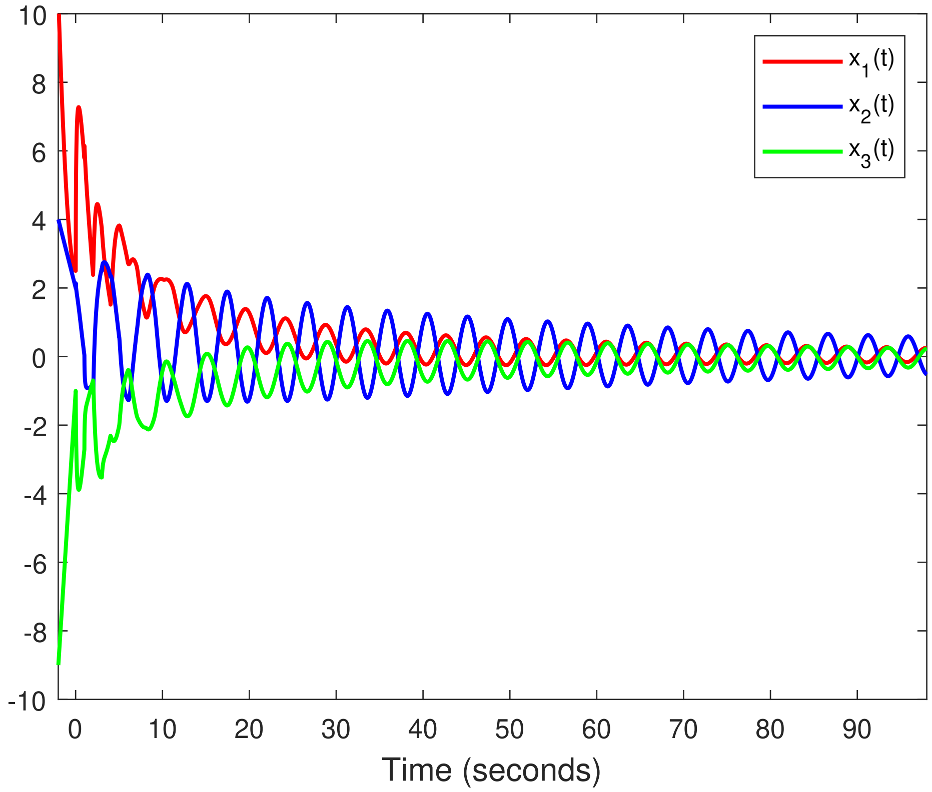


Figure 5: Orbits of the solution of the system (56) with the initial condition $x(t) = (2t^2 + 2.5, -t + 2, 4t - 1)^{\mathrm{T}}$ on the interval $[-2, 0]$.

by combining the observations above, we obtain

$$(\alpha_1 + \alpha_2 + \alpha_3)\frac{\pi}{2} - \Theta < \Delta \arg Q(i\omega)\big|_0^\infty < (\alpha_1 + \alpha_2 + \alpha_3)\frac{\pi}{2} + \Theta.$$

Following from Theorem 3.8 and Theorem 2.6, the system (56) is asymptotically stable. In Figure 5, we depict the asymptotic behavior of its solution with the initial condition $x(t) = (2t^2 + 2.5, -t + 2, 4t - 1)^{\mathrm{T}}$, $\forall t \in [-2, 0]$.

Example 3.13. Consider the system

$$^{C}D^{\hat{\alpha}}_{0^+}\left(x(t) + Ax(t-1)\right) = B_0 x(t) + B_1 x(t-2),\ t > 0, \tag{60}$$

where $\hat{\alpha} = (3/4, \frac{\pi}{4}, \frac{2}{\sqrt{5}})$, and

$$A = \begin{pmatrix} -0.3 & -0.3 & -0.2 \\ 0.3 & -0.2 & 0.3 \\ -0.2 & -0.1 & 0.2 \end{pmatrix},\ B_0 = \begin{pmatrix} -5 & -1 & -2 \\ -3 & -5 & -4 \\ -1 & -2 & -5 \end{pmatrix},\ B_1 = \begin{pmatrix} 1 & 1 & -3 \\ 1.5 & -2 & -1 \\ -1 & 1.5 & 1 \end{pmatrix}.$$

Then, its characteristic polynomial is

$$\begin{aligned}Q(s) &= s^{2.4298}(1 - 0.3e^{-s} + 0.04e^{-2s} + 0.053e^{-3s}) + s^{1.6798}(5 + 0.1e^{-s} - 1.31e^{-2s} \\ &+ 0.9e^{-3s} - 0.08e^{-4s}) + s^{1.6444}(5 + 0.8e^{-s} + 1.86e^{-2s} - 0.55e^{-3s} - 0.29e^{-4s}) \\ &+ s^{1.5354}(5 - 2.9e^{-s} - 0.32e^{-2s} + 1.15e^{-3s} - 0.325e^{-4s}) + s^{0.8944}(22 + 7e^{-s} \\ &+ 9.5e^{-2s} + 5.8e^{-3s} - 3.5e^{-4s} + 1.05e^{-5s}) + s^{0.7854}(23 - 7.6e^{-s} - 15e^{-2s} \\ &+ 8.55e^{-3s} - 2e^{-4s} - 2e^{-5s}) + s^{0.75}(17 - 2e^{-s} + 9e^{-2s} - 4.45e^{-3s} - 0.5e^{-4s} \\ &+ 0.25e^{-5s}) + (76 + 28.5e^{-2s} - 70.5e^{-4s} + 1.75e^{-6s}).\end{aligned}$$

Thus, in this case, we have $N = 7$, $\sum_{k=1}^{3} |c_{kN}| = 0.393 < 1$ and $\sum_{k=0}^{3} c_{k0} = 76 + 28.5 - 70.5 + 1.75 = 35.75 > 0$. Moreover, $\Theta = \arcsin(\sum_{k=1}^{3} |c_{kN}|) \approx \frac{\pi}{7.7783}$. Take $s = i\omega$, $0 \le \omega < \infty$, we see that

$$\begin{aligned}Q(i\omega) &= \omega^{2.4298}(-0.7806 - i0.625)[1 - 0.3(\cos\omega - i\sin\omega) + 0.04(\cos(2\omega) - i\sin(2\omega)) \\ &+ 0.053(\cos(3\omega) - i\sin(3\omega))] + \omega^{1.6798}(-0.8762 + i0.428)[5 + 0.1(\cos\omega - i\sin\omega) \\ &- 1.31(\cos(2\omega) - i\sin(2\omega)) + 0.9(\cos(3\omega) - i\sin(3\omega)) - 0.08(\cos(4\omega) - i\sin(4\omega))] \\ &+ \omega^{1.6444}(-0.848 + i0.53)[5 + 0.8(\cos\omega - i\sin\omega) + 1.86(\cos(2\omega) - i\sin(2\omega)) \\ &- 0.55(\cos(3\omega) - i\sin(3\omega)) - 0.29(\cos(4\omega) - i\sin(4\omega))] + \omega^{1.5354}(-0.7453 \\ &+ i0.6667)[5 - 2.9(\cos\omega - i\sin\omega) - 0.32(\cos(2\omega) - i\sin(2\omega)) + 1.15(\cos(3\omega) \\ &- i\sin(3\omega)) - 0.325(\cos(4\omega) - i\sin(4\omega))] + \omega^{0.8944}(0.1651 + i0.9863)[22 + 7(\cos\omega \\ &- i\sin\omega) + 9.5(\cos(2\omega) - i\sin(2\omega)) + 5.8(\cos(3\omega) - i\sin(3\omega)) - 3.5(\cos(4\omega) \\ &- i\sin(4\omega)) + 1.05(\cos(5\omega) - i\sin(5\omega))] + \omega^{0.7854}(0.3307 + i0.9437)[23 - 7.6(\cos\omega \\ &- i\sin\omega) - 15(\cos(2\omega) - i\sin(2\omega)) + 8.55(\cos(3\omega) - i\sin(3\omega)) - 2(\cos(4\omega) \\ &- i\sin(4\omega)) - 2(\cos(5\omega) - i\sin(5\omega))] + \omega^{0.75}(0.3827 + i0.9239)[17 - 2(\cos\omega - i\sin\omega) \\ &+ 9(\cos(2\omega) - i\sin(2\omega)) - 4.45(\cos(3\omega) - i\sin(3\omega)) - 0.5(\cos(4\omega) - i\sin(4\omega)) \\ &+ 0.25(\cos(5\omega) - i\sin(5\omega))] + [76 + 28.5(\cos(2\omega) - i\sin(2\omega)) - 70.5(\cos(4\omega) \\ &- i\sin(4\omega)) + 1.75(\cos(6\omega) - i\sin(6\omega))].\end{aligned}$$

Put

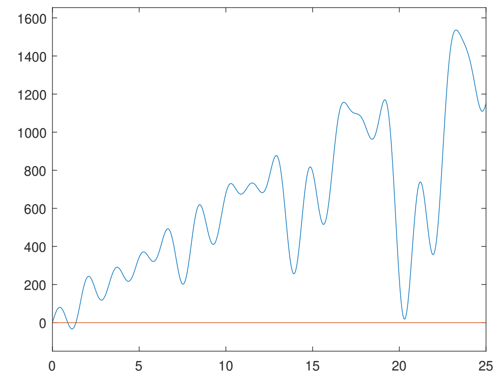
$$\begin{aligned} h_1(\omega) :&= \Re Q(i\omega) = \omega^{2.4298}(-0.7806 + 0.2342\cos\omega + 0.1875\sin\omega - 0.0312\cos(2\omega) \\ &- 0.025\sin(2\omega) - 0.0414\cos(3\omega) - 0.0331\sin(3\omega)) + \omega^{1.6798}(-4.381 \\ &- 0.0876\cos\omega + 0.0482\sin\omega + 1.1478\cos(2\omega) - 0.6314\sin(2\omega) - 0.7886\cos(3\omega) \\ &+ 0.4338\sin(3\omega) + 0.07\cos(4\omega) - 0.0386\sin(4\omega)) + \omega^{1.6444}(-4.24 - 0.6784\cos\omega \\ &+ 0.424\sin\omega - 1.5773\cos(2\omega) + 0.9858\sin(2\omega) + 0.4664\cos(3\omega) - 0.2915\sin(3\omega) \\ &+ 0.2459\cos(4\omega) - 0.1537\sin(4\omega)) + \omega^{1.5354}(-3.7265 + 2.1614\cos\omega - 1.9334\sin\omega \\ &+ 0.2385\cos(2\omega) - 0.2133\sin(2\omega) - 0.8571\cos(3\omega) + 0.7667\sin(3\omega) \\ &+ 0.2422\cos(4\omega) + 0.2167\sin(4\omega)) + \omega^{0.8944}(3.6322 + 1.1557\cos\omega + 6.9041\sin\omega \\ &+ 1.5685\cos(2\omega) + 9.3699\sin(2\omega) + 0.9576\cos(3\omega) + 5.7205\sin(3\omega) - 0.5779\cos(4\omega) \\ &- 3.4521\sin(4\omega) + 0.1734\cos(5\omega) + 1.0356\sin(5\omega)) + \omega^{0.7854}(7.6061 - 2.5133\cos\omega \\ &- 7.1721\sin\omega - 4.9605\cos(2\omega) - 14.1555\sin(2\omega) + 2.8275\cos(3\omega) + 8.0686\sin(3\omega) \\ &- 0.6614\cos(4\omega) - 1.8874\sin(4\omega) - 0.6614\cos(5\omega) - 1.8874\sin(5\omega)) + \omega^{0.75}\times \\ &\times (6.5059 - 0.7654\cos\omega - 1.8478\sin\omega + 3.4443\cos(2\omega) + 8.3151\sin(2\omega) \\ &- 1.703\cos(3\omega) - 4.1114\sin(3\omega) - 0.1914\cos(4\omega) - 0.462\sin(4\omega) + 0.0957\times \\ &\times \cos(5\omega) + 0.231\sin(5\omega)) + 76 + 28.5\cos(2\omega) - 70.5\cos(4\omega) + 1.75\cos(6\omega), \end{aligned}$$

and

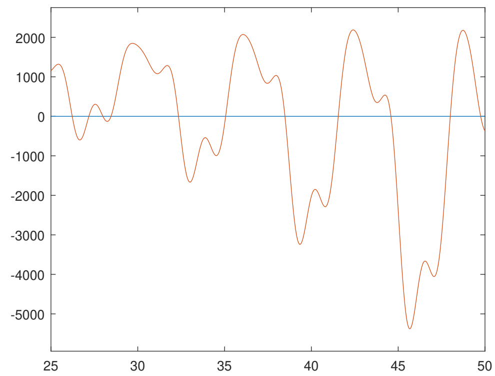
$$\begin{aligned} h_2(\omega) :&= \Im Q(i\omega) = \omega^{2.4298}(-0.625 + 0.1875\cos\omega - 0.2342\sin\omega - 0.025\cos(2\omega) \\ &+ 0.0312\sin(2\omega) - 0.0331\cos(3\omega) + 0.0414\sin(3\omega)) + \omega^{1.6798}(2.41 \\ &+ 0.0482\cos\omega + 0.0876\sin\omega - 0.6314\cos(2\omega) - 1.1478\sin(2\omega) + 0.4338\cos(3\omega) \\ &+ 0.7886\sin(3\omega) - 0.0386\cos(4\omega) - 0.07\sin(4\omega)) + \omega^{1.6444}(2.65 + 0.424\cos\omega \\ &+ 0.6784\sin\omega + 0.9858\cos(2\omega) + 1.5773\sin(2\omega) - 0.2915\cos(3\omega) - 0.4664\sin(3\omega) \\ &- 0.1537\cos(4\omega) - 0.2459\sin(4\omega)) + \omega^{1.5354}(3.3335 - 1.9334\cos\omega - 2.1614\sin\omega \\ &- 0.2133\cos(2\omega) - 0.2385\sin(2\omega) + 0.7667\cos(3\omega) + 0.8571\sin(3\omega) \\ &- 0.2167\cos(4\omega) + 0.2422\sin(4\omega)) + \omega^{0.8944}(21.6986 + 6.9041\cos\omega - 1.1557\sin\omega \\ &+ 9.3699\cos(2\omega) - 1.5685\sin(2\omega) + 5.7205\cos(3\omega) - 0.9576\sin(3\omega) - 3.4521\cos(4\omega) \\ &+ 0.5779\sin(4\omega) + 1.0356\cos(5\omega) - 0.1734\sin(5\omega)) + \omega^{0.7854}(21.7051 - 7.1721\cos\omega \\ &+ 2.5133\sin\omega - 14.1555\cos(2\omega) + 4.9605\sin(2\omega) + 8.0686\cos(3\omega) - 2.8275\sin(3\omega) \\ &- 1.8874\cos(4\omega) + 0.6614\sin(4\omega) - 1.8874\cos(5\omega) + 0.6614\sin(5\omega)) + \omega^{0.75}\times \\ &\times (15.7063 - 1.8478\cos\omega + 0.7654\sin\omega + 8.3151\cos(2\omega) - 3.4443\sin(2\omega) \\ &- 4.1114\cos(3\omega) + 1.703\sin(3\omega) - 0.462\cos(4\omega) + 0.1914\sin(4\omega) + 0.231\times \\ &\times \cos(5\omega) - 0.0957\sin(5\omega)) - 28.5\sin(2\omega) + 70.5\sin(4\omega) - 1.75\sin(6\omega). \end{aligned}$$

By the bisection method, the approximating solutions of the equation $h_2(\omega) = 0$ in the interval $(0, \infty)$ within the accuracy of 10^{-4} are

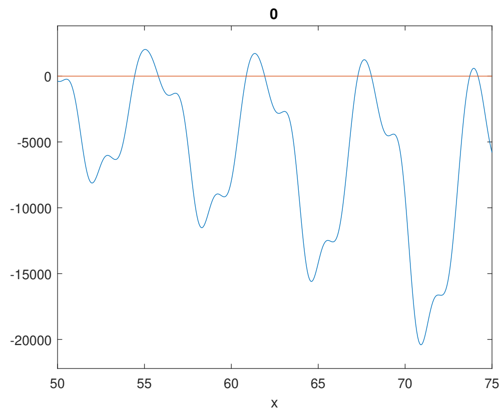
$$\begin{aligned} &\omega_1 \approx 0.8843,\ \omega_3 \approx 1.3525,\ \omega_8 \approx 26.2244,\ \omega_9 \approx 27.1831,\ \omega_{10} \approx 27.9934,\ \omega_{11} \approx 28.4585, \\ &\omega_{12} \approx 32.3448,\ \omega_{13} \approx 35.0607,\ \omega_{14} \approx 38.4778,\ \omega_{15} \approx 41.5478,\ \omega_{16} \approx 44.5826,\ \omega_{17} \approx 47.9995, \\ &\omega_{18} \approx 49.75,\ \omega_{19} \approx 54.4327,\ \omega_{20} \approx 55.8006.\ \omega_{21} \approx 60.8561,\ \omega_{22} \approx 61.9218,\ \omega_{23} \approx 67.2784, \\ &\omega_{24} \approx 68.0627,\ \omega_{25} \approx 73.7181.\ \omega_{26} \approx 74.1942. \end{aligned} \tag{61}$$



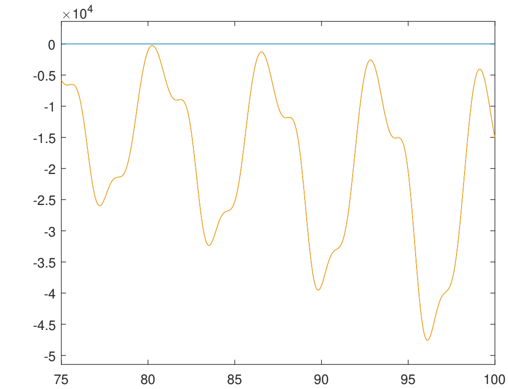
(a) The graph on the interval from 0 to 25.



(b) The graph on the interval from 25 to 50.

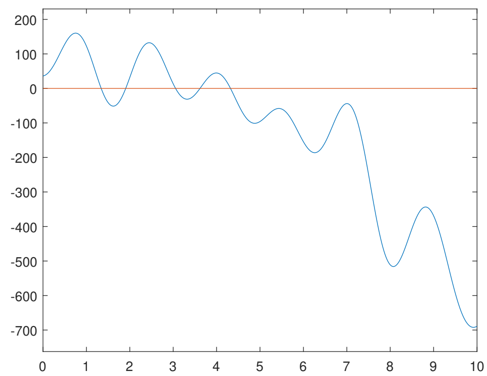


(c) The graph on the interval from 50 to 75.

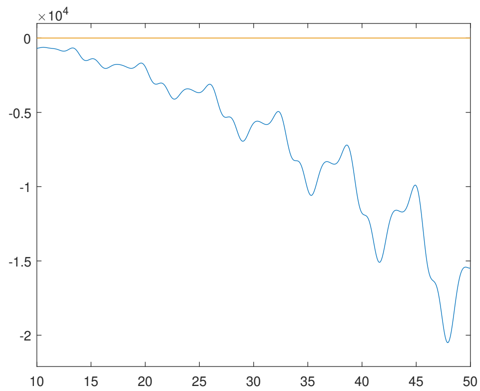


(d) The graph on the interval from 75 to 100.

Figure 6: The graph of $h_2(\omega)$ on the interval $[0, 100]$.



(a) The graph on the interval from 0 to 10.



(b) The graph on the interval from 10 to 50.

Figure 7: The graph of $h_1(\omega)$ on the interval $[0, 50]$.

Similarly, we can find the approximating solutions of the equation $h_1(\omega) = 0$ in the interval $(0, \infty)$ within the accuracy of 10^{-4} as follows:

$$\omega_2 \approx 1.3411,\ \omega_4 \approx 1.9059,\ \omega_5 \approx 3.0547,\ \omega_6 \approx 3.6175,\ \omega_7 \approx 4.3204. \tag{62}$$

From (61)–(62), we obtain

$$\Delta argQ(i\omega)\big|_0^\infty = \Delta argQ(i\omega)\big|_0^{\omega_1} + \sum_{j=1}^{25} \Delta argQ(i\omega)\big|_{\omega_j}^{\omega_{j+1}} + \Delta argQ(i\omega)\big|_{\omega_{26}}^\infty. \tag{63}$$

On the interval $(0, \omega_1)$, $h_1(\omega) > 0$ and $h_2(\omega) > 0$, hence $Q(i\omega)$ starts from the point $(35.75, 0)$, moves in the open part of the first quadrant and then returns to intersect the real axis at $(149.3315, 0)$ when ω increases from 0 to ω_1. This implies that $\Delta argQ(i\omega)\big|_0^{\omega_1} = 0$. Since $h_1(\omega) > 0$ and $h_2(\omega) < 0$ on (ω_1, ω_2), we observe that $Q(i\omega)$ initiates at $(149.3315, 0)$, moves in the open part of the fourth quadrant, and intersects the imaginary axis at $(0, -3.3034)$ in this interval. Thus, $\Delta argQ(i\omega)\big|_{\omega_1}^{\omega_2} = -\frac{\pi}{2}$. On the interval (ω_2, ω_3), $h_1(\omega) < 0$ and $h_2(\omega) < 0$, the graph of $Q(i\omega)$ will change as follows: it enters the open part of the third quadrant from the point $(0, -3.3034)$ and intersect the real axis at $(-3.7729, 0)$. Based on this fact, $\Delta argQ(i\omega)\big|_{\omega_2}^{\omega_3} = -\frac{\pi}{2}$. Similarly, we conclude $\Delta argQ(i\omega)\big|_{\omega_3}^{\omega_4} = -\frac{\pi}{2}$, $\Delta argQ(i\omega)\big|_{\omega_j}^{\omega_{j+1}} = 0$, $j = 4, 5, 6$, $\Delta argQ(i\omega)\big|_{\omega_7}^{\omega_8} = \frac{\pi}{2}$, $\Delta argQ(i\omega)\big|_{\omega_j}^{\omega_{j+1}} = 0$, $j = 8, 9, \ldots, 25$, and thus

$$\Delta argQ(i\omega)\big|_0^{\omega_1} + \sum_{j=1}^{25} \Delta argQ(i\omega)\big|_{\omega_j}^{\omega_{j+1}} = -\pi. \tag{64}$$

We now focus on the case $\omega \in (\omega_{26}, \infty)$. Notice that $h_1(\omega) < 0$ and $h_2(\omega) < 0$ for all $\omega \in (\omega_{26}, \infty)$. It implies that $Q(i\omega)$ initiates at $(-39599.2692, 0)$ and stays in the open part of the third quadrant. Furthermore, for $\omega > \omega_{26}$, then

$$\begin{aligned}-1.1137 < -0.7806 &+ 0.2342\cos\omega + 0.1875\sin\omega - 0.0312\cos(2\omega)\\ &- 0.025\sin(2\omega) - 0.0414\cos(3\omega) - 0.0331\sin(3\omega) < -0.4621,\end{aligned} \tag{65}$$

and

$$\begin{aligned}-0.9693 < -0.625 &+ 0.1875\cos\omega - 0.2342\sin\omega - 0.025\cos(2\omega)\\ &+ 0.0312\sin(2\omega) - 0.0331\cos(3\omega) + 0.0414\sin(3\omega) < -0.2911.\end{aligned} \tag{66}$$

From (65)–(66), for all $\omega > \omega_{26}$, we have

$$0.2613 < \frac{h_4(\omega)}{h_3(\omega)} < 2.0976,$$

here $h_3(\omega) = -0.7806 + 0.2342\cos\omega + 0.1875\sin\omega - 0.0312\cos(2\omega) - 0.025\sin(2\omega) - 0.0414\cos(3\omega) - 0.0331\sin(3\omega)$, and $h_4(\omega) = -0.625 + 0.1875\cos\omega - 0.2342\sin\omega - 0.025\cos(2\omega) + 0.0312\sin(2\omega) - 0.0331\cos(3\omega) + 0.0414\sin(3\omega)$. Thus, for $\omega > \omega_{26}$ large enough,

$$0.2613 < \frac{h_2(\omega)}{h_1(\omega)} < 2.0976,$$

and

$$0.2556 < \arctan\left(\frac{h_2(\omega)}{h_1(\omega)}\right) < 1.1259.$$

Due to $\arg Q(i\omega) = \arctan\frac{h_2(\omega)}{h_1(\omega)} - \pi$ for $\omega > \omega_9$, we see that

$$\frac{\pi}{12.2911} - \pi < \arg Q(i\omega) < \frac{\pi}{2.7902} - \pi.$$

It means that

$$\frac{\pi}{12.2911} < \Delta \arg Q(i\omega)\Big|_{\omega_{26}}^{\infty} < \frac{\pi}{2.7902},$$

which together with (63)–(64) lead to that

$$-\frac{\pi}{1.0886} < \Delta \arg Q(i\omega)\Big|_{0}^{\infty} < -\frac{\pi}{1.5585}.$$

Combining (3.13) with the following estimate

$$(\alpha_1 + \alpha_2 + \alpha_3)\frac{\pi}{2} - \Theta \approx \frac{\pi}{0.9205},$$

we obtain

$$\Delta \arg Q(i\omega)\Big|_{0}^{\infty} < (\alpha_1 + \alpha_2 + \alpha_3)\frac{\pi}{2} - \Theta.$$

By Theorem 3.8, the system (60) is not stable. In Figure 8, we simulate the orbits of the solution with the initial condition $x(t) = (0.1t^2 + 0.1, -0.1t + 0.1, 0.2t - 0.1)^{\mathrm{T}}$ on the interval $[-2, 0]$.

4 Conclusions

In this paper, we study non-commensurate fractional-order neutral differential systems with delays by the modified frequency domain analysis. In particular, we have established a new Mikhailov stability criterion. To do this, we have used Rouché's theorem and the argument principle from complex analysis. Then, based on the obtained result, we have proposed a three-step scheme to check the asymptotic stability of the solutions of these systems.

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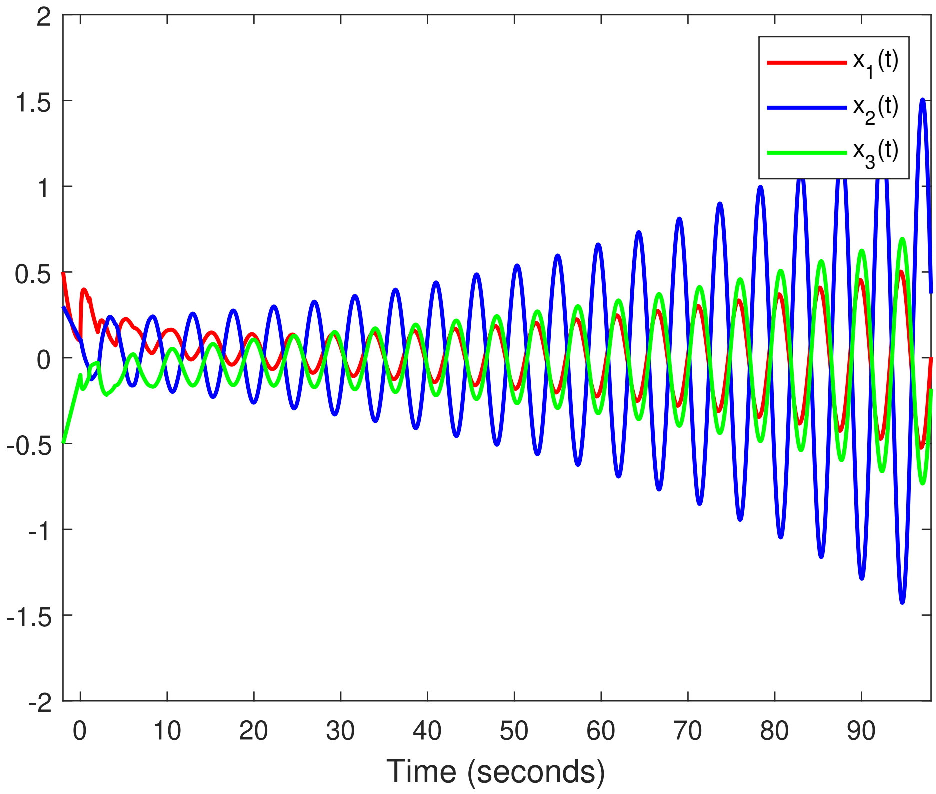

Figure 8: Orbits of the solution of the system (60) with the initial condition $x(t) = (0.1t^2 + 0.1, -0.1t + 0.1, 0.2t - 0.1)^{\mathrm{T}}$ on $[-2, 0]$.

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