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# Attractors of Caputo semi-dynamical systems

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**Abstract** The Volterra integral equation associated with autonomous Caputo fractional differential equation (FDE) of order $\alpha \in (0, 1)$ in $\mathbb{R}^d$ was shown by the authors [4] to generate a semi-group on the space $\mathfrak{C}$ of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^d$ with the topology uniform convergence on compact subsets. It serves as a semi-dynamical system for the Caputo FDE when restricted to initial functions $f(t) \equiv id_{x_0}$ for $x_0 \in \mathbb{R}^d$. Here it is shown that this semi-dynamical system has a global Caputo attractor in $\mathfrak{C}$, which is closed, bounded, invariant and attracts constant initial functions, when the vector field function in the Caputo FDE satisfies a dissipativity condition as well as a local Lipschitz condition.

**Keywords** Caputo fractional differential equations · dissipativity condition · Caputo semi-group · absorbing sets · global attractor

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## 1 Introduction

Consider an autonomous Caputo fractional differential equation (FDE) of order $\alpha \in (0, 1)$ in $\mathbb{R}^d$

$$^{C}D^{\alpha}_{0+}x(t) = g(x(t)), \tag{1.1}$$

where $g : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous and satisfies a growth bound. The Caputo FDE (1.1) with the initial condition $x(0) = x_0$ is essentially equivalent

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to the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s))ds. \tag{1.2}$$

The solutions of such equations are nonlocal. Hence cannot generate a semi-group on the space $\mathbb{R}^d$. This means, in particular, that without a semi-dynamical system on $\mathbb{R}^d$, there can be no attractor on $\mathbb{R}^d$. Nevertheless, when the vector field $g$ satisfies a dissipative condition, they do have omega limit sets in $\mathbb{R}^d$, which attract all future dynamics.

Let $\mathfrak{C}$ be the space of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^d$ with the topology uniform convergence on compact subsets, which is metrized by the metric

$$\rho(f,h) := \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(f,h), \text{ where } \rho_n(f,h) := \frac{\sup_{t\in[0,n]} \|f(t)-h(t)\|}{1+\sup_{t\in[0,n]} \|f(t)-h(t)\|}. \tag{1.3}$$

Define the operators $T_t : \mathfrak{C} \to \mathfrak{C}$, $t \in \mathbb{R}^+$, by

$$(T_t f)(\theta) = f(t+\theta) + \frac{1}{\Gamma(\alpha)} \int_0^t (t+\theta-s)^{\alpha-1} g(x_f(s))\, ds, \qquad \theta \in \mathbb{R}^+, \tag{1.4}$$

where $x_f$ is a solution of the singular Volterra integral equation for this $f$, i.e.,

$$x_f(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x_f(s))\, ds. \tag{1.5}$$

Doan & Kloeden [4] that the operators $T_t$, $t \in \mathbb{R}^+$, form a semi-group on the space $\mathfrak{C}$. This semi-group represents the Caputo FDE (1.1) as an autonomous semi-dynamical system on the space $\mathfrak{C}$, when the $f$ are restricted the identity functions $f(t) \equiv id_{x_0}$ for $x_0 \in \mathbb{R}^d$.

The aim is this paper is to show that when the vector field $g$ has a dissipativity property then this Caputo semi-group does have a global Caputo attractor in a Banach subspace $\mathfrak{C}_\alpha$ of $\mathfrak{C}$. This "attractor" is somewhat unusual in that it attracts only a restricted class of initial values and must be defined directly in terms of an omega limit set in $\mathfrak{C}_\alpha$, which results in some unconventional properties.

The relationship with the omega limit set in $\mathbb{R}^d$ will also be shown.

## 2 Dissipative vector fields

Tuan & Trinh [11, Theorem 2] showed that the solutions of the Caputo FDE (1.1) satisfy

$$^{C}D^{\alpha}_{0+} \|x(t)\|^2 \le 2 \left\langle x(t), {}^{C}D^{\alpha}_{0+} x(t) \right\rangle.$$

Hence, if the vector field $g$ of (1.1) satisfies the dissipativity condition

$$\langle x, g(x) \rangle \le a - b\|x\|^2, \tag{2.1}$$

where $a, b > 0$, then along the solutions of (1.1)

$$^{C}D^{\alpha}_{0+}\|x(t)\|^2 \leq 2\,\langle x(t), g(x(t))\rangle \leq 2a - 2b\|x(t)\|^2.$$

It then follows [9] that these solutions satisfy the inequality

$$\|x(t,x_0)\|^2 \leq \|x_0\|^2 E_\alpha(-2bt^\alpha) + \frac{a}{b}\left(1 - E_\alpha(-2bt^\alpha)\right), \tag{2.2}$$

where $E_\alpha$ is the Mittag-Leffler function [3,7] defined by

$$E_\alpha(\tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{\Gamma(k\alpha+1)}$$

with $\alpha > 0$.

It follows from this inequality that $\|x(t,x_0)\| \leq R$ for all $t \geq 0$ when $\|x_0\| \leq R$ and $R^2 \geq \frac{a}{b}$. [1]

It was also shown in [9] that the set

$$\mathcal{B}^* := \left\{ x \in \mathbb{R}^d \,:\, \|x\|^2 \geq 1 + \frac{a}{b} =: R_*^2 \right\}$$

is a positive invariant absorbing set for the solutions of the Caputo FDE (1.1). In particular, there exists $T_R \geq 0$ such that $\|x(t,x_0)\| \in \mathcal{B}^*$, i.e., $\|x(t,x_0)\| \leq R^*$ for all $t \geq T_R$ and $\|x_0\| \leq R$.

Since the absorbing set $\mathcal{B}^*$ is compact in $\mathbb{R}^d$, the corresponding omega limit set

$$\Omega^* = \overline{\{y \in \mathbb{R}^d \,:\, \exists\,\{x_{0,n}\}_{n\in\mathbb{N}} \text{ bnded}, t_n \to \infty \text{ such that } x(t_n, x_{0,n}) \to y\}}$$

is a nonempty compact subset of $\mathcal{B}^*$. Moreover, it attracts all of the future dynamics of the Caputo FDE (1.1) and contains all of the steady state solutions.

In general, $\Omega_{\mathcal{B}^*}$ cannot be considered as the attractor of the autonomous Caputo FDE (1.1), since the corresponding semi-dynamical system is defined on the function metric space $(\mathfrak{C}, \rho)$ and not on $\mathbb{R}^d$. Nevertheless, it will seen below that $\Omega_{\mathcal{B}^*}$ represents the *observable* part (in $\mathbb{R}^d$) of an attractor $\mathfrak{A}$ in $\mathfrak{C}$ of this Caputo semi-dynamical system and, essentially, determines it.

## 3 Attractors of semi-dynamical systems

The theory of autonomous semi-dynamical systems [8,10] implies the existence of a global attractor of a semi-dynamical system under appropriate assumptions.

[1] This implies the existence and uniqueness of solutions of the Caputo FDE (1.1) when the vector field $g$ is continuously differentiable, hence locally Lipschitz, and satisfies the dissipativity condition (2.1). See [9].

**Theorem 1** *Suppose that the semi-dynamical system $\{\phi_t, t \in \mathbb{R}^+\}$ on a Banach space $\mathcal{X}$ has a closed and bounded positively invariant absorbing set $\mathcal{B}$ in $\mathcal{X}$ and is asymptotically compact. Then the semi-dynamical system $\{\phi_t, t \in \mathbb{R}^+\}$ has a global attractor given by*

$$\mathcal{A} = \bigcap_{t\geq 0} \phi_t(\mathcal{B}).$$

Unfortunately, this theorem cannot be applied to the Caputo semi-group $\{T_t, t \in \mathbb{R}^+\}$ here since the attracting property is restricted to constant initial functions $f(t) \equiv id_{x_0}$ corresponding to initial values $x_0 \in \mathbb{R}^d$. An alternative approach will be used below.

Another difficulty is how to apply the dissipativity condition (2.1) to the vector field $g$ inside the integral equations (1.4) defining the Caputo semi-group for $\theta > 0$ to establish the existence of an absorbing set in the space $\mathfrak{C}$. In fact this can be circumvented.

Restricting to constant initial functions $f(t) \equiv id_{x_0}$ corresponding to initial values $x_0 \in \mathbb{R}^d$, the dissipativity condition (2.1) can be used in the case $\theta = 0$, which corresponds to the Caputo FDE (1.1) with the initial condition $x(0) = x_0$, using the inequality (2.2), and leads to

$$x(t, x_0) \in \mathcal{B}^* \ \Leftrightarrow\ \|x\| \leq \sqrt{1 + \frac{a}{b}} =: R_*, \quad t \geq T_R, \|x_0\| \leq R \tag{3.1}$$

for all $R \geq R_*^2$.

These bounds can then be used to estimate the integrals for the integral equations (1.4) with $\theta > 0$. Essentially, the integral equations (1.4) have a skew-product like structure with the solution of

$$x(t, x_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s, x_0))\, ds \tag{3.2}$$

inserted into

$$(T_t id_{x_0})(\theta) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t+\theta-s)^{\alpha-1} g(x(s, x_0))\, ds, \qquad \theta > 0. \tag{3.3}$$

Note that for $\theta = 0$

$$(T_t id_{x_0})(0) = x(t, x_0), \quad t \geq 0.$$

For technical reasons to be revealed in the proofs below a subspace of the space $\mathfrak{C}$ will be used with a weighted norm characterising uniform convergence on bounded intervals. In particular, consider the weighted norm on $\mathfrak{C}([0,\infty), \mathbb{R}^d)$ defined

$$\|f\|_\alpha := \|f(0)\| + \sum_{N=1}^{\infty} \frac{1}{2^N N^\alpha} \|f\|_N,$$

where

$$\|f\|_N := \sup_{t\in[N^{-1},N]} \|f(t)\|, \quad N = 1, 2, \cdots .$$

Let $\mathfrak{C}_\alpha$ be the subspace of $\mathfrak{C}([0,\infty),\mathbb{R}^d)$ consisting of functions $f$ with $\|f\|_\alpha < \infty$. Then $(\mathfrak{C}_\alpha, \|\cdot\|_\alpha)$ is a Banach space and the $(T_t,\, t \geq 0$, form a semi-group on $\mathfrak{C}_\alpha$.

**Theorem 2** *Suppose that the vector field $g$ is locally Lipschitz and satisfies the uniform dissipativity condition* (2.1)*. Then the semi-group $\{T_t\}_{t\in\mathbb{R}^+}$ on the space $\mathfrak{C}_\alpha$ corresponding to the integral equations* (1.5) *has an attracting set $\mathfrak{A} \subset \mathfrak{C}_\alpha$, which is closed, bounded and invariant, and attracts bounded subsets of constant initial value functions $f(t) \equiv id_{x_0}$ corresponding to initial values $x_0 \in \mathbb{R}^d$. In particular*

$$\mathfrak{A} = \overline{\bigcup_{\substack{D\subset\mathbb{R}^d \\ \text{bnded}}} \bigcap_{t\geq s} \bigcup_{s\geq 0} T_s(id_D)},$$

*where $id_D := \{id_{x_0} \in \mathfrak{C}_\alpha \,:\, x_0 \in D\}$.*

The set $\mathfrak{A}$ will be called the Caputo attractor. It is a bounded as a subset of the bounded absorbing set $\mathfrak{B}^*$ (see below) and contains the bounded functions $f(t) \equiv id_{\bar{x}}$, where $g(\bar{x})$ =0, i.e., a steady state solution the Caputo FDE (1.1). It contains no other constant functions.

The proof of the existence of the attractor in Theorem 2 is given in remaining sections of the paper. Let $x(t, x_0)$ be the solution of the Caputo FDE (1.1) satisfying the dissipativity condition (2.1) with the initial condition $x(0, x_0) = x_0$. This solution satisfies the bounds (3.1) and the following bounds hold:

$$B_R := \sup_{t\geq 0, \|x_0\|\leq R} \|x(t, x_0)\| < \infty, \qquad B_R^g := \sup_{\|x\|\leq B_R} \|g(x)\| < \infty,$$

where the continuity of the vector field $g$ has been used in the second bound. These are valid for $R = R_*$ provided $t \geq T_R$.

It will be shown that the semi-group $\{T_t\}_{t\in\mathbb{R}^+}$ is asymptotically compact and that the closed and bounded subset $\mathfrak{B}^*$ of $\mathfrak{C}_\alpha$ defined by

$$\mathfrak{B}^* := \left\{\chi \in \mathfrak{C}_\alpha \,:\, \|\chi\|_\alpha \leq 2R_* + \frac{B_{R_*}^g}{\alpha\Gamma(\alpha)} =: \widehat{R}_*\right\}$$

absorbs under the operators $T_t$ bounded sets of constant initial data functions $\|id_{x_0}\|_\alpha \leq \|x_0\| \leq R$ in the time $t \geq T_R$.

Note that the absorbing set $\mathcal{B}^*$ and omega limit set $\Omega_{\mathcal{B}^*}$ in $\mathbb{R}^d$ satisfy

$$\mathcal{B}^* = \left\{\chi(0) \in \mathbb{R}^d \,:\, \chi \in \mathfrak{B}^*\right\}, \qquad \Omega_{\mathcal{B}^*} = \left\{\chi(0) \in \mathbb{R}^d \,:\, \chi \in \mathfrak{A}\right\}.$$

## 4 Proof of Theorem 2

The proof requires some basic lemmas including the following elementary lemma. The proof is given here for completeness.

**Lemma 1** *[6, Lemma 3.1] Let $\theta > 0$. The function $r_\theta(t) := (t+\theta)^\alpha - t^\alpha$, $t \geq 0$, is monotonically decreasing from the maximum value $\theta^\alpha$. In particular, $0 < r_\theta(t) \leq \theta^\alpha$ for all $t \geq 0$.*

*Proof* Note that $r_\theta(0) = \theta^\alpha$ and that the derivative $r'_\theta(t) = \frac{1}{\alpha}\left((t+\theta)^{\alpha-1} - t^{\alpha-1}\right) < 0$ with $r'_\theta(t) \to 0^-$ as $t \to \infty$. □

The next result restates the Hölder continuity of solutions [6, Lemma 3.5] in the dissipative case considered here.

**Lemma 2** *The solution of the integral equation* (1.1) *is Hölder continuous with exponent $\alpha$. In particular,*

$$\|x(t+\theta, x_0) - x(t, x_0)\| \leq \frac{B^g_{R_*}}{\alpha\Gamma(\alpha)}\theta^\alpha$$

*for $\|x_0\| \leq R$ and $t \geq T_R$.*

*Proof* Let $t \geq 0$ and $\theta > 0$. Then, subtracting the integral expressions (1.2) for the solutions $x(t+\theta, x_0)$ and $x(t, x_0)$ gives

$$\begin{aligned} x(t+\theta, x_0) - x(t, x_0) &= \frac{1}{\Gamma(\alpha)}\int_0^{t+\theta}(t+\theta-\tau)^{\alpha-1}g(x(\tau,x_0))\,d\tau \\ &\quad -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g(x(\tau,x_0))\,d\tau, \\ &= \frac{1}{\Gamma(\alpha)}\int_t^{t+\theta}(t+\theta-\tau)^{\alpha-1}g(x(\tau,x_0))\,d\tau \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t \Big((t+\theta-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\Big)g(x(\tau,x_0))\,d\tau, \end{aligned}$$

Then

$$\begin{aligned} \left\|\int_t^{t+\theta}(t+\theta-\tau)^{\alpha-1}g(x(\tau,x_0))\,d\tau\right\| &\leq \int_t^{t+\theta}(t+\theta-\tau)^{\alpha-1}\,\|g(x(\tau,x_0))\|\,d\tau \\ &\leq B^g_R\int_t^{t+\theta}(t+\theta-\tau)^{\alpha-1}d\tau \leq \frac{B^g_R}{\alpha}\theta^\alpha. \end{aligned}$$

Similarly,

$$\left\|\int_0^t \Big((t+\theta-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}\Big)g(x(\tau,x_0))\,d\tau\right\| \leq \frac{B^g_R}{\alpha}\Big((t+\theta)^\alpha - t^\alpha - \theta^\alpha\Big).$$

Thus,

$$\|x(t+\theta,x_0)-x(t,x_0)\| \le \frac{B_R^g}{\alpha\Gamma(\alpha)}\Big((t+\theta)^\alpha - t^\alpha\Big) \le \frac{B_R^g}{\alpha\Gamma(\alpha)}\theta^\alpha$$

for all $t \ge 0$, see Lemma 1 above.

Thus the solution $x(t,x_0)$ is Hölder continuous with exponent $\alpha$. For $t \ge T_R$, the constant $B_R^g$ can be replaced by $B_{R_*}^g$. □

### 4.1 Growth bounded

**Lemma 3**

$$\|(T_t id_{x_{0,n}})(\theta)\| \le \frac{B_{R_*}^g}{\alpha\Gamma(\alpha)}\theta^\alpha + R_*, \quad t \ge T_R, \|x_{0,n}\| \le R. \tag{4.1}$$

*Proof* It follows from (3.2) that

$$x(t+\theta,x_{0,n}) = x_{0,n} + \frac{1}{\Gamma(\alpha)}\int_0^{t+\theta}(t+\theta-s)^{\alpha-1}g(x(s,x_{0,n}))\,ds.$$

Hence

$$\begin{aligned} x(t+\theta,x_{0,n}) &= x_{0,n} + \frac{1}{\Gamma(\alpha)}\int_0^{t+\theta}(t+\theta-s)^{\alpha-1}g(x(s,x_{0,n}))\,ds \\ &= x_{0,n} + \frac{1}{\Gamma(\alpha)}\int_0^{t}(t+\theta-s)^{\alpha-1}g(x(s,x_{0,n}))\,ds \\ &\qquad + \frac{1}{\Gamma(\alpha)}\int_t^{t+\theta}(t+\theta-s)^{\alpha-1}g(x(s,x_{0,n}))\,ds \\ &= (T_t id_{x_{0,n}})(\theta) + \frac{1}{\Gamma(\alpha)}\int_t^{t+\theta}(t+\theta-s)^{\alpha-1}g(x(s,x_{0,n}))\,ds. \end{aligned}$$

It follows that

$$\begin{aligned} \|x(t+\theta,x_{0,n}) - (T_t id_{x_{0,n}})(\theta)\| &= \frac{1}{\Gamma(\alpha)}\Big\|\int_t^{t+\theta}(t+\theta-s)^{\alpha-1}g(x(s,x_{0,n}))\,ds\Big\| \\ &\le \frac{1}{\Gamma(\alpha)}B_R^g\Big|\int_t^{t+\theta}(t+\theta-s)^{\alpha-1}ds\Big| \le \frac{B_R^g}{\alpha\Gamma(\alpha)}\theta^\alpha, \end{aligned}$$

i.e.,

$$\|x(t+\theta,x_{0,n}) - (T_t id_{x_{0,n}})(\theta)\| \le \frac{B_R^g}{\alpha\Gamma(\alpha)}\theta^\alpha, \quad t \ge 0. \tag{4.2}$$

Hence

$$\|(T_t id_{x_{0,n}})(\theta)\| \le \frac{B_R^g}{\alpha\Gamma(\alpha)}\theta^\alpha + \|x(t+\theta,x_{0,n})\| \le \frac{B_R^g}{\alpha\Gamma(\alpha)}\theta^\alpha + R.$$

Note that these estimates are uniform in $t \ge 0$.

Then, using the fact that $\|x(t,x_{0,n})\| \in \mathcal{B}^*$, i.e., $\|x(t,x_{0,n})\| \le R_*$, for all $t \ge T_R$ and $\|x_{0,n}\| \le R$, gives the sharper the inequality (4.1). □

It follows from (4.1) that

$$\|(T_t id_{x_{0,n}})(\theta)\| \le \frac{B^g_{R_*}}{\alpha\Gamma(\alpha)} N^\alpha + R_*, \quad t \ge T_R, 0 \le \theta \le N. \tag{4.3}$$

4.2 Boundedness of $\theta$-derivatives

The integrand in the integral

$$\int_0^t (t+\theta-s)^{\alpha-1}\, ds, \quad \theta > 0,$$

is non-singular, so the integral is in fact a classical Riemann integral. Hence we can different by the parameter $\theta$ to obtain

$$\frac{d}{d\theta}\int_0^t (t+\theta-s)^{\alpha-1}\, ds = (\alpha-1)\int_0^t (t+\theta-s)^{\alpha-2}\, ds = \theta^{\alpha-1} - (t+\theta)^{\alpha-1}.$$

Similarly

$$\frac{d}{d\theta}\int_0^t (t+\theta-s)^{\alpha-1} g(x(s,x_{0,n}))\, ds = (\alpha-1)\int_0^t (t+\theta-s)^{\alpha-2} g(x(s,x_{0,n}))\, ds.$$

**Lemma 4**

$$\left\|\frac{d}{d\theta}(T_t id_{x_{0,n}})(\theta)\right\| \le \frac{B^g_R}{\Gamma(\alpha)}\frac{1}{\theta^{1-\alpha}} \tag{4.4}$$

*for all $t \ge 0$, $\|x_{0,n}\| \le R$ and $\theta > 0$.*

*Proof*

$$\begin{aligned}\left\|\frac{d}{d\theta}\int_0^t (t+\theta-s)^{\alpha-1} g(x(s,x_{0,n}))\, ds\right\| &= (1-\alpha)\left\|\int_0^t (t+\theta-s)^{\alpha-2} g(x(s,x_{0,n}))\, ds\right\| \\ &\le (1-\alpha)B^g_R\left|\int_0^t (t+\theta-s)^{\alpha-2}\, ds\right| \\ &\le B^g_R\left(\theta^{\alpha-1} - (t+\theta)^{\alpha-1}\right).\end{aligned}$$

This gives

$$\left\|\frac{d}{d\theta}(T_t id_{x_{0,n}})(\theta)\right\| \le \frac{B^g_R}{\Gamma(\alpha)}\left(\theta^{\alpha-1} - (t+\theta)^{\alpha-1}\right) \le \frac{B^g_R}{\Gamma(\alpha)}\frac{1}{\theta^{1-\alpha}}$$

for all $t \ge 0$ and $\theta > 0$. □

4.3 Applying Ascoli's theorem

Write $\chi_n(t,\theta) := (T_t id_{x_{0,n}})(\theta)$, so $\chi_n(0,\theta) := x_{0,n}$. By estimate (4.3),

$$\|\chi_n(t,\theta)\| \leq \frac{B^g_{R^*}}{\alpha\Gamma(\alpha)}N^\alpha + R^*, \tag{4.5}$$

uniformly in $\theta \in [0,N]$ for all $N > 0$ and $t \geq T_R$.

In addition, by estimate (4.4)

$$\left\|\frac{d}{d\theta}\chi_n(t,\theta)\right\| \leq \frac{B^g_R}{\Gamma(\alpha)}\frac{1}{\theta^{1-\alpha}}, \quad t \geq T_R,$$

so for any $0 < \varepsilon \ll 1$,

$$\left\|\frac{d}{d\theta}\chi_n(t,\theta)\right\| \leq \frac{B^g_R}{\Gamma(\alpha)}\frac{1}{\varepsilon^{1-\alpha}}$$

uniformly in $\theta \in [\varepsilon,\infty)$ for all $t \geq T_R$. The $\chi_n(t,\cdot)$ are thus equi-Lipschitz uniformly in $\theta \in [\varepsilon,\infty)$ for all $t \geq 0$.

This means the Ascoli theorem can be applied on each interval of the form $[\varepsilon,N]$, i.e., in the space $\mathfrak{C}([\varepsilon,N],\mathbb{R}^d)$ of continuous functions $f : [\varepsilon,N]) \to \mathbb{R}^d$. Thus there are (sub)sequences $t_n \to \infty$ and a function $\chi^* \in \mathfrak{C}([\varepsilon,N],\mathbb{R}^d)$ osuch that

$$\chi_n(\theta) := \chi_n(t_n,\theta) \to \chi^*(\theta), \quad t_n \to \infty,$$

uniformly in $\theta \in [\varepsilon,N]$ for each $N \in \mathbb{N}$.

Set $\varepsilon = N^{-1}$. By increasing $N$, the interval $[N^{-1},N]$ and using a diagonal subsequence it follows that $\chi^*(\theta)$ is defined for all $\theta > 0$.

4.4 Continuity of $\chi^*(\theta)$ at $\theta = 0$

It follows from Lemma 2 and the dissipativity condition that

$$\|x(t+\theta,x_{0,n}) - x(t,x_{0,n})\| \leq \frac{B^g_R}{\alpha\Gamma(\alpha)}\theta^\alpha,$$

for all $t \geq 0$ and $\theta \geq 0$. Let $t_n \geq T_R$. Then

$$\|x(t_n+\theta,x_{0,n}) - x(t_n,x_{0,n})\| \leq \frac{B^g_{R_*}}{\alpha\Gamma(\alpha)}\theta^\alpha,$$

for all $\theta \geq 0$.

For each $\theta > 0$ there is a convergent subsequence (of the subsequence used above to obtain $\chi^*$) such that limits exist and satisfy

$$\|x^*(\theta) - x^*\| \leq \frac{B^g_{R_*}}{\alpha\Gamma(\alpha)}\theta^\alpha,$$

It is also clear from estimate (4.2) that

$$\|x^*(\theta) - \chi^*(\theta)\| \le \frac{B^g_{R_*}}{\alpha\Gamma(\alpha)}\theta^\alpha, \quad \theta \ge 0.$$

Thus

$$\begin{aligned}\|\chi^*(\theta) - x^*\| &\le \|\chi^*(\theta) - x^*(\theta)\| + \|x^* - x^*(\theta)\| \\ &\le \frac{2B^g_{R_*}}{\alpha\Gamma(\alpha)}\theta^\alpha \to 0 \quad \text{as } \theta \to 0.\end{aligned}$$

Summarising,

**Lemma 5** $\chi^* \in \mathfrak{C}_\alpha([0,\infty), \mathbb{R}^d)$.

Thus the operator $(T_t id_{x_{0,n}})(\cdot)$ is asymptotically compact on bounded intervals, i.e., for every sequence $t_n \to \infty$ and $\|x_{0,n}\| \le R$ and there is a subsequence $t_n \to \infty$ such that $\chi_n(t_n, \cdot) = (T_{t_n} id_{x_{0,n}})(\cdot) \to \chi^*(\cdot) \in \mathfrak{C}_\alpha([0,\infty), \mathbb{R}^d)$.

### 4.5 Estimates in the weighted norm

In terms of the weighted norm $\|\cdot\|_\alpha$ on $\mathfrak{C}_\alpha$, the bound (4.5) becomes

$$\begin{aligned}\|\chi(t,\cdot)\|_\alpha &= \|\chi_n(t,0)\| + \sum_{N=1}^\infty \frac{1}{2^N N^\alpha}\|\chi_n(t,\theta)\|_N, \\ &\le R_* + \sum_{N=1}^\infty \frac{1}{2^N N^\alpha}\left(\frac{B^g_{R_*}}{\alpha\Gamma(\alpha)}N^\alpha + R_*\right)\end{aligned}$$

for all $t \ge T_R$. Hence

$$\|\chi_n(t,\cdot)\|_\alpha \le R_*\left(1 + \sum_{N=1}^\infty \frac{1}{2^N N^\alpha}\right) + \frac{B^g_{R_*}}{\alpha\Gamma(\alpha)}\sum_{N=1}^\infty \frac{1}{2^N} \le 2R_* + \frac{B^g_{R_*}}{\alpha\Gamma(\alpha)} =: \widehat{R}^*$$

for all $t \ge T_R$.

### 4.6 Absorbing set and global attractor

The existence of an absorbing set and an attractor will now be established in the space $\mathfrak{C}_\alpha$ on which $(T_t id_{x_0})(\cdot)$, $t \ge 0$, forms a semi-group.

Define the closed and bounded subset $\mathfrak{B}^*$ of $\mathfrak{C}_\alpha$ by

$$\mathfrak{B}^* := \left\{\chi \in \mathfrak{C}_\alpha \,:\, \|\chi\|_\alpha \le 2R^* + \frac{B^g_{R^*}}{\alpha\Gamma(\alpha)} =: \widehat{R}^*\right\}.$$

This set an absorbing set for the Caputo semi-group $(T_t id_{x_0})(\cdot)$ in $\mathfrak{C}_\alpha$, i.e., it absorbs bounded sets of constant initial data $\|id_{x_0}\|_\alpha \leq 2\|x_0\| \leq 2R$ in time $t \geq T_R$. Moreover, the semi-group is asymptotically compact.

Hence, the set $\mathfrak{A}$ given in Theorem 2 is a nonempty, closed and bounded subset of $\mathfrak{B}^*$ and attracts the Caputo semi-group $T_t(\cdot)$ for all constant initial value functions $id_{x_0}$.

It remains to show that the set $\mathfrak{A}$ is invariant under the semi-group $T_t(\cdot)$. First let $f \in \mathfrak{A}$. Then there is a bounded sequence $\{x_{0,n}\}_{n\in\mathbb{N}}$ and $t_n \to \infty$ such that $T_{t_n} id_{x_{0,n}} \to f$. Let $\tau > 0$ be arbitrary. Then, using the semi-group property and continuity,

$$T_{\tau+t_n} id_{x_{0,n}} = T_\tau(T_{t_n} id_{x_{0,n}}) \to T_\tau f,$$

which means that $T_\tau\mathfrak{A} \subset \mathfrak{A}$. Alternatively, write $t_n = \tau + s_n$. Then

$$T_{t_n} id_{x_{0,n}} = T_{\tau+s_n} id_{x_{0,n}} = T_\tau(T_{s_n} id_{x_{0,n}}) \to T_\tau f,$$

By asymptotic compactness, $T_{s_n} id_{x_{0,n}} \to g$ (or a subsequence thereof). Hence, by continuity $T_\tau(T_{s_n} id_{x_{0,n}}) \to T_\tau g$. But $T_{t_n} id_{x_{0,n}} \to f$, so $T_\tau g = f$. Hence, $\mathfrak{A} \subset T_\tau\mathfrak{A}$. Together $\mathfrak{A} = T_\tau\mathfrak{A}$.

This completes the proof of Theorem 2.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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