.

# A DIRICHLET TYPE PROBLEM FOR NON-PLURIPOLAR COMPLEX MONGE-AMPÈRE EQUATIONS

THAI DUONG DO[1], HOANG-SON DO[2], VAN TU LE[3], NGOC THANH CONG PHAM[4]

ABSTRACT. In this paper, we study a Dirichlet type problem for the non-pluripolar complex Monge - Ampère equation with prescribed singularity on a bounded domain of $\mathbb{C}^n$. We provide a local version for an existence and uniqueness theorem proved by Darvas, Di Nezza and Lu in [11]. Our work also extends a result of Åhag, Cegrell, Czyż and Pham in [2].

CONTENTS



## 1. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. For each smooth plurisubharmonic function $u$ on $\Omega$, the complex Monge-Ampère operator of $u$ is defined by

$$(dd^c u)^n = C_n \det(Hu) dV,$$

where $Hu$ is the complex Hessian of $u$, $dV$ is the standard volume form and $C_n > 0$ is a constant depending only on $n$.

Bedford and Taylor [3, 4] have extended the concept of the complex Monge-Ampère operator for bounded plurisubharmonic function, whereby $(dd^c u)^n$ is a Radon measure satisfying the following property: If $u_j$ is a sequence of smooth plurisubharmonic functions decreasing to $u$ then $(dd^c u_j)^n$ converges weakly to $(dd^c u)^n$. The set $\mathscr{D}(\Omega)$ of plurisubharmonic functions whose Monge-Ampère operator can be defined as above is called the domain of definition of Monge-Ampère operator. The characteristics of the domain of definition of Monge-Ampère operator were studied by Cegrell [9] and Blocki [6].

---

*Date*: July 17, 2024
2020 *Mathematics Subject Classification.* 32U15, 32W20.
*Key words and phrases.* complex Monge - Ampère equations, pluripolar sets, non-pluripolar measures, model plurisubharmonic functions, the comparison principle.

In [5], Bedford and Taylor have studied the plurifine topology, which was first introduced by Fuglede in [14] as the weakest topology in which all plurisubharmonic functions are continuous, and defined the non-pluripolar complex Monge-Ampère measure (also known as "the non-pluripolar part of Monge-Ampère operator") for every plurisubharmonic function. If $u$ is a negative plurisubharmonic function then its non-pluripolar Monge-Ampère measure $\mathrm{NP}(dd^c u)^n$ is defined as the limit of the sequence of measures $\mathbb{1}_{\{u>-M\}}(dd^c \max\{u,-M\})^n$ as $M \to \infty$. This measure is a Borel measure which puts no mass on pluripolar subsets and may have a locally unbounded mass. If $u$ is a negative plurisubharmonic function belonging to the domain of the complex Monge-Ampère operator then $\mathrm{NP}(dd^c u)^n = \mathbb{1}_{\{u>-\infty\}}(dd^c u)^n$ (see [8]).

The idea behind the definition of the non-pluripolar complex Monge-Ampère measure in the local setting has been adapted to the case of Kähler manifold [15, 7]. Consider a complex compact Kähler manifold $(X,\omega)$ and let $\theta$ be a closed smooth real $(1,1)$-form on $X$ such that its cohomology class is big. In [7], Boucksom, Eyssidieux, Guedj and Zeriahi have defined the non-pluripolar complex Monge-Ampère measure $(\theta + dd^c u)^n$ for every $\theta$-plurisubharmonic function $u$. In [10, 11], Darvas, Di Nezza and Lu have studied the complex Monge-Ampère equation with prescribed singularity type:

$$\begin{cases} (\theta + dd^c u)^n = f\omega^n, \\ [u] = [\phi], \end{cases} \tag{1.1}$$

where $\phi$ is a given $\theta$-plurisubharmonic function and $f \geq 0$ is a $L^p$ function $(p>1)$ satisfying $\int_X f\omega^n = \int_X \theta_\phi^n > 0$. They have introduced the notion of the model potential, the model-type singularity and shown that this equation is well-posed only for potentials $\phi$ with model type singularities, i.e., $[\phi] = [P_\theta[\phi]]$, where

$$P_\theta[\phi] = (\sup\{\psi \in PSH(X,\theta) : \psi \leq 0, \psi \leq \phi + O(1)\})^*.$$

They have also emphasized that requiring $\phi$ to be a model potential is not only sufficient, but also a necessary condition for the solvability of (1.1) for every choice of $f$. Furthermore, Darvas, Di Nezza and Lu have shown the existence and uniqueness (up to a constant) of solution to the following problem, which is a general form of (1.1) (see [11, Theorem 4.7]):

$$\begin{cases} (\theta + dd^c u)^n = \mu, \\ P_\theta[u] = \phi, \end{cases} \tag{1.2}$$

where $\phi = P_\theta[\phi]$ is a model $\theta$-plurisubharmonic function and $\mu$ is a non-pluripolar positive Radon measure on $X$ satisfying $\int_X \theta_\phi^n = \int_X d\mu > 0$. Here, we say that a measure $\mu$ is non-pluripolar if it vanishes on every pluripolar set.

Inspired of [11], we say that a function $u \in \mathrm{PSH}^-(\Omega)$ is model if $u = P[u]$, where

$$P[u] = \Big(\sup\{v \in \mathrm{PSH}^-(\Omega) :\ v \leq u + O(1) \text{ on } \Omega,\ \liminf_{\Omega\setminus N \ni z \to \xi_0} (u(z) - v(z)) \geq 0\ \forall \xi_0 \in \partial\Omega\}\Big)^*,$$

and $N = \{u = -\infty\}$. A negative plurisubharmonic function $\phi$ is model iff $\mathrm{NP}(dd^c\phi)^n = 0$. Moreover, if $u$ is a negative plurisubharmonic function then the smallest model plurisubharmonic majorant of $u$ is $P[u]$. We refer the reader to Theorem 5.4 below for more details.

In this paper, we study the existence and uniqueness of solution to the following Dirichlet type problem for the non-pluripolar Monge-Ampère equation

$$\begin{cases} \mathrm{NP}(dd^c u)^n = \mu, \\ P[u] = \phi, \end{cases} \tag{1.3}$$

where $\mu$ is a non-pluripolar positive Borel measure on $\Omega$ and $\phi$ is a model plurisubharmonic function on $\Omega$.

Denote by $\mathcal{N}_{NP}(\Omega)$ (or $\mathcal{N}_{NP}$ for short) the set of negative plurisubharmonic functions $u$ on $\Omega$ with smallest model plurisubharmonic majorant identically zero (i.e., $P[u]=0$). The main result of this paper is as follows:

**Theorem 1.1.** *Assume that there exists $v \in PSH^-(\Omega)$ such that $\mathrm{NP}(dd^c v)^n \geq \mu$ and $P[v]=\phi$. Denote*

$$S = \{w \in PSH^-(\Omega) : w \leq \phi, \mathrm{NP}(dd^c w)^n \geq \mu\}.$$

*Then $u_S := (\sup\{w : w \in S\})^*$ is a solution of the problem* (1.3). *Moreover, if there exists $\psi \in \mathcal{N}_{\mathrm{NP}}$ such that $\mathrm{NP}(dd^c \psi)^n \geq \mu$ then $u_S$ is the unique solution of* (1.3).

In the case where $\Omega$ is hyperconvex, Cegrell has shown that if $\mu$ is a non-pluripolar positive Radon measure on $\Omega$ with $\mu(\Omega) < \infty$ then there exists a unique function $u \in \mathcal{F}$ satisfying $(dd^c u)^n = \mu$ (see [9, Lemma 5.14]). Here, the class $\mathcal{F}$ is defined as in [9, Definition 4.6]. Actually, $\mathcal{F}(\Omega)$ is the set of all the functions in $\mathcal{D}(\Omega)$ with smallest maximal plurisubharmonic majorant identically zero and with finite total Monge-Ampère mass (see, for example, [13, page 17]). By Remark 5.5 below, if $u \in \mathcal{F}$ and $(dd^c u)^n$ vanishes on pluripolar sets then $u \in \mathcal{N}_{\mathrm{NP}}$.

Using Theorem 1.1 and [9, Lemma 5.14], we obtain immediately the following result which can be seen as a local version of [11, Theorem 4.7]:

**Corollary 1.2.** *Assume that $\Omega$ is hyperconvex and $\mu$ is a non-pluripolar positive Radon measure on $\Omega$ satisfying $\mu(\Omega) < \infty$. Then, there exists a unique plurisubharmonic function $u$ satisfying* (1.3). *Moreover, $\phi + v \leq u \leq \phi$ for some $v \in \mathcal{F}^a(\Omega)$.*

For every $H \in \mathrm{PSH}^-(\Omega)$, we denote

$$\mathcal{N}_{NP}(H) = \{w \in \mathrm{PSH}^-(\Omega) : \text{there exists } v \in \mathcal{N}_{NP} \text{ such that } v + H \leq w \leq H\},$$

and

$$\mathcal{N}^a(H) = \{w \in \mathrm{PSH}^-(\Omega) : \text{there exists } v \in \mathcal{N}^a \text{ such that } v + H \leq w \leq H\},$$

where $\mathcal{N}^a$ is the set of functions $v \in \mathcal{D}(\Omega)$ with smallest maximal plurisubharmonic majorant identically zero and with $(dd^c v)^n$ vanishes on pluripolar sets. It is easy to check that $\mathcal{N}^a \subset \mathcal{N}_{\mathrm{NP}}$. The following result, which has been proven first by Åhag-Cegrell-Czyż-Pham, can be considered as a corollary of Theorem 1.1:

**Corollary 1.3.** [2, Theorem 3.7] *Assume that $\mu$ is a non-negative measure defined on $\Omega$ by $\mu = (dd^c \varphi)^n$ for some $\varphi \in \mathcal{N}^a$. Then, for every $H \in \mathcal{D}(\Omega)$ with $(dd^c H)^n \leq \mu$, there exists a unique function $u \in \mathcal{N}^a(H)$ such that $(dd^c u)^n = \mu$ on $\Omega$.*

The paper is organized as follows. In Section 2, we recall auxiliary facts about the plurifine topology and the non-pluripolar Monge-Ampère measure. In Sections 3 and 4, we introduce some important tools for the proof of the existence of solution to (1.3). In Section 5, we prove two Xing-type comparison principles and some related results. Theorem 1.1 and Corollary 1.3 are proved in Section 6.

## 2. Preliminaries

In this section, we recall some basic concepts and properties about the plurifine topology and the non-pluripolar Monge-Ampère measure. The reader can find more details in [5].

2.1. **The plurifine topology.** The plurifine topology on an open set $\Omega$ in $\mathbb{C}^n$ is the smallest topology on $\Omega$ for which all the plurisubharmonic functions are continuous. A basis $\mathscr{B}$ of the plurifine topology on $\Omega$ consists of the sets of the following form:

$$U \cap \{u > 0\}$$

where $U$ is an open subset in $\Omega$, $u \in \mathrm{PSH}(U)$.

The plurifine topology has the following quasi-Lindelöf property:

**Theorem 2.1.** [5, Theorem 2.7] *An arbitrary union of plurifine open subsets differs from a countable subunion by at most a pluripolar set.*

By the quasi-Lindelöf property, one get the following lemma:

**Lemma 2.2.** *Let $\mathscr{O}$ be a plurifine open subset of $\Omega$. Then there exists a decreasing sequence $\{V_l\}_l$ of open subsets of $\Omega$ such that $V_l$ contains $\mathscr{O}$ for every $l$ and $\cap_{l=1}^{\infty} V_l \setminus \mathscr{O}$ is a pluripolar set.*

*Proof.* Since we can write $\mathscr{O} = \cup\{\mathscr{O}_i \in \mathscr{B}, i \in I\}$, it follows from Theorem 2.1 that there exist a sequence $\{\mathscr{O}_j\}_j \subset \mathscr{B}$ and a pluripolar set $N$ such that

$$\mathscr{O} = \cup_{j=1}^{\infty} \mathscr{O}_j \cup N. \tag{2.1}$$

By the definition of $\mathscr{B}$, for each $j$, there exist an open subset $U_j$ of $\Omega$ and a plurisubharmonic function $u_j \in \mathrm{PSH}(U_j)$ such that

$$\mathscr{O}_j = \{z \in U_j : \ u_j(z) > 0\}.$$

Since $u_j$ is quasi-continuous on $U_j$, for every $l \in \mathbb{Z}^+$, there exists an open subset $W_{j,l}$ of $U_j$ such that $\mathrm{Cap}(W_{j,l}, U_j) < 2^{-1-l-j}$ and $u_j \in C(U_j \setminus W_{j,l})$. By Tietze's theorem, we can find a continuous extension $f_{j,l}$ of $u_j$ on $U_j$. Set

$$V_{j,l} = \bigcup_{s=l}^{\infty} W_{j,s}. \tag{2.2}$$

Then, the sequence $\{V_{j,l}\}_l$ is decreasing and

$$\mathrm{Cap}(V_{j,l}, U_j) \le \sum_{s=l}^{\infty} \mathrm{Cap}(W_{j,s}, U_j) \le \sum_{s=l}^{\infty} \frac{1}{2^{1+s+j}} = 2^{-j-l} \sum_{s=1}^{\infty} \frac{1}{2^s} = 2^{-j-l}. \tag{2.3}$$

Observe that

$$\begin{aligned}
\mathscr{O}_j \cup V_{j,l} =& V_{j,l} \cup \{z \in U_j : \ u_j(z) > 0\} \\
=& V_{j,l} \cup \{z \in U_j \setminus V_{j,l} : \ u_j(z) > 0\} \\
=& V_{j,l} \cup \{z \in U_j \setminus V_{j,l} : \ f_{j,l}(z) > 0\} \\
=& V_{j,l} \cup \{z \in U_j : \ f_{j,l}(z) > 0\},
\end{aligned}$$

which implies that $\mathscr{O}_j \cup V_{j,l}$ is open.

Let $u \in \mathrm{PSH}^-(\Omega)$ such that

$$N \subset \{u = -\infty\}. \tag{2.4}$$

For each $l \in \mathbb{Z}^+$, we denote $N_l = \{u < -l\}$. We have $N_l$ is open and

$$\lim_{l \to \infty} \mathrm{Cap}(N_l, \Omega) = 0. \tag{2.5}$$

Now, for every $l \in \mathbb{Z}^+$, we define

$$V_l = N_l \cup_{j=1}^{\infty} (\mathscr{O}_j \cup V_{j,l}).$$

Then $\{V_l\}_l$ is a decreasing sequence of open sets. By (2.1) and (2.4), $\cap_{l=1}^{\infty} V_l$ contains $\mathscr{O}$. Moreover, by (2.3), for every $l_0 \in \mathbb{Z}^+$, we have

$$\begin{aligned}
\mathrm{Cap}(\cap_{l=1}^{\infty} V_l \setminus \mathscr{O}, \Omega) \leq \mathrm{Cap}(V_{l_0} \setminus \mathscr{O}, \Omega) &\leq \mathrm{Cap}(N_{l_0} \cup_{j=1}^{\infty} V_{j,l_0}, \Omega) \\
&\leq \mathrm{Cap}(N_{l_0}, \Omega) + \sum_{j=1}^{\infty} \mathrm{Cap}(V_{j,l_0}, \Omega) \\
&\leq \mathrm{Cap}(N_{l_0}, \Omega) + \sum_{j=1}^{\infty} \mathrm{Cap}(V_{j,l_0}, U_j) \\
&\leq \mathrm{Cap}(N_{l_0}, \Omega) + \sum_{j=1}^{\infty} 2^{-(l_0+j)} \\
&= \mathrm{Cap}(N_{l_0}, \Omega) + 2^{-l_0}.
\end{aligned}$$

Letting $l_0 \to \infty$ and using (2.5), we obtain

$$\mathrm{Cap}(\cap_{l=1}^{\infty} V_l \setminus \mathscr{O}, \Omega) = 0.$$

Hence $\cap_{l=1}^{\infty} V_l \setminus \mathscr{O}$ is a pluripolar set.

The proof is completed. □

2.2. **The non-pluripolar complex Monge-Ampère measure.** We recall the definition of the non-pluripolar complex Monge-Ampère measures.

**Definition 2.3.** [5] *If $u \in PSH(\Omega)$ then the non-pluripolar complex Monge-Ampère measure of $u$ is the measure* $\mathrm{NP}(dd^c u)^n$ *satisfying*

$$\int_E \mathrm{NP}(dd^c u)^n = \lim_{j\to\infty} \int_{E\cap\{u>-j\}} (dd^c \max\{u, -j\})^n,$$

*for every Borel set $E \subset \Omega$.*

**Remark 2.4.** i. *If $E \subset \{u > -k\}$, then it follows from* [5, Corollary 4.3] *that*

$$\int_E (dd^c \max\{u, -j\})^n = \int_E (dd^c \max\{u, -k\})^n, \textit{ for every } j \geq k.$$

*In particular,*

$$\int_{E\cap\{u>-k\}} \mathrm{NP}(dd^c u)^n = \int_{E\cap\{u>-k\}} (dd^c \max\{u, -k\})^n,$$

*for every $k > 0$ and for every Borel set $E \subset \Omega$.*

ii. $\mathrm{NP}(dd^c u)^n$ *vanishes on every pluripolar sets.*

iii. *If $\Omega$ is the open unit ball and $u$ is defined by*

$$u(z) = (-\log|z_1|)^{1/n} (|z_2|^2 + \ldots + |z_n|^2 - 1),$$

*then* $\mathrm{NP}(dd^c u)^n$ *is not locally finite (see* [16]*).*

The following results are classical. We present the proof here for the convenience of the reader.

**Lemma 2.5.** *Let $u, v \in PSH^-(\Omega)$ and $\mu$ be a positive Borel measure that vanishes on pluripolar sets. If* $\mathrm{NP}(dd^c u)^n \geq \mu, \mathrm{NP}(dd^c v)^n \geq \mu$ *then* $\mathrm{NP}(dd^c \max\{u, v\})^n \geq \mu$.

*Proof.* Since $\mu$ is Borel which does not charge the set $\{u+v=-\infty\}$, we only need to show that

$$\int\limits_E \mathrm{NP}(dd^c \max\{u,v\})^n \geq \int\limits_E d\mu,$$

for every Borel set $E \subset \{u+v>-\infty\}$. Note that $E=\bigcup\limits_{j\geq 1} E_j$ where $E_j = E\cap\{u+v>-j\}$. We will show that

$$\int\limits_{E_{j_0}} \mathrm{NP}(dd^c \max\{u,v\})^n \geq \int\limits_{E_{j_0}} d\mu,$$

for every $j_0 \geq 1$.

Since $\max\{u,v\} \geq \min\{u,v\} \geq u+v$, we have

$$E_{j_0} \subset \{u+v>-j_0\} \subset \{u>-j\}\cap\{v>-j\} \subset \{\max\{u,v\}>-j\},$$

for every $j>j_0$. Hence, by Definition 2.3 and Remark 2.4 (i), we have

$$\int\limits_{E_{j_0}} \mathrm{NP}(dd^c w)^n = \int\limits_{E_{j_0}} \Big(dd^c \max\{w,-j\}\Big)^n, \tag{2.6}$$

for $w \in \{u,v,\max\{u,v\}\}$ and for every $j>j_0$.

Denote $u_j = \max\{u,-j\}$, $v_j = \max\{v,-j\}$ and $\phi_j = \max\{\max\{u,v\},-j\}$. Observe that $\phi_j = \max\{u_j,v_j\}$. By applying [12, Proposition 11.9] (see also [18, Proposition 4.3]), we have

$$(dd^c\phi_j)^n \geq \mathbb{1}_{\{u_j\geq v_j\}}(dd^c u_j)^n + \mathbb{1}_{\{u_j<v_j\}}(dd^c v_j)^n. \tag{2.7}$$

Note $E_{j_0}\cap\{u_j\geq v_j\} = E_{j_0}\cap\{u\geq v\}$ and $E_{j_0}\cap\{u_j<v_j\} = E_{j_0}\cap\{u<v\}$ for every $j>j_0$. Hence, it follows from (2.7) that

$$\int\limits_{E_{j_0}} (dd^c\phi_j)^n \geq \int\limits_{E_{j_0}\cap\{u\geq v\}} (dd^c u_j)^n + \int\limits_{E_{j_0}\cap\{u<v\}} (dd^c v_j)^n, \tag{2.8}$$

for every $j>j_0$.

Combining (2.6) and (2.8), we get

$$\int\limits_{E_{j_0}} \mathrm{NP}(dd^c \max\{u,v\})^n \geq \int\limits_{E_{j_0}\cap\{u\geq v\}} \mathrm{NP}(dd^c u)^n + \int\limits_{E_{j_0}\cap\{u<v\}} \mathrm{NP}(dd^c v)^n.$$

Thus, by the facts $\mathrm{NP}(dd^c u)^n \geq \mu$ and $\mathrm{NP}(dd^c v)^n \geq \mu$, we have

$$\int\limits_{E_{j_0}} \mathrm{NP}(dd^c \max\{u,v\})^n \geq \int\limits_{E_{j_0}} d\mu.$$

Letting $j_0 \to \infty$, we obtain

$$\int\limits_E \mathrm{NP}(dd^c \max\{u,v\})^n \geq \int\limits_E d\mu.$$

The proof is completed. □

**Lemma 2.6.** *Let $u,v \in PSH^-(\Omega)$. Then* $\mathrm{NP}(dd^c(u+v))^n \geq \mathrm{NP}(dd^c u)^n + \mathrm{NP}(dd^c v)^n$.

*Proof.* We need to show that

$$\int\limits_E \mathrm{NP}(dd^c(u+v))^n \geq \int\limits_E \mathrm{NP}(dd^c u)^n + \int\limits_E \mathrm{NP}(dd^c v)^n,$$

for every Borel set $E \subset \Omega \setminus \{u+v=-\infty\}$. For $j_0 \in \mathbb{Z}^+$, we denote $E_{j_0} = E \cap \{u+v > -j_0\}$. Note that

$$E_{j_0} \subset \{u+v>-j\} \subset \{u>-j\} \cap \{v>-j\}, \tag{2.9}$$

for every $j > j_0$. Hence, by Definition 2.3 and Remark 2.4 (i), we have

$$\int\limits_{E_{j_0}} \mathrm{NP}(dd^c w)^n = \int\limits_{E_{j_0}} \Big(dd^c \max\{w,-j\}\Big)^n, \tag{2.10}$$

for $w \in \{u, v, u+v\}$ and for every $j > j_0$.

Denote $u_j = \max\{u,-j\}$, $v_j = \max\{v,-j\}$ and $\phi_j = \max\{u+v,-j\}$. For every $z \in \{u+v > -j\}$, we have $u_j(z) = u(z)$, $v_j(z) = v(z)$ and $\phi_j(z) = u(z)+v(z)$. Hence

$$\phi_j = u_j + v_j,$$

on the plurifine open set $\{u+v>-j\}$. Hence, it follows from [5, Corollary 4.3] that

$$(dd^c\phi_j)^n|_{\{u+v>-j\}} = (dd^c(u_j+v_j))^n|_{\{u+v>-j\}} \geq \Big((dd^c u_j)^n + (dd^c v_j)^n\Big)|_{\{u+v>-j\}}. \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we have

$$\begin{aligned}\int\limits_{E_{j_0}} \mathrm{NP}(dd^c(u+v))^n &= \int\limits_{E_{j_0}} (dd^c\phi_j)^n \\ &\geq \int\limits_{E_{j_0}} (dd^c u_j)^n + \int\limits_{E_{j_0}} (dd^c v_j)^n \\ &= \int\limits_{E_{j_0}} \mathrm{NP}(dd^c u)^n + \int\limits_{E_{j_0}} \mathrm{NP}(dd^c v)^n,\end{aligned}$$

for every $j > j_0$.

Letting $j_0 \to \infty$, we obtain

$$\int\limits_{E} \mathrm{NP}(dd^c(u+v))^n \geq \int\limits_{E} \mathrm{NP}(dd^c u)^n + \int\limits_{E} \mathrm{NP}(dd^c v)^n.$$

The proof is completed. □

## 3. Stability of subsolutions and supersolutions

The goal of this section is to prove Lemmas 3.2 and 3.3 which are important tools for the proof of the main theorem. First, we need the following lemma:

**Lemma 3.1.** *Let $u, u_j$ ($j \in \mathbb{Z}^+$) be negative plurisubharmonic functions on $\Omega$ such that $\{u_j\}_{j\geq 1}$ is monotone and $u = (\lim_{j\to\infty} u_j)^*$. Assume $f, f_j$ are bounded, quasi-continuous on $\Omega$ satisfying $0 \leq f, f_j \leq 1$, $f_j$ converges monotonically to $f$ quasi-everywhere. Suppose that $\{f \neq 0\} \subset \{u \geq -M\}$ and $\{f_j \neq 0\} \subset \{u_j \geq -M\}$ for every $j$, where $M > 0$ is a constant. Then $f_j \mathrm{NP}(dd^c u_j)^n$ converges weakly to $f\mathrm{NP}(dd^c u)^n$ as $j \to \infty$.*

*Proof.* By the definition, we have

$$\mathbb{1}_{\{u>-M-1\}}\mathrm{NP}(dd^c u)^n = \mathbb{1}_{\{u>-M-1\}}(dd^c \max\{u,-k\})^n,$$

for every $k \geq M+1$. Since $\{f \neq 0\} \subset \{u > -M-1\}$, it follows that

$$f\mathrm{NP}(dd^c u)^n = f(dd^c \max\{u, -M-1\})^n. \tag{3.1}$$

Similar, we also have

$$f_j\mathrm{NP}(dd^c u_j)^n = f_j(dd^c \max\{u_j, -M-1\})^n \text{ for every } j. \tag{3.2}$$

Since $u_j$ converges monotonically to $u$, we have $(dd^c \max\{u_j, -M-1\})^n$ converges weakly to $(dd^c \max\{u, -M-1\})^n$ as $j \to \infty$. Hence, it follows from [5, Theorem 3.2(4 $\Rightarrow$ 3)] that

$$f_j(dd^c \max\{u_j, -M-1\})^n \xrightarrow{\text{w}} f(dd^c \max\{u, -M-1\})^n. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we get

$$f_j \text{NP}(dd^c u_j)^n \xrightarrow{\text{w}} f \text{NP}(dd^c u)^n,$$

as desired. □

**Lemma 3.2.** *Let $u_j$ be a monotone sequence of negative plurisubharmonic functions on $\Omega$ and let $\mu$ be a positive Borel measure on $\Omega$ such that $\text{NP}(dd^c u_j)^n \geq \mu$ for every $j \in \mathbb{Z}^+$. Assume that $u := \left(\lim_{j\to\infty} u_j\right)^*$ is not identically $-\infty$. Then $\text{NP}(dd^c u)^n \geq \mu$.*

*Proof.* We give the proof for the case where $(u_j)_j$ is increasing. The case of decreasing sequence is similar and we leave it for the readers.

For each $k \in \mathbb{Z}^+$, we denote

$$f_k = \min\{\max\{u_1 + k + 1, 0\}, 1\}.$$

Then $0 \leq f_k \leq 1$, $f_k|_{\{u_1 \geq -k\}} = 1$, $f_k|_{\{u_1 \leq -k-1\}} = 0$ and $f_k$ is continuous in plurifine topology. Since $u_1 \leq u_2 \leq ... \leq u_k \leq ... \leq u$, we have $\{f_k \neq 0\} \subset \{u_j > -k-1\} \cap \{u > -k-1\}$ for every $j$. Hence, it follows from Lemma 3.1 that $f_k \text{NP}(dd^c u_j)^n$ converges weakly to $f_k \text{NP}(dd^c u)^n$ as $j \to \infty$. Then, by the assumption $\text{NP}(dd^c u_j)^n \geq \mu$, we have

$$f_k \text{NP}(dd^c u)^n \geq f_k \mu.$$

Letting $k \to \infty$, we get

$$\text{NP}(dd^c u)^n \geq \mathbb{1}_{\{u_1 > -\infty\}} \mu. \tag{3.4}$$

Moreover, the assumption $\text{NP}(dd^c u_j)^n \geq \mu$ implies that $\mu$ vanishes on pluripolar sets. In particular,

$$\mu = \mathbb{1}_{\{u_1 > -\infty\}} \mu. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\text{NP}(dd^c u)^n \geq \mu.$$

The proof is completed. □

**Lemma 3.3.** *Let $u_j$ be a monotone sequence of negative plurisubharmonic functions on $\Omega$ such that $u := \left(\lim_{j\to\infty} u_j\right)^*$ is not identically $-\infty$. Let $\mu$ be a positive Borel measure on $\Omega$. Assume that there exists a plurifine open subset $U$ of $\Omega$ such that*

$$\mathbb{1}_U \text{NP}(dd^c u_j)^n \leq \mu,$$

*for every $j$. Then*

$$\mathbb{1}_U \text{NP}(dd^c u)^n \leq \mu.$$

*Proof.* We give the proof for the case where $(u_j)_j$ is decreasing. The case of increasing sequence is similar and we leave it for the readers.

By the quasi-Lindelöf property of plurifine topology (see Theorem 2.1) and by the fact that $\mathscr{B}$ is a basis of plurifine topology, the problem is reduced to the case $U \in \mathscr{B}$, i.e.,

$$U = \{z \in V : v(z) > 0\},$$

where $V$ is an open subset of $\Omega$ and $v$ is a plurisubharmonic function on $V$.

Let $\chi \in C_c(V)$ and denote

$$g_{\chi,k} = \chi \max\{\min\{4^k v - 2^k, 1\}, 0\},$$

for every $k \in \mathbb{Z}^+$. We have $g_{\chi,k}$ is a quasi continuous function on $\Omega$.

Denote

$$f_k = \min\{\max\{u + k + 1, 0\}, 1\}.$$

Then $0 \leq f_k \leq 1$, $f_k|_{\{u \geq -k\}} = 1$, $f_k|_{\{u \leq -k-1\}} = 0$ and $f_k$ is quasi-continuous. Since $u_1 \geq u_2 \geq \ldots \geq u_k \geq \ldots \geq u$, we have $\{f_k \neq 0\} \subset \{u_j > -k-1\} \cap \{u > -k-1\}$ for every $j$. Hence, it follows from Lemma 3.1 that $f_k g_{\chi,k} \mathrm{NP}(dd^c u_j)^n$ converges weakly to $f_k g_{\chi,k} \mathrm{NP}(dd^c u)^n$ as $j \to \infty$. Moreover, since $\mathrm{supp} g_{\chi,k} \subset U$ and $0 \leq f_k, g_{\chi,k} \leq 1$, we have $f_k g_{\chi,k} \mathrm{NP}(dd^c u_j)^n \leq \mu$ for every $j$. Then

$$f_k g_{\chi,k} \mathrm{NP}(dd^c u)^n \leq \mu.$$

Letting $k \to \infty$ and $\chi \nearrow \mathbb{1}_V$, we get

$$\mathbb{1}_U \mathrm{NP}(dd^c u)^n \leq \mu.$$

The proof is completed. $\square$

## 4. An envelope of plurisubharmonic functions

The main result of this section is as follows:

**Theorem 4.1.** *Let $\mu$ be a positive Borel measure on $\Omega$ and let $U \subset \Omega$ be a plurifine open set such that*

$$\mathrm{NP}(dd^c \varphi)^n \geq \mathbb{1}_U \mu,$$

*for some $\varphi \in PSH^-(\Omega)$. Denote*

$$u = \left(\sup\{w \in PSH^-(\Omega) : w \leq H \text{ on } \Omega \setminus U, \mathrm{NP}(dd^c w)^n \geq \mathbb{1}_U \mu\}\right)^*,$$

*where $H$ is a negative plurisubharmonic function on $\Omega$. Then $\mathbb{1}_U \mathrm{NP}(dd^c u)^n = \mathbb{1}_U \mu$.*

In order to prove the above theorem, we need the following lemmas:

**Lemma 4.2.** *Let $S$ be a family of negative plurisubharmonic functions on $\Omega$ and let $\mu$ be a positive Borel measure on $\Omega$ such that $\mathrm{NP}(dd^c w)^n \geq \mu$ for every $w \in S$. Denote*

$$u_S = (\sup\{w : w \in S\})^*.$$

*Then $\mathrm{NP}(dd^c u_S)^n \geq \mu$.*

*Proof.* By Choquet's lemma [17, Lemma 2.3.4], there exists a sequence $\{u_j\}_{j \in \mathbb{Z}^+} \subset S$ such that

$$u_S = \left(\sup\{u_j : j \in \mathbb{Z}^+\}\right)^*.$$

For every $j \in \mathbb{Z}^+$, we denote

$$v_j = \max\{u_1, u_2, \ldots, u_j\}.$$

Then $\{v_j\}$ is an increasing sequence and $u_S = (\lim_{j \to \infty} v_j)^*$. Moreover, it follows from Lemma 2.5 that for every $j \in \mathbb{Z}^+$,

$$\mathrm{NP}(dd^c v_j)^n \geq \mu.$$

Hence, by Lemma 3.2, we have

$$\mathrm{NP}(dd^c u_S)^n \geq \mu.$$

$\square$

**Lemma 4.3.** *Let $S$ be a family of negative plurisubharmonic functions on $\Omega$. Assume that there exist a set $W \subset \Omega$ and a function $H : W \to \mathbb{R}$ such that $w|_W \leq H$ for every $w \in S$. Put*

$$u_S = (\sup\{w : w \in S\})^*.$$

*Then, there exists a pluripolar set $N \subset \Omega$ such that $u_S \leq H$ on $W \setminus N$.*

*Proof.* Set $v_S = \sup\{w : w \in S\}$. Since negligible sets are pluripolar, we have $\{u_S > v_S\}$ is pluripolar. By Josefson's theorem, there exists $\psi \in \mathrm{PSH}^-(\Omega)$ such that $\{u_S > v_S\} \subset \{\psi = -\infty\}$. Therefore, for all $\varepsilon > 0$,

$$u_S + \varepsilon\psi \leq v_S.$$

Since $v_S \leq H$ on $W$, it follows that $u_S + \varepsilon\psi \leq H$ on $W$. Letting $\varepsilon \searrow 0$, we get $u_S \leq H$ on $W \setminus \{\psi = -\infty\}$.

The proof is completed. □

**Lemma 4.4.** *Let $u$ be a bounded, negative plurisubharmonic function on $\Omega$ and let $D \Subset \Omega$ be an open ball. Denote by $u_D$ the smallest maximal plurisubharmonic majorant of $u$ in $D$. Assume that $\mu$ is a non-pluripolar positive Radon measure on $D$ such that $\mu(D) < +\infty$. Then, there exists $v \in \mathscr{F}(D, u_D)$ such that $(dd^c v)^n = \mu$. Here, $\mathscr{F}(D, u_D)$ is the set of plurisubharmonic functions $\varphi$ on $D$ satisfying $u_D + w \leq \varphi \leq u_D$ for some $w \in \mathscr{F}(D)$.*

*Proof.* This lemma is an immediate corollary of [2, Theorem 3.7]. Here we will give a proof that does not use [2, Theorem 3.7].

Let $u_j$ be a sequence of smooth plurisubharmonic functions decreasing to $u$ on a neighborhood $V$ of $\overline{D}$. It is classical that for every $j$, there exists a unique maximal plurisubharmonic function $u_{j,D}$ on $D$ such that $\lim_{D \ni z \to z_0} u_{j,D}(z) = u_j(z_0)$ for every $z_0 \in \partial D$. It is easy to check that $u_{j,D}$ is decreasing to $u_D$ as $j \to \infty$.

By [1, Theorem 3.4], there exists a unique $v_j \in \mathscr{F}(D, u_{j,D})$ such that $(dd^c v_j)^n = \mu$. Moreover, it follows from the comparison principle [1, Theorem 3.2] that $v_j$ is a decreasing sequence and

$$v_0 + u_{j,D} \leq v_j \leq u_{j,D},$$

where $v_0$ is the unique function in $\mathscr{F}(D)$ satisfying $(dd^c v_0)^n = \mu$. Denote $v = \lim_{j\to\infty} v_j$. We have $v_0 \leq u_D \leq v \leq u_D$ and $(dd^c v)^n = \mu$.

This finishes the proof. □

Now we begin to prove Theorem 4.1. We first consider the case where $H$ is bounded and $\mu(\Omega) < +\infty$.

**Theorem 4.5.** *Let $\mu$ be a non-pluripolar positive Radon measure on $\Omega$ such that $\mu(\Omega) < +\infty$. Let $U$ be a plurifine open subset of $\Omega$ and $H \in L^\infty(\Omega)$. Denote*

$$u = \Big(\sup\{w \in PSH^-(\Omega) \colon w \leq H \text{ on } \Omega \setminus U, \mathrm{NP}(dd^c w)^n \geq \mathbb{1}_U \mu\}\Big)^*.$$

*Then $u \in \mathscr{D}(\Omega)$ and $\mathbb{1}_U (dd^c u)^n = \mathbb{1}_U \mu$.*

*Proof.* We first show that $u$ is well-defined, i.e., the family

$$S := \{w \in \mathrm{PSH}^-(\Omega) \colon w \leq H \text{ on } \Omega \setminus U, \mathrm{NP}(dd^c w)^n \geq \mathbb{1}_U \mu\}$$

is non-empty.

Let $D \Subset \Omega$ be an open ball. By [9, Theorem 5.14], there exists $\varphi \in \mathscr{F}(D)$ such that $(dd^c \varphi)^n = \mathbb{1}_U \mu$. Put $M = -\inf_\Omega H$. We have $\varphi|_\Omega - M \in S$. Hence, $u$ is well-defined. Moreover, since $\varphi \in \mathscr{F}(D)$ and $\varphi|_\Omega - M \leq u$, we have $u \in \mathscr{D}(\Omega)$.

It remains to show that $\mathbb{1}_U(dd^cu)^n = \mathbb{1}_U\mu$. We first consider the case where $U$ is open in the usual topology. In this case, we only need to show that $\mathbb{1}_B(dd^cu)^n = \mathbb{1}_B\mu$ for any open ball $B \Subset U$.

Since $\mu(\Omega) < \infty$, without loss of generality, we can assume that $\mu(\partial B) = 0$. Set

$$u_B = \Big(\sup\{w \in \mathrm{PSH}^-(\Omega) \colon w \leq u \text{ on } \Omega \setminus B\}\Big)^*.$$

Then, $u_B$ is maximal on $B$. In particular, $\mathbb{1}_B(dd^cu_B)^n = 0 \leq \mathbb{1}_B\mu$. By Lemma 4.4, there exists $w_B \in \mathcal{N}(B, u_B)$ such that

$$(dd^cw_B)^n = \mathbb{1}_B\mu.$$

Here, the notation $w_B \in \mathcal{N}(B, u_B)$ means that there exists a function $\psi \in \mathcal{N}(B)$ ( $\mathcal{N}(B)$ is the set of functions belonging in $\mathcal{D}(B)$ with smallest maximal plurisubharmonic majorant identically zero) such that

$$u_B + \psi \leq w_B \leq u_B \quad \text{on } B.$$

By Lemma 4.2, we have $(dd^cu)^n \geq \mathbb{1}_U\mu$. Then, it follows from the comparison principle [2, Theorem 3.1] (see also [2, Corollary 3.2]) that $u|_B \leq w_B$. On the other hand, for every $z_0 \in \partial B$, we have

$$\limsup_{B \ni z \to z_0} w_B(z) \leq \limsup_{B \ni z \to z_0} u_B(z) = \limsup_{U \setminus B \ni z \to z_0} u_B(z) = \limsup_{U \setminus B \ni z \to z_0} u(z) = u(z_0).$$

Hence, the function

$$\overline{u}_B := \begin{cases} u \text{ on } \Omega \setminus B, \\ w_B \text{ on } B, \end{cases}$$

is plurisubharmonic on $\Omega$. Moreover,

$$(dd^c\overline{u}_B)^n \geq \mathbb{1}_B(dd^cw_B)^n + \mathbb{1}_{\Omega \setminus \bar{B}}(dd^cu)^n \geq \mathbb{1}_{U \setminus \partial B}\mu = \mathbb{1}_U\mu.$$

Hence, $\overline{u}_B \in S$. Consequently, $\overline{u}_B \leq u$ on $\Omega$. Recall that $w_B \geq u$ on $B$, thus $\overline{u}_B \geq u$ on $\Omega$. Then $\overline{u}_B = u$ on $\Omega$ and it follows that

$$(dd^cu)^n|_B = (dd^c\overline{u}_B)^n|_B = (dd^cw_B)^n = \mathbb{1}_B\mu.$$

Now, we consider the general case where $U$ is plurifine open. By Lemma 2.2, there exists a decreasing sequence of open subset $U_j$ of $\Omega$ such that $U \subset \bigcap_{j\geq 1} U_j$ and $\bigcap_{j\geq 1} U_j \setminus U$ is pluripolar. For every $j$, we denote

$$S_j = \{w \in \mathrm{PSH}^-(\Omega) \colon w \leq H \text{ on } \Omega \setminus U_j, \mathrm{NP}(dd^cw)^n \geq \mathbb{1}_U\mu\},$$

and

$$u_j = \big(\sup\{w : w \in S_j\}\big)^*.$$

By using the case where $U$ is open, we have $u_j \in \mathcal{D}(\Omega)$ and

$$\mathbb{1}_{U_j}(dd^cu_j)^n = \mathbb{1}_U\mu. \tag{4.1}$$

Since $U \subset U_{j+1} \subset U_j$ for every $j$, we have $u_j$ is a decreasing sequence and $u_j \geq u$ for every $j$. Hence

$$\bar{u} := \lim_{j\to\infty} u_j \geq u. \tag{4.2}$$

Moreover, using (4.1) and applying Lemmas 3.2 and 3.3 (replace $\mu$ by $\mathbb{1}_U\mu$), we get

$$\mathbb{1}_U(dd^c\bar{u})^n = \mathbb{1}_U\mu. \tag{4.3}$$

It remains to show that $u = \bar{u}$. By Lemma 4.3, for every $j$, there exists a pluripolar set $N_j$ such that $u_j \leq H$ on $\Omega \setminus (U_j \cup N_j)$. Denote $N = \cup_{j=1}^{\infty} N_j$. We have $N$ is pluripolar and $\bar{u} \leq H$ on

$\Omega \setminus (U \cup N)$. By Josefson's theorem, there exists a negative plurisubharmonic function $\psi$ on $\Omega$ such that $N \subset \{\psi = -\infty\}$. Then, we have $\bar{u} + \varepsilon\psi \leq H$ on $\Omega \setminus U$ and, by Lemma 2.6,

$$\mathrm{NP}(dd^c \bar{u} + \varepsilon\psi)^n \geq (dd^c \bar{u})^n \geq \mathbb{1}_U \mu.$$

Hence $\bar{u} + \varepsilon\psi \in S$ for every $\varepsilon > 0$. As a consequence, we have

$$u \geq \left( \limsup_{\varepsilon \to 0} (\bar{u} + \varepsilon\psi) \right)^* = \bar{u} \tag{4.4}$$

Combining (4.2) and (4.4), we get $\bar{u} = u$. Therefore, by (4.3), we obtain $\mathbb{1}_U (dd^c u)^n = \mathbb{1}_U \mu$.

The proof is completed. □

*End of the proof of Theorem 4.1.* For every $j, k \in \mathbb{Z}^+$, we denote

$$U_j = \{z \in U : d(z, \partial\Omega) > 2^{-j}, \varphi(z) > -2^j\},$$

and

$$H_k = \max\{H, -k\}.$$

We also define

$$u_{j,k} = \left(\sup\{w \in \mathrm{PSH}^-(\Omega) : w \leq H_k \text{ on } \Omega \setminus U, \mathrm{NP}(dd^c w)^n \geq \mathbb{1}_{U_j} \mu\}\right)^*,$$

and

$$u_j = \left(\sup\{w \in \mathrm{PSH}^-(\Omega) : w \leq H \text{ on } \Omega \setminus U, \mathrm{NP}(dd^c w)^n \geq \mathbb{1}_{U_j} \mu\}\right)^*.$$

It is clear that the sequence $\{u_{j,k}\}_{k \in \mathbb{Z}^+}$ is decreasing for every $j$ and

$$u_{j,k} \geq u_j, \tag{4.5}$$

for every $j, k$. The assumption $\mathrm{NP}(dd^c \phi)^n \geq \mathbb{1}_U \mu$ implies that $\int_{U_j} d\mu < \infty$. It follows from Theorem 4.5 that

$$\mathbb{1}_U \mathrm{NP}(dd^c u_{j,k})^n = \mathbb{1}_{U_j} \mu,$$

for every $j, k$. Letting $k \to \infty$ and using Lemmas 3.2 and 3.3, we get

$$\mathbb{1}_U \mathrm{NP}(dd^c \bar{u}_j)^n = \mathbb{1}_{U_j} \mu, \tag{4.6}$$

where $\bar{u}_j = \lim_{k \to \infty} u_{j,k}$.

Now we will prove $\bar{u}_j = u_j$. By Lemma 4.3, for every $k$, there exists a pluripolar set $N_{j,k}$ such that $u_{j,k} \leq H_k$ on $\Omega \setminus (U \cup N_{j,k})$. Denote $N_j = \cup_{k=1}^{\infty} N_{j,k}$. We have $N_j$ is pluripolar and $\bar{u}_j \leq H$ on $\Omega \setminus (U \cup N_j)$. By Josefson's theorem, there exists a negative plurisubharmonic function $\psi_j$ on $\Omega$ such that $N \subset \{\psi_j = -\infty\}$. Then, we have $\bar{u} + \varepsilon\psi_j \leq H$ on $\Omega \setminus U$ and, by Lemma 2.6,

$$\mathrm{NP}(dd^c \bar{u}_j + \varepsilon\psi_j)^n \geq \mathrm{NP}(dd^c \bar{u}_j)^n \geq \mathbb{1}_{U_j} \mu.$$

By the definition of $u_j$, we have $\bar{u}_j + \varepsilon\psi_j \leq u_j$ for every $\varepsilon > 0$. Hence

$$\bar{u}_j = \left( \lim_{\varepsilon \to 0} (\bar{u}_j + \varepsilon\psi_j) \right)^* \leq u_j. \tag{4.7}$$

Combining (4.5) and (4.7), we get $\bar{u}_j = u_j$. Then, by (4.6), we have

$$\mathbb{1}_U \mathrm{NP}(dd^c u_j)^n = \mathbb{1}_{U_j} \mu.$$

Letting $j \to \infty$ and using Lemmas 3.2 and 3.3, we get $\mathbb{1}_{U_{j_0}} \mathrm{NP}(dd^c \bar{u})^n = \mathbb{1}_{U_{j_0}} \mu$ for every $j_0 \in \mathbb{Z}^+$, where $\bar{u} = \lim_{j \to \infty} u_j$. By the same argument as above, we also have $\bar{u} = u$. Hence

$$\mathbb{1}_{U_{j_0}} \mathrm{NP}(dd^c u)^n = \mathbb{1}_{U_{j_0}} \mu, \tag{4.8}$$

for every $j_0 \in \mathbb{Z}^+$. Observe that $\cup_{j_0=1}^{\infty} U_{j_0} = U \setminus \{\phi = -\infty\}$ and $\mu(\{\phi = -\infty\}) = 0$. Hence, by using (4.8) and letting $j_0 \to \infty$, we have

$$\mathbb{1}_U \mathrm{NP}(dd^c u)^n = \mathbb{1}_U \mu.$$

This finishes the proof. □

The following result is a corollary of Theorem 4.1 and Lemma 3.3:

**Proposition 4.6.** *Let $u \in PSH^-(\Omega)$. Then there exists $\bar{u} \in PSH^-(\Omega)$ such that $u \le \bar{u} \le P[u]$ and $\mathrm{NP}(dd^c\bar{u})^n = 0$. In particular, if $u$ is model then $\mathrm{NP}(dd^cu)^n = 0$.*

*Proof.* For every $j \in \mathbb{Z}^+$, we denote

$$V_j = \{z \in \Omega : \ d(z, \partial\Omega) > 2^{-j}, \ u(z) > -2^j\},$$

and

$$u_j = \left(\sup\left\{v \in \mathrm{PSH}(\Omega) : \ v \le u \text{ on } \Omega \setminus V_j\right\}\right)^*.$$

By using Theorem 4.1, we have $\mathbb{1}_{V_j}\mathrm{NP}(dd^cu_j)^n = 0$ for every $j$. Then, it follows from Lemma 3.3 that $\mathbb{1}_{V_j}\mathrm{NP}(dd^c\bar{u})^n = 0$ for every $j$, where $\bar{u} = (\lim_{j\to\infty} u_j)^*$. Since $\cup_{j=1}^\infty V_j = \Omega \setminus \{u = -\infty\}$ and $\mathrm{NP}(dd^c\bar{u})^n$ vanishes on pluripolar sets, it follows that

$$\mathrm{NP}(dd^c\bar{u})^n = 0.$$

Moreover, since $u \le u_j \le P[u]$ for every $j$, we also have $u \le \bar{u} \le P[u]$.

The proof is completed. □

## 5. Xing-type comparison principles

In [19], Xing provided a strong comparison principle for bounded plurisubharmonic functions. Xing's theorem then has been generalized by Nguyen-Pham [18] and by Åhag-Cegrell-Czyż-Pham [2]. In this section, we introduce two new Xing-type theorems (Theorems 5.1 and 5.6) and some applications.

**Theorem 5.1.** *Let $u, v \in PSH^-(\Omega)$ such that*

i, $\liminf\limits_{\Omega\setminus N \ni z \to \xi_0} (u(z) - v(z)) \ge 0$ *for every $\xi_0 \in \partial\Omega$, where $N = \{v = -\infty\}$;*

ii, $v \le u + O(1)$ *on $\Omega$.*

*Let $w_j \in PSH(\Omega, [-1, 0])$, $j = 1, \ldots, n$, and denote $T = dd^cw_1 \wedge \ldots \wedge dd^cw_n$. Then*

$$\frac{1}{n!}\int\limits_{\{u<v\}} (v-u)^nT + \int\limits_{\{u<v\}} (-w_1)\mathrm{NP}(dd^cv)^n \le \int\limits_{\{u<v\}} (-w_1)\mathrm{NP}(dd^cu)^n. \tag{5.1}$$

*Moreover, if $\mathrm{NP}(dd^cu)^n \le \mathrm{NP}(dd^cv)^n + \mu$ for some positive Borel measure $\mu$ then*

$$\frac{1}{n!}\int\limits_{\{u<v\}} (v-u)^nT \le \int\limits_{\{u<v\}} (-w_1)d\mu. \tag{5.2}$$

*Proof.* For each $M > 0$, we denote

$$u_M = \max\{u, -M\} \quad \text{and} \quad v_M = \max\{v, -M\}.$$

By the assumption $(i)$, we have

$$\liminf_{\Omega\setminus N \ni z \to \xi_0} (u_M(z) - (1+\varepsilon)v_M(z)) \ge \liminf_{\Omega\setminus N \ni z \to \xi_0} (u_M(z) - v_M(z)) \ge 0,$$

for every $\xi_0 \in \partial\Omega$ and $M \in \mathbb{Z}^+$. Hence, by using [18, Theorem 4.9] (observe that, in this theorem, the condition $\Omega$ *is hyperconvex* is not necessary), we have

$$\frac{1}{n!}\int\limits_{E_M} ((1+\varepsilon)v_M - u_M - \varepsilon)^nT \le \int\limits_{E_M} (-w_1)\left((dd^cu_M)^n - (dd^c((1+\varepsilon)v_M))^n\right), \tag{5.3}$$

where $E_M = \{u_M < (1+\varepsilon)v_M - \varepsilon\} \Subset \Omega$.

Note that if $z \in E_M$ then $v(z) > -\frac{M}{1+\varepsilon}$ and $v(z) > (1+\varepsilon)v(z) > u_M(z) \geq u(z)$. Moreover, by the assumption $(ii)$, there exists $K > 1$ such that $v \leq u + K$. Hence, we have

$$E_M \subset \left\{ v > -\frac{M}{1+\varepsilon} \right\} \cap \{u < v\} \subset \{u > -M\} \cap \{u < v\},$$

for every $M \geq \frac{(1+\varepsilon)K}{\varepsilon}$. In particular,

$$\mathbb{1}_{E_M}(dd^c u_M)^n = \mathbb{1}_{E_M}\mathrm{NP}(dd^c u)^n \leq \mathbb{1}_{\{u<v\}}\mathrm{NP}(dd^c u)^n,$$

for every $M \geq \frac{(1+\varepsilon)K}{\varepsilon}$. Hence,

$$\int_{E_M} (-w_1)(dd^c u_M)^n \leq \int_{\{u<v\}} (-w_1)\mathrm{NP}(dd^c u)^n, \tag{5.4}$$

for $M \gg 1$.

By the fact $E_M \subset \{v > -M\}$, we also have

$$\int_{E_M} (-w_1)(dd^c((1+\varepsilon)v_M))^n = \int_{E_M} (-w_1)\mathrm{NP}(dd^c(1+\varepsilon)v)^n \geq \int_{E_M} (-w_1)\mathrm{NP}(dd^c v)^n. \tag{5.5}$$

Combining (5.3), (5.4) and (5.5), we get

$$\frac{1}{n!}\int_{E_M} ((1+\varepsilon)v_M - u_M - \varepsilon)^n T + \int_{E_M} (-w_1)\mathrm{NP}(dd^c v)^n \leq \int_{\{u<v\}} (-w_1)\mathrm{NP}(dd^c u)^n,$$

for every $M \gg 1$.

Letting $M \to \infty$ and using the monotone convergence theorem (observer that $\{\mathbb{1}_{E_M}((1+\varepsilon)v_M - u_M - \varepsilon)^n\}_{M\in\mathbb{Z}^+}$ is an increasing sequence), we have

$$\frac{1}{n!}\int_{\{u<(1+\varepsilon)v-\varepsilon\}} ((1+\varepsilon)v - u - \varepsilon)^n T + \int_{\{u<(1+\varepsilon)v-\varepsilon\}} (-w_1)\mathrm{NP}(dd^c v)^n \leq \int_{\{u<v\}} (-w_1)\mathrm{NP}(dd^c u)^n.$$

Letting $\varepsilon \searrow 0$, we obtain the inequality (5.1).

It remains to prove (5.2). For every $M \geq \frac{(1+\varepsilon)K}{\varepsilon}$, by the fact $E_M \subset \{u > -M\} \cap \{v > -M\}$, we have

$$((dd^c u_M)^n - (dd^c((1+\varepsilon)v_M))^n)|_{E_M} = (\mathrm{NP}(dd^c u)^n - \mathrm{NP}(dd^c((1+\varepsilon)v))^n)|_{E_M} \leq \mu|_{E_M}.$$

Hence, it follows from (5.3) that

$$\frac{1}{n!}\int_{E_M} ((1+\varepsilon)v_M - u_M - \varepsilon)^n T \leq \int_{E_M} (-w_1)d\mu,$$

for every $M \gg 1$. Letting $M \to \infty$, we get

$$\frac{1}{n!}\int_{\{u<(1+\varepsilon)v-\varepsilon\}} ((1+\varepsilon)v - u - \varepsilon)^n T \leq \int_{\{u<(1+\varepsilon)v-\varepsilon\}} (-w_1)d\mu.$$

Letting $\varepsilon \searrow 0$, we obtain (5.2). This finishes the proof. □

**Corollary 5.2.** *Let $u, v \in PSH^-(\Omega)$ such that*

i, $\liminf\limits_{\Omega\setminus N\ni z\to\xi_0} (u(z) - v(z)) \geq 0$ *for every $\xi_0 \in \partial\Omega$, where $N = \{v = -\infty\}$;*

ii, $v \leq u + O(1)$ *on $\Omega$.*

*If* $\mathrm{NP}(dd^c u)^n \leq \mathrm{NP}(dd^c v)^n$ *then $u \geq v$.*

*Proof.* By the last assertion of Theorem 5.1, we have

$$\int_{\{u<v\}} (v-u)^n (dd^c w)^n = 0,$$

for every $w \in \mathrm{PSH}(\Omega, [-1,0])$. It follows that $v \le u$ a.e., and thus $v \le u$ everywhere in $\Omega$. □

**Corollary 5.3.** *If $u \in PSH^-(\Omega)$ and $\mathrm{NP}(dd^c u)^n = 0$ then $u$ is model.*

*Proof.* Let $v \in \mathrm{PSH}^-(\Omega)$ such that $v \le u + O(1)$ on $\Omega$ and $\liminf_{\Omega\setminus N \ni z \to \xi_0} (u(z) - v(z)) \ge 0$ for every $\xi_0 \in \partial\Omega$, where $N = \{u = -\infty\} \subset \{v = -\infty\}$. For $\varepsilon > 0$, we denote

$$v_\varepsilon(z) = v(z) + \varepsilon(\|z\|^2 - M),$$

where $M = \sup_{\overline{\Omega}} \|z\|^2$. Then $v_\varepsilon \in \mathrm{PSH}^-(\Omega)$, $v_\varepsilon \le u + O(1)$ on $\Omega$ and $\liminf_{\Omega\setminus N \ni z \to \xi_0} (u(z) - v_\varepsilon(z) \ge 0$ for every $\xi_0 \in \partial\Omega$. It follows from Corollary 5.2 that $u \ge v_\varepsilon$ for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain $u \ge v$. Taking the supremum over all such $v$ and taking the upper semi-continuous regularization yields $u \ge P[u]$ almost everywhere in $\Omega$, hence $u \ge P[u]$ everywhere in $\Omega$. It is clear that $u \le P[u]$. Therefore $u = P[u]$, which means $u$ is model, as desired. □

**Theorem 5.4.** *Suppose $u \in PSH^-(\Omega)$. Then*
(i) *$P[u]$ is a model plurisubharmonic function;*
(ii) *$u$ is model iff $\mathrm{NP}(dd^c u)^n = 0$.*

*Proof.* (ii) is an immediate corollary of Proposition 4.6 and Corollary 5.3. It remains to prove (i).

By Proposition 4.6, there exists $\bar{u} \in \mathrm{PSH}^-(\Omega)$ such that $u \le \bar{u} \le P[u]$ and $\mathrm{NP}(dd^c \bar{u})^n = 0$. Then, by Corollary 5.3, we have $\bar{u}$ is model and it follows that

$$u \le \bar{u} = P[\bar{u}] \le P[u].$$

Moreover, the condition $u \le \bar{u}$ implies that $P[u] \le P[\bar{u}]$. Hence

$$\bar{u} = P[\bar{u}] = P[u].$$

Thus, $P[u]$ is model. □

**Remark 5.5.** (i) *If $u$ is a negative maximal plurisubharmonic function then it follows directly from the definitions that $u$ is model. However, the converse is not true. For example, $u = \log|z|$ is a model plurisubharmonic function which is not maximal on the unit ball.*
(ii) *Let $u \in PSH^-(\Omega) \cap \mathscr{D}(\Omega)$ such that $(dd^c u)^n$ vanishes on pluripolar sets. If $v \ge u$ is a model plurisubharmonic function then it follows from* [6, Theorem 1.2] *and* [2, Lemma 4.1] *that $v \in \mathscr{D}(\Omega)$ and $(dd^c v)^n$ vanishes on pluripolar sets. Hence, by Theorem 5.4, we have $(dd^c v)^n = 0$, i.e., $v$ is maximal. Consequently, if $\Omega$ is a hyperconvex domain then*

$$\{w \in \mathscr{F}(\Omega) : (dd^c w)^n \text{ vanishes on pluripolar sets}\} \subset \mathscr{N}_{\mathrm{NP}}(\Omega).$$

**Theorem 5.6.** *Let $u, v, H \in PSH^-(\Omega)$ such that $u \in \mathscr{N}_{\mathrm{NP}}(H)$ and $v \le H$. Assume $w_j \in PSH(\Omega, [-1,0])$, $j = 1, ..., n$, and denote $T = dd^c w_1 \wedge ... \wedge dd^c w_n$. Then*

$$\frac{1}{n!} \int_{\{u<v\}} (v-u)^n T + \int_{\{u<v\}} (-w_1)\mathrm{NP}(dd^c v)^n \le \int_{\{u<v\}} (-w_1)\mathrm{NP}(dd^c u)^n.$$

*Moreover, if* $\mathrm{NP}(dd^c u)^n \le \mathrm{NP}(dd^c v)^n + \mu$ *for some positive Borel measure* $\mu$ *then*

$$\frac{1}{n!}\int\limits_{\{u<v\}} (v-u)^n T \le \int\limits_{\{u<v\}} (-w_1)d\mu.$$

*Proof.* We will use the same idea as in the proof of [2, Theorem 3.1]. Recall that

$$\mathscr{N}_{NP}(H) = \{w \in \mathrm{PSH}^-(\Omega) : \text{there exists } v \in \mathscr{N}_{NP} \text{ such that } v + H \le w \le H\},$$

where $\mathscr{N}_{NP}$ is the set of negative plurisubharmonic functions $u$ satisfying $P[u] = 0$.

Let $\varphi \in \mathscr{N}_{\mathrm{NP}}$ such that $H \ge u \ge H + \varphi$. For every $j \in \mathbb{Z}^+$, we denote

$$V_j = \{z \in \Omega : \ d(z, \partial\Omega) > 2^{-j}, \ \varphi(z) > -2^j, \ H(z) > -2^j\},$$

and

$$\varphi_j = (\sup\{\psi \in \mathrm{PSH}^-(\Omega) : \ \psi \le \varphi \text{ on } \Omega \setminus V_j\})^*.$$

Since $v \le H$, we have, for every $j \in \mathbb{Z}^+$,

$$u \ge H + \varphi = H + \varphi_j \ge \varphi_j + v \text{ on } \Omega \setminus \overline{V_j}, \tag{5.6}$$

which implies

$$\liminf_{\Omega\setminus N \ni z \to \partial\Omega} [u - (\varphi_j + v)] \ge 0, \tag{5.7}$$

where $N = \{u = -\infty\} \subset \{\varphi + v = -\infty\}$. We also have,

$$u \ge H + \varphi \ge -2^{j+1} \ge (\varphi_j + v) - 2^{j+1} \text{ on } \overline{V_j}. \tag{5.8}$$

By the inequalities (5.6) and (5.8), we have

$$\varphi_j + v - \varepsilon \le u + 2^{j+1} \text{ on } \Omega. \tag{5.9}$$

By using the inequalities (5.7) and (5.9), and applying Theorem 5.1, we have

$$\frac{1}{n!}\int\limits_{\{u<\varphi_j+v\}} (\varphi_j + v - u)^n T + \int\limits_{\{u<\varphi_j+v\}} (-w_1)\mathrm{NP}(dd^c(\varphi_j + v))^n \le \int\limits_{\{u<\varphi_j+v\}} (-w_1)\mathrm{NP}(dd^c u)^n.$$

Hence, by Lemma 2.6, we obtain

$$\frac{1}{n!}\int\limits_{\{u<\varphi_j+v\}} (\varphi_j + v - u)^n T + \int\limits_{\{u<\varphi_j+v\}} (-w_1)\mathrm{NP}(dd^c v)^n \le \int\limits_{\{u<\varphi_j+v\}} (-w_1)\mathrm{NP}(dd^c u)^n.$$

Then, by the monotone convergence theorem, we have

$$\frac{1}{n!}\int\limits_{\{u<\lim\limits_{j\to\infty}\varphi_j+v\}} (\lim_{j\to\infty}\varphi_j + v - u)^n T + \int\limits_{\{u<\lim\limits_{j\to\infty}\varphi_j+v\}} (-w_1)\mathrm{NP}(dd^c v)^n \le \int\limits_{\{u<\lim\limits_{j\to\infty}\varphi_j+v\}} (-w_1)\mathrm{NP}(dd^c u)^n.$$

By the same argument as in the proof of Proposition 4.6, we have $\mathrm{NP}(dd^c(\lim\limits_{j\to\infty}\varphi_j)^*)^n = 0$, and then it follows from Corollary 5.3 that $(\lim\limits_{j\to\infty}\varphi_j)^*$ is model. Hence, by the condition $\varphi \in$

$\mathcal{N}_{\rm NP}$ and the fact $\varphi \leq \varphi_j$ for every $j$, we have $(\lim\limits_{j\to\infty} \varphi_j)^* = 0$ and hence $\lim\limits_{j\to\infty} \varphi_j = 0$ outside a pluripolar set. It thus follows that

$$\frac{1}{n!}\int\limits_{\{u<v\}} (v-u)^n T + \int\limits_{\{u<v\}} (-w_1)\mathrm{NP}(dd^c v)^n \leq \int\limits_{\{u<v\}} (-w_1)\mathrm{NP}(dd^c u)^n.$$

Now, assume that $\mathrm{NP}(dd^c u)^n \leq \mathrm{NP}(dd^c v)^n + \mu$. Since $\mathrm{NP}(dd^c v)^n \leq \mathrm{NP}(dd^c(\varphi_j + v))^n$, we have $\mathrm{NP}(dd^c u)^n \leq \mathrm{NP}(dd^c(\varphi_j+v))^n + \mu$. By using the inequalities (5.7) and (5.9), and applying Theorem 5.1, we have

$$\frac{1}{n!}\int\limits_{\{u<\varphi_j+v\}} (\varphi_j + v - u)^n T \leq \int\limits_{\{u<\varphi_j+v\}} (-w_1) d\mu.$$

Letting $j\to\infty$ and using the fact $(\lim\limits_{j\to\infty}\varphi_j)^* = 0$, we obtain

$$\frac{1}{n!}\int\limits_{\{u<v\}} (v-u)^n T \leq \int\limits_{\{u<v\}} (-w_1) d\mu.$$

The proof is completed. □

Similar to Corollary 5.2, we have the following result:

**Corollary 5.7.** *Let $H \in PSH^-(\Omega)$ and $u, v \in \mathcal{N}_{\rm NP}(H)$. Assume that $\mathrm{NP}(dd^c u)^n \geq \mathrm{NP}(dd^c v)^n$. Then $u \leq v$.*

## 6. Proofs of Theorem 1.1 and Corollary 1.3

### 6.1. Proof of Theorem 1.1.

For the reader's convenience, we recall the statement of Theorem 1.1.

**Theorem 6.1.** *Assume that there exists $v \in PSH^-(\Omega)$ such that $\mathrm{NP}(dd^c v)^n \geq \mu$ and $P[v] = \phi$. Denote*

$$S = \{w \in PSH^-(\Omega) : w \leq \phi, \mathrm{NP}(dd^c w)^n \geq \mu\}.$$

*Then $u_S := (\sup\{w : w \in S\})^*$ is a solution of the problem*

$$\begin{cases} \mathrm{NP}(dd^c u)^n = \mu, \\ P[u] = \phi, \end{cases} \tag{6.1}$$

*Moreover, if there exists $\psi \in \mathcal{N}_{\rm NP}$ such that $\mathrm{NP}(dd^c\psi)^n \geq \mu$ then $u_S$ is the unique solution of* (6.1).

*Proof.* By the assumption, we have $v \leq u_S \leq \phi$ and $P[v] = \phi$. Therefore, $P[u_S] = \phi$. We need to show that $\mathrm{NP}(dd^c u_S)^n = \mu$.

For every $j \geq 1$, we denote

$$\Omega_j = \{z \in \Omega \colon d(z, \partial\Omega) > 2^{-j}\},$$

and

$$U_j = \{z \in \Omega_j \colon v + \phi > -2^j\}.$$

We also define

$$S_{j,k} = \{w \in \mathrm{PSH}^-(\Omega) \colon w \leq \phi \text{ on } \Omega \setminus U_k, \mathrm{NP}(dd^c w)^n \geq \mathbb{1}_{U_j}\mu\},$$

for all $k, j \geq 1$. It is easy to see that $v \in S_{j,k}$, hence $u_{j,k} := (\sup\{w \in S_{j,k}\})^*$ is well-defined. Since $S \subset S_{j,k}$, we also have

$$u_S \leq u_{j,k} \tag{6.2}$$

Recall that

$$P[\phi]=\Big(\sup\{w\in \mathrm{PSH}^-(\Omega)\colon w\leq \phi+O(1) \text{ on } \Omega, \liminf_{\Omega\setminus\{\phi=-\infty\}\ni z\to\xi}(\phi(z)-w(z))\geq 0 \forall \xi\in\partial\Omega\}\Big)^*.$$

By the definition of $S_{j,k}$, we have $u_{j,k}\leq\phi$ on $\Omega\setminus\overline{U_k}$ and $\phi\geq v+\phi\geq -2^k$ on $\overline{U_k}$. Hence, $\phi+O(1)\geq u_{j,k}$ on $\Omega$ and $\liminf_{\Omega\setminus\{\phi=-\infty\}\ni z\to\xi}(\phi(z)-u_{j,k}(z))\geq 0$ for all $\xi\in\partial\Omega$. Consequently, we have, $u_{j,k}\leq P[\phi]$. Since $\phi$ is model, it follows that

$$u_{j,k}\leq\phi, \forall k,j\geq 1. \tag{6.3}$$

Moreover, it follows from Theorem 4.1 that

$$\mathbb{1}_{U_k}\mathrm{NP}(dd^cu_{j,k})^n=\mathbb{1}_{U_k}(\mathbb{1}_{U_j}\mu)=\mathbb{1}_{U_j}\mu, \tag{6.4}$$

for every $k\geq j\geq 1$.

Note that if $j_1\leq j_2$ and $k_1\geq k_2$ then $S_{j_1,k_1}\leq S_{j_2,k_2}$. Hence

$$u_{j_1,k_1}\leq u_{j_2,k_2}, \forall j_1\leq j_2, k_1\geq k_2. \tag{6.5}$$

Put

$$u_j=(\lim_{k\to\infty}u_{j,k})^*.$$

It follows from (6.2) and (6.3) that

$$u_S\leq u_j\leq\phi. \tag{6.6}$$

In particular, $u_j\not\equiv-\infty$. By using (6.4) and applying Lemmas 3.2 and 3.3, we get

$$\mathbb{1}_{U_j}\mathrm{NP}(dd^cu_j)^n=\mathbb{1}_{U_j}\mu, \forall j\geq 1. \tag{6.7}$$

It follows from (6.5) that $(u_j)_{j\geq 1}$ is a decreasing sequence. Set

$$\bar{u}=\lim_{j\to\infty}u_j.$$

By (6.6), we have

$$u_S\leq\bar{u}\leq\phi. \tag{6.8}$$

By using (6.7) and applying Lemmas 3.2 and 3.3, we deduce that

$$\mathbb{1}_{U_{j_0}}\mathrm{NP}(dd^c\bar{u})^n=\mathbb{1}_{U_{j_0}}\mu,$$

for every $j_0\geq 0$. Letting $j_0\to\infty$, we obtain

$$\mathbb{1}_{\bigcup\limits_{j\geq 1}U_j}\mathrm{NP}(dd^c\bar{u})^n=\mathbb{1}_{\bigcup\limits_{j\geq 1}U_j}\mu. \tag{6.9}$$

By definition, $\Omega\setminus\bigcup\limits_{j\geq 1}U_j=\{v+\phi=-\infty\}$ is a pluripolar set. Therefore, (6.9) implies that

$$\mathrm{NP}(dd^c\bar{u})^n=\mu.$$

This combined with (6.8) gives

$$u_S\leq\bar{u}\leq(\sup\{w\in\mathrm{PSH}^-(\Omega): w\leq\phi, \mathrm{NP}(dd^cw)^n\geq\mu\})^*=u_S.$$

Hence, $u_S=\bar{u}$ and $\mathrm{NP}(dd^cu_S)^n=\mu$. Thus, $u_S$ is a solution of (6.1).

Now, assume that there exists $\psi\in\mathcal{N}_{NP}$ and $\mathrm{NP}(dd^c\psi)^n\geq\mu$. We need to show that $u_S$ is the unique solution of the problem (6.1). Note that $v:=\psi+\phi$ satisfies the conditions $\mathrm{NP}(dd^cv)^n\geq\mathrm{NP}(dd^c\psi)^n\geq\mu$ and $P[v]=\phi$. Hence $u_S$ is a solution (6.1) satisfying

$$\phi+\psi\leq u_S\leq\phi.$$

In particular $u_S\in\mathcal{N}_{NP}(\phi)$.

Let u be an arbitrary solution of (6.1). We will show that $u\in\mathcal{N}_{NP}(\phi)$.

Denote

$$V_j = \{z \in \Omega_j, \mathfrak{u} > -2^j\},$$

and

$$\mathfrak{u}_j = \Big(\sup\{w \in \mathrm{PSH}^-(\Omega)\colon w \le \mathfrak{u} \text{ on } \Omega \setminus V_j\}\Big)^*.$$

By the same argument as in the proof of Proposition 4.6, we have

$$\Big(\lim_{j\to\infty} \mathfrak{u}_j\Big)^* = P[u] = \phi. \tag{6.10}$$

It is easy to see that

$$\mathfrak{u} \le \mathfrak{u}_j, \tag{6.11}$$

on $\Omega$. Moreover, $\mathfrak{u}_j + \psi$ satisfying the conditions

- $\mathfrak{u}_j + \psi \le \mathfrak{u}_j = \mathfrak{u} \quad \text{on} \quad \Omega \setminus \overline{V_j}$;
- $\mathfrak{u}_j + \psi \le \mathfrak{u}_j \le \mathfrak{u} + 2^j \quad \text{on} \quad \overline{V_j}$;
- $\mathrm{NP}(dd^c(\mathfrak{u}_j + \psi))^n \ge \mathrm{NP}(dd^c\psi)^n \ge \mu$.

Then, it follows from Corollary 5.2 that

$$\mathfrak{u}_j + \psi \le \mathfrak{u}. \tag{6.12}$$

Combining (6.10), (6.11) and (6.12), we get

$$\phi + \psi = \Big(\lim_{j\to\infty} (\mathfrak{u}_j + \psi)\Big)^* \le \mathfrak{u} \le \Big(\lim_{j\to\infty} \mathfrak{u}_j\Big)^* = \phi.$$

In particular, $\mathfrak{u} \in \mathcal{N}_{NP}(\phi)$. By Corollary 5.7, we have $\mathfrak{u} = u_S$. Thus, $u_S$ is the unique solution of (6.1).

This finishes the proof. □

6.2. **Proof of Corollary 1.3.** In order to prove Corollary 1.3, we need the following lemma:

**Lemma 6.2.** *Let $u, v, h \in \mathcal{D}(\Omega)$ such that $u + v \le h$. Assume that $(dd^c u)^n$ and $(dd^c v)^n$ vanish on pluripolar sets. Then $(dd^c h)^n$ vanishes on pluripolar set.*

*Proof.* Since the problem is local, we can assume that $\Omega$ is hyperconvex and $u, v, h$ are negative. In particular, $u, v, h \in \mathcal{E}(\Omega)$ (see [9, Theorem 4.5] and [6, Theorem 1.2]). Replacing $\Omega$ by a relative compact subset of $\Omega$, we can also assume that $\int_\Omega (dd^c w)^n < \infty$ for $w = u, v, h$.

Let $A \subset \Omega$ be a pluripolar set. By [2, Lemma 4.4] and by the assumption $\int_A (dd^c u)^n = \int_A (dd^c v)^n = 0$, we have

$$\int_A (dd^c u)^k \wedge (dd^c v)^{n-k} \le \Big(\int_A (dd^c u)^n\Big)^{k/n} \wedge \Big(\int_A (dd^c v)^n\Big)^{(n-k)/n} = 0,$$

for every $k = 0, 1, ..., n$. Therefore,

$$\int_A (dd^c(u+v))^n = \sum_{k=0}^n \binom{n}{k} \int_A (dd^c u)^k \wedge (dd^c v)^{n-k} = 0.$$

Since $A$ is arbitrary, we have $(dd^c(u+v))^n$ vanishes on every pluripolar set. Thus, it follows from [2, Lemma 4.1] that $(dd^c h)^n$ vanishes on pluripolar sets. □

Now we begin to prove Corollary 1.3. We recall its statement for the reader's convenience.

**Corollary 6.3.** *Assume that $\mu$ is a non-negative measure defined on $\Omega$ by $\mu = (dd^c\varphi)^n$ for some $\varphi \in \mathcal{N}^a(\Omega)$. Then, for every $H \in \mathcal{D}(\Omega)$ with $(dd^c H)^n \le \mu$, there exists a unique function $u \in \mathcal{N}^a(H)$ such that $(dd^c u)^n = \mu$ on $\Omega$.*

*Proof.* Put $\phi = P[H]$ and $u = (\sup\{w : w \in S\})^*$, where

$$S = \{w \in \mathrm{PSH}^-(\Omega) : P[w] = \phi, (dd^c w)^n \geq \mu\}.$$

Since $\phi + \varphi \in S$, we have

$$u \geq \phi + \varphi \geq H + \varphi. \tag{6.13}$$

By the definitions of $\mathcal{N}^a$ and $\mathcal{N}_{\mathrm{NP}}$, we have $\mathcal{N}^a \subset \mathcal{N}_{\mathrm{NP}}$. In particular, $\varphi \in \mathcal{N}_{\mathrm{NP}}$. Then, it follows from Theorem 1.1 that $u$ is the unique solution to the problem

$$\begin{cases} \mathrm{NP}(dd^c w)^n = \mu, \\ P[w] = \phi. \end{cases} \tag{6.14}$$

Moreover, it follows from [6, Theorem 1.2] and Lemma 6.2 that $u \in \mathcal{D}(\Omega)$ and $(dd^c u)^n$ vanishes on pluripolar sets. Hence, we have

$$(dd^c u)^n = \mu. \tag{6.15}$$

Denote $\nu = (dd^c H)^n$. Then $H$ is a solution of the problem

$$\begin{cases} \mathrm{NP}(dd^c w)^n = \nu, \\ P[w] = \phi. \end{cases} \tag{6.16}$$

Moreover, by Theorem 1.1, the problem (6.16) has a unique solution. Hence

$$H = \left(\sup\{w \in \mathrm{PSH}^-(\Omega) : P[w] = \phi, (dd^c w)^n \geq \nu\}\right)^* \geq u. \tag{6.17}$$

Combining (6.13) and (6.17), we get $u \in \mathcal{N}^a(H)$. This combined with (6.15) gives that $u$ is a solution of the problem

$$\begin{cases} w \in \mathcal{N}^a(H), \\ (dd^c w)^n = \mu. \end{cases} \tag{6.18}$$

It remains to show the uniqueness of solution of the problem (6.18). Assume that $v$ is a solution of (6.18). Then there exists $\psi \in \mathcal{N}^a$ such that

$$H + \psi \leq v \leq H.$$

Since $\mathcal{N}^a \subset \mathcal{N}_{\mathrm{NP}}$, it follows that

$$P[H] = P[H] + P[\psi] \leq P[H + \psi] \leq P[v] \leq P[H].$$

Then $P[v] = P[H] = \phi$. Moreover, since $\mu = (dd^c \varphi)^n$ vanishes on pluripolar sets, the condition $(dd^c v)^n = \mu$ implies that $\mathrm{NP}(dd^c v)^n = \mu$. Hence, $v$ is a solution of the problem (6.14). By the uniqueness of solution of (6.14), we have $v = u$. Thus, $u$ is the unique solution of (6.18).

The proof is completed. □

**Acknowledgments.** Thai Duong Do was supported by the MOE (Singapore) grant MOE-T2EP20120-0010. Hoang-Son Do was supported in part by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2021.16.

**Conflict of interest.** The authors declare that there is no conflict of interest.

## REFERENCES


[1] P. Åhag, *A Dirichlet problem for the complex Monge-Ampère operator in $\mathscr{F}(f)$*. Michigan Math. J. **55** (2007), no. 1, 123–138.

[2] P. Åhag, U. Cegrell, R. Czyż, H. -H Pham, *Monge-Ampère measures on pluripolar sets*, J. Math. Pures Appl. (9) **92** (2009), no. 6, 613–627.

[3] E. Bedford, B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. **37** (1976), no. 1, 1–44.

[4] E. Bedford, B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), no. 1-2, 1–40.

[5] E. Bedford, B. A. Taylor, *Fine topology, Šilov boundary, and $(dd^c)^n$*, J. Funct. Anal. **72** (1987), no. 2, 225–251.

[6] Z. Blocki, *The domain of definition of the complex Monge-Ampère operator*, Amer. J. Math. **128** (2006), no. 2, 519–530.

[7] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, *Monge-Ampère equations in big cohomology classes*, Acta Math. **205** (2010), no. 2, 199–262.

[8] S. Benelkourchi, V. Guedj and A. Zeriahi, "Plurisubharmonic functions with weak singularities", in *Complex analysis and digital geometry,* Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., vol. 86, Univ. Uppsala, 2009, p. 57–74.

[9] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 1, 159–179.

[10] T. Darvas, E. Di Nezza, H.-C. Lu, *Monotonicity of nonpluripolar products and complex Monge-Ampère equations with prescribed singularity*, Anal. PDE **11** (2018), no. 8, 2049–2087.

[11] T. Darvas, E. Di Nezza, H.-C. Lu, *Log-concavity of volume and complex Monge-Ampère equations with prescribed singularity*, Math. Ann. **379** (2021), no. 1-2, 95–132.

[12] J.-P. Demailly, *Potential theory in several complex variables*, Lecture notes, ICPAM, Nice, 1989.

[13] H-S. Do, T. D. Do, *Some remarks on the Cegrell class $\mathscr{F}$*. Ann. Polon. Math. **125** (2020), no. 1, 13–24.

[14] B. Fuglede, *Fonctions finement holomorphes de plusieurs variables - un essai*, Séminaire d'Analyse P. Lelong-P. Dolbeault-H. Skoda, 1983/85, pp. 133–145, Lecture Notes in Math. **1198**, Springer, Berlin, 1986.

[15] V. Guedj, A. Zeriahi, *The weighted Monge-Ampère energy of quasiplurisubharmonic functions*, J. Funct. Anal. **250** (2007), no. 2, 442–482.

[16] C. O. Kiselman, Sur la définition de l'opérateur de Monge-Ampère complexe, *Proc. Analyse Complexe, Toulouse 1983, Lecture Notes in Math.,* vol. 1094, Springer Verlag, Berlin, 1984, pp. 139–150.

[17] M. Klimek: *Pluripotential theory,* London Math. Soc. Monogr. (N.S.), 6 Oxford Sci. Publ. The Clarendon Press, Oxford University Press, New York, 1991. xiv+266 pp.

[18] V. K. Nguyen, H. -H. Pham, *A comparison principle for the complex Monge-Ampère operator in Cegrell's classes and applications*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5539–5554.

[19] Y. Xing, *Continuity of the Complex Monge-Ampère Operator*, Proc. Amer. Math. Soc. **124** (1996), no. 2, 457–467.



[1] DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE - 10, LOWER KENT RIDGE ROAD - SINGAPORE 119076[5]

*E-mail address*: duongdothai.vn@gmail.com

[2] INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18, HOANG QUOC VIET, HANOI, VIET NAM

*E-mail address*: hoangson.do.vn@gmail.com, dhson@math.ac.vn

[3] FACULTY OF MATHEMATICS AND INFORMATICS, HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO. 1 DAI CO VIET, HAI BA TRUNG, HA NOI, VIETNAM

*E-mail address*: tu.levan@hust.edu.vn

[4] INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18, HOANG QUOC VIET, HANOI, VIET NAM

*E-mail address*: cong.pnt.math@gmail.com, phamngocthanhcong1997@gmail.com


[5]On leave from Institute of Mathematics, Vietnam Academy of Science and Technology