LOG CONTINUITY OF SOLUTIONS OF COMPLEX MONGE-AMPÈRE EQUATIONS

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ABSTRACT. Let X be a compact Kähler manifold with semipositive anticanonical line bundle. Let L be a big and semi-ample line bundle on X and α be the Chern class of L. We prove that the solution of the complex Monge-Ampère equations in α with L^p right-hand side ($p > 1$) is $\log^M$-continuous for every constant $M > 0$. As an application, we show that every singular Ricci-flat metric in a semi-ample class in a projective Calabi-Yau manifold X is globally $\log^M$-continuous with respect to a smooth metric on X.

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1. INTRODUCTION

Let (X, ω) be a compact Kähler manifold. A cohomology $(1,1)$-class α is said to be semi-positive if α contains a semi-positive smooth form. Let θ be a smooth closed $(1,1)$-form in a big and semi-positive cohomology class. We consider the following complex Monge-Ampère equation

$$(dd^c u + \theta)^n = f\omega^n, \quad \sup_X u = 0, \tag{1.1}$$

where $f \in L^p$ ($p > 1$) is a nonnegative function so that $\int_X f\omega^n = \int_X \theta^n$. The regularity of solutions of (1.1) is well-known if θ is Kähler thanks to pioneering works by Yau [44] and Kołodziej [29], and many subsequent papers. We refer to [16, 18, 19, 31, 30, 35, 36, 37, 40, 41, 42] and references therein for details on Hölder continuity of solutions when θ is Kähler.

The focus of our work is the case where θ belongs to a semi-positive and big cohomology class. In this general setting, it is well-known by [7] that the solution u is smooth outside the non-Kähler locus of the cohomology class of θ. By [21, 8] or [17], we know that the equation (1.1) admits a unique continuous solution u on X if the cohomology class of θ is integral (see [23] for more information). The aim of this paper is to quantify this continuity property of solutions. The methods in [21, 8] or [17] seem to be only qualitative. To state our results, we need to introduce some notions.

Let $M > 0$ be a constant. We fix a smooth Riemannian metric $\operatorname{dist}(\cdot,\cdot)$ on X. A function u on X is said to be $\log^M$-continuous if there exists a constant $C_M > 0$ such that

$$|u(x) - u(y)| \leq \frac{C_M}{|\log \operatorname{dist}(\cdot,\cdot)|^M},$$

for every $x, y \in X$. Let K_X be the canonical line bundle of X. Recall that X is Calabi-Yau if $c_1(K_X) = 0$, and X is Fano if $K_X < 0$. A line bundle L on X is said to be semi-ample if L^k

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is base-point free for some large enough integer $k \geq 1$. It is a well-known fact that (see [38, Section 2] for a summary), L is automatically semi-ample if X is a projective Calabi-Yau manifold and L is big and nef.

Here is our main result in this work giving a partial answer to the above question.

Theorem 1.1. *Let (X, ω) be a compact Kähler manifold such that K_X^* is semi-positive (where K_X^* is the dual of the canonical line bundle K_X) and let L be a big and semi-ample line bundle on X. Assume f is a L^p function for some constant $p > 1$ and $\theta \in c_1(L)$ is a smooth form. Then the unique solution u of (1.1) is $\log^M$-continuous for every constant $M > 0$.*

As far as we know, Theorem 1.1 is probably the first known quantitative (global) regularity for solutions of complex Monge-Ampère equations in a semi-positive class. We would like to notice that it was proved in [27] that the solution of the equation $(dd^c u+\omega)^n = e^F\omega^n$ for $e^F \in L^1(\log L)^p$ is $\log^M$-continuous for $M := \min\{\frac{p-n}{n}, \frac{p}{n+1}\}$; see also [24] for a recent development. As far as we can see, the method in [27] or [24] uses crucially the fact that ω is Kähler and it is not clear if this can be extended to semi-positive classes to obtain a $\log^M$-continuity for solutions of (1.1).

Assume that X, L, ω are as in the statement of Theorem 1.1. Hence the non-Kähler locus N of $c_1(L)$ is a proper analytic subset in X; see [5]. Let F be a smooth function on X such that $\int_X e^F\omega^n = \int_X (c_1(L))^n$ and denote by ω_F the (singular) positive $(1,1)$-form on X such that $\omega_F^n = e^F\omega^n$. Recall that ω_F is a genuine Kähler metric on $X\backslash N$.

Corollary 1.2. *Assume that X, L, ω, N and F are as above. Then for every constant $M > 0$, there exists a constant $C_M > 0$ such that*

$$d_{\omega_F}(x, y) \leq C_M |\log \operatorname{dist}(x, y)|^{-M},$$

for every $x, y \in X\backslash N$, where d_{ω_F} is the distance induced by ω_F on $X\backslash N$.

In the case where θ is in a Kähler class, one has better estimates; see [24, 33, 43] for details. We are not aware of any previous result similar to Corollary 1.2 for merely semi-ample and big classes. As an immediate consequence of Corollary 1.2, we get the following.

Corollary 1.3. *Let (X, ω) be a compact Kähler manifold such that K_X^* is semi-positive and let L be a big and semi-ample line bundle on X. Assume ω_0 is a (singular) Kähler-Einstein metric in $c_1(L)$. Then ω_0 has a $\log^M$-continuous potential. Moreover, if d_{ω_0} denotes the distance induced by ω_0 on $X\backslash N$ then for every constant $M > 0$ there is a constant $C_M > 0$ so that*

$$d_{\omega_0}(x, y) \leq C_M |\log \operatorname{dist}(x, y)|^{-M},$$

for every $x, y \in X\backslash N$.

One can apply Corollary 1.3 to the case where X is Calabi-Yau. In this case ω_0 is the Ricci-flat metric in $c_1(L)$ which always exists uniquely (see [21]).

We now explain main ideas in the proof of Theorem 1.1. We will need to approximate our smooth solution u by smooth quasi-psh function $(u_\epsilon)_\epsilon$. Using [14] or [15, Theorem 4.12] (analytic approximation for general closed positive $(1,1)$-currents), one obtains $(\theta+\epsilon\omega)$-psh functions u_ϵ so that u_ϵ converges to u in a quantitative way in L^1. However $\|\nabla u_\epsilon\|_{L^\infty}$ grows like $e^{1/\epsilon}$. The fact that u_ϵ is only $(\theta + \epsilon\omega)$-psh and a bad control on $\|\nabla u_\epsilon\|_{L^\infty}$ is not usable in our approach. For this reason, we have to restrict ourselves to the line bundle setting for which a more precise approximation procedure is available. Precisely we will need a modified version of Demailly's analytic approximation of singular (not necessarily Kähler) Hermitian metrics for a line bundle (Theorem 5.11). This together with Kołodziej's capacity technique will give us a weak Log continuity property for u (see Lemma 6.2). Our second ingredient (Sections 2 and 3) is to say that a function satisfying this weak Log continuity property is indeed Log continuous as desired.

The paper is organized as follows. In Sections 2 and 3, we present important facts about log continuity of functions. In Section 4, we recall some facts about Hölder continuous measures. In Section 5, we present a modified version of Demailly's analytic approximation. The rest of the paper is devoted to the proof of main results.

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2. Log continuity of pseudometrics

Let Z be a topological space and $d : Z^2 \to \mathbb{R}_{\geq 0}$ be a function. Let $B \geq 1$ be a constant. We say that d is *a B-pseudometric on Z* if the following holds:

(i) d is symmetric, i.e, $d(x,y) = d(y,x)$,

(ii) d is continuous on Z^2,

(iii) for every $x_1, \ldots, x_m \in Z$, one has

$$d(x_1, x_m) \leq B \sum_{j=1}^{m-1} d(x_j, x_{j+1}).$$

Lemma 2.1. *Let $U \subset \mathbb{R}^m$ be a bounded convex domain ($m \geq 2$). Let $B \geq 1$ be a constant. Let $d : U \times U \to [0,\infty)$ be a B-pseudometric satisfying the following condition: there exist constants $\alpha > 0$, $D > 1$ and $C_0 > 0$ such that*

$$d(x,y) \leq \frac{C_0}{|\log|x-y||^{\alpha}}, \tag{2.1}$$

for every $x, y \in U$ with $|x-y|^D \leq \min\{\operatorname{dist}(x,\partial U), \operatorname{dist}(y,\partial U)\}$, where

$$\operatorname{dist}(w,\partial U) = \inf\{|w-\xi| : \xi \in \partial U\}.$$

Then, there exists a constant $C > 0$ depending only on B, C_0, α, D and U such that

$$d(x,y) \leq \frac{C}{|\log|x-y||^{\alpha}},$$

for every $x, y \in U$.

Proof. Without loss of generality, we can assume that $diam(U) \leq 1$. In particular, $|x-y|^D \leq |x-y|$ for every $x, y \in U$.

Fix $a \in U$ and denote $r = \operatorname{dist}(a,\partial U)$. The desired assertion is clear if we have either $|x-y| \geq r/2$ or

$$\min\{\operatorname{dist}(x,\partial U), \operatorname{dist}(y,\partial U)\} \geq r/2 \geq |x-y|$$

(by (2.1)). Consider now the case where

$$|x-y|\leq r/2 \quad \text{and} \quad \min\{\operatorname{dist}(x,\partial U),\operatorname{dist}(y,\partial U)\}<r/2.$$

Thus, we have $\max\{|x-a|,|y-a|\}>|x-y|$. Without loss of generality, we can assume that $|x-a|>|x-y|:=\delta>0$. Set

$$x_0=\frac{(|x-a|-\delta)x}{|x-a|}+\frac{\delta a}{|x-a|}.$$

In other words, x_0 is a point in $[x,a]$ satisfying $|x-x_0|=\delta$. Since U is convex, we have

$$\operatorname{dist}(x_0,\partial U)\geq\frac{(|x-a|-\delta)\operatorname{dist}(x,\partial U)}{|x-a|}+\frac{\delta\operatorname{dist}(a,\partial U)}{|x-a|}\geq r\delta. \tag{2.2}$$

For every $k\in\mathbb{Z}^+$, we denote by x_k the point in $[x,x_0]$ satisfying $|x-x_k|=\delta^{D^k}$. Then, we have

$$\operatorname{dist}(x_k,\partial U)\geq\frac{\operatorname{dist}(x_0,\partial U)|x_k-x|}{|x-x_0|}\geq r\delta^{D^k} \quad \text{and} \quad |x_k-x_{k-1}|\leq\delta^{D^{k-1}}.$$

Put $M=\left[\dfrac{1}{r}\right]+1$, where $[\cdot]$ is the greatest integer function. For every $l=0,...,M$ and $k\in\mathbb{Z}^+$, we denote

$$x_{k,l}=x_{k-1}+\frac{l(x_k-x_{k-1})}{M}.$$

Then $|x_{k,l}-x_{k,l+1}|\leq\dfrac{\delta^{D^{k-1}}}{M}$. Moreover, since $\operatorname{dist}(.,\partial U)$ is a concave function on U, we have

$$\operatorname{dist}(x_{k,l},\partial U)\geq\min\{\operatorname{dist}(x_k,\partial U),\operatorname{dist}(x_{k-1},\partial U)\}\geq r\delta^{D^k}\geq\frac{\delta^{D^k}}{M}.$$

Therefore, by the condition (2.1), we get

$$d(x_{k,l},x_{k,l+1})\leq\frac{C_0}{|\log|x_{k,l}-x_{k,l+1}||^{\alpha}}\leq\frac{C_0}{D^{(k-1)\alpha}|\log\delta|^{\alpha}}.$$

Thus, we have

$$d(x_k,x_{k-1})\leq B\sum_{l=0}^{M-1}d(x_{k,l},x_{k,l+1})\leq\frac{BC_0M}{D^{(k-1)\alpha}|\log\delta|^{\alpha}}.$$

Hence

$$d(x_k,x_0)\leq B\sum_{j=1}^{k}d(x_j,x_{j-1})\leq\frac{B^2C_0M}{|\log\delta|^{\alpha}}\sum_{j=1}^{k}D^{-(j-1)\alpha}\leq\frac{C_1}{|\log\delta|^{\alpha}},$$

where $C_1=\dfrac{B^2C_0M\,D^{-\alpha}}{1-D^{-\alpha}}=\dfrac{B^2C_0M}{D^{\alpha}-1}$.

Since d is continuous on $U\times U$, one gets

$$d(x,x_0)=\lim_{k\to\infty}d(x_k,x_0)\leq\frac{C_1}{|\log\delta|^{\alpha}}. \tag{2.3}$$

Since $|y-x_0|\leq|x-y|+|x-x_0|\leq 2\delta$, by using the same argument as above, we also have

$$d(y,x_0)\leq\frac{C_2}{|\log\delta|^{\alpha}}, \tag{2.4}$$

where $C_2 > 0$ depends only on B, C_0, M, D and α.

Combining (2.3) and (2.4), we get

$$d(x,y) \le B(d(x,x_0) + d(y,x_0)) \le \frac{B(C_1+C_2)}{|\log\delta|^\alpha} = \frac{B(C_1+C_2)}{|\log|x-y||^\alpha}.$$

The proof is completed. □

Lemma 2.2. *Let $U \subset \mathbb{R}^m$ be a bounded convex domain ($m \ge 2$). Let $B \ge 1$ be a constant. Assume $d : U \times U \to [0,\infty)$ is a B-pseudometric satisfying the following condition: there exist constants $\alpha > 1$ and $C_0 > 0$ such that*

$$d(x,y) \le \frac{C_0}{|\log|x-y||^\alpha}, \tag{2.5}$$

for every $x, y \in U$ with $|x-y| \le \min\{\operatorname{dist}(x,\partial U), \operatorname{dist}(y,\partial U)\}$, where

$$\operatorname{dist}(w,\partial U) = \inf\{|w-\xi| : \xi \in \partial U\}.$$

Then, there exists a constant $C > 0$ depending only on B, C_0, α, D and U such that

$$d(x,y) \le \frac{C}{|\log|x-y||^{\alpha-1}},$$

for every $x, y \in U$.

Proof. We will use the same method as in the proof of Lemma 2.1. Without loss of generality, we can assume that there exists $a \in U$ such that $r = \operatorname{dist}(a,\partial U) \ge 1$. We only need to consider the case where $|x-y| \le r/2$ and $\min\{\operatorname{dist}(x,\partial U), \operatorname{dist}(y,\partial U)\} < r/2$. In this case, we have $\max\{|x-a|, |y-a|\} > |x-y|$. We can assume that $|x-a| > |x-y| := \delta > 0$. Set

$$x_0 = \frac{(|x-a|-\delta)x}{|x-a|} + \frac{\delta a}{|x-a|}.$$

In other words, x_0 is a point in $[x,a]$ satisfying $|x-x_0| = \delta$. Since U is convex, we have

$$\operatorname{dist}(x_0,\partial U) \ge \frac{(|x-a|-\delta)\operatorname{dist}(x,\partial U)}{|x-a|} + \frac{\delta \operatorname{dist}(a,\partial U)}{|x-a|} \ge r\delta. \tag{2.6}$$

For every $k \in \mathbb{Z}^+$, we denote by x_k the point in $[x,x_0]$ satisfying $|x-x_k| = 2^{-k}\delta$. Then, we have

$$\operatorname{dist}(x_k,\partial U) \ge \frac{\operatorname{dist}(x_0,\partial U)|x_k-x|}{|x-x_0|} \ge r2^{-k}\delta \quad \text{and} \quad |x_k-x_{k-1}| \le 2^{-k}\delta.$$

By the condition (2.5), we have

$$d(x_k,x_{k-1}) \le \frac{C_0}{|\log|x_k-x_{k-1}||^\alpha} \le \frac{C_0}{(|\log\delta| + k\log 2)^\alpha},$$

for every $k \in \mathbb{Z}^+$. Hence

$$d(x_k,x_0) \le B\sum_{j=1}^{k} d(x_j,x_{j-1}) \le \sum_{j=1}^{k} \frac{BC_0}{(|\log\delta| + j\log 2)^\alpha} \le \frac{BC_0}{\log 2}\int_{\log|\delta|}^{\infty} \frac{dt}{t^\alpha} \le \frac{C_1}{|\log\delta|^{\alpha-1}},$$

where $C_1 = \frac{BC_0}{(\alpha-1)\log 2}$.

Since d is continuous on $U \times U$, one has

$$d(x, x_0) = \lim_{k\to\infty} d(x_k, x_0) \le \frac{C_1}{|\log\delta|^{\alpha-1}}. \tag{2.7}$$

Since $|y - x_0| \le |x - y| + |x - x_0| \le 2\delta$, by using the same argument as above, we also have

$$d(y, x_0) \le \frac{C_2}{|\log\delta|^{\alpha-1}}, \tag{2.8}$$

where $C_2 > 0$ depends only on B, C_0 and α. Combining (2.7) and (2.8), we get

$$d(x, y) \le B(d(x, x_0) + d(y, x_0)) \le \frac{B(C_1 + C_2)}{|\log\delta|^{\alpha-1}} = \frac{B(C_1 + C_2)}{|\log|x - y||^{\alpha-1}}.$$

The proof is completed. □

Proposition 2.3. *Let $N_1, N_2 ..., N_p$ be affine subspaces of $\mathbb{R}^m$ such that $\operatorname{codim}(N_j) \ge 2$ for every $j = 1, ..., p$. Denote $N = \cup_{j=1}^p N_j$. Let $B \ge 1$ be a constant. Let $\alpha > 0$, $D \ge 1$ and $C_0 > 0$ be constants. Let d be a B-pseudometric on $\mathbb{B}^m \setminus N$ satisfying one of the following conditions*

(i) *$D > 1$ and*

$$d(x, y) \le \frac{C_0}{|\log|x - y||^{\alpha}}, \tag{2.9}$$

for every $x, y \in \mathbb{B}^m \setminus N$ with $|x - y|^D \le \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}$.

(ii) *$D = 1$ and*

$$d(x, y) \le \frac{C_0}{|\log|x - y||^{\alpha+1}}, \tag{2.10}$$

for every $x, y \in \mathbb{B}^m \setminus N$ with $|x - y| \le \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}$.

Then, there exists $C > 0$ depending only on B, C_0, α, D, N and m such that

$$d(x, y) \le \frac{C}{|\log|x - y||^{\alpha}},$$

for every $x, y \in \mathbb{B}^m \setminus N$.

Proof. We will give the proof for the first case where (i) is satisfied. The second case is similar (use Lemma 2.2 in place of Lemma 2.1). Recall that d is a continuous functionon $(\mathbb{B}^m \setminus N) \times (\mathbb{B}^m \setminus N)$. Let H_j be a hyperplane containing N_j for $j = 1, ..., p$, and denote $H = \cup_{j=1}^p H_j$. Observe that the connected components of $\mathbb{B}^m \setminus H$ are bounded convex subsets of $\mathbb{R}^m$. Moreover, if U is a connected component of $\mathbb{B}^m \setminus H$ then by (2.9), u satisfies the condition (2.1) in Lemma 2.1.

Let $x, y \in \mathbb{B}^m \setminus N$. We distinguish into three cases.

Case 1: there exists a connected component U of $\mathbb{B}^m \setminus H$ such that $x, y \in \overline{U} \setminus N$.
In this case, by Lemma 2.1 and by the continuity of d, we have

$$d(x, y) \le \frac{C_U}{|\log|x - y||^{\alpha}},$$

where $C_U > 0$ is a constant depending only on B, C_0, α, D and U.

Case 2: $[x, y] \cap H \ne \emptyset$ but $[x, y] \cap N = \emptyset$.
In this case, there exist connected components $U_1, U_2, ..., U_k$ of $\mathbb{B}^m \setminus H$ and $x_0, x_1, x_2, ..., x_k \in$

$[x,y]$ such that $x_0 = x \in \overline{U_1}$, $x_k = y \in \overline{U_k}$ and $x_j \in \partial U_j \cap \partial U_{j+1}$ for every $j = 1, ..., k-1$. Using the result in Case 1, we have

$$\begin{aligned} d(x,y) \le B\sum_{j=1}^{k} d(x_j, x_{j-1}) &\le \sum_{j=1}^{k} \frac{BC_{U_j}}{|\log|x_j - x_{j-1}||^\alpha} \\ &\le \frac{kBC_1}{|\log|x-y||^\alpha} \\ &\le \frac{(p+1)BC_1}{|\log|x-y||^\alpha}, \end{aligned}$$

where $C_1 = \sup\{C_U : U \text{ is a connected component of } \mathbb{B}^m \setminus N\}$.

Case 3: $[x,y] \cap N \neq \emptyset$.

Denote $f(t) = tx + (1-t)y$, $0 \le t \le 1$. Then, there exist $0 < k \le p$ and $0 < t_1 < t_2 < ... < t_k < 1$ such that

$$[x,y] \cap N = \{f(t_j) : j = 1, ..., k\}.$$

By Lemma 2.4 below, for every $j = 1, ..., k$ and for every $0 < \epsilon \ll 1$, there exists a pieceweise linear curve $l = a_0 a_1 ... a_{4^p}$ with $a_0 = f(t_j + \epsilon)$ and $a_{4^p} = f(t_j - \epsilon)$ such that l does not intersect N and

$$L(l) \le C_2 |f(t_j + \epsilon) - f(t_j - \epsilon)| = 2C_2\epsilon |x-y|,$$

where $C_2 \ge 1$ is a constant depending only on p. Therefore, by the result in Case 2, we have

$$d(f(t_j + \epsilon), f(t_j - \epsilon)) = O(\epsilon). \tag{2.11}$$

Denote $t_0 = 0$ and $t_{k+1} = 1$. By Case 2 and by (2.11), we have

$$\begin{aligned} d(x,y) &= \lim_{\epsilon \to 0+} d(f(t_0 + \epsilon), f(t_{k+1} - \epsilon)) \\ &\le \limsup_{\epsilon \to 0+} B \sum_{j=0}^{k} d(f(t_j + \epsilon), f(t_{j+1} - \epsilon)) + \limsup_{\epsilon \to 0+} B \sum_{j=1}^{k} d(f(t_j + \epsilon), f(t_j - \epsilon)) \\ &\le \limsup_{\epsilon \to 0+} \sum_{j=0}^{k} \frac{B^2 C_1 (p+1)}{|\log|f(t_j + \epsilon) - f(t_{j+1} - \epsilon)||^\alpha} \\ &\le \frac{B^2 C_1 (p+1)^2}{|\log|x-y||^\alpha}. \end{aligned}$$

The proof is completed. □

The following lemma plays also an important role in our proof later.

Lemma 2.4. *Let $N_1, N_2, ..., N_k$ be affine subspaces of $\mathbb{R}^m$ such that $\operatorname{codim}(N_j) \ge 2$ for every $j = 1, ..., k$. Denote $N = \cup_{j=1}^k N_j$. Then, there is a constant $C \ge 1$ depending only on k (and m) satisfying the following property: for every $x, y \in \mathbb{R}^m \setminus N$, there is a polygonal chain $l = a_0 a_1 ... a_{4^k}$ with $a_0 = x$ and $a_{4^k} = y$ such that*

$$C \operatorname{dist}(\xi, N) \ge \min\{\operatorname{dist}(x,N), \operatorname{dist}(y,N)\},$$

for every $\xi \in \cup_{s=0}^{4^k-1}[a_s, a_{s+1}]$*, and*

$$L(l) \leq C|x-y|,$$

where $L(l) = |a_0 - a_1| + |a_1 - a_2| + ... + |a_{4^k-1} - a_{4^k}|$ *is the length of* l*.*

In order to prove Lemma 2.4, we need the following elementary lemma:

Lemma 2.5. *Let* N *be an affine subspace of* $\mathbb{R}^m$ *with* $\operatorname{codim} N \geq 2$*. Let* $r \geq 1$ *be a constant. Then, for every* $x, y \in \mathbb{R}^m \setminus N$*, there exists* $w \in \mathbb{R}^m \setminus N$ *such that*

$$|x-w| + |w-y| \leq 3|x-y|,$$

and

$$2r \operatorname{dist}(\xi, N) \geq \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\} \geq r \operatorname{dist}(\xi, [x, y]),$$

for every $\xi \in [x, w] \cup [w, y]$*.*

Proof. Observe that the function $\operatorname{dist}(\cdot, N)$ is convex on $\mathbb{R}^m$. Indeed, for every $a, b \in \mathbb{R}^m$, there exist $a_0, b_0 \in \mathbb{N}$ such that $|a - a_0| = \operatorname{dist}(a, N)$ and $|b - b_0| = \operatorname{dist}(b, N)$. Hence, if $\eta = \alpha a + (1-\alpha)b$ for some $\alpha \in [0, 1]$ then

$$\begin{aligned}\alpha \operatorname{dist}(a, N) + (1-\alpha)\operatorname{dist}(b, N) &= \alpha|a - a_0| + (1-\alpha)|b - b_0| \\ &\geq |\alpha(a - a_0) + (1-\alpha)(b - b_0)| \\ &= |\eta - (\alpha a_0 + (1-\alpha)b_0)| \geq \operatorname{dist}(\eta, N).\end{aligned}$$

Let $R := \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}$. If $\operatorname{dist}(\eta, N) \geq R/(2r)$ for every $\eta \in [x, y]$, then $w := x$ satisfies the desired property. Assume, from now on, that there is a point $\eta \in [x, y]$ such that $\operatorname{dist}(\eta, N) \leq R/(2r) \leq R/2$. We deduce that

$$|x-y| = |x-\eta| + |\eta - y| \geq \operatorname{dist}(x, N) - \operatorname{dist}(\eta, N) + \operatorname{dist}(y, N) - \operatorname{dist}(\eta, N) \geq R. \tag{2.12}$$

We distinguish into three cases

Case 1: Either $[x, y]$ is parallel to N or the line passing through x, y intersects N but $[x, y] \cap N = \varnothing$.
In this case, we can take $w := x$.

Case 2: $[x, y] \cap N \neq \varnothing$.
Since $\operatorname{codim} N \geq 2$, there exists a hyperplane $\tilde{N}$ containing x, y, N. Let $w_0 = [x, y] \cap N$. Let w be a point in $\mathbb{R}^m \setminus \tilde{N}$ so that $|w - w_0| = \frac{R}{r}$ and $[w, w_0]$ is orthogonal to $\tilde{N}$. We have

$$|w-x| + |w-y| \leq |x - w_0| + 2|w - w_0| + |y - w_0| \leq |x-y| + 2R \leq 3|x-y|,$$

where the last estimate holds due to (2.12).

Moreover, if $\xi \in [x, w] \cup [y, w]$ then

$$\operatorname{dist}(\xi, [x, y]) \leq \operatorname{dist}(w, [x, y]) = |w - w_0| = \frac{R}{r}.$$

Let $x_0 \in N$ such that $|x - x_0| = \operatorname{dist}(x, N)$. If $\xi \in [x, w]$ then $\xi = \alpha x + (1-\alpha)w$ for some $\alpha \in [0, 1]$. Since $x - x_0 \perp N$ and $w - w_0 \perp \tilde{N}$, we have

$$\operatorname{dist}(\xi, N) = |\xi - \alpha x_0 - (1-\alpha)w_0| = \sqrt{\alpha^2|x - x_0|^2 + (1-\alpha)^2|w - w_0|^2} \geq \frac{R}{2r}.$$

Similarly, if $\xi \in [y,w]$ then we also have $\mathrm{dist}(\xi, N) \geq \frac{R}{2r}$. Then, w satisfies the desired properties.

Case 3: $[x,y]$ is not parallel to N and the line d passing through x, y does not intersect N.
In this case, there exist $w_1 \in N$, $w_2 \in d$ such that $[w_1, w_2]$ is orthogonal to N and d, and $|w_1 - w_2| = \min\{|z - z'| : z \in d, z' \in N\}$. Using the convexity of $d(\cdot, N)$ and the fact that there exists $\eta \in [x,y]$ with $d(\eta, N) < R/2$, we deduce that $d(\xi, N) > R/2$ for every $\xi \in d \backslash [x,y]$. Consequently $w_2 \in [x,y]$.

Let w be the point in the line passing through w_1, w_2 such that w_2 lies between w_1 and w, and $|w - w_2| = R/r$. We check that w satisfies the required properties. Let $\xi \in [w,x]$. Write $\xi = \alpha x + (1-\alpha)w$ for some constant $\alpha \in [0,1]$. Let ξ_0, x_0 be points in N such that $[x, x_0]$ and $[\xi, \xi_0]$ are orthogonal to N. We have

$$\xi_0 = \alpha x_0 + (1-\alpha) w_1.$$

Compute

$$\begin{aligned}|\xi - \xi_0|^2 &= |\alpha(x - x_0) + (1-\alpha)(w - w_1)|^2 \\ &= \alpha^2 |x - x_0|^2 + (1-\alpha)^2 |w - w_1|^2 + 2\alpha(1-\alpha)\langle x - x_0, w - w_1\rangle.\end{aligned}$$

Recall that $w - w_1$ is both orthogonal to N and d. It follows that

$$\langle x - x_0, w - w_1\rangle = \langle x - w_2, w - w_1\rangle + \langle w_2 - w_1, w - w_1\rangle + \langle w_1 - x_0, w - w_1\rangle$$

which is equal to $\langle w_2 - w_1, w - w_1\rangle \geq 0$. Hence we obtain

$$\begin{aligned}|\xi - \xi_0|^2 &\geq \alpha^2 |x - x_0|^2 + (1-\alpha)^2 |w - w_1|^2 \\ &\geq \alpha^2 R^2 + (1-\alpha)^2 R^2/r^2 \geq \frac{R^2}{2r^2}.\end{aligned}$$

Since $\mathrm{dist}(\xi, N) = |\xi - \xi_0|$, we infer

$$2\,\mathrm{dist}(\xi, N) \geq R/r.$$

On the other hand, we have

$$\mathrm{dist}(\xi, [x,y]) \leq \mathrm{dist}(w, [x,y]) \leq |w - w_2| = R/r.$$

We obtain a similar inequalities if $\xi \in [w,y]$. Finally, observe

$$|w-x| + |w-y| \leq |w - w_2| + |x - w_2| + |w - w_2| + |w_2 - y| = 2R + |x-y| \leq 3R \leq 3|x-y|,$$

because $w_2 \in [x,y]$ and we used here (2.12). Thus w satisfies the desired properties.

This finishes the proof. □

Proof of Lemma 2.4. We will use induction in k. The case $k = 1$ is an immediate corollary of Lemma 2.5. Assume that Lemma 2.4 is true for $k = k_0$. We will show that it is also true for $k = k_0 + 1$.

Denote $N' = N_1 \cup N_2 \cup ... \cup N_{k_0}$ and $N = N_1 \cup N_2 \cup ... \cup N_{k_0+1}$. Let $x, y \in \mathbb{R}^m \setminus N$, $x \neq y$. By the induction assumption, there exists a polygonal chain $l_0 = a_0 a_1 ... a_{4^{k_0}}$ with $a_0 = x$ and $a_{4^{k_0}} = y$ such that

$$L(l_0) \leq C_0 |x - y|, \tag{2.13}$$

and

$$C_0 \operatorname{dist}(\xi, N') \geq \min\{\operatorname{dist}(x, N'), \operatorname{dist}(y, N')\}, \tag{2.14}$$

for every $\xi \in \cup_{s=1}^{4^{k_0}}[a_{s-1}, a_s]$, where $C_0 \geq 1$ is a constant depending only on k_0 and m.

We will construct a polygonal chain $l = b_0 b_1 ... b_{4^{k_0}+1}$ satisfying the conditions as in Lemma 2.4. Denote

$$A := \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}.$$

If $2C_0 \operatorname{dist}(\xi, N) \geq A$ for every $\xi \in \cup_{s=1}^{4^{k_0}}[a_{s-1}, a_s]$ then we can choose $l = l_0$ and $C = 2C_0$. It remains to consider the case where $2C_0 \operatorname{dist}(\xi_0, N) < A$ for some $\xi_0 \in \cup_{s=1}^{4^{k_0}}[a_{s-1}, a_s]$. In this case, we have

$$L(l_0) \geq |x-\xi_0|+|y-\xi_0| \geq (\operatorname{dist}(x, N)-\operatorname{dist}(\xi_0, N))+(\operatorname{dist}(y, N)-\operatorname{dist}(\xi_0, N)) \geq A. \tag{2.15}$$

For every $s = 0, ..., 4^{k_0}$, we define b_{4s} as follows

- If $2C_0 \operatorname{dist}(a_s, N) \geq A$ then we put $b_{4s} = a_s$;
- If $2C_0 \operatorname{dist}(a_s, N) < A$ then we choose $b_{4s} \in \mathbb{R}^m$ such that the vector $b_{4s} - a_s$ is perpendicular to N_{k_0+1} and

$$\operatorname{dist}(b_{4s}, N_{k_0+1}) = |a_s - b_{4s}| + \operatorname{dist}(a_s, N_{k_0+1}) = \frac{A}{2C_0}. \tag{2.16}$$

Thus we have

$$|b_{4s} - b_{4s+4}| \leq |a_s - a_{s+1}| + |a_s - b_{4s}| + |a_{s+1} - b_{4s+4}| \leq |a_s - a_{s+1}| + \frac{A}{C_0}, \tag{2.17}$$

and

$$\operatorname{dist}(\xi, [a_s, a_{s+1}]) \leq \max\{|b_{4s} - a_s|, |b_{4s+4} - a_{s+1}|\} \leq \frac{A}{2C_0}, \tag{2.18}$$

for each $\xi \in [b_{4s}, b_{4s+4}]$ and for every $s = 0, 1, ..., 4^{k_0} - 1$.

Combining (2.14) and (2.18), we get

$$\operatorname{dist}(\xi, N') \geq \inf_{\eta \in [a_s, a_{s+1}]} \operatorname{dist}(\eta, N') - \operatorname{dist}(\xi, [a_s, a_{s+1}]) \geq \frac{A}{2C_0}, \tag{2.19}$$

for each $\xi \in [b_{4s}, b_{4s+4}]$ and for every $s = 0, 1, ..., 4^{k_0} - 1$.

We will find b_{4s+1}, b_{4s+2} and b_{4s+3} such that

(i) $\sum_{j=4s}^{4s+3} |b_j - b_{j+1}| \leq 3|a_s - a_{s+1}| + \frac{2A}{C_0}$;
(ii) $\operatorname{dist}(\xi, N) \geq \frac{A}{8C_0}$ for every $\xi \in \cup_{j=4s}^{4s+3}[b_j, b_{j+1}]$.

We distinguish into three cases.

Case 1: $\operatorname{dist}(\xi, N_{k_0+1}) \geq \frac{A}{4C_0}$ for all $\xi \in [b_{4s}, b_{4s+4}]$.
In this case, we put $b_{4s+1} = b_{4s+2} = b_{4s+3} = b_{4s+4}$. It follows from (2.17) and (2.19) that the conditions (i) and (ii) are satisfied.

Case 2: $\operatorname{dist}(\xi_0, N_{k_0+1}) < \frac{A}{4C_0}$ for some $\xi_0 \in [b_{4s}, b_{4s+4}]$ and either $a_s \neq b_{4s}$ or $a_{s+1} \neq b_{4s+4}$.
In this case, we have

$$\min\{\operatorname{dist}(b_{4s}, N_{k_0+1}), \operatorname{dist}(b_{4s+4}, N_{k_0+1})\} = \frac{A}{2C_0}.$$

By Lemma 2.5, we can choose $q \in \mathbb{R}^n$ such that

$$|b_{4s} - q| + |b_{4s+4} - q| \leq 3|b_{4s} - b_{4s+4}| \tag{2.20}$$

and

$$4\,\mathrm{dist}(\xi, N_{k_0+1}) \geq \frac{A}{2C_0} \geq 2\,\mathrm{dist}(\xi, [b_{4s}, b_{4s+4}]), \tag{2.21}$$

for every $\xi \in [b_{4s}, q] \cup [q, b_{4s+4}]$.

By (2.19) and (2.21), we have

$$\mathrm{dist}(\xi, N') \geq \inf_{\eta \in [b_{4s}, b_{4s+4}]} \mathrm{dist}(\eta, N') - \mathrm{dist}(\xi, [b_{4s}, b_{4s+4}]) \geq \frac{A}{4C_0}. \tag{2.22}$$

for every $\xi \in [b_{4s}, q] \cup [q, b_{4s+4}]$.

Put $b_{4s+1} = b_{4s+2} = b_{4s+3} = q$. It follows from (2.17) and (2.20) that (i) is satisfied. By (2.21) and (2.22), we also get (ii).

Case 3: $\mathrm{dist}(\xi_0, N_{k_0+1}) < \frac{A}{4C_0}$ for some $\xi_0 \in [b_{4s}, b_{4s+4}]$ and $a_j = b_{4j}$ for $j = s, s+1$.
In this case, we choose $b_{4s+2} \in \mathbb{R}^m$ such that the vector $b_{4s} - a_s$ is perpendicular to N_{k_0+1} and

$$\mathrm{dist}(b_{4s+2}, N_{k_0+1}) = |b_{4s+2} - b_{4s}| + \mathrm{dist}(\xi_0, N_{k_0+1}) = \frac{A}{4C_0}. \tag{2.23}$$

Similar to (2.17) and (2.18) (and note that $a_j = b_{4j}$ for $j = s, s+1$), we have

$$|b_{4s} - b_{4s+2}| + |b_{4s+2} - b_{4s+4}| \leq |b_{4s} - b_{4s+4}| + \frac{A}{2C_0} = |a_s - a_{s+1}| + \frac{A}{2C_0}, \tag{2.24}$$

and

$$\mathrm{dist}(\xi, [a_s, a_{s+1}]) = \mathrm{dist}(\xi, [b_{4s}, b_{4s+4}]) \leq \frac{A}{4C_0}, \tag{2.25}$$

for every $\xi \in [b_{4s}, b_{4s+2}] \cup [b_{4s+2}, b_{4s+4}]$.

Combining (2.14) and (2.25), we get

$$\mathrm{dist}(\xi, N') \geq \frac{3A}{4C_0}, \tag{2.26}$$

for every $\xi \in [b_{4s}, b_{4s+2}] \cup [b_{4s+2}, b_{4s+4}]$.

By using Lemma 2.5 for $[b_{4s}, b_{4s+2}]$ and $[b_{4s+2}, b_{4s+4}]$, we can choose b_{4s+1} and b_{4s+3} such that

$$\sum_{j=4s}^{4s+3} |b_j - b_{j+1}| \leq 3(|b_{4s} - b_{4s+2}| + |b_{4s+2} - b_{4s+4}|), \tag{2.27}$$

and

$$2\,\mathrm{dist}(\xi, N_{k_0+1}) \geq \frac{A}{4C_0} \geq \mathrm{dist}(\xi, [b_{4s}, b_{4s+2}] \cup [b_{4s+2}, b_{4s+4}]), \tag{2.28}$$

for every $\xi \in \cup_{j=4s}^{4s+3} [b_j, b_{j+1}]$.

By (2.26) and (2.28), we have

$$\mathrm{dist}(\xi, N') \geq \inf_{\eta \in I} \mathrm{dist}(\eta, N') - \mathrm{dist}(\xi, I) \geq \frac{A_0}{2C_0}, \tag{2.29}$$

for every $\xi \in \cup_{j=4s}^{4s+3}[b_j, b_{j+1}]$, where $I = [b_{4s}, b_{4s+2}] \cup [b_{4s+2}, b_{4s+4}]$. It follows from (2.28) and (2.29) that (ii) is satisfied. By (2.24) and (2.27), we also obtain (i).

Now, let $l_0 = b_0 b_1 ... b_{4^{k_0+1}}$. By (ii), we have

$$\operatorname{dist}(\xi, N) \geq \frac{1}{8C_0} \min\{\operatorname{dist}(x, N), \operatorname{dist}(y, N)\}.$$

By (2.13), (2.15) and (i), we have

$$\begin{aligned} L(l) = \sum_{j=0}^{4^{k_0+1}-1} |b_j - b_{j+1}| &\leq \sum_{s=0}^{4^{k_0}-1} \left(3|a_s - a_{s+1}| + \frac{2A}{C_0}\right) \\ &= 3L(l_0) + 4^{k_0}\frac{2A}{C_0} \\ &\leq 3\left(1 + \frac{4^{k_0}}{C_0}\right) L(l_0) \\ &\leq 3\left(C_0 + 4^{k_0}\right)|x - y|. \end{aligned}$$

Choosing $C = 8\left(C_0 + 4^{k_0}\right)$, we see that l and C satisfy the desired conditions. Thus, Lemma 2.4 is true in the case $k = k_0 + 1$. This completes the proof. □

3. Log continuity preserved under blowups

Let $f : X \to Y$ be a smooth surjective map between compact differentiable manifolds. Let g_X, g_Y be Riemannian metrics on X, Y respectively. Let d_{g_X}, d_{g_Y} denote the distances induced by g_X, g_Y on X, Y respectively. For $E \subset X$, let $d_X(a, E) := \inf_{b \in E} d_X(a, b)$. For every $a, b \in Y$, we define

$$d_{g_X, f}(a, b) := \inf_{a' \in f^{-1}(a), b' \in f^{-1}(b)} d_{g_X}(a', b').$$

We note the last function is in general not a metric on Y. Observe

$$d_{g_Y} \leq C d_{g_X, f} \tag{3.1}$$

for some constant $C > 0$ because the differential Df is bounded uniformly on X.

Lemma 3.1. *Let $X_0, \dots, X_m$ be compact complex manifolds and $f : X_j \to X_{j-1}$ be the blow up along a smooth submanifold $V_{j-1} \subset X_{j-1}$ in X_{j-1} for $1 \leq j \leq m$. Let $f := f_m \circ \cdots \circ f_0 : X_m \to X_0$. Let g_j be a Riemannian metric on X_j for $1 \leq j \leq m$. Let $A > 0, \beta \in (0, 1]$ be constants. Let u be a function on X_0 and $M > 0$ be a constant. Then if $u \circ f_m$ is a $\log^M$-continuous function, then so is u.*

We note that a similar property for Hölder continuity was proved in [26]. The following proof is more or less similar.

Proof. This is indeed implicitly in the proof of Lemma 3.4 in [26] if $m = 1$. The general case follows from an immediate induction on m. For readers' convenience, we reprove below the case where $m = 1$.

Let $f_1 : X_1 \to X_0$ be the blow up along a smooth submanifold V in X_0. Let $n := \dim X_0$ and $l := \dim V$. Let $a \in V$ and let $\big(U, x = (x_1, \dots, x_n)\big)$ be a local chart around a such that V is given by $\{x_j = 0, 1 \leq j \leq n - l\}$. Thus $f_1^{-1}(U)$ is biholomorphic to the submanifold of $U \times \mathbb{CP}^{n-l-1}$ defined by the equations $x_j v_s = v_j x_s$ for $1 \le j, s \le n-l$, where $v := [v_1 : \cdots : v_{n-l}]$ are the homogeneous coordinates on $\mathbb{CP}^{n-l-1}$. One can cover $f_1^{-1}(U)$ by $(n-l)$ open subsets

$$U_j := \big\{(x, v) \in f_1^{-1}(U) : v_j \neq 0\big\}.$$

In U_j, we have

$$f_1(x, v) = (v_1 x_j / v_j, v_2 x_j / v_j, \ldots, v_{n-l} x_j / v_j, x_{n-l+1}, \ldots, x_n).$$

Now let $a, b \in X_0$. It suffices to consider a, b close to each other and both close to V (because f_1 is biholomorphic outside V). We split the proof into several cases. Firstly observe that if $a, b \in V$, then since $f_1 : f_1^{-1}(V) \to V$ is a submersion, one gets

$$C d_{g_0}(a, b) \ge d_{g_1, f_1}(a, b),$$

for some constant $C > 0$ independent of a and b. Hence

$$\begin{aligned} |u(a) - u(b)| &= \inf_{a' \in f_1^{-1}(a), b' \in f_1^{-1}(b)} |u \circ f_1(a') - u \circ f_1(b')| \\ &\lesssim |\log d_{g_1, f_1}(a, b)|^{-M} \lesssim |\log d_{g_0}(a, b)|^{-M}. \end{aligned} \tag{3.2}$$

Note that in the last inequality, we only consider a and b close to each other, hence $\log d_{g_0}(a, b) < 0$.

Case 1. Consider now $b \in V$ and $a \notin V$ but close to b. Then there is a local chart (U, x) on X_0 containing b, a such that V is given by $\{x_j = 0, 1 \le j \le n-l\}$. We use now the Euclidean distance on that local chart.

Without loss of generality, we can assume that $b = 0$, $a = (x_1', \ldots, x_n')$ with $x_1' \neq 0$ and $h(t) := (t x_1', \ldots, t x_n') \in U$ for every $t \in [0, 1]$. We see that

$$\hat{h}(t) := f_1^{-1} \circ h(t) = (t x_1', \ldots, t x_n', [x_1' : \ldots : x_n']),$$

for $t > 0$. Letting $t \to 0$ gives

$$\hat{h}(0) := \lim_{t \to 0} f_1^{-1} \circ h(t) = (0, \ldots, 0, [x_1' : \ldots : x_n']) \in f_1^{-1}(b).$$

We infer that

$$d_{g_1}\big(\hat{h}(1), \hat{h}(0)\big) \lesssim |x_1'| + \cdots + |x_n'| \lesssim |a - b|.$$

It follows that

$$d_{g_1}\big(\hat{h}(1), f^{-1}(b)\big) \lesssim |a - b| \lesssim d_{g_0}(a, b).$$

Hence $d_{g_1, f_1}(a, b) \lesssim d_{g_0}(a, b)$. Thus we get an estimate similar to (3.2).

Case 2. Consider now $a, b \notin V$ but close to V. Direct computations show that $|D f_1^{-1}(a)| \lesssim |d_{g_0}(a, V)|^{-2}$. Thus we get

$$d_{g_1, f_1}(a, b) \lesssim \max\{d_{g_0}(a, V)^{-2}, d_{g_0}(b, V)^{-2}\} d_{g_0}(a, b).$$

Hence if $\min\{d_{g_0}(a, V)^2, d_{g_0}(b, V)^2\} \ge d_{g_0}(a, b)^{1/2}$, then

$$d_{g_1, f_1}(a, b) \lesssim d_{g_0}(a, b)^{1/2}.$$

We treat now the case where $\min\{d_{g_0}(a, V)^2, d_{g_0}(b, V)^2\} \le d_{g_0}(a, b)^{1/2}$. Without loss of generality, we can assume that $d_{g_0}(b, V) \le d_{g_0}(a, b)^{1/4}$. Then

$$d_{g_0}(a, V) \le d_{g_0}(a, b) + d_{g_0}(b, V) \lesssim d_{g_0}(a, b)^{1/4}.$$

Now we consider a local chart (U, x) containing a, b. We use now the Euclidean metric. Let a_V, b_V be the projection of a, b to V respectively.

$$|a_V - b_V| \lesssim |a - b|$$

and

$$|a - a_V| \lesssim |a - b|^{1/4}, |b - b_V| \lesssim |a - b|^{1/4}.$$

Now applying Case 1 to $(a, a_V), (b, b_V)$ and (a_V, b_V), one obtains

(3.3)

$$d_{g_1,f_1}(a, a_V) + d_{g_1,f_1}(a_V, b_V) + d_{g_1,f_1}(b_V, b) \lesssim |a - a_V| + |a_V - b_V| + |b - b_V| \lesssim |a - b|^{1/4}.$$

Put $x_1 := |u(a) - u(a_V)|$, $x_2 := |u(a_V) - u(b_V)|$ and $x_3 := |u(b_V) - u(b)|$. By previous parts of the proof, we see that

$$d_{g_0}(a, a_V) \gtrsim e^{-x_1^{-\frac{1}{M}}}, \quad d_{g_0}(a_V, b_V) \gtrsim e^{-x_2^{-\frac{1}{M}}},$$

and

$$d_{g_0}(b_V, b) \gtrsim e^{-x_3^{-\frac{1}{M}}}.$$

This combined with (3.1) and (3.3) gives

$$|a - b|^{1/4} \gtrsim \sum_{j=1}^{3} e^{-x_j^{-\frac{1}{M}}} \gtrsim \exp\left\{ -\left(\frac{x_1 + x_2 + x_3}{3} \right)^{-1/M} \right\}.$$

It follows that

$$x_1 + x_2 + x_3 \lesssim \left| \log \frac{|a - b|}{C} \right|^{-M} \lesssim |\log |a - b||^{-M},$$

for some constant $C > 0$ independent of a and b. The left-hand side of the last inequality is $\geq |u(a) - u(b)|$. Hence $|u(a) - u(b)| \lesssim |\log |a - b||^{-M}$. This finishes the proof. □

4. HÖLDER CONTINUOUS MEASURES

Let η be a closed smooth semi-positive $(1,1)$-form in a big (semi-positive) cohomology class. Let K be a Borel subset of X. The *capacity* of K is given by

$$\text{cap}_\eta(K) := \sup \Big\{ \int_K \eta_\varphi^n : 0 \leq \varphi \leq 1, \varphi \ \eta\text{-psh} \Big\}.$$

The above notion was introduced in [21] generalizing those in [2, 29]; see [11, 34] and references therein for various generalizations of capacity.

Let $A, \beta > 0$. We say that a Borel measure μ on X satisfies the condition $\mathcal{H}(\beta, A, \eta)$ if

$$\mu(K) \leq A \left(\text{cap}_\eta(K) \right)^{1+\beta},$$

for every Borel set $K \subset X$.

Fix a Kähler form ω on X. Let μ be a measure on X. Recall that μ is said to be a Hölder continuous measure with the Hölder constant A and the Hölder exponent $\gamma \in (0, 1]$ if for every ω-psh function φ_1, φ_2 with $\int_X \varphi_j \omega^n = 0$ for $j = 1, 2$ there holds

$$\int_X (\varphi_1 - \varphi_2) d\mu \leq A \|\varphi_1 - \varphi_2\|_{L^1(\omega^n)}.$$

Let $\mathcal{M}(A,\gamma)$ be the set of Hölder continuous measures with the Hölder constant A and the Hölder exponent $\gamma \in (0,1]$. By [19, Lemma 3.3], a measure $\mu \in \mathcal{M}(A,\gamma)$ if there is a constant $C > 0$ depending only on A such that for every ω-psh function φ_1, φ_2, we have

$$\int_X |\varphi_1 - \varphi_2| d\mu \leq C \max\left\{ \|\varphi_1 - \varphi_2\|^{\gamma}_{L^1(X)}, \|\varphi_1 - \varphi_2\|_{L^1(X)} \right\}. \tag{4.1}$$

Note that if μ is a Hölder continuous measure then it follows from [19, Proposition 2.4 and Proposition 4.4] that for every constant $\beta > 0$, there exists a constant $A_\beta > 0$ such that μ satisfies the condition $\mathcal{H}(\beta, A_\beta, \omega)$. Therefore, by the comparison of capacities (see [20, Theorem 3.17]), for every $\beta > 0$, there exists $A_\beta > 0$ such that μ satisfies the condition $\mathcal{H}(\beta, A_\beta, \eta)$. Alternatively, one can prove the last property by using results in [11].

The following proposition is a special case of [25, Proposition 5.3] (replace φ and ψ by w_1 and w_2, respectively):

Proposition 4.1. *Let η be a big semi-positive closed smooth $(1,1)$-form and let w_1, w_2 be negative η-psh functions such that w_1 is of full Monge-Ampère mass (i.e, $\int_X \eta^n_{w_1} = \int_X \eta^n$). Denote $\mu_1 = (\eta + dd^c w_1)^n$. Assume that the following conditions hold*

(i) *there exists a constant $M > 0$ such that*
$$-M \leq \max\{w_1, w_2\} \leq 0;$$

(ii) *there exist constants $A, \beta > 0$ such that μ_1 satisfies the condition $\mathcal{H}(\beta, A, \eta)$.*

Then, for every constant $r > 0$, there exists a constant $C > 0$ depending on $\omega, \eta, M, A, \beta$ and r such that
$$w_1 - w_2 \geq -C \|w_1 - w_2\|^{\frac{\beta r}{n+\beta(n+r)}}_{L^r(\mu_1)}.$$
In particular, if $\mu_1 \in \mathcal{M}(B,\alpha)$ for some $B > 0$ and $0 < \alpha \leq 1$ then for every $r, \gamma > 0$, there exists a constant $C' > 0$ depending on $\omega, \eta, B, \alpha, \gamma$ and r such that
$$w_1 - w_2 \geq -C' \|w_1 - w_2\|^{\frac{\gamma r}{n+\gamma(n+r)}}_{L^r(\mu_1)}.$$

We will apply Proposition 4.1 to the case where r is large enough, this means the exponent $\frac{\beta r}{n+\beta(n+r)}$ is close to be 1.

Corollary 4.2. *Let η be a big semi-positive closed smooth $(1,1)$-form and let w be a negative η-psh function of full Monge-Ampère mass with $\sup_X w = 0$. Assume that $(\eta + dd^c w)^n \in \mathcal{M}(B,\alpha)$ for some $B > 0$ and $0 < \alpha \leq 1$. Then $\|w\|_{L^\infty} \leq C$, where $C > 0$ is a constant depending on ω, η, B and α.*

By [19], a measure μ of mass $\int_X \omega^n$ is Hölder continuous if and only if $\mu = (dd^c u + \omega)^n$ for some Hölder continuous ω-psh function u on X. The following will be important for us.

Corollary 4.3. *Let $X_0, \ldots, X_m$ be compact complex manifolds and $f_j : X_j \to X_{j-1}$ be the blow up along a smooth submanifold $V_{j-1} \subset X_{j-1}$ in X_{j-1} for $1 \leq j \leq m$. Let $f := f_m \circ \cdots \circ f_0 : X_m \to X_0$. Let μ be a Hölder continuous measures on X_m. Then $f_*\mu$ is also Hölder continuous.*

Proof. By induction, it suffices to prove the desired assertion for $m = 1$. Let u_1, u_2 be ω_0-psh functions on X_0 for $j = 1, 2$, where ω_0 is a Kähler form on X_0. Put $u'_j := f_1^* u_j$. Let ω_1 be a Kähler form on X_1. Using Hölder continuity of μ, we obtain
$$\|u_1 - u_2\|_{L^1((f_1)_*\mu)} = \|u'_1 - u'_2\|_{L^1(\mu)} \lesssim \|u'_1 - u'_2\|^{\gamma}_{L^1(\omega_1^n)} + \|u'_1 - u'_2\|_{L^1(\omega_1^n)}.$$

Standard computations using local coordinates for blowups show that there exists a function $g \in L^p(\omega_0^n)$ for some constant $p > 1$ satisfying $(f_1)_*\omega_1^n = g\omega_0^n$. Hence

$$\|u_1' - u_2'\|_{L^1(\omega_1^n)} = \int_{X_0} |u_1 - u_2| g\omega_0^n \lesssim \|u_1 - u_2\|_{L^q(\omega_0^n)}$$

where $1/q + 1/p = 1$. By [19, Lemma 2.2], one has

$$\|u_1 - u_2\|_{L^q(\omega_0^n)} \lesssim \|u_1 - u_2\|_{L^1(\omega_0^n)}^{1/(2q)}.$$

Hence $(f_1)_*\mu$ is Hölder continuous. □

5. Regularization of psh functions

5.1. L^2-estimates. We recall first the L^2-estimates for $\bar\partial$ and discuss some of its variants.

Theorem 5.1. *(see* [15, Corollary 5.3]*) Let (X, ω) be a compact Kähler manifold. Let $\epsilon > 0$ be a constant. Let L be a holomorphic line bundle on X together with a singular Hermitian metric h satisfying*

$$c_1(L, h) \geq \epsilon\omega.$$

Then for every $g \in L^2_{n,1}(X, L)$ with $\bar\partial g = 0$, there exists $u \in L^2_{n,0}(X, L)$ such that $\bar\partial u = g$, and

$$\int_X |u|^2_{h,\omega}\omega^n \leq \epsilon^{-1}\int_X |g|^2_{h,\omega}\omega^n,$$

where $|g(x)|_{h,\omega}$ denotes the norm of g with respect to the norm induced by the Hermitian metric h on L and the Riemannian metric on X associated to ω.

We deduce from the above result the following more or less standard consequence.

Theorem 5.2. *Let (X, ω) be a compact Kähler manifold. Let $\epsilon > 0$ be a constant. Let K_X^* be the dual of the canonical line bundle, and let $h_{K_X^*}$ denote the metric induced by ω on K_X^*. Let L be a holomorphic line bundle on X together with a singular Hermitian metric h. Assume that there exists a singular metric $\tilde h_{K_X^*}$ on K_X^* so that*

$$c_1(L, h) + c_1(K_X^*, \tilde h_{K_X^*}) \geq \epsilon\omega.$$

Then for every $g \in L^2_{0,1}(X, L)$ with $\bar\partial g = 0$, there exists $u \in L^2_{0,0}(X, L)$ such that $\bar\partial u = g$, and

$$\int_X |u|^2_h e^{-2\vartheta}\omega^n \leq \epsilon^{-1}\int_X |g|^2_{h,\omega}e^{-2\vartheta}\omega^n,$$

where ϑ is a quasi-psh function defined by $\tilde h_{K_X^*} = e^{-2\vartheta}h_{K_X^*}$.

Proof. Set $L' := L \otimes K_X^*$. Thus $L = L' \otimes K_X$. Let h' be the singular Hermitian metric on L' given by $h' = h \otimes \tilde h_{K_X^*}$. For every $0 \leq q \leq n$, we have a natural isometry

$$\Psi_q : \Lambda^{0,q}(T^*X) \otimes L \to \Lambda^{n,q}(T^*X) \otimes L',$$

e.g., see the proof of [9, Corollary 4.3]), where we use the metric h on L, and $h \otimes h_{K_X^*}$ on L'. The map Ψ commutes with $\partial, \bar\partial$ operators. Thus $\Psi_1(g) \in L^2_{n,1}(X, L')$ with $\bar\partial\Psi_1(g) = 0$. Since Ψ_1 is an isometry, one gets

$$|\Psi_1(g)|^2_{h'} = |\Psi_1(g)|^2_{h\otimes h_{K_X^*}} e^{-2\vartheta} = |g|^2_h e^{-2\vartheta}.$$

The desired assertion now follows from Theorem 5.1. □

In particular we obtain the following.

Corollary 5.3. *Let (X,ω) be a compact Kähler manifold so that the Chern class of K_X^* contains a closed positive $(1,1)$-current of bounded potentials,* i.e*, there exists a bounded η_ω-psh function ϑ on X, where η_ω is the Chern form of the metric on K_X^* induced by ω. Let L be a holomorphic line bundle on X together with a singular Hermitian metric h such that*

$$c_1(L,h) \geq \epsilon\omega.$$

Then for every $g \in L^2_{0,1}(X,L)$ with $\bar\partial g = 0$, there exists $u \in L^2_{0,0}(X,L)$ such that $\bar\partial u = g$, and

$$\int_X |u|_h^2 \omega^n \leq \frac{e^{4\|\vartheta\|_{L^\infty}}}{\epsilon} \int_X |g|^2_{h,\omega}\omega^n.$$

We also use the following consequence of Corollary 5.3.

Corollary 5.4. *Let X,ω,ϑ be as in Corollary 5.3. Let θ be a semi-positive form on X with $\theta \leq \omega$ such that θ is Kähler in an open dense Zariski subset W in X. Assume that there exist a weakly pseudoconvex manifold U' with a smooth Kähler metric θ', an open connected subset U in X and a biholomorphic map $\Phi : U \to U'$ such that $\theta := \Phi^*\theta'$ on U. Let L be a holomorphic line bundle on X together with a singular Hermitian metric h such that*

$$c_1(L,h) \geq \epsilon\omega.$$

Let $f \in H^0(U,L)$ and let χ' be a smooth function with compact support in U' and χ is constant on some open subset Z' in U'. Set $\chi := \chi' \circ \Phi$. Then there exists a smooth real section u of L over W such that $\bar\partial u = \bar\partial(\chi f)$ on W, and

$$\int_X |u|_h^2 \theta^n \leq \frac{M^2 e^{4\|\vartheta\|_{L^\infty}}}{\epsilon} \int_{X\setminus\Phi^{-1}(Z')} |f|_h^2\theta^n,$$

where $M := \sup_{x'\in U'} |\partial\chi'(x')|_{\theta'}$.

The crucial point here is that we obtain a version of L^2-estimates for a possibly degenerate volume form θ^n.

Proof. Let $r > 0$ be a small constant and let $\theta_r := \theta + r\omega \leq (1+r)\omega$. Hence $c_1(L,h) \geq (1+r)^{-1}\theta_r$. Applying Corollary 5.3 to θ_r, we obtain

$$\int_X |u_r|_h^2 \theta_r^n \leq \frac{e^{4\|\vartheta\|_{L^\infty}}}{\epsilon} \int_X |\bar\partial(\chi g)|_h^2 \theta_r^n.$$

We compute

$$|\bar\partial(\chi f)|_{h,\theta_r} = |g\bar\partial\chi|_{h,\theta_r} = |f|_h|\bar\partial\chi|_{\theta_r}$$

Since $|\bar\partial\chi(\Phi^{-1}(x'))|_{\theta_r} \to |\bar\partial\chi'(x')|_\theta$ which is $\leq M$, we infer that

$$\limsup_{r\to 0} |\bar\partial(\chi f)|_{h,\theta_r} \leq M|f|_h \mathbf{1}_{\Phi^{-1}(Z')}.$$

The desired estimate thus follows from Corollary 5.3 applied to θ_r. We infer that

$$\limsup_{r\to\infty} \int_X |u_r|_h^2\theta^n \leq \frac{M^2 e^{4\|\vartheta\|_{L^\infty}}}{\epsilon} \int_{X\setminus\Phi^{-1}(Z')} |f|_h^2\theta^n.$$

Thus, extracting a subsequence if necessary, we can assume that u_r converges weakly to some u in the Hilbert space $L^2(X,\theta^n)$. Consequently,

$$\int_X |u|_h^2\theta^n \le \frac{M^2e^{4\|\vartheta\|_{L^\infty}}}{\epsilon}\int_{X\setminus\Phi^{-1}(Z')} |f|_h^2\theta^n.$$

On the other hand, since θ is Kähler on W, we infer that u_r converges weakly to u as currents on W. It follows that $\bar\partial u = \lim_{r\to 0}\bar\partial u_r = \bar\partial(\chi f)$ on W, hence, u is in particular smooth on W because it is a solution of $\bar\partial$-equation with smooth right-hand side. □

We recall now a special case of the Ohsawa-Takegoshi extension theorem (see [15, Theroem 13.6]).

Theorem 5.5. *Let X be a weakly pseudoconvex n-dimensional manifold with a Kähler metric ω. Let y be a point in X. Let (L,h) be a line bundle on X and let E be the trivial holomorphic vector bundle of rank n equipped with the trivial Hermitian metric such that there exists a global section s of E with $y=\{s=0\}$ so that $\Lambda^n ds(y)\neq 0$ and $|s|\le e^{-1}$. Then for every $(n,0)$-form f with values in L at y, there exist a $\bar\partial$-closed $(n,0)$-form F with values in L on X such that $F(y)=f(y)$ and*

$$\int_X \frac{|F|^2_{h,\omega}}{|s|^{2n}(-\log|s|)^2}\omega^n \le C_n\frac{|f|^2_{h,\omega}}{|\Lambda^n ds(y)|^2_\omega},$$

where C_n is a numerical constant depending only on n.

We note that since E and its Hermitian metric are trivial, the curvature of the metric of E vanishes everywhere, and s is nothing but a collection of n holomorphic functions on X.

Corollary 5.6. *Let X,ω,L,h,E,y,s be as in Theorem 5.5. Let $h_{K_X^*}$ be the Hermitian metric on K_X^* induced by ω. Assume furthermore that there exists a singular Hermitian metric $\tilde h_{K_X^*}$ on K_X^* such that $\tilde h_{K_X^*}=h_{K_X^*}e^{-2\vartheta}$, for some bounded η_ω-psh function ϑ on X, where η_ω is the Chern form of $h_{K_X^*}$. Then for every section f of L at y, there exists $F\in H^0(X,L)$ such that $F(y)=f(y)$ and*

$$\int_X \frac{|F|^2_{h}}{|s|^{2n}(-\log|s|)^2}\omega^n \le C_ne^{4\|\vartheta\|_{L^\infty}}\frac{|f|^2_{h}}{|\Lambda^n ds(y)|^2_\omega},$$

where C_n is a numerical constant depending only on n.

Proof. Let $L':=L\otimes K_X^*$ and $h':=h\otimes\tilde h_{K_X^*}$. The desired inequality follows from Theorem 5.5 applied to (L',h') and arguments as in the proof of Theorem 5.2. □

We deduce the following degenerate version of the above extension.

Corollary 5.7. *Let the notations and assumptions be as in Corollary 5.6. Let θ be a semi-positive form on X which is Kähler on an open Zariski dense subset W in X. Assume that there exist a manifold U' with a smooth Kähler metric θ', an open connected subset U in X and a biholomorphic map $\Phi: U\to U'$ such that $\theta:=\Phi^*\theta'$ on U and $y\in U$. Then for every for every section f of L at y, there exists $F\in H^0(W,L)$ such that $F(y)=f(y)$ and*

$$\int_X \frac{|F|^2_{h}}{|s|^{2n}(-\log|s|)^2}\theta^n \le C_ne^{4\|\vartheta\|_{L^\infty}}\frac{|f|^2_{h}}{|\Lambda^n ds'(y')|^2_{\theta'}}, \tag{5.1}$$

where C_n is a numerical constant depending only on n, where $s':=s\circ\Phi^{-1}$ and $y':=\Phi^{-1}(y)$.

Proof. Let $\theta_r := \theta + r\omega$ which is a Kähler form on X. Let $\omega' := \Phi_*\omega$ and $\theta'_r := \theta' + r\omega'$. We see that the norm $|\Lambda^n ds'(y')|_{\theta'_r}$ converges to $|\Lambda^n ds'(y')|_{\theta'}$ as $r \to 0$, and one has

$$|\Lambda^n ds(y)|_{\theta_r} = |\Lambda^n ds'(y')|_{\theta'_r}.$$

Applying Corollary 5.6 to θ_r gives

$$\int_X \frac{|F_r|_h^2}{|s|^{2n}(-\log|s|)^2}\theta_r^n \leq C_n^{4\|\vartheta\|_{L^\infty}} \frac{|f|_h^2}{|\Lambda^n ds(y)|^2_{\theta_r}},$$

for some $F_r \in Hp(X, L)$ with $F_r(y) = f(y)$ Letting $r \to 0$ and arguing as in the end of the proof of Corollary 5.4 show that after extracting a subsequence if necessary, F_r converges weakly to a function $F \in L^2_{loc}(W)$ and the estimate (5.1) holds. Furthermore since F_r is holomorphic (hence $\bar\partial F_r = 0$), we infer that F is indeed holomorphic on W. □

It is a good moment to mention a result about the extension of holomorphic functions which is used later in the paper: every holomorphic function on the complement of an analytic subset of codimension at least 2 in a normal complex space is automatically extended to a global holomorphic function on that space (see [22]).

5.2. **Analytic regularisation of psh functions.** Let (X, ω) be a compact Kähler manifold. From now on we assume the following hypothesis:

(H) The Chern class of K_X^* contains a closed positive $(1,1)$-current of bounded potentials, *i.e,* there exists a bounded η_ω-psh function ϑ on X, where η_ω is the Chern form of the Hermitian metric on K_X^* induced by ω.

In particular, this assumption is fulfilled if K_X^* is semi-positive. Let L be a big and semi-ample line bundle on X (hence X is forced to be projective by Moishezon's theorem). Since L is big, by Demailly [15], there exists a negative θ-psh function ρ such that locally

$$\rho = \log\big(\sum_{j=1}^r |f_j|\big) + O(1),$$

for some local holomorphic functions $f_1, \dots, f_r$, and

$$dd^c\rho + \theta \geq \delta_0\omega,$$

where $\delta_0 > 0$ is a constant. We can choose ρ so that $N := \{\rho = -\infty\}$ is equal to the non-Kähler locus of $c_1(L)$, see [5]. Recall that the non-Kähler locus of $c_1(L)$ is equal to the augmented base locus of L (see [39, Theorem 2.3] or [4]).

Let $d_k := \dim H^0(X, L^k)$ and $\{s_1, \dots, s_{d_k}\}$ be a basis of $H^0(X, L^k)$. We define $\Phi_k : X \to \mathbb{CP}^{d_k - 1}$ by putting

$$\Phi_k(x) := [s_1(x) : \cdots : s_{d_k}(x)].$$

Observe that Φ_k is a well-defined map outside $B(kL) := \cap_{s \in H^0(X, L^k)}\{s = 0\}$.

We recall $dd^c := i/\pi\partial\bar\partial$. Since L is semi-ample, there is $k' > 0$ sufficiently large so that $B(k'L) = \varnothing$. Hence $\Phi_{k'} : X \to \mathbb{CP}^{d_k}$ is a holomorphic map. Since L is big, we can find $k'' > 0$ so that $\Phi_{k''}$ is of maximal rank. Let $k_L := k'k''$. It follows that Φ_{k_L} is a holomorphic map of maximal rank. Let $X' := \Phi_{k_L}(X)$ which is an irreducible analytic subset of dimension n in $\mathbb{CP}^{d_{k_L}-1}$. By increasing k_L if necessary, we also have that Φ_{k_L} is an algebraic fibre space, *i.e*, the fibers of Φ_{k_L} are connected, and X' is a normal variety (see

[32, Theorem 2.1.27]), moreover Φ_{k_L} is bihomorphic outside the non-Kähler locus N, see [6, Theorem A].

Let

$$\theta := \frac{1}{2k_L} dd^c \log \sum_{j=1}^{d_{k_L}} |s_j|^2$$

which is smooth closed form in $c_1(L)$. Hence θ is the pull-back of the Fubini-Study form in $\mathbb{CP}^{d_{k_L}-1}$ under Φ_{k_L}. Let h_0 be a smooth Hermitian metric on L with $c_1(L, h_0) = \theta$.

Fix a smooth Riemannian metric on X and let $\mathbb{B}(x, r)$ be the ball of radius r with respect to this metric. Let $r_X > 0$ be a constant so that for every $x \in X$ the closure of the ball $\mathbb{B}(x, r_X)$ is contained in a local chart of X which is biholomorphic to a ball in $\mathbb{C}^n$.

Lemma 5.8. *There exist constants $C_0 > 0$, $r_0 > 0$ small enough such that for every $y \in X$, there exist global negative θ-psh functions u_y on X so that*

$$u_y(x) \leq \log|y - x| + C_0$$

on $\Phi_{k_L}^{-1}\big(\mathbb{B}(y', r_0)\big)$ where $y' := \Phi_{k_L}(y)$ and by abuse of notation, for every $r > 0$, we denote by $\mathbb{B}(y', r)$ the ball of radius r centered at $y' \in \mathbb{CP}^{d_{k_L}-1}$ with respect to a fixed smooth metric on $\mathbb{CP}^{d_{k_L}-1}$. Furthermore, for every constant $\epsilon > 0$, we have

$$u_y \geq \log \epsilon - C \tag{5.2}$$

outside $\Phi_{k_L}^{-1}\big(\mathbb{B}(y', \epsilon)\big)$ for some constant C independent of y, ϵ.

Proof. Let $y' := \Phi_{k_L}(y)$ and let $v_y(z)$ be a ω_{FS}-psh function on $\mathbb{CP}^{d_{k_L}-1}$ given by

$$v_y(z) := \log|z - y'|$$

where we use homogeneous coordinates for z, y', and ω_{FS} is the Fubini-Study form on $\mathbb{CP}^{d_{k_L}-1}$. Thus, for every $\epsilon > 0$, there holds $v_y \geq C \log \epsilon$ outside $\mathbb{B}(y', \epsilon)$ for some constant C independent of y, ϵ.

Since $\Phi_{k_L}^* \omega_{FS} = \theta$, we infer $u_y := \Phi_{k_L}^* v_y$ and $\tilde{u}_y := \Phi_{k_L}^* \tilde{v}_y$ are θ-psh and satisfies that

$$u_y(x) = \log|\Phi_{k_L}(y) - \Phi_{k_L}(x)| \leq \log|y - x| + C_0$$

on $\Phi_{k_L}^{-1}\big(\mathbb{B}(y', r_0)\big)$. Moreover one also has (5.2) because of the smoothness of v_y outside y'. □

Let $\mathbb{B}_r(y)$ be the ball of radius r centered at y in $\mathbb{C}^{k_L-1}$. If $y = 0$, then we write $\mathbb{B}_r$ for $\mathbb{B}_r(y)$. Put $N' := \Phi_{k_L}(N)$ which is an analytic subset in X'. Let $U_1', \ldots, U_l'$ be open subsets in $\mathbb{CP}^{d_{k_L}-1}$ such that the following properties hold:

(i) $U_j' \Subset U_j''$ which is biholomorphic to the ball $\mathbb{B}_3$ in $\mathbb{C}^{d_{k_L}-1}$ under a map Ψ_j for every $1 \leq j \leq l$ and U_j is biholomorphic to $\mathbb{B}_2$ under Ψ_j,

(ii) $X' \subset \cup_{j=1}^l U_j'$,

(iii) There is a hyperplane H_j on $\mathbb{CP}^{d_{k_L}-1}$ such that H_j does not intersect U_j'' for every $1 \leq l \leq j$.

By our choice of U_j', we see that U_j' is hyperconvex (hence weakly pseudoconvex), i.e, there is a smooth psh function w_j on U_j' such that $\{w_j < c\}$ is relatively compact in U_j' for every constant $c < 0$ and every j. Let

$$U_j := \Phi_{k_L}^{-1}(U_j').$$

Note that U_j is also hyperconvex and L is trivial over U_j because $L = \Phi_{k_L}^* \mathcal{O}(1)$ and $\mathcal{O}(1)$ is trivial over $X'\backslash H_j$.

Lemma 5.9. *Let $h := h_0 e^{-2\phi}$ be a singular positively curved metric on L. Fix $1 \le j \le l$. Let $y \in U_j\backslash N$. Let e be a local holomorphic frame of L over U_j. Then for every $a \in \mathbb{C}$, there exists a section $f \in H^0(U_j, L)$ such that $f(y) = ae(y)$ and*

$$\int_{U_j} |f|_{h_0}^2 e^{-2\phi}\theta^n \le C|a|^2|e(y)|_{h_0}^2 e^{-2\phi(y)}.$$

where $C > 0$ is a constant independent of y and a.

Proof. Let $L|_{U_j}$ be the restriction of L to U_j, and $E := L|_{U_j} \oplus \cdots \oplus L|_{U_j}$ (n times). Since L is trivial on U_j, so is E. Equip E with the trivial Hermitian metric. Hence a section of E is simply a collection of n holomnorphic functions on U_j. Let $z = (z_1, \dots, z_{k_L - 1})$ be the local coordinates on $U_j'' \approx \mathbb{B}_3$. We can assume that y' is the origin in these local coordinates. Let $s_E'(z) := z$. Observe that $s_E := s_E' \circ \Phi_{k_L}^{-1}$ is a section of E on U_j and vanishes only at y. Recall that $\theta = \Phi_{k_L}^* \omega_{FS}$, where ω_{FS} is the Fubini-Study form on $\mathbb{CP}^{k_L - 1}$.

Let $N' := \Phi_{k_L}(N)$. Then $X'\backslash N'$ is smooth and Φ_{k_L} is a biholomoprhism from $U_j\backslash N$ to $U_j'\backslash N'$. Let $X'' := X' \cap \mathbb{B}_2$, we have a natural inclusion $\xi : X'' \to U_j' \approx \mathbb{B}_2$. Let Ψ be an orthogonal change of coordinates on $\mathbb{C}^{k_L - 1}$ so that $\Psi_* \xi_* T_y X''$ is given by the subspace $\{z_1, \dots, z_n, 0, \dots, 0\}$ at 0 in $\mathbb{C}^{k_L - 1}$. Write $\Psi = (\Psi_1, \dots, \Psi_{k_L})$. Let

$$s_E' := (\Psi_1, \dots, \Psi_n) \circ \xi$$

regarded as a section of $\Phi_{k_L - 1}^* E$. Let $Y := X'' \cap \{s_E' = 0, \det J_{s_E'} \ne 0 : 1 \le k \le n\}$ contains 0 as an isolated point, where $\xi_y(z) := (z_{j_1}, \dots, z_{j_n})$ for $z \in X''$ (note that Y may not be connected). Note that ω_0 is preserved under Ψ. By the choice of s_E', there is a constant $\epsilon_0 > 0$ independent of y such that

$$|\Lambda^n ds_E'(y')|_{\xi^*\omega_{FS}} \ge \epsilon_0. \tag{5.3}$$

Indeed, by the choice of Ψ, the norm $|\Lambda^n ds_E'(y')|_{\xi^*\omega_0}$ (which is the norm of $\det J_{s_E'}$ with respect to $\xi^*\omega_0$) is equal to the absolute value of the determinant of the (n, n)-submatrix of the Jacobian of $(\Psi_1, \dots, \Psi_n)$ given by the first n rows. Hence $|\Lambda^n ds_E'(y)|_{\xi^*\omega_0} = 1$. Since ω_{FS} and ω_0 are equivalent on U', we get (5.3).

Let $s_E := s_E' \circ \Phi_{k_L}$. Applying Corollary 5.7 to $U_j, \theta, s_E, \Phi_{k_L}, y$ implies that there exists a section $f \in H^0(U_j\backslash N, L)$ such that $f(y) = ae(y)$ and

$$\int_{U_j} |f|_{h_0}^2 e^{-2\phi}\theta^n \le C|a|^2|e(y)|_{h_0}^2 e^{-2\phi(y)} |\Lambda^n ds_E'(y')|_{\omega_{FS}}^{-2} \lesssim C|a|^2|e(y)|_{h_0}^2 e^{-2\phi(y)}$$

by (5.3), where $C > 0$ is a constant independent of y and a. This finishes the proof. □

Let h be a positive Hermitian metric on L. Let e_L is a local holomorphic frame for L (i.e., e_L is a local holomorphic section of L and $e_L \ne 0$ everywhere). Write $h = h_0 e^{-2\varphi}$. Thus by hypothesis one gets

$$0 \le c_1(L, h) = -dd^c \log |e_L|_h = dd^c\varphi + \theta.$$

In other words, φ is θ-psh function. By multiplying a large constant with h_0, without loss of generality we can assume that $\varphi \le 0$. We assume from now on that φ is bounded.

For every constant $\delta \in (0,1)$, define

$$\varphi_\delta := (1-\delta)\varphi + \delta\rho, \quad h_\delta := h_0 e^{-2\varphi_\delta}.$$

We have $dd^c\varphi_\delta + \theta \geq \delta\delta_0\omega$. Let $m \in \mathbb{N}$ and $d_m := \dim H^0(X, L^m)$ which is $\approx m^n$ as $m \to \infty$. Let $m_0 := 2n+3$. Let $a_0 \in (0, 1/2)$ be a constant such that

$$\int_X e^{-2a_0\rho}\omega^n < \infty.$$

For $m > m_0$, $\delta \in (0, a_0/m)$ and $s, s' \in H^0(X, L^m)$, we put

$$\langle s, s'\rangle_{L^2} = \langle s, s'\rangle_{L^2,m,\delta} := \int_X \langle s, s'\rangle_{h_0^m} e^{-2(m-m_0)\varphi_\delta}\theta^n$$

which is finite because the boundedness of φ and the choice of a_0. To give readers a hint what we do with δ, we remark that we will choose later $\delta := m^{-2D}$ for some constant $D \geq 1$ (see the proof of Lemma 6.2 below), thus the condition $\delta < a_0/m$ is automatically satisfied for $m \geq a_0^{-1}$.

Let $\{\sigma_1, \ldots, \sigma_{d_m}\}$ be an orthonormal basis of $H^0(X, L^m)$ with respect to L^2-product, and let

$$\psi_{m,\delta} := \frac{1}{2m}\log\left(\sum_{j=1}^{d_m}|\sigma_j|^2_{h_0^m}\right) = \frac{1}{2m}\sup_{s\in H^0(X,L^m):\|s\|_{L^2}=1}\log|s|^2_{h_0^m}.$$

Since $\log|\sigma_j|_{h_0^m}$ is $m\theta$-psh, we infer that ψ_m is θ-psh.

Lemma 5.10. *Let $\xi := \omega^n/\theta^n$. There exists a constant $p_0 > 1$ such that*

$$\int_X \xi^{p_0}\theta^n < \infty. \tag{5.4}$$

Proof. Direct computations show that on a small enough local chart U, one has

$$\xi^{-1} = |f_0|(1 + \sum_{1\leq j\leq k_L-1}|f_j|^2)$$

for some holomorphic functions $f_0, \ldots, f_{k_L-1}$, e.g., see the proof of [10, Proposition 4.36]. Let $\psi := -\log\xi$. One sees that ψ is quasi-psh on U, hence ψ is quasi-psh function on X. Now observe

$$\int_X \xi^{p_0}\theta^n = \int_X \xi^{p_0-1}\omega^n = \int_X e^{-(p_0-1)\psi}\omega^n$$

which is finite for $p_0 - 1 > 0$ small enough because ψ is quasi-psh. This finishes the proof. □

The following result is a variant from [15, Theorem 14.21]. Recall $N = \{\rho = -\infty\}$.

Theorem 5.11. *There exists a constant $C > 0$ such that for every $\delta \in (0, a_0/m)$ and every $m \geq m_0 + 1$ there holds:*

(i)

$$\frac{m-m_0}{m}\varphi_\delta(x) - \frac{C + |\log\delta|}{2m} \leq \psi_{m,\delta} \leq \frac{m-m_0}{m}\sup_{x'\in\mathbb{B}(x,r)}\varphi_\delta(x') + Cr + C\frac{|\log r|}{m},$$

for every $x \in X$ and $r > 0$.

(ii)

$$|\nabla \psi_{m,\delta}(x)| \le C + \frac{C}{m\delta^{1/2}r^{n+1}} e^{(m-m_0)(\sup_{\mathbb{B}(x,r)}\varphi_\delta - \varphi_\delta(x)) + C(m-m_0)r},$$

for every $x \in X\backslash N$ *and* $r > 0$.

Proof. We check the second inequality in (i). Let U be a small local chart around x. We trivialize L over U and let $e_{L,U}$ be a nowhere vanishing holomorhic section of L over U. Hence we can identify $h_0 = e^{-2\phi_0}$ for some smooth function ϕ_0, and sections of L^m are identified with holomorphic functions on U.

Let $s \in L^m$ with $\|s\|_{L^2} = 1$. By abuse of notation we still denote by s the holomorphic function corresponding to a section s of L^m. Thus $|s|^2_{h_0^m} = |s|^2 e^{-2m\phi_0}$. Put $q = 2(m - m_0)\varphi_\delta + 2m\phi_0$ and $\xi := \omega^n/\theta^n$. Let p_0 be the constant in Lemma 5.10 and put $q_0 := p_0/(p_0-1)$ and $\epsilon_0 := 1/q_0$. By the submean inequality and Lemma 5.10, one gets

$$\begin{aligned}
|s(x)|^{2\epsilon_0} &\lesssim r^{-2n}\int_{\mathbb{B}(x,r)} |s|^{2\epsilon_0} d\,\mathrm{Leb}_{\mathbb{C}^n}\\
&\lesssim r^{-2n} e^{2\epsilon_0 \sup_{\mathbb{B}(x,r)} q}\int_{\mathbb{B}(x,r)} |s|^{2\epsilon_0} e^{-2\epsilon_0 q}\xi\theta^n\\
&\le r^{-2n} e^{2\epsilon_0 \sup_{\mathbb{B}(x,r)} q}\left(\int_{\mathbb{B}(x,r)} |s|^2 e^{-2q}\theta^n\right)^{1/q_0}\left(\int_X \xi^{p_0}\theta^n\right)^{1/p_0}\\
&\lesssim r^{-2n} e^{2\epsilon_0 \sup_{\mathbb{B}(x,r)} q}\|s\|^{2/q_0}_{L^2}.
\end{aligned}$$

Thus

$$|s(x)|^2_{h_0^m} = |s(x)|^2 e^{-2m\phi_0(x)} \lesssim r^{-2n/\epsilon_0} e^{2(m-m_0)(\sup_{\mathbb{B}(x,r)}(\varphi_\delta + \phi_0) - \phi_0(x))}.$$

By this and the fact that $\phi_0 \in \mathcal{C}^1$ we infer that

$$|s(x)|^2_{h_0^m} \lesssim r^{-2n} e^{2(m-m_0)\sup_{\mathbb{B}(x,r)}\varphi_\delta + 2(m-m_0)C_1 r},$$

for some constant $C_1 > 0$ independent of δ, m, φ, s, for $s \in H^0(X, L^m)$ with $|s|_{L^2} = 1$. It follows that

$$e^{2m\psi_{m,\delta}} = \sup_{s\in H^0(X,L^m):\|s\|_{L^2}=1} |s(x)|^2_{h_0^m} \le e^{C_2} r^{-2n/\epsilon_0} e^{2(m-m_0)\sup_{\mathbb{B}(x,r)}\varphi_\delta + 2(m-m_0)C_1 r}, \tag{5.5}$$

where $C_1, C_2 > 0$ are constants independent of δ, m, φ, s. Hence we obtain

$$\psi_{m,\delta} \le \frac{m-c_0}{m}\sup_{x'\in\mathbb{B}(x,r)}\varphi_\delta(x') + C_1 r + \frac{2n\epsilon_0^{-1}|\log r| + C_2}{2m} \le \frac{m-m_0}{m}\sup_{x'\in\mathbb{B}(x,r)}\varphi_\delta(x') + C_3 r + \frac{C_3|\log r|}{m},$$

for every $x \in X$ and $r > 0$, where $C_3 = C_1\epsilon_0^{-1} + C_2 e + n$.

The remaining inequality of (i) requires the L^2-estimate. It suffices to consider $x \notin N$. Let $U_1, \dots, U_l$ be the open cover of X defined above. Without loss of generality we can assume that $x \in U_1$. Choose $U := U_1$. We can modify the coordinates on $U_1' \approx \mathbb{B}_2$ so that $\Phi_{k_L}(x)$ is the center of U_1'. By Lemma 5.9, there are a constant $B_1 > 0$ independent of x such that for every $a \in \mathbb{C}$, there is a $f \in H^0(U_1, L^m)$ so that $f(x) = a e^m_{L,U}$ and

$$\int_{U_1} |f|^2_{h_0^m} e^{-2(m-m_0)\varphi_\delta}\theta^n \le B_1 |a|^2 |e_{L,U}(x)|^{2m}_{h_0} e^{-2(m-m_0)\varphi_\delta(x)}.$$

Fix a cut-off function χ' supported on U_1' and equal to 1 on $\mathbb{B}_{1/2}$ (recall that $U_1' \approx \mathbb{B}_2$ and $\Phi_{k_L}(x) = 0$ is the origin). Put $\chi := \chi' \circ \Phi_{k_L}^{-1}$. By Lemma 5.8, there exist a constant $C_4 > 0$ independent of x and a negative θ-psh function u_x satisfying $u_x(z) \leq \log|z - x| + C_4$, and $u_x(z) \geq -C_4$ for $z \notin \Phi_{k_L}^{-1}(\mathbb{B}_{1/2}(x'))$ ($x' := \Phi_{k_L}(x)$). Let

$$w := \big(m - m_0\big)\varphi_\delta + m_0 u_x.$$

Observe that

$$dd^c w + m\theta \geq \big(m - m_0\big)\delta\delta_0\omega.$$

Let

$$h_{u_x} := h_0 e^{-2u_x}, \quad \tilde{h} := h_0^m e^{-2w}$$

which is a singular Hermitian metric on L^m. Thanks to the hypothesis about the semi-positivity of K_X^*, we can apply Corollary 5.4 to $\tilde{h}$ and $g := \bar{\partial}(\chi f)$ which is smooth. Hence we find a smooth section v of L over $X\backslash N$ so that $\bar{\partial} v = g$ and

$$\int_X |v|^2_{h_0^m} e^{-2w}\theta^n \leq \frac{1}{(m - m_0)\delta\delta_0} \int_{X\backslash \Phi_{k_L}^{-1}(\mathbb{B}_{1/2}(x'))} |f|^2_{h_0^m} e^{-2w}\theta^n \tag{5.6}$$

because $\sup_{U_1'} |\partial\chi'|_{\omega_{FS}}$ is bounded by a constant independent of x. This combined with the fact that $u_x(x') \geq C_4$ outside $\Phi_{k_L}^{-1}(\mathbb{B}_{1/2}(x'))$ yields

$$\begin{aligned}\int_X |v|^2_{h_0^m} e^{-2w}\theta^n \leq \frac{1}{(m - m_0)\delta\delta_0} \int_{X\backslash \Phi_{k_L}^{-1}(\mathbb{B}_{1/2}(x'))} |f|^2_{h_0^m} e^{-2(m-m_0)\varphi_\delta}\theta^n \\ |a|^2 |e_{L,U}(x)|^{2m}_{h_0} e^{-2(m-m_0)\varphi_\delta(x)}.\end{aligned} \tag{5.7}$$

Note that since g vanishes near x, one gets that $\bar{\partial} v = 0$ near x. Thus v is holomorphic near x. By properties of u_x, observe that

$$e^{-2w(x')} \gtrsim \frac{1}{|x' - x|^{2n+2}}$$

which in turn implies that $v(x) = 0$ because $\int_X |v|^2_{h_0^m} e^{-2w}\omega^n$ is finite. This together with (5.7) gives

$$\int_X |v|^2_{h_0^m} e^{-2w}\theta^n \lesssim \frac{|a|^2}{(m - m_0)\delta} e^{-2(m-m_0)\varphi_\delta(x) - 2m\phi_0(x)}. \tag{5.8}$$

Let $\tilde{v} := \chi f - v \in H^0(X\backslash N, L^m)$. The function $\tilde{v}$ extends to a global holomorphic section of L on X because $\tilde{v} \circ \Phi_{k_L}^{-1}$ is holomorphic on $X'\backslash N'$, X' is normal and N' is of codimension 2 in X'. Since $u_x \leq 0$, using (5.8) and the choice of f, we obtain

$$\int_X |\tilde{v}|^2_{h_0^m} e^{-2(m-m_0)\varphi_\delta}\theta^n \leq \frac{B_2 |a|^2 |e_{L,U}(x)|^{2m}_{h_0}}{(m - m_0)\delta} e^{-2(m-m_0)\varphi_\delta(x)},$$

for some constant $B_2 > 0$ independent of x, a, m, δ. Choose

$$a := B_2^{-1/2} |e_{L,U}(x)|^{-m}_{h_0} \delta^{1/2} (m - m_0)^{1/2} e^{(m-m_0)\varphi_\delta(x)}.$$

We see that

$$\int_X |\tilde{v}|^2_{h_0^m} e^{-2(m-m_0)\varphi_\delta}\theta^n \leq 1.$$

and

$$\tilde{v}(x) = f(x) - v(x) = f(x) = a\, e^m_{L,U}(x).$$

It follows that

$$e^{2m\psi_{m,\delta}(x)} \geq |\tilde{v}(x)|^2_{h_0^m} \geq \frac{\delta(m-m_0)}{B_2} e^{2(m-m_0)\varphi_\delta(x)}. \tag{5.9}$$

Thus

$$\begin{aligned}\psi_{m,\delta}(x) &\geq \frac{m-m_0}{m}\varphi_\delta(x) + \frac{1}{2m}\log\frac{\delta(m-m_0)}{B_2}\\ &\geq \frac{m-m_0}{m}\varphi_\delta(x) - \frac{|\log\delta| + \log B_2}{2m}.\end{aligned}$$

This finishes the proof for (i).

We now check (ii). We work in a small local chart (U, x) and write $h_0 = e^{-2\phi_0}$ as above. We identify sections with holomorphic functions on a trivialization of L over this local chart. We have

$$\psi_{m,\delta} = \frac{1}{2m}\log\sum_{j=1}^{d_m}|\sigma_j|^2 - \phi_0.$$

Direct computations give

$$\partial\psi_{m,\delta} = \frac{1}{2m}\frac{\sum_{j=1}^{d_k}\bar{\sigma}_j\partial\sigma_j}{\sum_{j=1}^{d_k}|\sigma_j|^2} - \partial\phi_0.$$

Hence

$$|\partial\psi_{m,\delta}| \leq \frac{1}{2m}\frac{\left(\sum_{j=1}^{d_k}|\partial\sigma_j|^2\right)^{1/2}}{\left(\sum_{j=1}^{d_k}|\sigma_j|^2\right)^{1/2}} + |\partial\phi_0|. \tag{5.10}$$

By (5.9), we have

$$\sum_{j=1}^{d_k}|\sigma_j(x)|^2 = e^{2m(\psi_{m,\delta}+\phi_0)} \geq \frac{\delta(m-m_0)}{B_2}e^{2(m-m_0)\varphi_\delta(x)+2m\phi_0(x)}. \tag{5.11}$$

On the other hand, since σ_j is homomorphic, it follows from Cauchy's integral formula that

$$\sum_{j=1}^{d_k}|\partial\sigma_j(x)|^2 \lesssim r^{-n-2}\sum_{j=1}^{d_k}\int_{x+\partial_0\Delta_r^n}|\sigma_j|^2 d\xi_1...d\xi_n \lesssim r^{-2}\sup_{x+\Delta_r^n}\sum_{j=1}^{d_k}|\sigma_j|^2 = r^{-2}\sup_{x+\Delta_r^n}e^{2m(\psi_{m,\delta}+\phi_0)},$$

for every $0 < r < \operatorname{dist}(x, \partial U)$, where Δ_r denotes the disk of radius r with center at 0 in $\mathbb{C}$, and $\partial_0\Delta_r^n := (\partial\Delta_r)^n$. Therefore, by (5.5) and the fact $\phi_0 \in \mathcal{C}^1$, we get

$$\sum_{j=1}^{d_k}|\partial\sigma_j(x)|^2 \lesssim r^{-2n-2}e^{2(m-m_0)(\sup_{\mathbb{B}(x,r)}\varphi_\delta+\phi_0)+C_5(m-m_0)r}. \tag{5.12}$$

Combining (5.10), (5.11) and (5.12), we get

$$|\partial\psi_{m,\delta}| \lesssim 1 + \frac{1}{m\delta^{1/2}r^{n+1}}e^{(m-m_0)(\sup_{\mathbb{B}(x,r)}\varphi_\delta - \varphi_\delta(x))+C_5(m-m_0)r}.$$

□

The new point here is that we approximate φ through the analytic approximation sequence for φ_δ with δ depending on m. We will choose δ to be very small compared to m.

Lemma 5.12. *Let u be a bounded negative psh function on the open unit ball $\mathbb{B}$ of $\mathbb{C}^n$ and let $K \Subset \mathbb{B}$. Let v be a Hölder continuous plurisubharmonic function on $\mathbb{B}$ and denote $\mu = (dd^c v)^n$. Then there exist constants $\alpha = \alpha(\mu, K)$ and $C = C(n, K, \mu, \|u\|_{L^\infty}) > 0$ such that*

$$\int_K \big| \sup_{x' \in \mathbb{B}(x,s)} u(x') - u(x)\big| d\mu \leq C s^\alpha,$$

for every $0 < s < r_0^3$, where $r_0 := \frac{1}{4}\inf_{w\in K} \operatorname{dist}(w, \partial\mathbb{B})$.

Proof. Denote $M = \|u\|_{L^\infty}$, $U = (1-3r_0)\mathbb{B}$ and $V = (1-2r_0)\mathbb{B}$. First, we prove that

$$\int_V \big| \sup_{y \in \mathbb{B}(x,s)} u(y) - u(x)\big| d\,\mathrm{Leb} \leq C_0 M s^{2/3}, \tag{5.13}$$

for every $0 < s < r_0^3$, where $C_0 > 0$ is a constant depending only on n and r_0.

For every $0 < r < r_0$ and $z \in U$, we denote

$$\hat{u}_r(z) = \frac{1}{\operatorname{vol}(\mathbb{B}(z,r))} \int_{\mathbb{B}(z,r)} u(\xi) dV(\xi),$$

and

$$\bar{u}_r(z) = \sup_{\xi \in \mathbb{B}(z,r)} u(\xi).$$

Let $z_0 \in V$ and $v_M := \frac{M}{r_0}(|z - z_0|^2 - 1)$. We have $v_M < u$ on $\mathbb{B}_{\sqrt{1/2 - r_0}}(z_0)$ (which contains $V + r_0\mathbb{B}$ because $r_0 < 1$). By the comparison principle for Laplace operator, one has

$$\int_{\{v_M < u\}} \Delta u \leq \int_{\{v_M < u\}} \Delta v_M \lesssim M/r_0.$$

It follows that there exists $C_1 > 0$ depending only on n such that

$$\int_{V + r_0\mathbb{B}} \Delta u \leq \frac{C_1 M}{r_0}.$$

Then, by Jensen formula (see, for example, [1, 16]), one has

$$\int_V |\hat{u}_r(z) - u(z)| d\,\mathrm{Leb} \leq C_2 M r^2, \tag{5.14}$$

for every $0 < r < r_0$, where $C_2 > 0$ depends only on n and r_0.

For every $z \in V$ and for every $0 < s < r$, there exists $\hat{z} \in \overline{\mathbb{B}(z,s)}$ such that

$$\bar{u}_s(z) = u(\hat{z}) \leq \hat{u}_r(\hat{z}).$$

Since u is negative, it follows that

$$\bar{u}_s(z) \leq \left(\frac{r-s}{r}\right)^{2n} \hat{u}_{r-s}(z). \tag{5.15}$$

Combining (5.14) and (5.15), we get

$$\begin{aligned}\int_V |\bar{u}_s(z) - u(z)| \,\mathrm{d\,Leb} &\leq \left(\frac{r-s}{r}\right)^{2n} \int_V |\hat{u}_{r-s}(z) - u(z)| d\,\mathrm{Leb} + \frac{r^{2n} - (r-s)^{2n}}{r^{2n}} \int_V |u(z)| d\,\mathrm{Leb} \\ &\leq C_3 \left(M(r-s)^2 + \frac{Ms}{r} \right),\end{aligned}$$

for every $0 < s < r < r_0$, where $C_3 > 0$ is a constant depending only on n and r_0. Choosing $s = r^3$, we obtain (5.13) (with $C_0 = 2C_3$).

Fix $s \in (0, r_0^3)$. For every $z \in V$, we denote $\psi(z) = \frac{M}{r_0}(|z|^2 - (1-2r_0)^2)$, $u' := \max\{u, \psi\}$ and $v' := \max\{\bar{u}_s, \psi\}$. We have $u = u'$ on K, $\bar{u}_s = v'$ on K and $u' = v' = \psi$ on $V \setminus U$. Let $\phi \in \mathscr{C}_0^\infty(V)$ such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on U. Put $\tilde{u} = \phi u'$ and $\tilde{v} = \phi v'$. By using the standard embedding $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$, one can extend $\tilde{u}$ and $\tilde{v}$ to $A\omega_{FS}$-plurisubharmonic functions on $\mathbb{CP}^n$, where $A \geq 1$ is a constant depending only on n, M and r_0. Since $\mu = (dd^c v)^n$, we have $\tilde{\mu} := \mathbf{1_V}\mu$ is a Hölder continuous measure on $\mathbb{CP}^n$. Therefore, there exist constants $\beta = \beta(\tilde{\mu}) > 0$ and $C_4 = C_4(\tilde{\mu}, M, A) > 0$ such that

$$\int_K |\bar{u}_s - u| d\mu \leq \|\tilde{u} - \tilde{v}\|_{L^1(\tilde{\mu})} \leq C_4 \|\tilde{u} - \tilde{v}\|^\beta_{L^1(\mathbb{CP}^n)} \leq C_4 \|\bar{u}_s - u\|^\beta_{L^1(V)}. \tag{5.16}$$

Combining (5.13) and (5.16), we get

$$\int_K |\bar{u}_s - u| d\mu \leq C_5 s^{2\beta/3},$$

where $C_5 > 0$ is a constant depending on $n, r_0, \tilde{\mu}$ and M. The proof is completed. □

Recall that $N = \{\rho = -\infty\}$. By the choice of ρ, and Lojasiewicz's inequality (e.g., see [3]), there exist constants $A_0, A_1 > 1$ such that

$$A_0 \log \operatorname{dist}(x, N) - A_1 \leq \rho(x) \leq \frac{1}{A_0} \log \operatorname{dist}(x, N) + A_1, \tag{5.17}$$

for every $x \in X$.

Theorem 5.13. *Let μ be a Hölder continuous measure on X and $p \geq 1$ be a constant. Assume that φ is bounded on X and $B := \|\varphi\|_{L^\infty}$. Then there exist a constant $C > 0$ and a family of θ-psh functions $\psi_{m,\delta}$ with $\delta \in (0, a_0/m), m \in \mathbb{Z}^+$ satisfying the following three properties:*

(i)

$$\|\psi_{m,\delta} - \varphi\|_{L^p(\mu)} \leq C \frac{|\log \delta| + \log m}{m} + C\delta,$$

(ii)

$$\psi_{m,\delta}(x) \geq \varphi(x) - \frac{Bm_0}{m} + A_0(\delta + m^{-1}) \log \operatorname{dist}(x, N) - C\left(\delta + \frac{|\log \delta|}{m}\right)$$

for every $x \in X$,

(iii)

$$|\nabla \psi_{m,\delta}(x)| \leq C\delta^{-1/2} e^{(B+1)m} e^{-A_0 m\delta \log \operatorname{dist}(x,N)}$$

for every $x \in X$.

Proof. We note that the assumption that φ is bounded implies that the Chern class of L is nef by Demailly's regularisation theorems. The property (ii) follows from Theorem 5.11 (i) and from (5.17). The property (iii) follows from Theorem 5.11 (ii) applied to $r = 1$ and from (5.17). It remains to prove (i).

Since $\varphi_\delta = (1-\delta)\varphi + \delta\rho$, we have

$$\sup_{\mathbb{B}(x,r)} \varphi_\delta - \varphi_\delta(x) \leq (1-\delta)(\sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)) + \delta|\rho(x)| \leq \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x) + \delta|\rho(x)|, \tag{5.18}$$

and

$$\left|\frac{m-m_0}{m}\varphi_\delta(x) - \varphi(x)\right| \leq \left(\frac{m_0}{m} + \delta\right)|\varphi(x)| + \delta|\rho(x)|, \tag{5.19}$$

for every $x \in X$, $m > m_0$ and $0 < \delta < 1$.

Using (5.18), (5.19) and Theorem 5.11 (i), we get

$$\begin{aligned}
|\psi_{m,\delta} - \varphi| &\leq \left|\psi_{m,\delta} - \frac{m-m_0}{m}\varphi_\delta\right| + \left|\frac{m-m_0}{m}\varphi_\delta - \varphi\right| \\
&\leq \sup_{B(x,r)} \varphi_\delta - \varphi_\delta(x) + C_1 r + C_1\frac{|\log r| + |\log\delta| + 1}{m} + \left(\frac{m_0}{m} + \delta\right)|\varphi(x)| + \delta|\rho(x)| \\
&\leq \sup_{\mathbb{B}(x,r)} \varphi - \varphi(x) + 2\delta|\rho(x)| + C_1 r + B\delta + C_2\frac{|\log r| + |\log\delta|}{m},
\end{aligned}$$

for every $m > m_0, r > 0$ and $0 < \delta < 1/2$, where $C_1, C_2 > 0$ are constants. Then we have

$$\|\psi_{m,\delta} - \varphi\|_{L^p(\mu)} \leq \|\sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^p(\mu)} + C_3\left(\frac{|\log\delta| + |\log r|}{m} + \delta + r + \delta\|\rho\|_{L^p(\mu)}\right). \tag{5.20}$$

It follows from [19, Proposition 4.4] that there exist constants $\epsilon, M > 0$ depending only on X, ω, θ and μ satisfying

$$\int_X e^{-\epsilon w} d\mu \leq M,$$

for every $w \in \mathrm{PSH}(X,\theta)$ with $\sup_X w = 0$. Then, by Hölder inequality, we have

$$\|\sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^p(\mu)} \leq \|\sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^1(\mu)}^{\frac{1}{2p}} \|\sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^{2p-1}(\mu)}^{\frac{2p-1}{2p}} \leq C_4\|\sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^1(\mu)}^{\frac{1}{2p}},$$

where $C_4 > 0$ is a constant depending only on M, ϵ, μ and p. This combined with Lemma 5.12 gives

$$\|\sup_{\mathbb{B}(x,r)} \varphi - \varphi(x)\|_{L^p(\mu)} \leq C_5 r^{\alpha/p}, \tag{5.21}$$

for every $0 < r < r_0$, where $r_0 = r_0(X,\omega), \alpha = \alpha(X,\omega,\mu)$ and $C_5 = C_5(n, X, \omega, \theta, \mu, B, p)$ are positive constants.

Combining (5.20) and (5.21), we get

$$\|\psi_{m,\delta} - \varphi\|_{L^p(\mu)} \leq C_6\left(r^{\alpha/p} + \frac{|\log\delta| + |\log r|}{m} + \delta + r\right),$$

for every $m > m_0$, $0 < r < r_0$ and $0 < \delta < 1/2$. Choosing $r = \frac{r_0}{m^{p/\alpha}}$, we obtain (i). The proof is completed. □

6. GOING UP TO THE DESINGULARISATION OF N

Let $\pi : \widehat{X} \to X$ be the composition of sequence of blowups along smooth centers over N such that $\widehat{N} := \pi^{-1}(N)$ is a simple normal crossing hypersurface in X'. By Lojasiewicz's inequality, one has

$$\operatorname{dist}(\pi(x), N) \lesssim \operatorname{dist}(x, \widehat{N}) \lesssim \operatorname{dist}^{\beta}(\pi(x), N), \tag{6.1}$$

for some constant $\beta > 0$ independent of $x \in \widehat{X}$. Let $\widehat{\varphi} := \pi^*\varphi$ which is $\widehat{\theta}$-psh, where $\widehat{\theta} := \pi^*\theta$.

Theorem 6.1. *Let μ be a Hölder continuous measure on $\widehat{X}$ and $p \geq 1$ be a constant. Assume that φ is bounded and let $B := \|\varphi\|_{L^\infty}$. Then there exist constants $A, C > 0$ and a family of $\widehat{\theta}$-psh functions $\widehat{\psi}_{m,\delta}$ with $\delta \in (0, a_0/m), m \in \mathbb{Z}^+$ satisfying the following three properties:*

(i)

$$\|\widehat{\psi}_{m,\delta} - \widehat{\varphi}\|_{L^p(\mu)} \leq C\frac{|\log\delta| + \log m}{m} + C\delta,$$

(ii)

$$\widehat{\psi}_{m,\delta}(x) \geq \widehat{\varphi}(x) - \frac{Bm_0}{m} + A\delta\log\operatorname{dist}(x, \widehat{N}) - C\left(\delta + \frac{|\log\delta|}{m}\right),$$

for every $x \in \widehat{X}$,

(iii)

$$|\nabla\widehat{\psi}_{m,\delta}(x)| \leq C\delta^{-1/2}e^{(B+1)m}e^{-Am\delta\log\operatorname{dist}(x,\widehat{N})},$$

for every $x \in \widehat{X}$.

Proof. Let $\psi_{m,\delta}$ be functions in Theorem 5.13. Let $\widehat{\psi}_{m,\delta} := \pi^*\psi_{m,\delta}$. The desired assertions (ii) and (iii) follow directly from (6.1) and Theorem 5.13. To see why (i) holds, we recall that π is a composition of successive blowups along smooth centers. Thus the desired inequality (i) is deduced by Theorem 5.13 and Corollary 4.3 applied to μ. The proof is complete. □

Lemma 6.2. *Assume that φ is bounded on X and $(dd^c\varphi + \theta)^n = \mu$ is a Hölder continuous measure. Let γ be an arbitrary constant in $(0, 1)$. Then for every constant $D > 1$, there is a constant $c_{D,\gamma} > 0$ so that*

$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \leq \frac{c_{D,\gamma}}{|\log\operatorname{dist}(x, y)|^{\gamma}},$$

for every $x, y \in \widehat{X}\backslash\widehat{N}$ with

$$(\operatorname{dist}(x, y))^D \leq \min\{\operatorname{dist}(x, \widehat{N}), \operatorname{dist}(y, \widehat{N})\}.$$

Proof. Without loss of generality, we can assume that $0 < \operatorname{dist}(x, y) < 1/2$. Let $p > 1$ be a constant. Denote $\gamma_0 := p/(p + 2n + 1)$ and $\gamma = p/(p + 2n + 2)$. Note that if $p \to \infty$, then $\gamma \to 1$. Let $\delta := m^{-2D}$. By Lemma 4.1 (we choose the constant $\gamma = 1$ in Lemma 4.1) and Theorem 6.1(i), one get

$$\widehat{\psi}_{m,\delta}(x) - \widehat{\varphi}(x) \lesssim_{\gamma_0} \|\widehat{\psi}_{m,\delta} - \widehat{\varphi}\|^{\gamma_0}_{L^p(\mu)} \lesssim_{\gamma_0} \left(\frac{\log m}{m}\right)^{\gamma_0},$$

for every $x \in \widehat{X}$, $m > m_0$. This combined with Theorem 6.1 (ii) yields

$$|\widehat{\psi}_{m,\delta}(x) - \widehat{\varphi}(x)| \lesssim \left(\frac{\log m}{m}\right)^{\gamma_0} + m^{-2D}(-\log \mathrm{dist}(x, \widehat{N}))_+ \tag{6.2}$$

for every $x \in \widehat{X}$, $m > m_0$. Here $(-\log \mathrm{dist}(x, \widehat{N}))_+ = \max\{-\log \mathrm{dist}(x, \widehat{N}), 0\}$.

Let $l_{x,y}$ be the curve chosen as in Lemma 2.4 (for $\widehat{N}$ in place of N). Now using (6.2) and Theorem 6.1 (iii), and Lemma 2.4, we estimate

$$\begin{aligned}|\widehat{\varphi}(x) - \widehat{\varphi}(y)| &\le |\widehat{\varphi}(x) - \widehat{\psi}_{m,\delta}(x)| + |\widehat{\varphi}(y) - \widehat{\psi}_{m,\delta}(y)| + |\widehat{\psi}_{m,\delta}(x) - \widehat{\psi}_{m,\delta}(y)| \\ &\lesssim \left(\frac{\log m}{m}\right)^{\gamma_0} + m^{-2D}(-\log \mathrm{dist}(x, \widehat{N}))_+ + m^{-2D}(-\log \mathrm{dist}(y, \widehat{N}))_+ \\ &+ \mathrm{dist}(x,y) m^D e^{m(B+1)} e^{-Am^{-2D+1}\log \mathrm{dist}(l_{x,y}(t), \widehat{N})},\end{aligned}$$

for some point $t \in [0,1]$. Since $\mathrm{dist}(l_{x,y}(t), \widehat{N}) \ge C^{-1}\min\{\mathrm{dist}(x, \widehat{N}), \mathrm{dist}(y, \widehat{N})\}$, we obtain

$$\begin{aligned}|\widehat{\varphi}(x) - \widehat{\varphi}(y)| &\lesssim \left(\frac{|\log \delta|}{m}\right)^{\gamma_0} + \delta^{\gamma_0} + \delta(-\log \min\{\mathrm{dist}(x, \widehat{N}), \mathrm{dist}(y, \widehat{N})\})_+ \\ &+ \mathrm{dist}(x,y)\delta^{-1/2} e^{m(B+1)} e^{-Am^{-2D+1}\log \min\{\mathrm{dist}(x,\widehat{N}), \mathrm{dist}(y,\widehat{N})\}}.\end{aligned}$$

Hence, if $(\mathrm{dist}(x,y))^D \le \min\{\mathrm{dist}(x, \widehat{N}), \mathrm{dist}(y, \widehat{N})\}$ then we have

$$\begin{aligned}|\widehat{\varphi}(x) - \widehat{\varphi}(y)| &\lesssim m^{-\gamma} - Dm^{-2D}\log \mathrm{dist}(x,y) \\ &+ \mathrm{dist}(x,y) m^D e^{m(B+1)} e^{-ADm^{-2D+1}\log \mathrm{dist}(x,y)}.\end{aligned}$$

By choosing

$$m := \max\left\{m_0 + 1, \frac{\gamma|\log \mathrm{dist}(x,y)|}{3(B+1)}\right\},$$

we get

$$|\widehat{\varphi}(x) - \widehat{\varphi}(y)| \le \frac{c_D}{|\log \mathrm{dist}(x,y)|^\gamma},$$

for every $x, y \in \widehat{X} \backslash \widehat{N}$ with

$$(\mathrm{dist}(x,y))^D \le \min\{\mathrm{dist}(x, \widehat{N}), \mathrm{dist}(y, \widehat{N})\}.$$

This finishes the proof. □

Proposition 6.3. *Assume that φ is bounded on X and θ_φ^n is a Hölder continuous Monge-Ampère measure. Then for every constant $\gamma \in (0,1)$, there exists a constant $C_\gamma > 0$ such that*

$$|\varphi(x) - \varphi(y)| \le \frac{C_\gamma}{|\log \mathrm{dist}(x,y)|^\gamma},$$

for every $x, y \in X \backslash N$.

Proof. By Lemma 6.2, we can apply Proposition 2.3 to $\widehat{\varphi}$, and we see that $\widehat{\varphi}$ is $\log^\gamma$-continuous on $\widehat{X}$. This combined with Lemma 3.1 yields that φ is $\log^\gamma$-continuous. □

Theorem 1.1 is a direct consequence of the following result.

Theorem 6.4. *Let (X,ω) be a compact Kähler manifold such that the Chern class of $-K_X$ contains closed positive current of bounded potentials. Let L be a big and semi-ample line bundle on X. Let θ be a smooth semi-positive form in $c_1(L)$. Let μ be a Hölder continuous measure on X of mass equal to $\int_X \theta^n$. Then the unique solution u to the equation $(dd^c u+\theta)^n = \mu$ is $\log^M$-continuous for every constant $M>0$.*

Proof. Throughout this proof, C_j ($j=1,2,3...$) is a constant independent of m,δ,x,r.

Let $\gamma\in(0,1)$. By Proposition 6.3, we have

$$\begin{aligned}\sup_{x'\in\mathbb{B}(x,r)} \varphi_\delta(x')-\varphi_\delta(x) &\le (1-\delta)\big(\sup_{x'\in\mathbb{B}(x,r)} \varphi(x')-\varphi(x)\big)-\delta\rho(x)\\ &\le C_1\left(|\log r|^{-\gamma}+\delta|\log \operatorname{dist}(x,N)|\right),\end{aligned}$$

for every $x\in X$ and $0<r,\delta<1/2$. This combined with Theorem 5.11 (ii) yields

$$|\nabla\psi_{m,\delta}(x)|\le C_2\delta^{-1/2}r^{-n-1}e^{C_2 m(|\log r|^{-\gamma}+\delta|\log\operatorname{dist}(x,N)|+r)},$$

for every $m>m_0$. Now choose

$$r:=e^{-m^{\frac{1}{1+\gamma}}}.$$

We obtain that

$$|\nabla\psi_{m,\delta}(x)|\le C_3\delta^{-1/2}e^{C_3 m^{\frac{1}{1+\gamma}}+C_3 m\delta|\log\operatorname{dist}(x,N)|}, \tag{6.3}$$

for every $x\in X$, $0<\delta<1/2$ and $m>m_0$.

Let $\pi:\widehat{X}\to X$ and $\widehat{N}$ be as above. Let $\widehat{\psi}_{m,\delta}:=\pi^*\psi_{m,\delta}$. Thanks to (6.3) one gets immediately the following property (which is a stronger version of Theorem 6.1 (iii)):

$$|\nabla\widehat{\psi}_{m,\delta}(x)|\le C_4\delta^{-1/2}e^{C_4 m^{\frac{1}{1+\gamma}}+C_4 m\delta|\log\operatorname{dist}(x,\widehat{N})|}, \tag{6.4}$$

for every $x\in\widehat{X}$.

Now arguing exactly as in the proofs of Lemma 6.2 (use (6.4) in place of Theorem 6.1 (iii)) with $\delta:=m^{-2D}$, we get

$$\begin{aligned}|\widehat{\varphi}(x)-\widehat{\varphi}(y)| &\lesssim m^{-\gamma}-Dm^{-2D}\log\operatorname{dist}(x,y)\\ &\quad+\operatorname{dist}(x,y)m^D e^{C_4 m^{\frac{1}{1+\gamma}}}e^{-C_5 m^{-2D+1}\log\operatorname{dist}(x,y)},\end{aligned}$$

for every $x,y\in\widehat{X}\backslash\widehat{N}$ with $(\operatorname{dist}(x,y))^D\le\min\{\operatorname{dist}(x,\widehat{N}),\operatorname{dist}(y,\widehat{N})\}$. Now letting

$$m:=\max\left\{m_0+1,\left(\frac{\gamma|\log\operatorname{dist}(x,y)|}{3C_4}\right)^{1+\gamma}\right\},$$

we obtain

$$|\widehat{\varphi}(x)-\widehat{\varphi}(y)|\lesssim|\log|x-y||^{-\gamma(1+\gamma)},$$

for every $x,y\in\widehat{X}\backslash\widehat{N}$ with $(\operatorname{dist}(x,y))^D\le\min\{\operatorname{dist}(x,\widehat{N}),\operatorname{dist}(y,\widehat{N})\}$. We note that if $\gamma\to1$, then $\gamma(1+\gamma)\to2$. Using again arguments from the proof of Proposition 6.3 we infer that Proposition 6.3 holds for $\gamma(1+\gamma)$ in place of γ. Applying now Proposition 2.3 to $\widehat{\varphi}$, we see that $\widehat{\varphi}$ is $\log^{\gamma(1+\gamma)}$-continuous on $\widehat{X}$. This combined with Lemma 3.1 yields that φ is $\log^{\gamma'}$-continuous for every $\gamma'\in(0,2)$. Repeating this procedure gives the desired assertion. □

7. Log continuity of Monge-Ampère metrics

In this section we prove Corollary 1.2. We start with some auxiliary results. We fix a smooth Kähler form ω on X which induces a distance on X.

Lemma 7.1. *Let v be a bounded ω-psh function. Then there exists a constant $C > 0$ such that for every $x \in X$ and $\epsilon \in (0, 1]$ one has*

$$\lambda(v, x, \epsilon) := \epsilon^{-2n+2} \int_{\mathbb{B}(x,\epsilon)} (dd^c v + \omega) \wedge \omega^{n-1} \leq C/|\log \epsilon|.$$

Proof. Fix $x_0 \in X$. Let ψ be a negative ω-psh function on X such that $\psi = c \log |x - x_0|$ on an open neighborhood of x_0 in X for some constant $c > 0$, and ψ is smooth outside x_0. For ϵ small enough, we see that $\psi(x) = c \log |x - x_0|$ on $\mathbb{B}(x, \epsilon)$. Hence $\psi \leq c \log \epsilon$ on $\mathbb{B}(x_0, \epsilon)$. Recall also that

$$\epsilon^{-2n+2} \int_{\mathbb{B}(x,\epsilon)} (dd^c v + \omega) \wedge \omega^{n-1} \lesssim \int_{\mathbb{B}(x,\epsilon)} (dd^c v + \omega) \wedge (dd^c \psi + \omega)^{n-1},$$

see [12, Page 159]. Hence we get

$$\begin{aligned} \lambda(v, x, \epsilon) &\lesssim |\log \epsilon|^{-1} \int_{\mathbb{B}(x,\epsilon)} -\psi (dd^c v + \omega) \wedge (dd^c \psi + \omega)^{n-1} \\ &\leq |\log \epsilon|^{-1} \int_X -\psi (dd^c v + \omega) \wedge (dd^c \psi + \omega)^{n-1} \\ &= |\log \epsilon|^{-1} \int_X -\psi \omega \wedge (dd^c \psi + \omega)^{n-1} \\ &\quad + |\log \epsilon|^{-1} \int_X -v dd^c \psi \wedge \omega \wedge (dd^c \psi + \omega)^{n-1} \\ &\lesssim |\log \epsilon|^{-1} (\|v\|_{L^\infty} + 1). \end{aligned}$$

This finishes the proof. □

Lemma 7.2. *Let u be a $\log^M$-continuous θ-psh function on X such that u is smooth outside N. Let $\delta \in (0, 1]$ be a constant. Then there exist a constant $C > 0$ independent of δ and a sequence of smooth $(\theta + \delta\omega)$-psh function $(u_\epsilon)_\epsilon$ so that u_ϵ converges uniformly to u and u_ϵ converges to u locally in the $\mathcal{C}^\infty$-topology on $X \backslash N$.*

Proof. This follows essentially from Demailly's regularisation of psh functions ([13, Section 8]). Let $\exp_z$ be the exponential map at $z \in X$ of (X, ω). Let χ be a cut-off function as in [13, Page 492]. We define

$$u_\epsilon(z) := \frac{1}{C\epsilon^{2n}} \int_{\zeta \in T_z X} u(\exp_z(\zeta)) \chi'(|\zeta|^2/\epsilon^2) d\lambda(\zeta),$$

where $d\lambda$ denotes the Lebesgue measure on the Hermitian space $T_z X$ and

$$C := \int_{\zeta \in T_z X} \chi'(|\zeta|^2) d\lambda(\zeta).$$

One sees immediately that u_ϵ is $\log^M$-continuous uniformly in ϵ because u is already $\log^M$-continuous.

By [13, Proposition 8.5 and Lemma 8.6], we know that there is a constant $A_1 > 0$ such that

$$dd^c u_\epsilon(x) + \theta(x) \geq \big(- A_1 \lambda(u, x, \epsilon) - A_1 \epsilon / |\log \epsilon| \big) \omega,$$

where

$$\lambda(u, x, \epsilon) := \epsilon^{-2n+2} \int_{\mathbb{B}(x,\epsilon)} (dd^c u + \omega) \wedge \omega^{n-1} \leq A_2 / |\log \epsilon|$$

for some constant A_2 independent of x by Lemma 7.1. Hence we infer that

$$dd^c u_\epsilon + \omega \geq -A_3 |\log \epsilon|^{-1} \omega$$

for some constant $A_3 > 0$ independent of ϵ. This finishes the proof. □

Lemma 7.3. *Let $\delta \in (0, 1]$, $M > 1$ and $C_0 > 0$ be constants. Let u be a smooth $(\theta + \delta\omega)$-psh functions such that*

$$|u(x) - u(y)| \leq C_0 |\log \operatorname{dist}(x, y)|^{-2M}$$

for every $x, y \in X$. Denote by $\tilde{d}$ the distance induced by $dd^c u + \theta + \delta\omega$. Then

$$\tilde{d}(x, y) \leq C |\log \operatorname{dist}(x, y)|^{-M+1}$$

for every $x, y \in X$, where $C > 0$ is a constant independent of u and δ.

Proof. Let $\Omega(r)$ be the modulus of continuity of u. By hypothesis, one has

$$\Omega(r) \leq C_0 |\log r|^{-2M} \tag{7.1}$$

for every $0 < r < 1$. We cover X by finitely many local charts (which are relatively compact in bigger local charts) and since the Kähler form ω is equivalent to the standard Kähler form on $\mathbb{C}^n$ in these local charts, we can assume that ω is equal to the standard form on $\mathbb{C}^n$ on these local charts.

Let $\mathbb{B}(x, r)$ denotes the ball of radius r with center at $x \in \mathbb{C}^n$. Fix $x^* \in X$ and a local chart U around x^* biholomorphic to $\mathbb{B}(0, 2)$ such that $x^* = 0$ in these local coordinates. Define $d(x) := \tilde{d}(x, 0)$. Recall that $\tilde{d}$ is the Riemannian metric induced by $dd^c u + \theta + \delta\omega$. For $x \in \mathbb{B}(0, 1)$, let

$$d_r(x) := \operatorname{vol}(\mathbb{B}(x, r))^{-1} \int_{x' \in \mathbb{B}(x,r)} d(x') \, \omega^n.$$

Arguing as in the proof of [27, Lemma 5] (see also [33] or [26]) and using (7.1), for every $x_0 \in \mathbb{B}(0, 1)$, one obtains

$$\int_{\mathbb{B}(x_0,r)} |\nabla d|^2_\omega \omega^n \leq C_1 r^{2n} + C_1 \int_{\mathbb{B}(x_0,3r/2)} |u(x) - u(x_0)| \omega^n \leq C_2 r^{2n-2} |\log r|^{-2M}.$$

Therefore, by Poincaré inequality, we infer

$$r^{-2n} \int_{\mathbb{B}(x_0,r)} |d(x) - d_r(x_0)|^2 \omega^n \leq C_3 |\log r|^{-2M},$$

where $C_3 > 0$ is a uniform constant independent of x_0, δ and r. This combined with Hölder inequality gives

$$r^{-2n} \int_{\mathbb{B}_\omega(x_0,r)} |d(x) - d_r(x_0)| \omega^n \leq C_3 |\log r|^{-M}. \tag{7.2}$$

We now use some arguments similar to the proof of Campanato's lemma. We follow the presentation in [28, Chapter 3]. Assume $0 < r_1 < r_2 < 1$ and $x_1, x_2 \in \mathbb{B}(0,1)$ with $\mathbb{B}(x_1, r_1) \subset \mathbb{B}(x_2, r_2)$. Observe that

$$|d_{r_1}(x_0) - d_{r_2}(x_0)| \le |d_{r_1}(x_0) - d(x)| + |d_{r_2}(x_0) - d(x)|$$

for every $x \in \mathbb{B}(x_1, r_1) \subset \mathbb{B}(x_2, r_2)$. It follows that

$$|d_{r_1}(x_1) - d_{r_2}(x_2)| \le \\ \mathrm{vol}(\mathbb{B}_\omega(x_1,r_1))^{-1}\bigg(\int_{\mathbb{B}_\omega(x_1,r_1)} |d_{r_1}(x_1) - d(x)|\omega^n + \int_{\mathbb{B}_\omega(x_2,r_2)} |d_{r_2}(x_2) - d(x)|\omega^n\bigg)$$

By (7.2), it follows that

$$|d_{r_1}(x_1) - d_{r_2}(x_2)| \lesssim r_1^{-2n}\big(r_1^{2n}|\log r_1|^{-M} + r_2^{2n}|\log r_2|^{-M}\big). \tag{7.3}$$

Applying the last inequality to $r_1 = r/2^{k+1}, r_2 = r/2^k$ and $x_1 = x_2 = x_0$ yields

$$|d_{2^{-k}r}(x_0) - d_{2^{-k-1}r}(x_0)| \le C_4\big(|\log r| + (k+1)\log 2\big)^{-M} \le \int_k^{k+1} \frac{C_4 dt}{(|\log r| + t\log 2)^{-M}},$$

for some uniform constant $C_4 > 0$ independent of $x_0, r, k, \epsilon, \delta$. Summing over $k = 0, 1, 2, \dots$ yields

$$\sum_{k=0}^{\infty} |d_{2^{-k}r}(x_0) - d_{2^{-k-1}r}(x_0)| \le \int_0^\infty \frac{C_4 dt}{(|\log r| + t\log 2)^{-M}} \le C_5|\log r|^{-M+1}.$$

Since d_r converges uniformly to d on $\mathbb{B}(0,1)$ as $r \searrow 0$, it follows that

$$|d(x_0) - d_r(x_0)| \le C_5|\log r|^{-M+1} \tag{7.4}$$

for every $x_0 \in \mathbb{B}(x^*, 1)$ (recall $x^* = 0$), and $0 < r < 1$. Let $x \in \mathbb{B}(x^*, 1/8)$ and $r := \mathrm{dist}(x, x^*) \le 1/8$. Using (7.4) and then applying (7.3) for $x_1 = x$, $x_2 = x^*$ and $r_2 = 3r_1 = 3r$, we get

$$\begin{aligned} d(x) &= d(x) - d(x^*) \\ &\le |d(x) - d_r(x)| + |d(x^*) - d_{3r}(x^*)| + |d_r(x) - d_{3r}(x^*)| \\ &\lesssim |\log r|^{-M+1} + |d_r(x) - d_{3r}(x^*)| \\ &\lesssim |\log r|^{-M+1} + |\log r|^{-M} \\ &\lesssim |\log r|^{-M+1}. \end{aligned}$$

This finishes the proof. □

Proof of Corollary 1.2. Write $\omega_F = dd^c u + \theta$, where u is $\log^M$-continuous θ-psh function for every constant $M > 1$ by Theorem 1.1. Fix $\delta \in (0,1]$. Let u_ϵ be as in Lemma 7.2 for u. Since $dd^c u_\epsilon + \theta + \delta\omega$ converges to $dd^c u + \theta + \delta\omega \ge dd^c u + \theta$ locally in the $\mathcal{C}^\infty$-topology in $X\backslash N$, one sees that the desired assertion follows from Lemma 7.3 by letting $\epsilon \to 0$. □

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