COMPUTATION OF THE ŁOJASIEWICZ EXPONENTS OF REAL BIVARIATE ANALYTIC FUNCTIONS

SĨ TIỆP ĐINH[1], FENG GUO[2], HỒNG ĐỨC NGUYỄN[3], AND TIẾN-SƠN PHẠM[4]

Abstract. The main goal of this paper is to present some explicit formulas for computing the Łojasiewicz exponent in the Łojasiewicz inequality comparing the rate of growth of two real bivariate analytic function germs.

1. Introduction

The Łojasiewicz inequalities and their variants play an important role in many branches of mathematics. For example, Łojasiewicz inequalities are very useful in the study of continuous regular functions, see [13, 18] for pioneering works and [19] for a survey. Also, Łojasiewicz inequalities, together with Nullstellensätz, are crucial tools for the study of the ring of (bounded) continuous semi-algebraic functions on a semi-algebraic set, see [10, 11, 12].

Let $f, g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be nonzero real analytic function germs. Assume that $0 \in \{f = 0\} \subset \{g = 0\}$. By the classical Łojasiewicz inequality on comparing the rate of growth, there exist positive constants C, r and α such that

$$|f(x)| \geqslant C|g(x)|^{\alpha} \quad \text{for} \quad |x| \leqslant r. \tag{1}$$

The infimum of such α is called the *Łojasiewicz exponent of f w.r.t. g* and denoted by $\mathscr{L}_g(f)$.

Note that several versions of the Łojasiewicz inequality have been studied for a special case where g is the distance function to the zero set of f, see [4, 5, 6, 7, 8, 9, 16, 17, 20]. Furthermore, the computation or estimation of Łojasiewicz exponents in this case has been considered in these works. In [3], the authors provided a global version of the Łojasiewicz inequality on comparing the rate of growth of two polynomial functions in the case the mapping defined by these functions is (Newton) non-degenerate at infinity. However, no computation or estimation of Łojasiewicz exponents has been given.

In this work, we will address partially to this problem by giving some explicit formulas for computing the Łojasiewicz exponent $\mathscr{L}_g(f)$ in the most general case when f and g are two arbitrary real bivariate analytic function germs. Moreover, our proof provides a new algorithm computing the limit of bivariate rational functions (See Corollary 5.1).

The rest of the paper is organized as follows. In Section 2, we recall the notions of Newton polygon relative to an arc and sliding due to Kuo and Parusiński which

Date: May 10, 2024.

[1,3,4]These authors are funded by International Centre for Research and Postgraduate Training in Mathematics (ICRTM) under grant number ICRTM04_2021.04.

[2]Feng Guo was supported by the Chinese National Natural Science Foundation under grant 11571350, the Fundamental Research Funds for the Central Universities.

are crucial in the proof of our formulas for the Łojasiewicz exponent, which are our main results (Theorem 3.1 and Theorem 3.2), whose statements, together with the proofs, will be given in Section 3.

2. The Newton polygon relative to an arc

The technique of Newton polygons plays an important role in this paper. It is well-known that Newton transformations which arise in a natural way when applying the Newton algorithm provide a useful tool for calculating invariants of singularities. For a complete treatment we refer to [1, 2, 24, 25]. In this section we recall the notion of Newton polygon relative to an arc due to Kuo and Parusiński [21] (see also, [14] and [15]).

Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{C}$ and let $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ denote a nonzero analytic function germ with Taylor expansion:

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots ,$$

where f_k is a homogeneous polynomial of degree k, and $f_m \not\equiv 0$. For the remainder of the paper, we will assume that f is *regular in x of order m* in the sense that $f_m(1, 0) \neq 0$. (This can be achieved by a linear transformation $x' = x, y' = y + cx$, where c is a generic number). Let ϕ be an analytic arc in $\mathbb{K}^2$, which is not tangent to the x-axis. Then it can be parametrized by

$$x = c_1 t^{n_1} + c_2 t^{n_2} + \cdots \in \mathbb{K}\{t\} \text{ and } y = t^N$$

and therefore can be identified with a *Puiseux series* (denoted also by ϕ for simplicity of notation)

$$x = \phi(y) = c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots \in \mathbb{K}\{y^{1/N}\}$$

with $N \leq n_1 < n_2 < \cdots$ being positive integers. The changes of variables $X := x - \phi(y)$ and $Y := y$ yield

$$F(X, Y) := f(X + \phi(Y), Y) := \sum c_{ij} X^i Y^{j/N}.$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a *Newton dot.* The set of Newton dots is called the *Newton diagram.* They generate a convex hull, whose boundary is called the *Newton polygon of f relative to ϕ*, to be denoted by $\mathbb{P}(f, \phi)$. Note that this is the Newton polygon of F in the usual sense. If ϕ is a *Newton–Puiseux root* of f (i.e., $f(\phi(y), y) = 0$), then there are no Newton dots on $X = 0$, and vice versa. Assume that ϕ is not a Newton–Puiseux root of f, then the exponents of the series $f(\phi(y), y) = F(0, Y)$ correspond to the Newton dots on the line $X = 0$. In particular, $\mathrm{ord} f(\phi(y), y) = h_0$, where $(0, h_0)$ is the lowest Newton dot on $X = 0$.

The *highest Newton edge*, denoted by E_H (or E_1) is defined as follows: If ϕ is a Newton–Puiseux root of f, then E_1 is the non-compact edge of the polygon $\mathbb{P}(f, \phi)$ parallel to the y-axis. If ϕ is not a Newton–Puiseux root of f, then E_1 is the compact edge of the polygon $\mathbb{P}(f, \phi)$ with a vertex being the lowest Newton dot on $X = 0$. The *Newton edges* $E_2, E_3, \ldots, E_s$ are compact adges of $\mathbb{P}(f, \phi)$. These edges and their associated *Newton angles* $\theta_2, \ldots, \theta_s$ are defined in an obvious way as illustrated in the following example.

Example 2.1. Take $f(x, y) := x^3 - y^5 + y^6$ and $\phi(y) := y^{5/3}$. We have

$$F(X, Y) := f(X + \phi(Y), Y) = X^3 + 3\,X^2 Y^{5/3} + 3\,XY^{10/3} + Y^6.$$

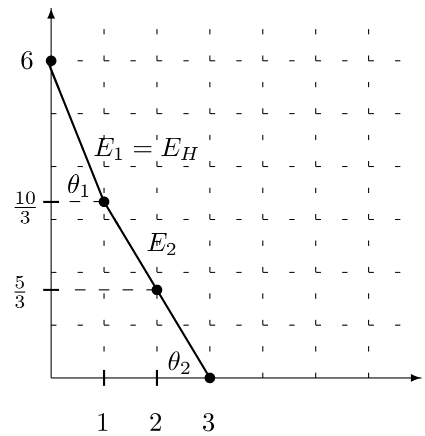

FIGURE 1.

By definition, the Newton polygon of f relative to ϕ has three edges E_1, E_2 with $\tan\theta_1 = 8/3$ and $\tan\theta_2 = 5/3$ (see Figure 1).

Take any edge E_s. The *associated polynomial* $\mathcal{E}_s(z)$ is defined to be $\mathcal{E}_s(z) := \mathcal{E}_s(z, 1)$, where

$$\mathcal{E}_s(X, Y) := \sum_{(i,j/N)\in E_s} c_{ij} X^i Y^{j/N}.$$

Next, let us recall the notion of *sliding* (see [21]). Suppose that ϕ is not a Newton–Puiseux root of f. Consider the Newton polygon $\mathbb{P}(f, \phi)$. Take any nonzero root c of $\mathcal{E}_1(z) = 0$, the polynomial equation associated to the highest Newton edge E_1. We call

$$\phi_1 \colon x = \phi(y) + cy^{\tan\theta_1}$$

a *sliding* of ϕ along f, where θ_1 is the angle associated to E_1. A recursive sliding

$$\phi \to \phi_1 \to \phi_2 \to \cdots$$

produces a limit, denoted by ϕ_∞, which is a Newton–Puiseux root of f. The series ϕ_∞ will be called a *final result of sliding* ϕ *along* f. Note that ϕ_∞ has the form

$$\phi_\infty \colon x = \phi(y) + cy^{\tan\theta_1} + \text{higher order terms},$$

due to the following technical lemma.

Lemma 2.1. *Let ϕ be a Puiseux series, which is not a Newton–Puiseux root of f. Let θ_1 and $\mathcal{E}_1$ be respectively the Newton angle and polynomial associated to the highest Newton edge E_1. Consider a series of the following form*

$$\psi \colon x = \phi(y) + cy^\rho + \textit{ higher order terms},$$

where $c \in \mathbb{K}$ and $\rho \in \mathbb{Q}, \rho > 0$. Then the following statements hold:

(i) *If either c or ρ is generic (i.e., $\arctan\rho$ is not a Newton angle of $\mathbb{P}(f, \phi)$; or $\arctan\rho$ is a Newton angle of $\mathbb{P}(f, \phi)$ but c is not a root of the polynomial associated to the Newton edge with Newton angle $\arctan\rho$), then*

$$\operatorname{ord} f(\psi(y), y) = \min\{a\rho + b \mid (a, b) \in \mathbb{P}(f, \phi)\}.$$

Furthermore,

$$\operatorname{ord} f(\psi(y), y) \leqslant \operatorname{ord} f(\phi(y), y).$$

In particular, if either $\tan\theta_H < \rho$ *or* $\tan\theta_H = \rho$ *and* $\mathcal{E}_H(c) \neq 0$ *then* $\mathbb{P}(f,\psi) = \mathbb{P}(f,\phi)$*, and therefore*

$$\mathrm{ord} f(\psi(y), y) = \mathrm{ord} f(\phi(y), y).$$

(ii) *If* $\tan\theta_H = \rho$ *and* $\mathcal{E}_H(c) = 0$ *then*

$$\mathrm{ord} f(\psi(y), y) > \mathrm{ord} f(\phi(y), y).$$

Proof. cf. [1, 2, 24]. For a detailed proof, we refer to [14]. In fact, the special case where $\psi(y) = \phi(y) + cy^{\tan\theta_1}$ was proved in [14, Lemma 2.1]. Then the lemma is deduced by applying the special case (possibly infinitely) many times. □

Definition 2.1. For each Puiseux series $\phi(y) = \sum_i a_i y^{\alpha_i}$ and for each positive real number ρ, the *ρ-approximation* of $\phi(y)$ is defined to be the series $\sum_{\alpha_i<\rho} a_i y^{\alpha_i} + cy^{\rho}$, where c is a generic real number. We associate to any Puiseux series ϕ its *real approximation* $\phi^{\mathbb{R}}(y)$ defined to be the ρ-approximation of ϕ, where ρ is the smallest exponent occurring in ϕ with non-real coefficient. It is clear that if φ is *real*, i.e., all coefficients of φ are real, then the real approximation of φ is itself. Now, for $f \in \mathbb{K}\{x,y\}$ which is regular in x, let $\mathcal{V}_{\mathbb{R}}(f)$ be the set of all real approximations of non-real Newton–Puiseux roots of f.

For any two distinct series ϕ_1, ϕ_2, their *approximation*, denoted by $\phi_{1,2}$, is defined to be the ρ-approximation of ϕ_1 where $\rho := \mathrm{ord}(\phi_1 - \phi_2)$. Let $\mathcal{V}_a(f)$ be the set of all approximations of ϕ_1 and ϕ_2 with $\phi_1 \neq \phi_2$ being Newton–Puiseux roots of f. Note that $\mathcal{V}_{\mathbb{R}}(f) \subset \mathcal{V}_a(f)$ if $f \in \mathbb{R}\{x,y\}$.

The following useful assertion is a direct consequence of Lemma 2.1:

Lemma 2.2. *Assume that* $\mathbb{K} = \mathbb{R}$*. Let* ϕ *be a Puiseux series and let* $E_1, \ldots, E_s$ *be the Newton edges of* $\mathbb{P}(f,\phi)$*. Let* θ_i *and* $\mathcal{E}_i$ *be the corresponding Newton angle and polynomial associated to* E_i*. Then by a permutation of indexes, we have*

$$\pi/2 \geqslant \theta_1 > \theta_2 > \ldots > \theta_s$$

and the following statements hold:

(i) *If* $\mathcal{E}_i$ *has two distinct roots, there exists* $\psi \in \mathcal{V}_a(f)$ *being of the form*

$$\psi(y) = \phi(y) + cy^{\tan\theta_i} + \text{ higher order terms},$$

where c *is a generic number.*

(ii) *If* $\theta \neq \theta_1$ *is a Newton angle, then there exists* $\psi \in \mathcal{V}_a(f)$ *being of the form*

$$\psi(y) = \phi(y) + cy^{\tan\theta} + \text{ higher order terms},$$

where c *is a generic number.*

(iii) *If* $\mathcal{E}_i$ *has a non-real root, then there exists* $\psi \in \mathcal{V}_{\mathbb{R}}(f)$ *being of the form*

$$\psi(y) = \phi(y) + cy^{\tan\theta_i} + \text{ higher order terms},$$

where c *is a generic number.*

3. Formulas for Łojasiewicz exponents

For the remainder of this section, let $f, g\colon (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be nonzero real analytic function germs, which are regular in x such that $0 \in \{f = 0\} \subset \{g = 0\}$. By the classical Łojasiewicz inequality (see, for instance [22]), there exists positive constants C, r and α such that

$$|f(x,y)| \geqslant C|g(x,y)|^{\alpha} \text{ for } |x| \leqslant r.$$

The infimum of such α is called the *Lojasiewicz exponent of f with respect to g* and denoted by $\mathscr{L}_g(f)$.

Take any analytic arc $\phi \in \mathbb{R}^2$ at the origin parametrized by $(x(t), y(t))$. If $g \circ \phi \not\equiv 0$, then we can define the following positive rational number

$$\ell(\phi) \quad := \quad \frac{\mathrm{ord} f(\phi(t))}{\mathrm{ord} g(\phi(t))}.$$

By the Curve Selection Lemma (see [23, Lemma 3.1]), it is not hard to show that

$$\mathscr{L}_g(f) \quad = \quad \sup_{\phi} \ell(\phi), \tag{2}$$

where the supremum is taken over all analytic arcs passing through the origin, which are not contained in the zero locus of g. Furthermore, since f and g are x-regular, the supremum in (2) can be taken over all real analytic arcs passing through the origin not contained in the zero locus of y.

Remark 3.1. Note that the supremum in (2) may not be attained, i.e., it is possible that there is no analytic arc $\phi \in \mathbb{R}^2$ at the origin such that $\mathscr{L}_g(f) = \ell(\phi)$. The following example is an illustration.

Let

$$f(x, y) = x^2 \qquad \text{and} \qquad g(x, y) = x(x^2 + y^2).$$

Then $f(x, y) \geqslant g^2(x, y)$ for (x, y) closed enough to the origin. So $\mathscr{L}_g(f) \leqslant 2$. On the other hand, for each positive integer k, let $\phi_k(t) = (t, t^{1/k})$, then we have

$$\ell(\phi_k) = \frac{2}{1 + \frac{2}{k}} \to 2 \qquad \text{as} \quad k \to +\infty.$$

Therefore $\mathscr{L}_g(f) = 2$. Now, for any analytic arc $\phi(t) = (x(t), y(t))$ at the origin, we have

$$\ell(\phi) = \frac{2\mathrm{ord}\ x(t)}{\mathrm{ord}\ x(t) + 2\min\{\mathrm{ord}\ x(t), \mathrm{ord}\ y(t)\}} < \frac{2\mathrm{ord}\ x(t)}{\mathrm{ord}\ x(t)} = 2,$$

i.e., $\mathscr{L}_g(f)$ is not attained for any analytic arc $\phi \in \mathbb{R}^2$ at the origin.

3.1. **First formula for the Łojasiewicz exponents.** Let $\beta_j,\ j = 1, \ldots, k$ be the common real Newton–Puiseux roots of f and g of multiplicities m_j and n_j respectively. Let $\mathcal{V}_a(fg)$ be the set given by Definition 2.1. For any $\phi \in \mathcal{V}_a(fg)$, we will write $\ell(\phi)$ instead of $\ell((\phi(y), y))$ for simplicity.

Theorem 3.1. *Define*

$$\mathcal{L}_g^+(f) := \max\left\{\ell(\phi), \frac{m_j}{n_j} \;\Big|\; \phi \in \mathcal{V}_a(fg),\ j = 1, \ldots, k\right\} \quad \textit{and} \quad \mathcal{L}_g^-(f) = \mathcal{L}_{\bar{g}}^+(\bar{f}),$$

where $\bar{f}(x, y) := f(x, -y)$ and $\bar{g}(x, y) := g(x, -y)$. Then the Lojasiewicz exponent of g w.r.t f is given by

$$\mathscr{L}_g(f) = \max\left\{\mathcal{L}_g^+(f), \mathcal{L}_g^-(f)\right\}.$$

Proof. We first show that

$$\mathscr{L}_g(f) \geqslant \max\left\{\mathcal{L}_g^+(f), \mathcal{L}_g^-(f)\right\}. \tag{3}$$

By (2), it is obvious that

$$\mathscr{L}_g(f) \geqslant \max\left\{\ell(\phi) \mid \phi \in \mathcal{V}_a(fg)\right\}.$$

Therefore we only need to show that

$$\mathscr{L}_g(f) \geqslant \frac{m_j}{n_j} \quad \text{for all} \quad j = 1, \ldots, k. \tag{4}$$

To do this, fix $j \in \{1, \ldots, k\}$ and consider the Newton polygons $\mathbb{P}(f, \beta_j)$ of f and $\mathbb{P}(g, \beta_j)$ of g relative to the arc β_j. Let $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ be respectively the vertices of $\mathbb{P}(f, \beta_j)$ and $\mathbb{P}(g, \beta_j)$ being closest to the y-axis. We will show that

$$x_1 = m_j \quad \text{and} \quad x_2 = n_j.$$

Indeed, in view of Puiseux's theorem (see, for example [24, page 98]), we can write

$$f(x, y) = (x - \beta_j(y))^{m_j} h(x, y), \tag{5}$$

where $h(\beta_j(y), y) \neq 0$ for all j. So

$$f(X + \beta_j(Y), Y) = X^{m_j} h(X + \beta_j(Y)Y).$$

This implies $x_1 = m_j$ and $y_1 = \text{ord } h(\beta_j(y), y))$ and similarly $x_2 = n_j$.

For each positive integer n, define a new arc

$$x = \phi_n(y) = \beta_j(y) + y^n.$$

By (5), we have

$$f(\phi_n(y), y) = y^{nx_1} h(\phi_n(y), y).$$

For n large enough, we have

$$y_1 = \text{ord } h(\beta_j(y), y)) = \text{ord } h(\phi_n(y), y)).$$

So this yields $\text{ord } f(\phi_n(y), y) = nx_1 + y_1$. By the same way, we also have $\text{ord } g(\phi_n(y), y) = nx_2 + y_2$. Consequently,

$$\ell(\phi_n) = \frac{nx_1 + y_1}{nx_2 + y_2}.$$

Note that

$$\lim_{n \to \infty} \ell(\phi_n) = \frac{x_1}{x_2} = \frac{m_j}{n_j},$$

so (4) follows from (2). Therefore, $\mathscr{L}_g(f) \geqslant \mathcal{L}_g^+(f)$. Similarly, one has $\mathscr{L}_g(f) \geqslant \mathcal{L}_g^-(f)$ and hence the inequality (3) holds. Now we need to show that the inequality in (3) is actually an equality.

Suppose for contradiction that

$$\mathscr{L}_g(f) > \max\left\{\mathcal{L}_g^+(f), \mathcal{L}_g^-(f)\right\}.$$

Then there is a real analytic arc ϕ passing through the origin and not lying in the x-axis such that $g \circ \phi \not\equiv 0$ and

$$\ell(\phi) > \max\left\{\mathcal{L}_g^+(f), \mathcal{L}_g^-(f)\right\}.$$

Note that ϕ can be parametrized by either

$$(x = \phi(t), y = t) \quad \text{or} \quad (x = \phi(t), y = -t),$$

where $\phi(t)$ is an element in $\mathbb{R}\{t^{1/N}\}$ for some positive integer N with $\phi(0) = 0$. Without loss of generality we may assume that ϕ can be parametrized by $(x = \phi(t), y = t)$. Denote by E_1 and E_2 the highest Newton edges of $\mathbb{P}(f, \phi)$ and $\mathbb{P}(g, \phi)$ respectively. Let $\mathcal{E}_i$ and θ_i be respectively its associated polynomial and Newton angle.

Claim 3.1. *We have* $\tan\theta_1 = \tan\theta_2$.

Proof. Assume for contradiction that, $\tan\theta_1 > \tan\theta_2$. Let ϕ_∞ be a final result of sliding ϕ along f. Write

$$\phi_\infty(y) = \phi(y) + \sum_{i\geqslant 1} a_i y^{\alpha_i},$$

where $a_i \in \mathbb{C}\setminus\{0\}$, $\tan\theta_1 = \alpha_1 < \alpha_2 < \cdots$. We will show that $a_i \in \mathbb{R}$ for all $i \geqslant 1$. In fact, if this is not the case, for each $n \geqslant 0$, define the series

$$\phi_0(y) := \phi(y), \quad \phi_n(y) := \phi(y) + \sum_{i=1}^{n} a_i y^{\alpha_i} \quad \text{for} \quad n \geqslant 1,$$

and let n_0 be the smallest index such that $a_{n_0} \notin \mathbb{R}$. Then $n_0 \geqslant 1$ and

$$\phi_{n_0}^{\mathbb{R}}(y) = \phi_{n_0-1}(y) + cy^{\alpha_{n_0}} + \text{ higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. By applying Lemma 2.1, we obtain

$$\text{ord } f(\phi_{n_0}^{\mathbb{R}}(y), y) = \text{ord } f(\phi_{n_0-1}(y), y) > \cdots > \text{ord } f(\phi(y), y)$$

and

$$\text{ord } g(\phi_{n_0}^{\mathbb{R}}(y), y) = \text{ord } g(\phi_{n_0-1}(y), y) = \cdots = \text{ord } g(\phi(y), y).$$

So

$$\ell(\phi_{n_0}^{\mathbb{R}}) = \ell(\phi_{n_0-1}) > \ldots > \ell(\phi) > \mathcal{L}_g^+(f),$$

a contradiction, since $\phi_{n_0}^{\mathbb{R}} \in \mathcal{V}_{\mathbb{R}}(f) \subset \mathcal{V}_{\mathbb{R}}(fg) \subset \mathcal{V}_a(fg)$. This shows that $a_n \in \mathbb{R}$ for all $n \geqslant 1$. But then this contradicts to the assumption that $\{f = 0\} \subset \{g = 0\}$ in $\mathbb{R}^2$, hence

$$\tan\theta_1 \leqslant \tan\theta_2.$$

Now assume for contradiction that $\tan\theta_1 < \tan\theta_2$. Note that, θ_1 and θ_2 are Newton angles of $\mathbb{P}(fg, \phi)$. Then by Lemma 2.2(ii), there exists $\psi \in \mathcal{V}_a(fg)$ being of the form

$$\psi(y) = \phi(y) + cy^{\tan\theta_1} + \text{ higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. It follows from Lemma 2.1(i) that

$$\text{ord } f\left(\psi(y), y\right) = \text{ord } f\left(\phi(y), y\right) \quad \text{ and } \quad \text{ord } g\left(\psi(y), y\right) \leqslant \text{ord } g\left(\phi(y), y\right),$$

and hence $\ell(\psi) \geqslant \ell(\phi) > \mathcal{L}_g^+(f)$. This contradiction finishes the claim. □

Claim 3.2. *The polynomial $\mathcal{E}_1\mathcal{E}_2$ has only one root.*

Proof. Assume for contradiction that $\mathcal{E}_1\mathcal{E}_2$ has two distinct roots. By Claim 3.1, $\theta_1 = \theta_2$, so $\mathcal{E}_1\mathcal{E}_2$ is the Newton polynomial associated to the highest Newton edge of $\mathbb{P}(fg, \phi)$ (with Newton angle θ_1). Then by Lemma 2.2(i), there exists $\psi \in \mathcal{V}_a(fg)$ being of the form

$$\psi(y) = \phi(y) + cy^{\tan\theta_1} + \text{ higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. Then Lemma 2.1(ii) yields

$$\text{ord } f\left(\psi(y), y\right) = \text{ord } f\left(\phi(y), y\right) \text{ and ord } g\left(\psi(y), y\right) = \text{ord } g\left(\phi(y), y\right).$$

Hence $\ell(\phi) = \ell(\psi)$ and so $\ell(\phi) \leqslant \mathcal{L}_g^+(f)$ which contradicts the assumption $\ell(\phi) > \mathcal{L}_g^+(f)$. □

Let $a \in \mathbb{R}$ be the unique root of the polynomial $\mathcal{E}_1\mathcal{E}_2$ and let $\widetilde{\phi}(y) := \phi(y) + ay^{\tan\theta_1}$. We denote by $\widetilde{E}_1$ and $\widetilde{E}_2$ the highest Newton edge of $\mathbb{P}(f,\widetilde{\phi})$ and $\mathbb{P}(g,\widetilde{\phi})$ respectively. For $i = 1, 2$, let $\widetilde{\theta}_i$ and $\widetilde{\mathcal{E}}_i$ be the Newton angle and the polynomial associated to $\widetilde{E}_i$. Recall that E_1 and E_2 are respectively the highest Newton edges of $\mathbb{P}(f,\phi)$ and $\mathbb{P}(g,\phi)$. Let $B_i = (x_i, y_i)$ be the vertex of E_i which is not contained in the y-axis.

Claim 3.3. *If $\widetilde{\phi}(y)$ is not a Newton–Puiseux root of f, then the following properties hold:*

(i) *B_i is a vertex of $\widetilde{E}_i$, therefore $\deg\widetilde{\mathcal{E}}_i = x_i = \deg\mathcal{E}_i$.*
(ii) $\tan\widetilde{\theta}_1 = \tan\widetilde{\theta}_2$.
(iii) *The polynomial $\widetilde{\mathcal{E}}_1\widetilde{\mathcal{E}}_2$ has only one root.*
(iv) $\ell(\widetilde{\phi}) \geqslant \ell(\phi)$.

Proof. (i) Let us define the function

$$\mu(t) = \frac{tx_1 + y_1}{tx_2 + y_2}.$$

We first claim that $x_1y_2 \geqslant x_2y_1$. In fact, if this is not the case, i.e., $x_1y_2 < x_2y_1$, then the function μ is strictly decreasing and $y_1 > 0$. Since f is regular in x, there exists a Newton edge E of $\mathbb{P}(f,\phi)$ which is different from E_1 and has B_1 as a vertex. Let θ be the Newton angle associated to E. Clearly $\theta < \theta_1 = \theta_2$, therefore, by Lemma 2.2(ii), there exists $\psi \in \mathcal{V}_a(f) \subset \mathcal{V}_a(fg)$ such that

$$\psi(y) = \phi(y) + cy^{\tan\theta} + \text{ higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. We have

$$\operatorname{ord} f(\phi(y),y) = x_1\tan\theta_1 + y_1 \quad \text{and} \quad \operatorname{ord} g\,(\phi(y),y) = x_2\tan\theta_2 + y_2 = x_2\tan\theta_1 + y_2,$$

Moreover, by Lemma 2.1(i) and the choice of the edge E,

$$\operatorname{ord} f(\psi(y),y) = x_1\tan\theta + y_1 \quad \text{and} \quad \operatorname{ord} g\,(\psi(y),y) \leqslant x_2\tan\theta + y_2.$$

Hence

$$\ell(\phi) = \mu\tan\theta_1 < \mu\tan\theta \leqslant \ell(\psi) \leqslant \mathcal{L}_g^+(f),$$

which is a contradiction. Hence we must have $x_1y_2 \geqslant x_2y_1$, i.e., the function μ is increasing.

Let $\widetilde{E}$ be the edge of $\mathbb{P}(f,\widetilde{\phi})$ such that B_1 is the vertex having larger x-coordinate. Let $\widetilde{\theta}$ be the Newton angle associated to $\widetilde{E}$. Since $\widetilde{\phi}(y)$ is not a Newton–Puiseux root of f, $\widetilde{E}$ is a compact edge and therefore $\theta_1 < \widetilde{\theta} < \pi/2$. If $\widetilde{E}$ is not the highest Newton edge of $\mathbb{P}(f,\widetilde{\phi})$, then by Lemma 2.2(ii), there exists $\varphi \in \mathcal{V}_a(f) \subset \mathcal{V}_a(fg)$ such that

$$\varphi(y) = \widetilde{\phi}(y) + cy^{\tan\widetilde{\theta}} + \text{ higher order terms},$$

where $c \in \mathbb{R}$ is a generic number. It follows from Lemma 2.1(i) that

$$\operatorname{ord} f(\varphi(y),y) = x_1\tan\widetilde{\theta} + y_1 \quad \text{and} \quad \operatorname{ord} g\,(\varphi(y),y) \leqslant x_2\tan\widetilde{\theta} + y_2.$$

Hence

$$\ell(\phi) = \mu(\tan\theta_1) \leqslant \mu(\tan\widetilde{\theta}) \leqslant \ell(\varphi) \leqslant \mathcal{L}_g^+(f).$$

This contradiction yields $\widetilde{E} \equiv \widetilde{E}_1$, i.e., B_1 is a vertex of $\widetilde{E}_1$. Similarly we can show that B_2 is a vertex of $\widetilde{E}_2$ and hence Item (i) follows.

(ii)–(iii) These can be proved by using completely the same argument as in Claims 3.1 and 3.2.

(iv) It follows from Items (ii) and (iii) that

$$\operatorname{ord} f(\widetilde{\phi}(y), y) = x_1 \tan\widetilde{\theta}_1 + y_1 \quad \text{and} \quad \operatorname{ord} g(\widetilde{\phi}(y), y) = x_2 \tan\widetilde{\theta}_2 + y_2 = x_2 \tan\widetilde{\theta}_1 + y_1,$$

i.e.,

$$\ell(\widetilde{\phi}) = \mu(\tan\widetilde{\theta}_1) \geqslant \mu(\tan\theta_1) = \ell(\phi).$$

This implies (iv) and hence the claim follows. □

We are now in position to complete the theorem. Applying Claim 3.3 (possibly infinitely) many times, we obtain ϕ_∞ as a final result of sliding of ϕ along f. This implies that, $x = \phi_\infty(y)$ is a common Newton–Puiseux root of f and g of multiplicities x_1 and x_2 respectively. Moreover, from the proof Claim 3.3, $x_1 y_2 \geqslant x_2 y_1$, it follows that

$$\ell(\phi) = \frac{x_1 \tan\theta_1 + y_1}{x_2 \tan\theta_1 + y_2} \leqslant \frac{x_1}{x_2} \leqslant \mathcal{L}_g^+(f).$$

This contradicts the assumption $\ell(\phi) > \mathcal{L}_g^+(f)$. The theorem is proved. □

3.2. **Second formula for the Łojasiewicz exponents.** Recall that $\mathcal{V}_{\mathbb{R}}(f)$ is the set of real approximations of non-real Newton–Puiseux roots of f as defined in Definition 2.1. Let $\beta_j,\ j = 1, \ldots, k$, be the common real Newton–Puiseux roots of f and g of multiplicity m_j and n_j respectively.

Theorem 3.2. *Define*

$$\mathscr{L}_g^+(f) := \max\left\{\ell(\gamma), \frac{m_j}{n_j} \;\Big|\; \gamma \in \mathcal{V}_{\mathbb{R}}(f),\ j = 1, \ldots, k\right\} \quad \textit{and} \quad \mathscr{L}_g^-(f) = \mathscr{L}_{\bar{g}}^+(\bar{f}),$$

where $\bar{f}(x, y) := f(x, -y)$ *and* $\bar{g}(x, y) := g(x, -y)$. *Then the Łojasiewicz exponent of* g *w.r.t* f *is given by*

$$\mathscr{L}_g(f) = \max\left\{\mathscr{L}_g^+(f), \mathscr{L}_g^-(f)\right\}.$$

Proof. Since $\mathcal{V}_{\mathbb{R}}(f) \subset \mathcal{V}_a(fg)$ and $\mathcal{V}_{\mathbb{R}}(\bar{f}) \subset \mathcal{V}_a(\bar{f}\bar{g})$, by Theorem 3.1,

$$\mathscr{L}_g(f) = \max\left\{\mathcal{L}_g^+(f), \mathcal{L}_g^-(f)\right\} \geqslant \max\left\{\mathscr{L}_g^+(f), \mathscr{L}_g^-(f)\right\}.$$

Arguing by contradiction, we assume that

$$\mathscr{L}_g(f) > \max\left\{\mathscr{L}_g^+(f), \mathscr{L}_g^-(f)\right\}.$$

It follows from Theorem 3.1 that there is an analytic arc ϕ passing through the origin and not lying in the x-axis such that $g \circ \phi \not\equiv 0$ and

$$\mathscr{L}_g(f) = \ell(\phi) > \max\left\{\mathscr{L}_g^+(f), \mathscr{L}_g^-(f)\right\}.$$

Note that ϕ can be parametrized by either

$$(x = \phi(t), y = t) \quad \text{or} \quad (x = \phi(t), y = -t),$$

where $\phi(t)$ is an element in $\mathbb{R}\{t^{1/N}\}$ for some positive integer number N with $\phi(0) = 0$. Let $\mathcal{E}_\phi$ be the polynomial associated to the highest Newton edge of $\mathbb{P}(f, \phi)$. With no loss of generality, we can assume that ϕ has the following property:

For any analytic arc $\widetilde{\phi}$ *passing through the origin not lying in the* x*-axis and having the parametrization* $(x = \widetilde{\phi}(t), y = t)$ *such that* $g \circ \widetilde{\phi} \not\equiv 0$ *and*

$$\mathscr{L}_g(f) = \ell(\widetilde{\phi}) > \max\left\{\mathscr{L}_g^+(f), \mathscr{L}_g^-(f)\right\},$$

if $\mathcal{E}_{\widetilde{\phi}}$ *is the polynomial associated to the highest Newton edge of* $\mathbb{P}(f, \widetilde{\phi})$*, then* $\deg \mathcal{E}_{\widetilde{\phi}} \geqslant \deg \mathcal{E}_{\phi}$.

Indeed, if there is an analytic arc $\widetilde{\phi}$ such that this property does not hold, i.e., $\deg \mathcal{E}_{\widetilde{\phi}} < \deg \mathcal{E}_{\phi}$, then it is enough to replace ϕ by $\widetilde{\phi}$ and repeat the process until the property is satisfied.

Let E_1 and E_2 be the highest Newton edges of $\mathbb{P}(f, \phi)$ and $\mathbb{P}(g, \phi)$ respectively. For each $i = 1, 2$, let $\mathcal{E}_i$ and θ_i be the associated polynomial and the Newton angle of E_i respectively. Let $B_i = (x_i, y_i)$ be the vertex of E_i which is not contained in the y-axis. Then the following statement holds.

Claim 3.4. *We have* $\tan \theta_1 = \tan \theta_2$.

Proof. Applying the same argument as in the proof of Claim 3.1, we get $\tan \theta_1 \leqslant \tan \theta_2$. Assume for contradiction that $\tan \theta_1 < \tan \theta_2$. Let

$$\psi(y) = \phi(y) + cy^{\tan \theta_1}$$

with a generic number c. It follows from Lemma 2.1 that

$$\mathrm{ord}\ f(\psi(y), y) = x_1 \tan \theta_1 + y_1 = \mathrm{ord}\ f(\phi(y), y)$$

and

$$\mathrm{ord}\ g(\psi(y), y) \leqslant x_2 \tan \theta_1 + y_2 < x_2 \tan \theta_2 + y_2 = \mathrm{ord}\ g(\phi(y), y).$$

These imply

$$\ell(\psi) > \ell(\phi) = \mathscr{L}_g(f),$$

which is a contradiction. □

Claim 3.5. *The polynomial* $\mathcal{E}_1$ *has only real roots.*

Proof. Assume for contradiction that $a \notin \mathbb{R}$ is a root of $\mathcal{E}_1$. It follows from Lemma 2.2(iii) that there exists $\psi \in \mathcal{V}_{\mathbb{R}}(f)$ of the form

$$\psi = \phi + cy^{\tan \theta_1} + \text{higher order terms}$$

with a generic real number c. Applying Lemma 2.1(i) we obtain

$$\mathrm{ord}\ f(\psi(y), y) = \mathrm{ord}\ f(\phi(y), y) \quad \text{and} \quad \mathrm{ord}\ g(\psi(y), y) = \mathrm{ord}\ g(\phi(y), y).$$

Therefore

$$\ell(\psi) = \ell(\phi) > \mathscr{L}_g^+(f),$$

which contradicts the definition of $\mathscr{L}_g^+(f)$. □

Claim 3.6. *We have*

(i) $x_1 y_2 = x_2 y_1$*, and therefore* $\ell(\phi) = \dfrac{x_1}{x_2} = \dfrac{tx_1 + y_1}{tx_2 + y_2}$ *for all* t.

(ii) *The polynomial* $\mathcal{E}_1 \mathcal{E}_2$ *has only one root.*

Proof. (i) First of all, let us prove

$$x_1 y_2 \geqslant x_2 y_1. \tag{6}$$

Indeed, if this is not the case, i.e., $x_1 y_2 < x_2 y_1$, then the function

$$\mu(t) := \frac{tx_1 + y_1}{tx_2 + y_2}$$

is strictly decreasing. Let $\psi(y) = \phi(y) + y^{\rho}$ with $0 < \rho < \tan\theta_1$ closed enough to $\theta_1 (= \theta_2$ by Claim 3.4) so that $\arctan\rho$ is larger than the other Newton angles of $\mathbb{P}(f, \phi)$ and $\mathbb{P}(g, \phi)$. Then

$$\text{ord } f(\psi(y), y) = x_1\rho + y_1 \quad \text{and} \quad \text{ord } g(\psi(y), y) = x_2\rho + y_2.$$

So we get

$$\ell(\psi) = \mu(\rho) > \mu\tan\theta_1 = \ell(\phi) = \mathscr{L}_g(f),$$

which contradicts the definition of $\mathscr{L}_g(f)$. Hence $x_1y_2 \geqslant x_2y_1$. Let us now prove that the equality always holds.

Let $c_j \in \mathbb{R}, j = 1, \ldots, q$, be the roots of $\mathcal{E}_1(z)$ of multiplicity x_1^j with $x_1^j > 0$ and $q \geqslant 1$. We write

$$\mathcal{E}_2(z) = a(z)\prod_{j=1}^{q}(z - c_j)^{x_2^j}$$

with $x_2^j \geqslant 0$ and $a(c_j) \neq 0$. Observe that

$$\sum_{j=1}^{q} x_1^j = \deg\mathcal{E}_1 = x_1 \quad \text{and} \quad \sum_{j=1}^{q} x_2^j + \deg a(z) = \deg\mathcal{E}_2 = x_2. \tag{7}$$

Let us denote by $A_i^j = (x_i^j, y_i^j)$ the intersection of the line $\{x = x_i^j\}$ with the edge E_i for each $i = 1, 2$. Set

$$\mu_j(t) := \frac{tx_1^j + y_1^j}{tx_2^j + y_2^j}.$$

Since $A_i^j \in E_i$, it follows that

$$\mu_j\tan\theta_1 = \ell(\phi) \quad \text{and} \quad y_i^j = y_i + (x_i - x_i^j)\tan\theta_1, \tag{8}$$

for all $i = 1, 2,\ j = 1\ldots, q$. We also notice that A_1^j is a vertex of the Newton polygon $\mathbb{P}(f, \widetilde{\phi}_j)$ with $\widetilde{\phi}_j(y) := \phi(y) + c_jy^{\theta_1}$. We shall show that

$$x_1^jy_2^j \leqslant x_2^jy_1^j \quad \text{for all } j = 1, \ldots, q. \tag{9}$$

In fact, by contradiction, assume that $x_1^jy_2^j > x_2^jy_1^j$, i.e., the function $\mu_j(t)$ is strictly increasing. From this and (8), for $\rho > \tan\theta_1$ sufficiently closed to $\tan\theta_1$, we have

$$\mathscr{L}_g(f) = \ell(\phi) = \mu_j\tan\theta_1 < \mu_j\rho = \ell\left(\widetilde{\phi}_j(y) + cy^{\rho}\right) \leqslant \mathscr{L}_g(f)$$

for every non-zero $c \in \mathbb{R}$, which is clearly a contradiction. Thus (9) must holds. Combining (8) and (9) yields

$$x_1^j[y_2 + \tan\theta_1(x_2 - x_2^j)] \leqslant x_2^j[y_1 + \tan\theta_1(x_1 - x_1^j)] \quad \text{for all } j = 1, \ldots, q.$$

Summing up we obtain

$$(y_2 + x_2\tan\theta_1)\sum_{j=1}^{q} x_1^j \leqslant (y_1 + x_1\tan\theta_1)\sum_{j=1}^{q} x_2^j.$$

Combining this with (7), we get

$$(y_2 + x_2\tan\theta_1)x_1 \leqslant (y_1 + x_1\tan\theta_1)(x_2 - \deg a(z)) \leqslant (y_1 + x_1\tan\theta_1)x_2.$$

Equivalently

$$x_1y_2 \leqslant x_2y_1.$$

By this and (6), we have $x_1y_2 = x_2y_1$ and Item (i) follows.

(ii) By Item (i), it follows that $x_1^j y_2^j = x_2^j y_1^j$ for all $j = 1, \ldots, q$ and $\deg a(z) = 0$. Hence the function $\mu_j(t)$ is constant. Consider, for each j, the curve $\psi_j(y) = \widetilde{\phi}_j + y^\rho$ for some $\rho > \tan\theta_1$ sufficiently closed to $\tan\theta_1$. Then

$$\ell(\psi_j) = \mu_j(\rho) = \mu_j \tan\theta_1 = \ell(\phi).$$

Moreover, it is not hard to check that A_1^j is a vertex of the Newton polygon $\mathbb{P}(f, \psi_j)$. So $\deg \mathcal{E}_{\psi_j} = x_1^j$ where $\mathcal{E}_{\psi_j}$ is the polynomial associated to the highest Newton edge of $\mathbb{P}(f, \psi_j)$. Then it follows from the choice of ϕ that $x_1^j \geqslant x_1$. This implies $q = 1$ and therefore, by the fact that $\deg a(z) = 0$, the polynomial $\mathcal{E}_1\mathcal{E}_2$ must have only one root. The claim is proved. $\square$

Let $a \in \mathbb{R}$ be the unique root of the polynomial $\mathcal{E}_1\mathcal{E}_2$ and let $\widetilde{\phi}(y) := \phi(y) + ay^{\tan\theta_1}$. Let $\widetilde{\mathbb{P}}_1 := \mathbb{P}(f, \widetilde{\phi})$ and $\widetilde{\mathbb{P}}_2 := \mathbb{P}(g, \widetilde{\phi})$. We denote by $\widetilde{E}_i$ the Newton edge of $\widetilde{\mathbb{P}}_i$ containing B_i as the vertex with the larger x-coordinate. Let $\widetilde{\theta}_i$ and $\widetilde{\mathcal{E}}_i$ be the Newton angle and the polynomial associated to $\widetilde{E}_i$ respectively.

Claim 3.7. *We have* $\tan\widetilde{\theta}_1 = \tan\widetilde{\theta}_2$.

Proof. Assume for contradiction that $\tan\widetilde{\theta}_1 > \tan\widetilde{\theta}_2$. Consider the curve

$$\psi(y) = \widetilde{\phi} + cy^{\tan\widetilde{\theta}_1}$$

with a generic number c. Then it follows from Lemma 2.1(i) that, for any $(u, v) \in \widetilde{E}_2$ such that $(u, v) \neq B_2$, we have

$$\text{ord } f(\psi(y), y) = x_1 \tan\widetilde{\theta}_1 + y_1$$

and

$$\text{ord } g(\psi(y), y) \leqslant u \tan\widetilde{\theta}_1 + v < x_2 \tan\widetilde{\theta}_1 + y_2.$$

Therefore

$$\ell(\psi) > \frac{x_1 \tan\widetilde{\theta}_1 + y_1}{x_2 \tan\widetilde{\theta}_1 + y_2} = \frac{x_1 \tan\theta_1 + y_1}{x_2 \tan\theta_1 + y_2} = \ell(\phi) = \mathscr{L}_g(f),$$

where the first equality follows from Claim 3.6(i). This is a contradiction.

Now, by contradiction, suppose that $\tan\widetilde{\theta}_1 < \tan\widetilde{\theta}_2$. Let us show that the polynomial $\widetilde{\mathcal{E}}_1$ has only real root. In fact, if this is not the case, then by Lemma 2.2(iii), there exists $\psi \in \mathcal{V}_{\mathbb{R}}(f)$ of the form

$$\psi(y) = \widetilde{\phi} + cy^{\tan\widetilde{\theta}_1}$$

with a generic number c. It then follows from Lemma 2.1(i) that

$$\text{ord } f(\psi(y), y) = x_1 \tan\widetilde{\theta}_1 + y_1 \quad \text{and} \quad \text{ord } g(\psi(y), y) \leqslant x_2 \tan\widetilde{\theta}_1 + y_2.$$

Therefore, in view of Claim 3.6(i),

$$\ell(\psi) \geqslant \frac{x_1 \tan\widetilde{\theta}_1 + y_1}{x_2 \tan\widetilde{\theta}_1 + y_2} = \frac{x_1 \tan\theta_1 + y_1}{x_2 \tan\theta_1 + y_2} = \ell(\phi) = \mathscr{L}_g(f),$$

which is a contradiction, because $\psi \in \mathcal{V}_{\mathbb{R}}(f)$.

We now take $0 \neq a \in \mathbb{R}$ such that $\widetilde{\mathcal{E}}_1(a) = 0$ and define $\gamma(y) = \widetilde{\phi} + ay^{\tan\widetilde{\theta}_1}$. Then it follows from Lemma 2.1(i) that

$$\text{ord } f(\gamma(y), y) > x_1 \tan\widetilde{\theta}_1 + y_1 \quad \text{and} \quad \text{ord } g(\gamma(y), y) \leqslant x_2 \tan\widetilde{\theta}_1 + y_2.$$

Therefore

$$\ell(\gamma) > \frac{x_1 \tan \widetilde{\theta}_1 + y_1}{x_2 \tan \widetilde{\theta}_1 + y_2} = \ell(\phi) = \mathscr{L}_g(f),$$

a contradiction. Hence $\tan \widetilde{\theta}_1 = \tan \widetilde{\theta}_2$. □

Claim 3.8. *If $\widetilde{\phi}(y)$ is not a Newton–Puiseux root of f, then it and the Newton polygons of f and g relative to it share the following properties with that of ϕ:*

(i) $\tan \widetilde{\theta}_1 = \tan \widetilde{\theta}_2$.

(ii) *The polynomial $\widetilde{\mathcal{E}}_1 \widetilde{\mathcal{E}}_2$ has only one root. In particular, for each $i = 1, 2$, $\widetilde{E}_i$ is the highest Newton edge of $\widetilde{\mathbb{P}}_i$.*

(iii) $\ell(\widetilde{\phi}) = \dfrac{x_1}{x_2} = \ell(\phi)$.

Proof. It is clear that Item (i) follows from Claim 3.7. Furthermore, Items (ii) and (iii) can be proved by using the same argument as in the proof of Claim 3.6. □

We are now in position to complete the theorem. Applying Claim 3.8 (possibly infinitely) many times, we obtain a final result ϕ_∞ of sliding of ϕ along f which is also that of g. This implies that, ϕ_∞ is a common Newton–Puiseux root of f and g of multiplicities x_1 and x_2 respectively. Therefore,

$$\ell(\phi) = \frac{x_1}{x_2} \leqslant \mathscr{L}_g^+(f).$$

This contradicts the assumption that $\ell(\phi) > \mathscr{L}_g^+(f)$. Hence the theorem follows. □

4. Algorithms

In this section we provide an algorithm verifying whether $\{f = 0\} \subset \{g = 0\}$ and computing the Lojasiewicz exponent $\mathscr{L}_g(f)$ if it is defined. Let $f \in \mathbb{R}[x, y]$ be regular in x and let $\mathcal{V}(f)$ be the set of all Newton–Puiseux roots $x = \gamma(y)$ of f. Let $x = \varphi(y)$ be a (complex) Puiseux series. The *contact order* of φ and f is defined as

$$\rho(\varphi, f) := \max\{\operatorname{ord}(\varphi(y) - \gamma(y)) \mid \varphi \neq \gamma \in \mathcal{V}(f)\}.$$

For each rational number q, the series φ is called a Newton–Puiseux root $\mod q+$ of f if there exists $\gamma \in \mathcal{V}(f)$ such that $\operatorname{ord}(\varphi(y) - \gamma(y)) > q$. Assume that

$$x = \gamma(y) = \sum c_\alpha y^\alpha$$

is a Newton–Puiseux root of f, then the series

$$\tilde{\gamma}(y) = \sum_{\alpha \leq \rho} c_\alpha y^\alpha,$$

where $\rho = \rho(\gamma, f)$, is called *a truncated Newton–Puiseux root* of f. We denote by $\widetilde{\mathcal{V}}(f)$ the set of truncated Newton–Puiseux roots f.

Remark 4.1. It follows from the definition that:

(i) If $\gamma(y)$ and $\gamma'(y)$ are distinct Newton–Puiseux roots of f, then $\tilde{\gamma} \neq \tilde{\gamma}'$. That is, the natural map $\mathcal{V}(f) \to \widetilde{\mathcal{V}}(f)$ is bijective.

(ii) If $\gamma(y)$ is a Newton–Puiseux root of f then

$$\operatorname{ord}(\tilde{\gamma}(y) - \gamma(y)) > \rho(\gamma, f) := \max\{\operatorname{ord}(\gamma(y) - \gamma'(y)) \mid \gamma \neq \gamma' \in \mathcal{V}(f)\}.$$

Theorem 4.1. *Let $f \in \mathbb{R}[x,y]$ be regular in x and let $x = \gamma(y)$ be a Newton–Puiseux root of f. Let $\rho = \rho(\gamma, f)$ the contact order of γ and f. Then*

(i) *If the truncated Newton–Puiseux root $\tilde{\gamma}$ of γ is real, then γ is a real Newton–Puiseux root of f.*

(ii) *We write*

$$f(X + \tilde{\gamma}(y), Y) = \sum c_{ij} X^i Y^{j/N}.$$

Then the multiplicity of γ, denoted by $\operatorname{mult}_\gamma f$, is equal to the minimum of i such that

$$i\rho + j/N = \operatorname{ord}(f(\tilde{\gamma}_\rho(y), y)) \text{ and } c_{ij} \neq 0, \tag{10}$$

where $\tilde{\gamma}_\rho$ is the ρ-approximation of $\tilde{\gamma}$.

(iii) *Let $g \in \mathbb{R}[x,y]$ be regular in x and let $h := \gcd(f,g)$. If $\tilde{\gamma}$ is a root $\operatorname{mod} \rho(\gamma, f)+$ of h then $\gamma(y)$ is also a root of g.*

Proof. (i) Assume for contradiction that γ is not real and write

$$\gamma(y) = \phi(y) + cy^\alpha + \text{ higher order terms},$$

where ϕ is the sum of terms of order lower than α with real coefficients and $c \in \mathbb{C} \backslash \mathbb{R}$. Since f is real, the conjugate

$$\bar{\gamma}(y) = \phi(y) + \bar{c}y^\alpha + \text{ higher order terms}$$

is also a Newton–Puiseux root of f. Thus

$$\begin{aligned} \rho(\gamma, f) &= \max\{\operatorname{ord}(\gamma(y) - \gamma'(y)) \mid \gamma \neq \gamma' \in \mathcal{V}(f)\} \\ &\geq \operatorname{ord}(\gamma(y) - \bar{\gamma}(y)) = \alpha. \end{aligned}$$

By definition of truncated Newton–Puiseux root, $\tilde{\gamma}$ contains the term cy^α so it is not real which is a contradiction. Consequently γ is real.

Let $\mathbb{P}(f, \tilde{\gamma})$ be the Newton polygon of f relative to $\tilde{\gamma}$ and let $E_1, \ldots, E_s$ be its Newton edges. Let θ_i and $\mathcal{E}_i$ be the Newton angle and the polynomial associated to E_i respectively. Consider a progress of recursive slidings

$$\tilde{\gamma} \to \tilde{\gamma}_1 \to \ldots \to \tilde{\gamma}_\infty$$

of $\tilde{\gamma}$ along f. The following claim is a direct consequence of Lemma 2.1.

Claim 4.1. *We have*

$$\operatorname{ord}(\tilde{\gamma}(y) - \tilde{\gamma}_\infty(y)) = \max\{\operatorname{ord}(\tilde{\gamma}(y) - \gamma'(y)) \mid \gamma' \in \mathcal{V}(f)\} = \theta_1.$$

and

$$\rho = \max\{\operatorname{ord}(\tilde{\gamma}(y) - \gamma'(y)) \mid \gamma \neq \gamma' \in \mathcal{V}(f)\} = \theta_2.$$

This together with Remark 4.1 implies that $\tilde{\gamma}_\infty = \gamma$. This means that, there is a unique progress of recursive slidings of $\tilde{\gamma}$ along f. Write

$$\tilde{\gamma}_\infty = \tilde{\gamma} + a_1 y^{\alpha_1} + a_2 y^{\alpha_2} + \ldots.$$

Then for all $n \geq 1$,

$$\tilde{\gamma}_n = \tilde{\gamma} + a_1 y^{\alpha_1} + a_2 y^{\alpha_2} + \ldots + a_n y^{\alpha_n}.$$

Since $\tilde{\gamma}_n$ is the only sliding of $\tilde{\gamma}_{n-1}$ along f, the polynomial $\mathcal{E}_H^{n-1}$ of associated to the highest Newton edges of $\mathbb{P}(f, \tilde{\gamma}_{n-1})$ has only one root a_n of multiplicity $\deg \mathcal{E}_H^{n-1} = \deg \mathcal{E}_H^0 = \deg \mathcal{E}_1$. Then the multiplicity $\operatorname{mult}_\gamma f$ of γ is equal to

$$\deg \mathcal{E}_1 = \operatorname{ord} \mathcal{E}_2 = \min\{i \mid (i, j/N) \in E_2\}.$$

Since

$$E_2 = \{(i, j/N) \in \text{supp}(f) \mid i\theta_2 + j/N = \text{ord}(f(\tilde{\gamma}_\rho(y), y))\}$$

it follows that

$$\text{mult}_\gamma f = \min\{i \mid i\rho + j/N = \text{ord}(f(\tilde{\gamma}_\rho(y), y)) \text{ and } c_{ij} \neq 0\},$$

which gives (ii). Now, we take a root ξ of h such that $\text{ord}\,(\xi(y) - \tilde{\gamma}(y)) > \rho(\gamma, f)$. Then, it follows from the definition of $\rho(\gamma, f)$ that $\xi = \gamma$. This completes (iii). □

As a consequence, we obtain the following algorithm for computing the Łojasiewicz exponent $\mathcal{L}_g(f)$.

Algorithm BiLojEx.

INPUT: Two polynomials f and g in $\mathbb{Q}[x, y]$ of positive orders.

OUTPUT: Decide whether or not $\{f = 0\} \subset \{g = 0\}$ and compute the Łojasiewicz exponent $\mathcal{L}_g(f)$.

Step 1. If one of the polynomials f, g is not x-regular, make a linear transformation, so that the new polynomials f, g are x-regular. Compute $h := \gcd(f, g)$.

Step 2. Compute the set $\tilde{\mathcal{V}}(f)$ of truncated roots of f. Compute the sets $\tilde{\mathcal{V}}_{\mathbb{R}}(f)$ and $\tilde{\mathcal{V}}_{\mathbb{R}}(h)$ of truncated real roots of f and h.

If $\sharp\, \tilde{\mathcal{V}}_{\mathbb{R}}(f) \leq \sharp\, \tilde{\mathcal{V}}_{\mathbb{R}}(h)$ then $\{f = 0\} \subset \{g = 0\}$ and proceed to the next step. Otherwise, the Łojasiewicz exponent $\mathcal{L}_g(f)$ is not defined and the algorithm stops.

Step 3. Compute for each $\gamma \in \tilde{\mathcal{V}}_{\mathbb{R}}(f)$ the multiplicies $\text{mult}_\gamma f$ and $\text{mult}_\gamma g$ by Formula (10).

Step 4. Compute the set $\tilde{\mathcal{V}}_a(f)$ of the real approximations of series in $\tilde{\mathcal{V}}(f) \setminus \tilde{\mathcal{V}}_{\mathbb{R}}(f)$ and compute

$$\mathscr{L}_g^+(f) := \max\left\{\ell(\gamma), \frac{\text{mult}_\gamma f}{\text{mult}_\gamma g} \;\Big|\; \gamma \in \tilde{\mathcal{V}}_a(f), \gamma \in \tilde{\mathcal{V}}_{\mathbb{R}}(f)\right\}.$$

Step 5. Set $\tilde{f}(x, y) := f(x, -y)$ and $\tilde{g}(x, y) := g(x, -y)$ and compute $\mathscr{L}_{\tilde{g}}^+(\tilde{f})$.

Step 6. $\mathscr{L}_g(f) := \max\{\mathscr{L}_g^-(f), \mathscr{L}_{\tilde{g}}^+(\tilde{f})\}$.

5. Applications

Computing limits of (real) multivariate functions at given points is one of the basic problems in computational mathematics. Let $\frac{g}{f}$ be a rational function with f, g real polynomials. It is well known that if the limit $\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)}$ exists, it can be easily computed by evaluating the limit along a ray R through $(0, 0)$. Therefore, replacing $\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)}$ by $\lim_{(x,y)\to(0,0)} \frac{g(x,y)-Lf(x,y)}{f(x,y)}$ with $L = \lim_{R\ni(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)}$ for some ray R, one reduces the problem to studying whether $\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)} = 0$. The following sufficient condition is straightforward.

Proposition 5.1. *(1) If $0 < \mathcal{L}_g(f) < 1$, then*

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)} = 0.$$

(2) If $\mathcal{L}_g(f) > 1$, then the limit $\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)}$ does not exist.

In the case when $\mathcal{L}_g(f) = 1$ the limit $\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)}$ may exist or not. However, we can deduce the following corollary from the proof of our main results (Theorem 3.2).

Corollary 5.1. *Let $f \in \mathbb{R}[x, y]$ be regular in x and $\mathcal{V}_{\mathbb{R}}(f)$ be the set of real approximations of non-real Newton–Puiseux roots of f as defined in Definition 2.1. Asume that f and g have no common factors, then*

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)} = 0$$

if and only if $f = 0$ has only isolated point $(0, 0)$ and

$$\lim_{y\to 0} \frac{g(\phi(y), y)}{f(\phi(y), y)} = 0$$

for all $\phi \in \mathcal{V}_{\mathbb{R}}(f)$.

This provides a new algorithsm, which are easy to implement, to determine whether the limit $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{g(x,y)}$ exists and compute the limit if it exists.

Acknowledgment. A part of this work was done while the first author and the third author were visiting at Vietnam Institute for Advanced Study in Mathematics (VIASM) in the spring of 2022. These authors would like to thank the Institute for hospitality and support.

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

[1] E. Brieskorn and H. Knörrer. *Plane algebraic curves (Translated from the German by John Stillwell)*. Birkhäuser Verlag, Basel, 1986.

[2] E. Casas-Alvero. *Singularities of plane curves*. London Mathematical Society Lecture Note Series, 276. Cambridge University Press, Cambridge, 2000.

[3] S. T. Dinh, F. Guo, and T. S. Pham. Global Łojasiewicz inequalities on comparing the rate of growth of polynomial functions. *J. Math. Anal. Appl.*, 499(2, 125057):24 pp, 2021.

[4] S. T. Dinh, H. V. Hà, and T. S. Phạm. Hölder-type global error bounds for non-degenerate polynomial systems. *Acta Math. Vietnam.*, 42(3):563–585, 2017.

[5] S. T. Dinh, H. V. Hà, and N. T. Thao. Łojasiewicz inequality for polynomial functions on non compact domains. *Internat. J. Math.*, 23(4):1250033 (28 pages), 2012.

[6] S. T. Dinh, H. V. Hà, N. T. Thao, and T. S. Phạm. Global Łojasiewicz-type inequality for non-degenerate polynomial maps. *J. Math. Anal. Appl.*, 410(2):541–560, 2014.

[7] S. T. Dinh, K. Kurdyka, and O. L. Gal. Łojasiewicz inequality on non-compact domains and singularities at infinity. *Internat. J. Math.*, 24(10):8 pp, 2013.

[8] S. T. Dinh, K. Kurdyka, and T. S. Pham. Global mixed Łojasiewicz inequalities and asymptotic critical values. *Annales Polonici Mathematici*, 123(1):259–266, 2019.

[9] S. T. Dinh and T. S. Phạm. Łojasiewicz-type inequalities with explicit exponents for the largest eigenvalue function of real symmetric polynomial matrices. *Internat. J. Math.*, 27(2):27pp, 2016.

[10] J. F. Fernando. On chains of prime ideals in rings of semialgebraic functions. *Q. J. Math.*, 65(3):893–930, 2014.

[11] J. F. Fernando and J. M. Gamboa. On Łojasiewicz's inequality and the nullstellensatz for rings of semialgebraic functions. *J. Algebra*, 399:475–488, 2014.

[12] J. F. Fernando and J. M. Gamboa. On the krull dimension of rings of continuous semialgebraic functions. *Rev. Mat. Iberoam.*, 31(3):753–766, 2015.
[13] G. Fichou, J. Huisman, F. Mangolte, and J.-P. Monnier. Fonctions régulues. *J. Reine Angew. Math.*, 718:103–151, 2016.
[14] H. V. Hà and H. D. Nguyen. On the Łojasiewicz exponent near the fibre of polynomial mappings. *Ann. Polon. Math.*, 94(1):43–52, 2008.
[15] H. V. Hà and H. D. Nguyen. Łojasiewicz inequality at infinity for polynomials in two real variables. *Math. Z.*, 266(2):243–264, 2010.
[16] H. V. Hà. Global Hölderian error bound for non-degenerate polynomials. *SIAM J. Optim.*, 23(2):917–933, 2013.
[17] H. V. Hà, V. N. Huỳnh, and T. S. Phạm. A global version of the classical Łojasiewicz inequality. *J. Math. Anal. Appl.*, 421(2):1559–1572, 2015.
[18] W. Kucharz. Rational maps in real algebraic geometry. *Adv. Geom.*, 9(4):517–539, 2009.
[19] W. Kucharz and K. Kurdyka. From continuous rational to regulous functions. *In B. Sirakov, P. N. de Souza, and M. Viana, editors, Proc. Int. Cong. of Math. (ICM 2018)*, pages 719–747, Rio de Janeiro, Brazil, 2019.
[20] T-C. Kuo, Computation of Lojasiewicz exponent of $f(x, y)$, Comment. Math. Helv. 49 (1974), 201-213.
[21] T.-C. Kuo and A. Parusiński. Newton polygon relative to an arc. *Real and Complex Singularities (São Carlos, 1998), Chapman & Hall Res. Notes Math.*, 412:76–93, 2000.
[22] S. Łojasiewicz. *Introduction to complex analytic geometry (Translated from the Polish by Maciej Klimek)*. Birkhäuser Verlag, Basel, 1991.
[23] J. Milnor. *Singular points of complex hypersurfaces*, volume 61 of *Ann. of Math. Stud.* Princeton University Press, 1968.
[24] R. J. Walker. *Algebraic Curves*, volume 13 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N. J, 1950.
[25] C. T. C. Wall. *Singular points of plane curves*. London Mathematical Society Student Texts, 63. Cambridge University Press, Cambridge, 2004.

Institute of Mathematics, VAST, 18, Hoang Quoc Viet Road, Cau Giay District 10307, Hanoi, Vietnam
Email address: dstiep@math.ac.vn

School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China
Email address: fguo@dlut.edu.cn

TIMAS, Thang Long University, Nghiem Xuan Yem, Hanoi, Vietnam

Institute of Mathematics, VAST, 18, Hoang Quoc Viet Road, Cau Giay District 10307, Hanoi, Vietnam
Email address: duc.nh@thanglong.edu.vn

Department of Mathematics, Dalat University, 1 Phu Dong Thien Vuong, Dalat, Vietnam
Email address: sonpt@dlu.edu.vn