Determining a source term in an elliptic equation in a cylinder from boundary observations

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Abstract

We consider the problem of determining a term in the right-hand side of an elliptic equation in a cylinder from boundary observations with constant and variable coefficients. Based on the special form of the considered equation in a cylinder, the solution of the direct and inverse problems can be represented by the Fourier series. As the problem is ill-posed, we regularize it by truncating the Fourier series. We prove error estimates of the method and present some numerical examples for showing its efficiency.

Keywords: Inverse source problem, elliptic equations, ill-posedness, regularization, truncated Fourier series

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1 Introduction and problem setting

As noted in our work [4], the problem of determining sources in elliptic equations has attracted researchers for several decades, see e.g., [2,5,7,10,12,14,15]. However, the inverse source problems with boundary observations are not many [1,2,5–10,12–18]. In this paper, we continue our research on these source inverse problems, but for a special elliptic equation with constant and variable coefficients in a cylinder. Namely, let Ω be a bounded, open set in $\mathbb{R}^n$ with Lipschitz boundary, $Q = (0,T) \times \Omega$, where T is a given positive number. Let $a_{ij}(x), i, j = 1, \dots n$, belong to $C^1(\bar{\Omega})$ and $a(x)$ belong to $C(\bar{\Omega})$ satisfying

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \mu\|\xi\|^2_{\mathbb{R}^2} \text{ for all } \xi \in \mathbb{R}^n, \text{ and } a(x) \geq m, \tag{1.1}$$

with given positive constants μ and m. Set

$$\mathcal{L}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + a(x)u.$$

Consider the Neumann problem

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} + \mathcal{L}u &= f(x) + g(x,t), x \in \Omega, t \in (0,T), \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega\times(0,T)} &= 0, \\ \frac{\partial u}{\partial t}|_{t=0} &= 0, \\ \frac{\partial u}{\partial t}|_{t=T} &= 0. \end{cases} \tag{1.2}$$

Here, $\partial/\partial\nu$ is the normal outer derivative defined on the boundary $\partial\Omega$.

Definition 1.1. *Let f and g be given. A weak solution in $H^1(Q)$ to (1.2) is a function $u \in H^1(Q)$ such that*

$$\int_Q \frac{\partial u}{\partial t}\frac{\partial v}{\partial t}dxdt + \int_Q \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt + \int_Q a(x)uvdxdt = \int_Q (f+g)dxdt \tag{1.3}$$

for all $v \in H^1(\Omega)$.

It is well-known that, for each pair $f \in L^2(\Omega)$ and $g \in L^2(Q)$, there exists a unique solution $u \in H^1(Q)$ to (1.2). Moreover, there is a positive number c independent of f and g such that

$$\|u\|_{H^1(Q)} \leq c\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(Q)}\right). \tag{1.4}$$

The inverse problem is that of determining f from an observation taken on a part of the boundary ∂Q. Namely, we consider the problem of determining the term f in the right-hand side of the first equation of (1.2) from the observation

$$u(x,0) = \varphi(x). \tag{1.5}$$

We will study the well-posedness of this problem and will suggest a stable method based on the truncating the Fourier series representing the solution to the direct problem (1.2).

Denote $\bar{u}$ and $\tilde{u}$ respectively the solutions to the following problems:

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} + \mathcal{L}u &= f(x), x \in \Omega, t \in (0,T), \\ \frac{\partial \bar{u}}{\partial N}|_{\partial\Omega\times(0,T)} &= 0, \\ \frac{\partial \bar{u}}{\partial t}|_{t=0} &= 0, \\ \frac{\partial \bar{u}}{\partial t}|_{t=T} &= 0, \end{cases} \tag{1.6}$$

and

$$\begin{cases} -\frac{\partial^2 \tilde{u}}{\partial t^2} + \mathcal{L}\tilde{u} &= g(x,t), x \in \Omega, t \in (0,T), \\ \frac{\partial \tilde{u}}{\partial N}|_{\partial\Omega\times(0,T)} &= 0, \\ \frac{\partial \tilde{u}}{\partial t}|_{t=0} &= 0, \\ \frac{\partial \tilde{u}}{\partial t}|_{t=T} &= 0, \end{cases} \tag{1.7}$$

Clearly, $u = \bar{u} + \tilde{u}$ is the solution to (1.2). Since $g \in L^2(Q)$ is given, problem (1.7) is well-posed. That is, there exists a unique $\tilde{u} \in H^1(Q)$ satisfying (1.7). So, the above inverse problem can be rewritten as the problem of determining f from

$$\begin{cases} -\frac{\partial^2 \bar{u}}{\partial t^2} + \mathcal{L}\bar{u} &= f(x), x \in \Omega, t \in (0,T), \\ \frac{\partial \bar{u}}{\partial N}|_{\partial\Omega\times(0,T)} &= 0, \\ \frac{\partial \bar{u}}{\partial t}|_{t=0} &= 0, \\ \frac{\partial \bar{u}}{\partial t}|_{t=T} &= 0, \end{cases} \tag{1.8}$$

with the observation

$$\bar{u}(x,0) = u(x,0) - \tilde{u}(x,0) = \varphi(x) - \tilde{u}(x,0) := \varphi(x) - \tilde{\varphi}(x) := \bar{\varphi}(x). \tag{1.9}$$

Suppose that φ is approximated by $\varphi^\epsilon \in L^2(\Omega)$ such that

$$\|\varphi - \varphi^\epsilon\|_{L^2(\Omega)} \le \epsilon \tag{1.10}$$

which ϵ being a noise level. Our aim is to determine f from the noisy data φ^ϵ.

Note that since u and $\tilde{u}$ are in $H^1(Q)$, we have $\bar{u}(x,0) = u(x,0) - \tilde{u}(x,0) = \bar{\varphi}(x)$ belongs to $H^{1/2}(\Omega)$. The operator A mapping $f \in L^2(\Omega)$ to the restriction of the solution u on Ω is linear and bounded. Since $H^{\frac{1}{2}}(\Omega)$ is compactly embedded on $L^2(\Omega)$, the operator $A : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact. So, the inverse problem (1.2)–(1.5) now becomes the linear compact operator equation

$$Af = \bar{\varphi}, \tag{1.11}$$

which is ill-posed.

1.1 A representation of the solution

We will find the solution to the direct problem (1.2) by the method of separation of variables (the Fourier method). Consider the eigenvalue problem

$$\begin{cases} \mathcal{L}u = \lambda u, & x \in \Omega, \\ \frac{\partial \bar{u}}{\partial N}|_{\partial\Omega} = 0. \end{cases} \tag{1.12}$$

It is known that [11, §3, pp. 174–181] $\mathcal{L}$ admits an orthonormal eigenbasis $\{\phi_k\}_{k\ge 0}$ in $L^2(\Omega)$ and the associated eigenvalues $m \le \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$, where λ_k tends to infinity as $k \to \infty$. Furthermore, the system

$$\left\{ \frac{\phi_1}{\sqrt{\lambda_0 - m + 1}}, \frac{\phi_2}{\sqrt{\lambda_1 - m + 1}}, \dots, \frac{\phi_n}{\sqrt{\lambda_n - m + 1}} \right\}$$

forms the orthonormal basis of $H^1(\Omega)$.

To find a solution $u(x,t)$ to (1.2), we formally represent it in a series

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\phi_k(x), \tag{1.13}$$

with $u_k(t)$ being sought.

Set

$$f_k = \langle f(\cdot), \phi_k(\cdot)\rangle_{L^2(\Omega)},$$

and

$$g_k(t) = \langle g(\cdot,t), \phi_k(\cdot)\rangle_{L^2(\Omega)}.$$

Then the series

$$f(x) = \sum_{k\ge 0} f_k \phi_k(x) \tag{1.14}$$

is convergent in $L^2(\Omega)$, and the series

$$g(x,t) = \sum_{k\geq 0} g_k(t)\phi_k(x) \tag{1.15}$$

is convergent in $L^2(Q)$. Substituting (1.13) into (1.2), we get

$$\begin{cases} -\sum\limits_{k\geq 0} u_k''(t)u_k\phi_k + \sum\limits_{k\geq 0} \lambda_k u_k(t)\phi_k &= \sum\limits_{k\geq 0} (f_k + g_k(t))\phi_k, \\ -\sum\limits_{k\geq 0} u_k'(0)\phi_k &= 0, \\ \sum\limits_{k\geq 0} u_k'(T)\phi_k &= 0, \end{cases} \tag{1.16}$$

Taking the $L^2(\Omega)$ inner product to the both sides of (1.16) with ϕ_k we obtain the second-order ordinary differential equations

$$-u_k''(t) + \lambda_k u_k(t) = f_k + g_k(t) \tag{1.17}$$

with the boundary conditions

$$u_k'(0) = 0,\ u_k'(T) = 0,\ k = 0, 1, 2, \ldots \tag{1.18}$$

The unique solution to problem (1.17)–(1.18) is

$$\begin{aligned} u_k(t) = \frac{f_k}{\lambda_k} &+ \frac{1}{\sqrt{\lambda_k}\sinh\sqrt{\lambda_k}T}\int_0^t g_k(t)\sinh\sqrt{\lambda_k}(\xi - t)dt \\ &+ \frac{\cosh\sqrt{\lambda_k}t}{\sqrt{\lambda_k}\sinh\sqrt{\lambda_k}T}\int_0^T g_k(t)\cosh\sqrt{\lambda_k}(T-\xi)dt. \end{aligned}$$

Therefore,

$$u_k(0) = \frac{1}{\sqrt{\lambda_k}\sinh\sqrt{\lambda_k}T}\int_0^T g_k(\xi)\cosh(\sqrt{\lambda_k}(T-\xi))d\xi + \frac{f_k}{\lambda_k},\ k = 0, 1, \ldots.$$

Thus,

$$u(x,0) = \sum_{k\geq 0}^{\infty} u_k(0)\phi_k(x) \tag{1.19}$$

$$= \sum_{k\geq 0}\left(\frac{1}{\sqrt{\lambda_k}\sinh\sqrt{\lambda_k}T}\int_0^T g_k(\xi)\cosh(\sqrt{\lambda_k}(T-\xi))d\xi + \frac{f_k}{\lambda_k}\right)\phi_k(x),\ k = 0, 1, \ldots \tag{1.20}$$

which converges in $L^2(\Omega)$.

Further, since $\varphi \in L^2(\Omega)$, we have

$$u(x,0) = \varphi(x) = \sum_{k\geq 0}\varphi_k\phi_k, \tag{1.21}$$

where $\varphi_k = \langle\varphi, \phi_k\rangle_{L^2(\Omega)},\ k = 0, 1, \ldots$

Comparing this series with (1.19), we get

$$\frac{1}{\sqrt{\lambda_k}\sinh\sqrt{\lambda_k}T}\int_0^T g_k(\xi)\cosh(\sqrt{\lambda_k}(T-\xi))d\xi + \frac{f_k}{\lambda_k} = \varphi_k,\ k = 0, 1, \ldots$$

Therefore,

$$f_k = \lambda_k \left(\varphi_k - \frac{1}{\sqrt{\lambda_k}\sinh\sqrt{\lambda_k}T} \int_0^T g_k(\xi)\cosh(\sqrt{\lambda_k}(T-\xi))d\xi \right),\ k = 0, 1, \dots \tag{1.22}$$

When $f = 0$, u is the solution $\tilde{u}$ to (1.7). Hence,

$$\begin{aligned}\tilde{u}(x,0) = \tilde{\varphi}(x) &= \sum_{k\geq 0}^{\infty} \tilde{\varphi}_k(0)\phi_k(x) \\ &= \sum_{k\geq 0} \frac{\phi_k(x)}{\sqrt{\lambda_k}\sinh\sqrt{\lambda_k}T} \int_0^T g_k(\xi)\cosh(\sqrt{\lambda_k}(T-\xi))d\xi,\ k = 0, 1, \dots\end{aligned}$$

which converges in $L^2(\Omega)$.

Thus, denoting $\bar{\varphi}_k = \int_\Omega \bar{\varphi}(x)\phi_k(x)dx$, which $\bar{\varphi}$ being defined by (1.9), we have

$$f(x) = \sum_{k\geq 0} \lambda_k(\varphi_k - \tilde{\varphi}_k)\phi_k(x) = \sum_{k\geq 0} \lambda_k \bar{\varphi}_k \phi_k(x). \tag{1.23}$$

Since λ_k tends to infinity as $k \to \infty$, this series does not always converge in $L^2(\Omega)$ if φ is approximately given. Thus, the problem of determining f from $u(x,0)$ is ill-posed. Therefore, a regularization method for it is desirable. We shall do it by truncating the Fourier series (1.22) ((1.23)).

1.2 Regularization by the truncated Fourier series

We now use the traditional truncated Fourier series method for regularizing the inverse problem (1.2)–(1.5). Denoting φ_k^ϵ the Fourier coefficient of φ^ϵ and $\bar{\varphi}_k^\epsilon = \varphi_k^\epsilon - \tilde{\varphi}_k$, we approximate f by truncating the Fourier series (1.23):

$$f^{N,\epsilon} = \sum_{k=0}^{N} \lambda_k \bar{\varphi}_k^\epsilon \phi_k. \tag{1.24}$$

We shall determine $N = N(\epsilon) \in \mathbb{N}$ such that $\|f^{N,\epsilon} - f\|_{L^2(\Omega)} \to 0$ as $\epsilon \to 0$. To do this we require some "smoothness" of f. Namely, we introduce the space $H^\alpha(\Omega), \alpha \geq 0$, which consists of all functions f such that the series $\sum_{k=0}^\infty \lambda_k^\alpha f_k \phi_k$ converges in $L^2(\Omega)$. We introduce the norm in this space by

$$\|f\|_{H^\alpha(\Omega)} = \Big(\sum_{k=0}^{\infty} \lambda_k^{2\alpha} |f_k|^2 \Big)^{1/2}.$$

We see that $H^0(\Omega) = L^2(\Omega)$.

Theorem 1.2. *Let α be a given positive number and $f \in H^\alpha(\Omega)$. Let further that there is a positive number E such that $\|f\|_{H^\alpha(\Omega)} \leq E$. Then, with*

$$N = N^* := \Big[\Big(\frac{E}{c_1 c_0^\alpha \epsilon} \Big)^{\frac{n}{2+2\alpha}} \Big]$$

with $[\gamma]$ being the entire part of a number γ, there exists a positive number $c_2 = c_2(E, n, \alpha)$ independent of ϵ such that

$$\|f - f^{N,\epsilon}\|_{L^2(\Omega)} \leq c_2 \epsilon^{\frac{\alpha}{\alpha+1}}$$

which tends to zero as ϵ tends to zero.

Proof. For $N \in \mathbb{N}$, we have

$$\|f - f^{N,\epsilon}\|_{L^2(\Omega)} \le \|f - f^N\|_{L^2(\Omega)} + \|f^N - f^{N,\epsilon}\|_{L^2(\Omega)} := A + B.$$

Here,

$$\begin{aligned} A^2 = \|f - f^N\|^2_{L^2(\Omega)} &= \Big\| \sum_{k \ge N+1} \lambda_k f_k \phi_k \Big\|^2_{L^2(\Omega)} \\ &= \sum_{k \ge N+1} f_k^2 = \sum_{k \ge N+1} \lambda_k^{-2\alpha} \lambda_k^{2\alpha} f_k^2 \\ &\le \lambda_{N+1}^{-2\alpha} \sum_{k \ge N+1} \lambda_k^{2\alpha} f_k^2 \\ &\le \lambda_{N+1}^{-2\alpha} \|f\|^2_{H^\alpha(\Omega)} \\ &\le \lambda_{N+1}^{-2\alpha} E^2. \end{aligned} \tag{1.25}$$

From [11, Theorem 5, p. 189], there exists constants c_0 and c_1, $0 < c_0 < c_1$ and a number N_0 such that

$$c_0 s^{\frac{2}{n}} \le \lambda_s \le c_1 s^{\frac{2}{n}}$$

for all $s \ge N_0$. Taking $N \ge N_0$, we then have

$$A \le c_0^{-\alpha} (N+1)^{\frac{-2\alpha}{n}} E \le c_0^{-\alpha} N^{\frac{-2\alpha}{n}} E. \tag{1.26}$$

Since $m \le \lambda_0 \le \lambda_1 \le \cdots \le \lambda_N$, we have

$$\begin{aligned} B^2 = \|f^N - f^{N,\epsilon}\|^2 &= \| \sum_{k=0}^{N} \lambda_k (\varphi_k - \varphi_k^\epsilon) \phi_k \|^2_{L^2(\Omega)} \\ &\le \lambda_N^2 \|\varphi - \varphi^\epsilon\|^2 \le \epsilon^2 \lambda_N^2 \\ &\le c_1^2 N^{\frac{4}{n}} \epsilon^2. \end{aligned} \tag{1.27}$$

Combining the last inequalities, we get

$$\|f - f^{N,\epsilon}\|_{L^2(\Omega)} \le c_0^{-\alpha} N^{\frac{-2\alpha}{n}} E + c_1 N^{\frac{2}{n}} \epsilon.$$

Choosing, for example,

$$N = N^* := \Big[\Big(\frac{E}{c_1 c_0^\alpha \epsilon} \Big)^{\frac{n}{2+2\alpha}} \Big], \text{ and } c_2 = 3 \Big(\frac{E}{c_0 c_1^\alpha} \Big)^{\frac{1}{1+\alpha}}$$

we see that

$$\|f - f^{N,\epsilon}\|_{L^2(\Omega)} \le c_2 \epsilon^{\frac{\alpha}{\alpha+1}}$$

which tends to zero as ϵ tends to zero.

□

2 A special case of equations with constant coefficients in a parallelepiped

In this paragraph, we consider a special case of the previous paragraph. We take $\Omega = (0,2)^{n-1}, n \geq 2$ and $Q = \Omega \times (0,2)$ and the elliptic in (1.2) with constant coefficients. Namely, we consider the Neumann problem for finding a function $u = u(x,t)$ satisfying

$$\begin{cases} -\Delta u + au &= f(x) + g(x,t) \ \text{ in } Q, \\ -\nabla u \cdot n &= 0 \ \text{ on } \partial Q. \end{cases} \tag{2.1}$$

Here, f and g are supposed to be in $L^2(\Omega)$ and $L^2(Q)$, respectively; a is a given positive constant.

As in the previous paragraph, a weak solution in $H^1(Q)$ to problem (2.1) is a function $u \in H^1(Q)$ such that

$$\int_Q \nabla u \nabla v dx dt + \int_Q auv dx = \int_Q (f+g) v dx dt$$

for all $v \in H^1(Q)$.

It is well-known that the exists a unique solution $u \in H^1(Q)$ to (2.1). We consider the inverse problem of determining f from the observation

$$u(x,0) = \varphi(x), \ x \in \Omega. \tag{2.2}$$

We have $\varphi \in H^{\frac{1}{2}}(\Omega) \subset L^2(\Omega)$. Denote $\bar{u}$ and $\tilde{u}$ the solutions to the following problems, respectively

$$\begin{cases} -\Delta \bar{u} + a\bar{u} &= f(x) \ \text{ in } Q, \\ -\nabla \bar{u} \cdot n &= 0 \ \text{ on } \partial Q. \end{cases} \tag{2.3}$$

and

$$\begin{cases} -\Delta \tilde{u} + a\tilde{u} &= g \ \text{ in } Q, \\ -\nabla \tilde{u} \cdot n &= 0 \ \text{ on } \partial Q. \end{cases} \tag{2.4}$$

Clearly, $u = \bar{u} + \tilde{u}$ is the solution to (2.1). Since $g \in L^2(Q)$ is given, problem (2.4) is well-posed. That is, $\tilde{u}$ uniquely exists $\in H^1(Q)$. Therefore, the inverse problem (2.1)–(2.2) can be rewritten as

$$\begin{cases} -\Delta \bar{u} + a\bar{u} &= f \ \text{ in } Q, \\ -\nabla \bar{u} \cdot n &= 0 \ \text{ on } \partial Q. \end{cases} \tag{2.5}$$

with observation

$$\bar{u}(x,0) = \varphi(x) - \tilde{u}(x,0) = \bar{\varphi}(x). \tag{2.6}$$

Suppose further that $\varphi \in L^2(\Omega)$ such that

$$\|\varphi - \varphi^\epsilon\| \leq \epsilon, \tag{2.7}$$

with ϵ being a given noise level. Our aim is to determine f^ϵ from the noisy data φ^ϵ. As previously, we solve the Neumann problem (2.1) by the Fourier method. However, for this special case, we proceed a little bit differently. Namely, we consider the eigenvalue problem

$$\begin{cases} -\Delta u + au &= \lambda u \ \text{ in } Q, \\ -\nabla u \cdot n &= 0 \text{ on } \partial Q. \end{cases} \tag{2.8}$$

A nonzero function $u \in H^1(Q)$ satisfying (2.8) is called an eigenfunction to this problem and the number λ is called the eigenvalue (corresponding to the eigenfunction u).

Set $k = (k_1, k_2, \ldots, k_{n-1}, k_n), k_i \in \mathbb{N}$, and

$$\begin{aligned} |k| &= k_1 + k_2 + \cdots + k_{n-1} + k_n, \\ k' &= (k_1, k_2, \cdots, k_{n-1}), \\ |k'| &= k_1 + k_2 + \cdots + k_{n-1}. \end{aligned}$$

We see that a is the smallest eigenvalue of (2.8) corresponding to the normalized in $L^2(Q)$ eigenfunction $\phi_{(0,0,\ldots,0)} = \frac{1}{\sqrt{2}}$. The other eigenvalues are

$$\lambda_k = a + \left(\frac{k_1\pi}{2}\right)^2 + \left(\frac{k_2\pi}{2}\right)^2 + \ldots \left(\frac{k_n\pi}{2}\right)^2$$

which correspond to the normalized in $L^2(Q)$ eigenfunction

$$\phi_k(x,t) = \cos\left(\frac{k_1\pi x_1}{2}\right)\cos\left(\frac{k_2\pi x_2}{2}\right)\cdots\cos\left(\frac{k_n\pi x_{n-1}}{2}\right)\cos\left(\frac{k_n\pi t}{2}\right). \tag{2.9}$$

The system $\{\phi_k\}_{|k|\geq 0}$ forms an orthonormal system in $L^2(Q)$ and the system $\{\tilde{\phi}_k\}_{|k|\geq 0}$, with

$$\tilde{\phi}_0 = \phi_0,\ \tilde{\phi}_k = \frac{\phi_k}{\sqrt{1 + \sum\limits_{i=1}^{n}\left(\frac{k_i\pi}{2}\right)^2}}$$

forms an orthonormal system in $H^1(Q)$, (see, e.g. [11, p. 174–181]).

Set

$$f_k = \int_\Omega f(x)\phi_k(x,t)dxdt \text{ and } g_k = \int_\Omega g(x,t)\phi_k(x,t)dxdt.$$

First we see that $g = \sum\limits_{|k|\geq 0} g_k\phi_k$ converges in $L^2(Q)$. Concerning $f(x)$, we have

$$\begin{aligned} f_k &= \int_Q f(x)\phi_k(x,t)dxdt \\ &= \int_\Omega f(x)\cos\frac{k_1\pi x_1}{2}\cos\frac{k_2\pi x_2}{2}\cdots\cos\frac{k_{n-1}\pi x_{n-1}}{2}dx_1dx_2\cdots dx_{n-1}\int_0^2\cos\frac{k_n\pi t}{2}dt \\ &= \int_\Omega f(x)\phi_{k'}(x)dx\int_0^2\cos\frac{k_n\pi t}{2}dt. \end{aligned}$$

Since

$$\int_0^2 \cos\frac{k_n\pi t}{2}dt = \begin{cases} 0, \text{ if } k_n \neq 0 \\ 2 \text{ if } k_n = 0, \end{cases}$$

the coefficient $f_k \neq 0$ only if $k_n = 0$, or $(x,t) = (x,0)$. Set

$$\phi_{k'}(x) = \cos\frac{k_1\pi x_1}{2}\cos\frac{k_2\pi x_2}{2}\ldots\cos\frac{k_{n-1}\pi x_{n-1}}{2}.$$

Then $\phi_k(x,t) = \phi_{k'}(x)$ and $f_k = 2\int_\Omega f(x)\phi_{k'}(x)dx = 2f_{k'}$ where

$$f_k' = \int_\Omega f(x)\phi_k'(x)dx.$$

Thus,

$$f(x) = \sum_{|k|\geq 0} f_k \phi_k(x) = 2 \sum_{|k'|\geq 0} f_{k'} \phi_{k'}(x) \tag{2.10}$$

which converges in $L^2(\Omega)$.

Formally representing

$$u(x,t) = \sum_{|k|\geq 0} u_k \phi_k(x,t),$$

then putting it into (2.1) and comparing the resulting equation with the series of f and g we have

$$u_k = \frac{1}{\lambda_k}(f_k + g_k).$$

Thus,

$$u(x,t) = \sum_{|k|\geq 0} \frac{1}{\lambda_k}(f_k + g_k)\phi_k(x,t) \tag{2.11}$$

which converges in $H^1(Q)$.

Substituting it into (2.2), we get

$$\varphi(x) = u(x,0) = 2 \sum_{|k'|\geq 0} \frac{1}{\lambda_k} f_{k'} \phi_{k'}(x) + \sum_{|k|\geq 0} \frac{1}{\lambda_k} g_k \phi_k(x). \tag{2.12}$$

Similarly to f, we represent φ into the series

$$\varphi(x) = \sum_{|k'|\geq 0} \varphi_{k'} \phi_{k'}(x).$$

This series converges in $L^2(\Omega)$.

Then, from (2.2), we have

$$\varphi(x) = 2 \sum_{|k'|\geq 0} \varphi_{k'} \phi_{k'}(x) = 2 \sum_{|k'|\geq 0} f_{k'} \phi_{k'}(x) + \sum_{|k|\geq 0} \frac{1}{\lambda_k} g_k \phi_k(x,0) \tag{2.13}$$

We note that

$$\begin{aligned} g_k &= g_{(k',k_n)} \\ &= \int_Q g(x_1, x_2, \cdots x_{n-1}, x_n, t) \times \\ &\cos\frac{k_1 x_1 \pi}{2} \cos\frac{k_2 \pi x_2}{2} \cdots \cos\frac{k_{n-1}\pi x_{n-1}}{2} \cos\frac{k_n \pi t}{2} dx_1 dx_2 \cdots dx_{n-1} dt, \end{aligned}$$

and

$$\phi_k(x,0) = \cos\frac{k_1 \pi x_1}{2} \frac{\cos k_2 \pi x_2}{2} \cdots \cos\frac{k_{n-1}\pi x_{n-1}}{2} = \phi_{k'}(x),$$

comparing the both sides of (2.13), we have

$$\varphi_{k'} = \sum_{k_n=0}^{\infty} \frac{1}{\lambda_{(k',k_n)}} g_{(k',k_n)} + 2\frac{1}{\lambda_{k'}} f_{k'}.$$

Hence,

$$f_{k'} = \frac{1}{2}\left(\lambda_{k'}\varphi_{k'} - \lambda_{k'}\sum_{k_n=0}^{\infty}\frac{1}{\lambda_{(k',k_n)}}g_{(k',k_n)}\right).$$

We denote $\tilde{u}(x,t) = \sum\limits_{|k|\geq 0}\frac{g_k}{\lambda_k}\phi_k(x,t)$, that means, $\tilde{u}(x,t)$ solves the problem

$$\begin{cases} -\Delta u + au &= g \ \text{ in } \Omega, \\ \nabla u\cdot n &= 0 \ \text{ on } \partial\Omega. \end{cases} \tag{2.14}$$

Set

$$\bar{\varphi}(x) = \varphi(x) - \tilde{u}(x,0)$$

and denote

$$\bar{\varphi}_{k'} = \int_{\Omega}\bar{\varphi}(x)\phi_{k'}(x)dx.$$

Hence,

$$f(x) = \sum_{|k'|\geq 0}\lambda_{k'}\bar{\varphi}_k\phi_{k'}(x). \tag{2.15}$$

Since

$$\lambda_{k'} = a + \left(\frac{k_1\pi}{2}\right)^2 + \left(\frac{k_2\pi}{2}\right)^2 + \dots\left(\frac{k_{n-1}\pi}{2}\right)^2$$

tends to infinity as $|k'|$ tends to infinity, we see from (2.15) that the problem of reconstructing f from φ is ill-posed, and we will use truncated Fourier series method for regularizing it.

Suppose that instead of φ we have only its approximate data $\varphi^\epsilon \in L^2(\Omega)$ which satisfies (2.7). Then we see that the series (2.15) may not converge for this data. To avoid it, we shall truncate this series. Namely, we take

$$f^{N,\epsilon}(x) = \sum_{|k'|\geq 0}^{N}\lambda_{k'}(\varphi^{\epsilon}_{k'} - \bar{\varphi}_{k'})\phi_{k'}(x) \tag{2.16}$$

$$f^{N}(x) = \sum_{|k'|\geq 0}^{N}\lambda_{k'}(\varphi_{k'} - \bar{\varphi}_{k'})\phi_{k'}(x) = \sum_{|k'|\geq 0}^{N}\lambda_{k'}\bar{\varphi}_{k'}\phi_{k'}(x). \tag{2.17}$$

The purpose of this regularization method is to determine an appropriate $N = N(\epsilon) \in \mathbb{N}$ such that $\|f^{N,\epsilon} - f\|_{L^2(\Omega)} \to 0$ as $\epsilon \to 0$.

Theorem 2.1. *Let α be a positive given number, f a function in $H^\alpha(\Omega)$. Furthermore, suppose that there is a positive constant E such that*

$$\|f\|_{H^\alpha(\Omega)} \leq E.$$

Then with

$$N = N^* = \Big[\Big(\frac{E}{\epsilon}\Big)^{\frac{1}{2+2\alpha}}\Big(\frac{4(n-1)}{\pi^2}\Big)^{\frac{\alpha}{2+2\alpha}}(a+\frac{\pi^2}{4})^{-\frac{1}{(2+2\alpha)}}\Big]$$

there exists a positive $c_3 = c_3(E,\alpha,n)$ independent of ϵ such that

$$\|f - f^{N,\epsilon}\|_{L^2(\Omega)} \leq c_3\epsilon^{\frac{\alpha}{1+\alpha}}$$

which tends to zero as ϵ tends to zero.

Proof. For $N \in \mathbb{N}$, we have,

$$\|f - f^{N,\epsilon}\|_{L^2(\Omega')} \leq \|f - f^N\|_{L^2(\Omega')} + \|f^N - f^{N,\epsilon}\|_{L^2(\Omega')} := A + B. \tag{2.18}$$

We have

$$\begin{aligned}
A^2 &= \Big\| \sum_{|k'| \geq N+1} f_{k'} \phi_{k'}(\cdot) \Big\|^2_{L^2(\Omega)} \\
&= \sum_{|k'| \geq N+1} f^2_{k'} = \sum_{|k'| \geq N+1} \lambda^{2\alpha}_{k'} f^2_{k'} \lambda^{-2\alpha}_{k'} \\
&\leq \lambda^{-2\alpha}_{N+1} \sum_{|k'| \geq N+1} \lambda^{2\alpha}_{k'} f^2_{k'} \\
&\leq \lambda^{-2\alpha}_{N+1} \|f\|^2_{H^\alpha(\Omega)} \\
&\leq \lambda^{-2\alpha}_{N+1} E^2.
\end{aligned}$$

Using the Cauchy-Bunyakovsky inequality, we have

$$\begin{aligned}
\lambda^{-\alpha}_{k'} &= \left(a + \left(\frac{k_1 \pi}{2} \right)^2 + \left(\frac{k_2 \pi}{2} \right)^2 + \ldots \left(\frac{k_{n-1} \pi}{2} \right)^2 \right)^{-\alpha} \\
&\leq \left(a + \frac{(k_1 + k_2 + \ldots k_{n-1})^2 \pi^2}{4(n-1)} \right)^{-\alpha} = \left(a + \frac{|k'|^2 \pi^2}{4(n-1)} \right)^{-\alpha} \\
&\leq \left(a + \frac{(N+1)^2 \pi^2}{4(n-1)} \right)^{-\alpha} < \Big(\frac{N^2 \pi^2}{4(n-1)} \Big)^{-\alpha}.
\end{aligned} \tag{2.19}$$

Thus,

$$A < E \left(\frac{4(n-1)}{\pi^2} \right)^\alpha N^{-2\alpha}. \tag{2.20}$$

On the other hand, we have

$$\lambda_N = a + \left(\frac{k_1 \pi}{2} \right)^2 + \left(\frac{k_2 \pi}{2} \right)^2 + \ldots \left(\frac{k_{n-1} \pi}{2} \right)^2$$

with $k_1 + k_2 + \cdots + k_{n-1} = N$. So,

$$\lambda_N \leq a + \frac{(k_1 + k_2 + \cdots + k_{n-1})^2 \pi^2}{4} = a + \frac{N^2 \pi^2}{4} \leq \left(a + \frac{\pi^2}{4} \right) N^2.$$

Since $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_N$, we have

$$\begin{aligned}
B^2 &= \| \sum_{|k'| \geq 0}^{N} \lambda_{k'} (\bar{\varphi}^\epsilon_{k'} - \bar{\varphi}_{k'}) \phi_{k'}(x') \|^2_{L^2(\Omega')} \\
&= \sum_{|k'| \geq 0}^{N} \lambda^2_{k'} \left((\varphi_{k'} - \tilde{u}_k(.,0)) - (\varphi^\epsilon_{k'} - \tilde{u}_k(.,0)) \right)^2 = \sum_{|k'| \geq 0}^{N} \lambda^2_{k'} \left(\varphi_{k'} - \varphi^\epsilon_{k'} \right)^2 \\
&\leq \lambda^2_N \epsilon^2 \leq \left(a + \frac{\pi^2}{4} \right)^2 N^4 \epsilon^2.
\end{aligned} \tag{2.21}$$

Therefore,

$$B \leq \left(a + \frac{\pi^2}{4}\right) N^2 \epsilon \tag{2.22}$$

From estimates (2.18), (2.20) and (2.22), we have

$$\|f - f^{N,\epsilon}\|_{L^2(\Omega)} \leq E \left(\frac{4(n-1)}{\pi^2}\right)^{\alpha} N^{-2\alpha} + \left(a + \frac{\pi^2}{4}\right) N^2 \epsilon. \tag{2.23}$$

By taking

$$N = N^* = \left[\left(\frac{E}{\epsilon}\right)^{\frac{1}{2+2\alpha}} \left(\frac{4(n-1)}{\pi^2}\right)^{\frac{\alpha}{1+2\alpha}} (a + \frac{\pi^2}{4})^{-\frac{1}{(2+2\alpha)}}\right]$$

and

$$c_3 = 3E^{\frac{1}{1+\alpha}} \left(\frac{4(n-1)}{\pi^2}\right)^{\frac{\alpha}{1+\alpha}} \left(a + \frac{\pi^2}{4}\right)^{\frac{\alpha}{1+\alpha}},$$

for example, we have

$$\|f - f^{N,\epsilon}\|_{L^2(\Omega')} \leq c_3 \epsilon^{\frac{\alpha}{1+\alpha}} \tag{2.24}$$

which tends to zero as ϵ tends to zero.

□

3 Numerical examples

In this section we apply the proposed method to some concrete examples for illustrating its efficiency. We wish to determine $f = f(x)$ in the problem

$$\begin{cases} \Delta u + 2u & = f(x) + g(x,y), (x,y) \text{ in } \Omega, \\ \nabla u \cdot n & = 0, (x,y) \text{ on } \partial\Omega \end{cases} \tag{3.1}$$

from the noisy observation on the boundary:

$$u(x,0) \approx \varphi^{\epsilon}(x) = \cos\frac{\pi x}{2}\Big(1 + p * rand(-1,1)\Big). \tag{3.2}$$

Here, $rand(-1,1)$ generates a random number in $(-1,1)$, p is the percentage of the error. So, the noise level is $\epsilon = p.\|u(.,0)\|$.

We test our method for three cases:
Example 1: f is a smooth function

$$g(x,y) = (1.25\pi^2 + 2)\cos\frac{\pi x}{2}\cos\frac{\pi y}{2} - 3\sin(\pi x) + 1).$$

The exact solution with $p = 0$ is

$$(u,f) = (\cos\frac{\pi x}{2}\cos\frac{\pi y}{2}, 3\sin(\pi x) + 1).$$

Example 2: f is a continuous but non-smooth function

$$g(x,y) = (1.25\pi^2 + 2)\cos\frac{\pi x}{2}\cos\frac{\pi y}{2} - |x-1|.$$

The exact solution with $p = 0$ is

$$(u,f) = (\cos\frac{\pi x}{2}\cos\frac{\pi y}{2}, |x-1|).$$

Example 3: f is a discontinuous function

$$g(x,y)=\begin{cases}(1.25\pi^2+2)\cos\frac{\pi x}{2}\cos\frac{\pi y}{2},\ (x,y)\in(0,\frac{1}{2})\times(0,2)\cup(\frac{3}{2},2)\times(0,2),\\(1.25\pi^2+2)\cos\frac{\pi x}{2}\cos\frac{\pi y}{2}-1,\ (x,y)\in[\frac{1}{2},\frac{3}{2}]\times 2.\end{cases}$$

In all examples, we use the Finite Difference Method with 80×80 nodes the domain and boundary. We compare the accuracy of the Finite Difference Method under different conditions: using the same number of Fourier coefficients with varying noise levels ($5\%, 7\%,$ and 10%) and using different numbers of coefficients ($M=7,10,15$ for Examples 1 and 2, and $\mathrm{M}=10,15,20$ for Example 3) with a fixed noise level.

The results of Example 1 are presented in Figures 1–4 and Table 1 while results of Example 2 and Examples 3 are shown in Figures 5–8, Table 2 and 9–12, 3, respectively. From these figures and tables we can see the decline in errors with decreasing noise: The errors in the computational solutions consistently decrease as the noise level drops from 10% to 5%, given a fixed number of Fourier coefficients. This pattern is evident across all examples. In Example 1, using 15 Fourier coefficients, the relative error drops from 0.1992 at a noise level of 10% to 0.0477 at a noise level of 5%. Increasing the number of Fourier coefficients also reduces errors: When employing the same noise level of 5% and increasing the number of Fourier coefficients from 7 to 15, the relative errors in Example 1 decrease from 0.0903 to 0.0477. This suggests that incorporating more coefficients can further enhance accuracy. Examples 2 and 3 exhibit similar trends, though their errors are generally larger than those in Example 1. This can be attributed to the underlying complexity of their exact solutions, which are non-smooth functions compared to the smooth solution in Example 1.

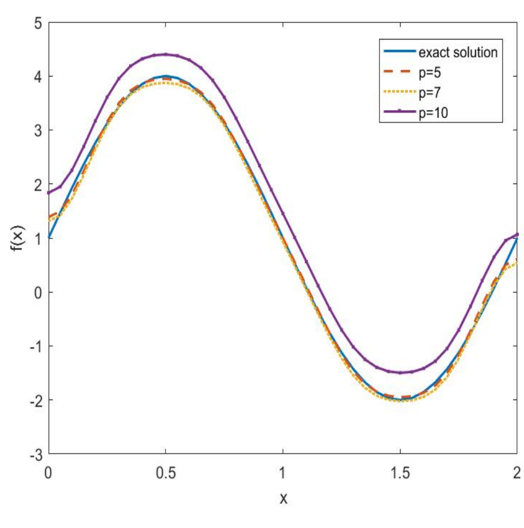

Figure 1: Example 1: Exact solution and numerical solutions with different perturbations in non-smooth case

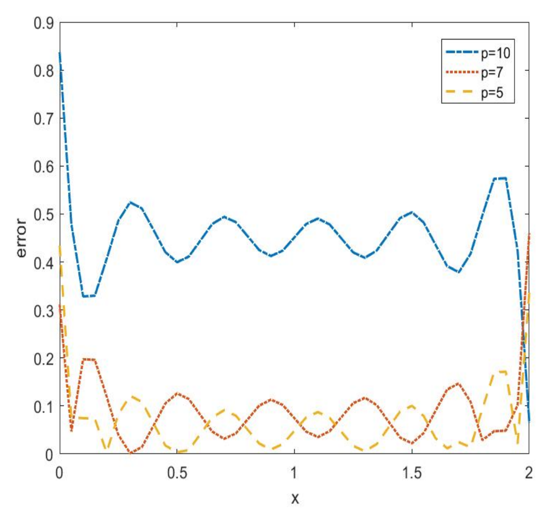

Figure 2: Example 1: Comparison of errors of numerical solutions with different perturbations

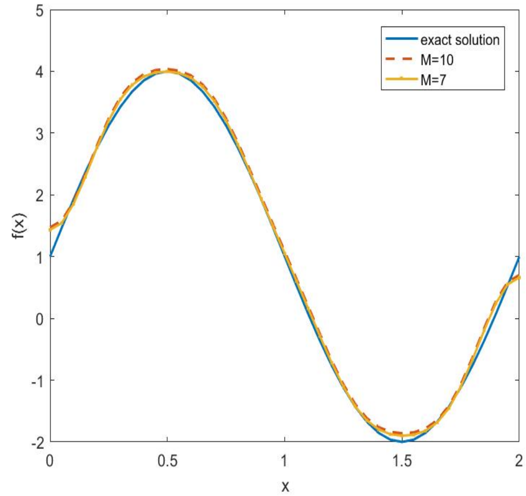

Figure 3: Example 1: Exact solution and numerical solutions with p=7 different number of Fourier coefficients

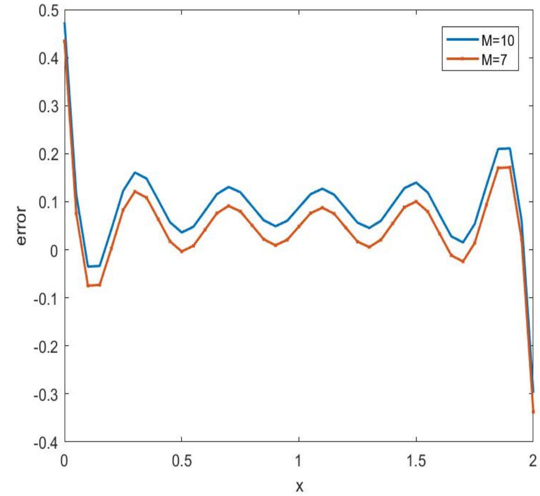

Figure 4: Example 1: Comparison errors of numerical solutions with different number of Fourier coefficients

Table 1: Example 1. The L^2-norm of relative errors

p	5%	7 %	10%
$M = 10$	0.0573	0.1427	0.3978
M=15	0.0477	0.0536	0.1992

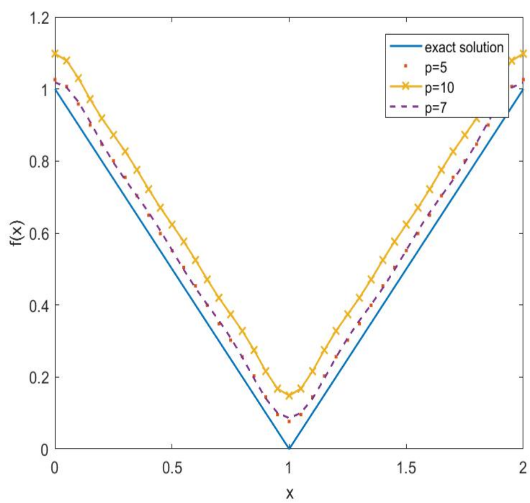

Figure 5: Example 2: Exact solution and numerical solutions with different perturbations in non-smooth case

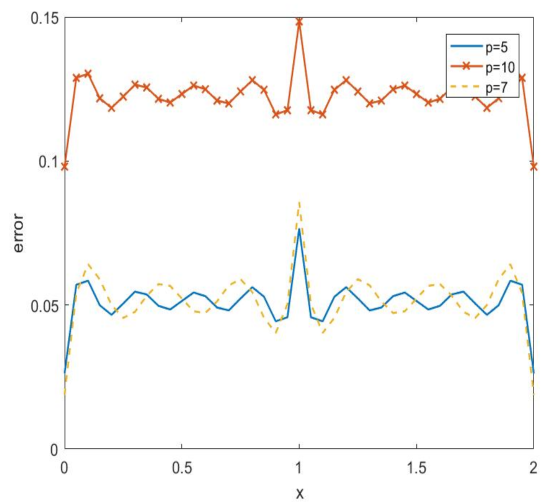

Figure 6: Example 2: Comparison of errors of numerical solutions with different perturbations

Table 2: Example 2. The L^2-norm of relative errors.

p	5%	7 %	10%
$M = 10$	0.1095	0.2077	0.3626
M=15	0.0868	0.0987	0.2077

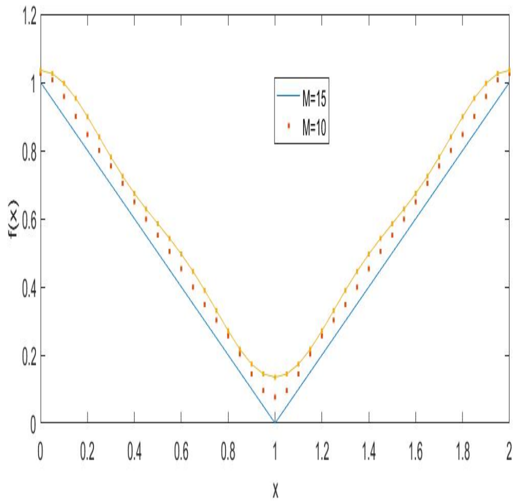

Figure 7: Example 2: Exact solution and numerical solutions with p=7 different number of Fourier coefficients

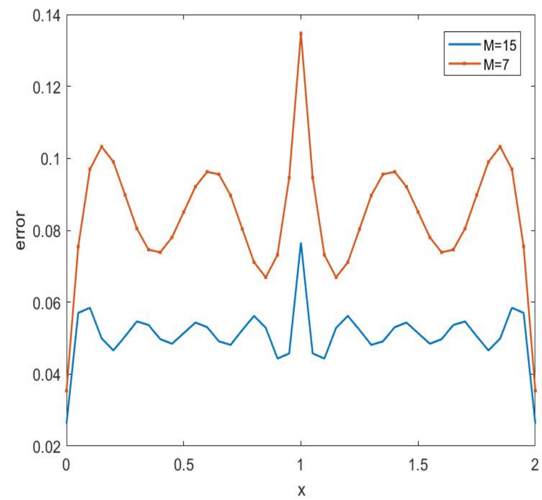

Figure 8: Example 2: Comparison errors of numerical solutions with different number of Fourier coefficients

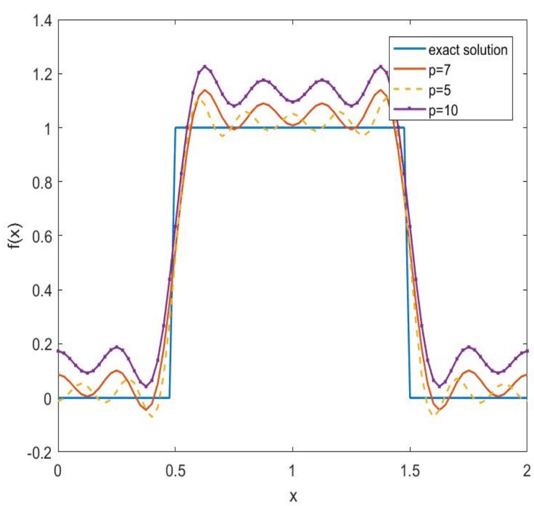

Figure 9: Example 3:Exact solution and numerical solutions with different perturbations

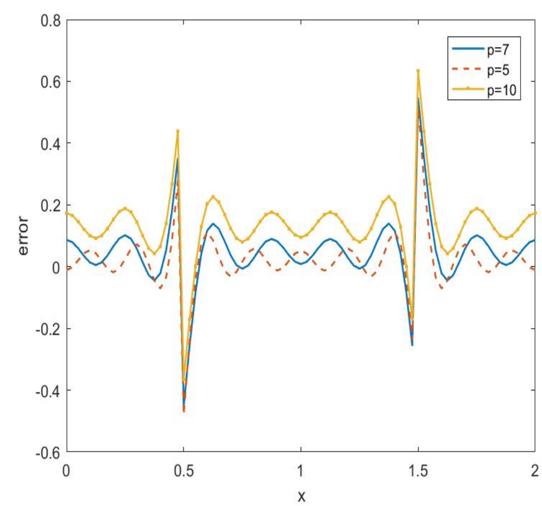

Figure 10: Example 3: Comparison of errors of numerical solutions with different perturbations

Table 3: Example 3. The L^2-norm of relative errors

p	5%	7 %	10%
$M = 15$	0.1730	0.1776	0.3894
M=20	0.1520	0.1494	0.2522

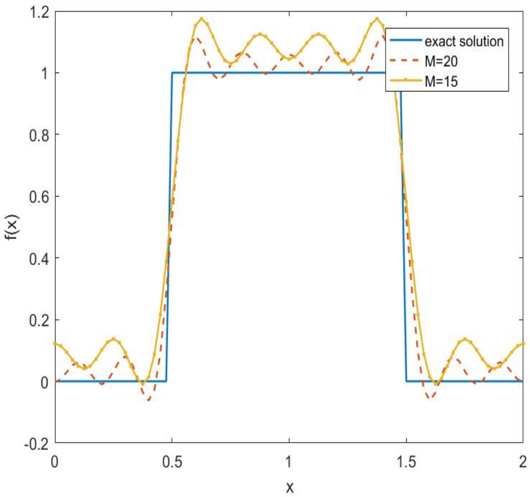

Figure 11: Example 3: Exact solution and numerical solutions with p=7 different number of Fourier coefficients

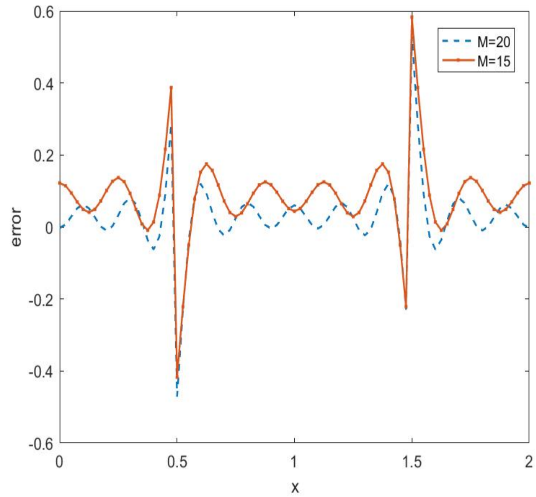

Figure 12: Example 3: Comparison errors of numerical solutions with different number of Fourier coefficients

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