IMAGE OF ITERATED POLYNOMIAL MAPS OF THE REAL PLANE

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Dedicate to the 60th birthday of Professor Osamu Saeki.

Abstract. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping. We consider the image of the compositions F^k of F. We prove that under some condition then the image of the iterated map F^k is stable when k is large.

1. Introduction

The Jacobian conjecture stated firstly by Keller in 1939 for a field k of characteristic 0 asserts that any polynomial mapping $f : k^n \to k^n$, for which the Jacobian determinant $J(f)$ is a nonzero constant, is bijective and the inverse mapping is also polynomial ([6]), we denote this assertion as $JC(k, n)$. The full Jacobian conjecture asserts that $JC(k, n)$ is true for all fields k of characteristic 0, and all integers $n > 0$; till now, no particular case of the conjecture has been proved for $n > 1$. We consider the *real Jacobian conjecture* which states that any polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ whose Jacobian determinant does not vanish anywhere has polynomial inverse. This real version of $JC(k, n)$ is disproved by Pinchuk ([12]), more precisely, Pinchuk constructed a polynomial mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is locally diffeomorphism but is not a global diffeomorphism (see also [3] for different examples). One natural question raised from those families of mappings is that which kind of global properties does a polynomial mapping have? In this paper, we do not deal with the Jacobian conjecture, but prove a global property on the iterated image of a polynomial mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, namely: we show that, under some condition, the iterated image of F is stable at some moment. The main result is the following.

Theorem 1.1. *Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a polynomial mapping satisfying the following conditions:*

1) *The complement $\mathbb{R}^2 \setminus F(\mathbb{R}^2)$ consists of finite number of points;*
2) *Each point outside the asymptotic variety of F has more than one preimage via F.*

Then, there exists a natural number N such that the image $F^k(\mathbb{R}^2)$ of F^k are equal for all $k \geqslant N$.

A similar observation has been considered in [11] for complex mappings, where the authors proved the stability of the iterated images of an open polynomial mapping from an affine complex algebraic set to itself. The proof of Theorem 1.1 is in the last section, while next section recalls some known results in real algebraic geometry.

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2. Preliminaries

We begin recalling the notion of the so-called asymptotic value of a polynomial mapping.

Definition 2.1. Given a mapping $F : k^n \longrightarrow k^m, k = \mathbb{R}, \mathbb{C}$. A point $y_0 \in k^m$ is called an *asymptotic value* of F if there exists a curve $\gamma : (0, \infty) \to k^n$, such that $\gamma(t) \to \infty$ as $t \to \infty$ and $F(\gamma(t)) \to y_0$ as $t \to \infty$. The curve γ is called an *asymptotic curve* with respect to y_0.

The set of all asymptotic values of F is denoted by S_F, we call it the *asymptotic variety* of F.

Example 2.2. Consider the polynomial mapping $(x^2, xy) : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$. By Definition 2.1, a point (a, b) is an asymptotic value of the mapping if there is a curve $\gamma = (\gamma_1, \gamma_2) : (0, \infty) \to \mathbb{R}^2$ such that, as $t \to \infty$, we have $\gamma_1^2(t)+\gamma_2^2(t) \to \infty, \gamma_1^2(t) \to a$ and $\gamma_1(t)\gamma_2(t) \to b$, then $\gamma_2(t) \to \infty$ and $\gamma_1(t) \to 0$ (otherwise, $\gamma_1(t)\gamma_2(t)$ could not have a finite limit). Hence $a = 0$.

On the other hand, any point $(0, \lambda), \lambda \in \mathbb{R}$ is an asymptotic value of the mapping with the asymptotic curve $\gamma(t) = (1/t, \lambda t), t > 0$. Thus, the set of asymptotic values of this mapping is the line $\{u = 0\}$ in the target plane.

It is clear that a proper map has the empty set as asymptotic variety, then asymptotic varieties play an important role in the study of polynomial mappings. In [5], Hadamard shows that a smooth map F from $\mathbb{R}^n$ to $\mathbb{R}^n$ is invertible if and only if its Jacobian determinant is nowhere vanishing and F is proper. Hence, the Jacobian Conjecture could be restated as: If the Jacobian determinant of a mapping is a nonzero constant then the mapping is proper.

There are many studies on the set of asymptotic values of polynomial mappings (for complex maps, see [7], for real maps, see [8]). For real mappings, the geometry of asymptotic variety is described by Z. Jelonek in [8] as follow.

Theorem 2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial mapping. Then the set S_f is closed, semi-algebraic and for every non-empty connected component $S \subset S_f$ we have $1 \leq \dim S \leq n - 1$.*

In the sequel, we also need the following property of fibers of morphisms between real algebraic sets. The original result is for regular functions between Zariski open sets, but we need only the restricted version for algebraic sets (see [1, Proposition 2.3.2]).

Proposition 2.4. *Let V and W be real algebraic sets with W irreducible. Let $f : V \to W$ be a regular function. Then there is a number $\delta(f) \in \{0, 1\}$ and a proper real algebraic subset X of W with $\dim X < \dim W$, such that for each $z \in W \setminus X$ we have*

$$\chi(f^{-1}(z)) = \delta(f) \quad \textit{modulo} \quad 2,$$

where χ denotes the Euler characteristic.

Definition 2.5. For a regular function f as in the previous proposition, we call $\delta(f)$ the *degree* of f.

3. Image of iterated maps on real plane

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping. We consider the iterated map

$$F^k := F \circ F \circ \cdots \circ F \quad (k \quad \text{composition of factors}).$$

One sees that

$$F(\mathbb{R}^2) \supseteq F^2(\mathbb{R}^2) \supseteq F^3(\mathbb{R}^2) \supseteq \ldots.$$

In this Section, we will look for the stability of this sequence. More precisely, we will prove Theorem 1.1.

Let F be a polynomial mapping satisfying the assumption of Theorem 1.1. Set

$$\mathbb{R}^2 \setminus F(\mathbb{R}^2) = \{x_1, ..., x_d\}$$

and, for each non-negative integer n, denote

$$A_n := \{x_1, x_2, ..., x_d; F(x_1), F(x_2), ..., F(x_d); ...; F^{n-1}(x_1), F^{n-1}(x_2), ..., F^{n-1}(x_d)\}.$$

Then $\mathbb{R}^2 \setminus F^n(\mathbb{R}^2) \subset A_n$.

We have the following lemmas.

Lemma 3.1. *With the assumption of Theorem 1.1 and assuming further that for each $i \in \{1, 2, \ldots, d\}$ there is a positive integer n_i, such that $F^{n_i}(x_i) \in F^{n_i+1}(\mathbb{R}^2)$. Then, there exists a positive integer N which $F^N(\mathbb{R}^2) = F^{N+1}(\mathbb{R}^2)$.*

Proof. For each $i = 1, 2, ..., d$, let $b_i \in \mathbb{N}$ be the smallest number such that $F^{n_i}(x_i) \in F^{n_i+1}(\mathbb{R}^2)$. Let

$$N = \max\{b_i \mid i = 1, 2, ..., d\}.$$

It is obvious that $F^{N+1}(\mathbb{R}^2) \subseteq F^N(\mathbb{R}^2)$. We will show that $F^N(\mathbb{R}^2) = F^{N+1}(\mathbb{R}^2)$.

Assume by contradiction that there is a point $y \in F^N(\mathbb{R}^2) \setminus F^{N+1}(\mathbb{R}^2)$. Hence $y = F^N(x)$ for some $x \in \mathbb{R}^2$. Since $y \notin F^{N+1}(\mathbb{R}^2)$, we get $x \notin F(\mathbb{R}^2)$, that means $x = x_i$ for some $i = 1, 2, ..., d$. We have

$$y = F^N(x_i) = F^{N-b_i}(F^{b_i}(x_i)) \in F^{N-b_i}(F^{b_i+1}(\mathbb{R}^2)) = F^{N+1}(\mathbb{R}^2),$$

this is a contradiction. Thus $F^N(\mathbb{R}^2) = F^{N+1}(\mathbb{R}^2)$. □

Lemma 3.2. *With the hypothesis of Theorem 1.1, if for some $i \in \{1, 2, \ldots, d\}$ one has $F^n(x_i) \notin F^{n+1}(\mathbb{R}^2)$ for all $n \geqslant 0$, then there does not exist sequence $0 < n_1 < n_2 < \ldots$ such that $\#F^{-1}(F^{n_k}(x_i)) \geqslant 2$.*

Proof. By contradiction, assume that there exist $i \in \{1, 2, \ldots, d\}$ and a sequence $\{n_k\}_{k \geqslant 0}$ satisfying that $F^n(x_i) \notin F^{n+1}(\mathbb{R}^2)$ for all $n \geqslant 0$ and $\#F^{-1}(F^{n_k}(x_i)) \geqslant 2$ for all $k \geqslant 0$.

Since $F^n(x_i) \notin F^{n+1}(\mathbb{R}^2)$, for all $n \geqslant 0$, for any pair of natural numbers $m < s$ we have $F^m(x_i) \neq F^s(x_i)$ (otherwise $F^m(x_i) = F^s(x_i) \in F^s(\mathbb{R}^2) \subseteq F^{m+1}(\mathbb{R}^2)$). Further, from the definition of A_{n_k}, there does not exist $x \in \mathbb{R}^2 \setminus A_{n_k}$ which $F(x) = F^{n_k}(x_i)$. Therefore, for each k, beside $F^{n_k-1}(x_i)$, the point $F^{n_k}(x_i)$ has another preimage via F which belongs to A_{n_k}. Denote that preimage by $F^{m_k}(x^{n_k}), (x^{n_k} \neq x_i, m_k \leqslant n_k - 1)$.

So, for each $k \geqslant 0$ there is a point $F^{m_k}(x^{n_k}) \in A_{n_k}$, such that

$$F^{m_k+1}(x^{n_k}) = F^{n_k}(x_i) \quad \text{and} \quad F^{m_k}(x^{n_k}) \neq F^{n_k-1}(x_i). \tag{3.1}$$

Since $x^{n_k} \in \{x_1, ..., x_d\}$ for all $k \geqslant 0$, there are numbers $k < l$, such that $x^{n_k} = x^{n_l}$. Therefore

$$\begin{cases} F^{m_k+1}(x^{n_k}) = F^{n_k}(x_i), \\ F^{m_l+1}(x^{n_k}) = F^{m_l+1}(x^{n_l}) = F^{n_l}(x_i). \end{cases}$$

Hence

$$F^{m_k+1}(F^{n_l}(x_i)) = F^{m_k+1}(F^{m_l+1}(x^{n_k})) = F^{m_l+1}(F^{m_k+1}(x^{n_k})) = F^{m_l+1}(F^{n_k}(x_i)).$$

That means

$$F^{m_k+n_l+1}(x_i) = F^{m_l+n_k+1}(x_i).$$

On the other hand, the sequence $\{F^n(x_i)\}_{n\geqslant 0}$ is not periodic then $m_k + n_l = m_l + n_k$, this means $m_l - m_k = n_l - n_k = h \geqslant 1$. Hence

$$F^{m_l}(x^{n_l}) = F^{m_l}(x^{n_k}) = F^{h+m_k}(x^{n_k}) = F^{h-1}(F^{m_k+1}(x^{n_k})) = F^{h-1}(F^{n_k}(x_i)) = F^{n_l-1}(x_i),$$

this contradicts to 3.1. The lemma is proved. $\square$

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. By Lemma 3.1 and Lemma 3.2, if there does not exist a positive integer N such that

$$F^N(\mathbb{R}^2) = F^{N+1}(\mathbb{R}^2)$$

then one can find a number $i \in \{1, 2, \dots, d\}$ satisfying the followings

i) $F^n(x_i) \notin F^{n+1}(\mathbb{R}^2)$, for all $n \geqslant 0$;

ii) $\#F^{-1}(F^n(x_i)) = 1$, for all $n \geqslant M$, where M is some non-negative integer.

Without the genericity, one can assume that $i = 1$. According to the hypothesis, every point outside the asymptotic variety S_F has more than one preimage via the map F then $F^n(x_1) \in S_F$, for all $n \geqslant M$.

By Theorem 2.3, the asymptotic variety S_F is a closed semi-algebraic set of dimension at most one. Therefore, its Zariski closure $\overline{S_F}$ is an algebraic set of dimension at most one. Let

$$\overline{S_F} = \bigcup_{i=1}^{s} \gamma_i$$

be the decomposition of $\overline{S_F}$ into irreducible components (γ_i is either a point or an irreducible algebraic curve in $\mathbb{R}^2$).

Since $F^n(x_1) \in \bigcup_{i=1}^{s} \gamma_i$ for all $n \geqslant M$, we have

$$F^n(x_1) \in (F^k)^{-1}(\bigcup_{i=1}^{s} \gamma_i) = \bigcup_{i=1}^{s} (F^k)^{-1}(\gamma_i)$$

for all $k \geqslant 0, n \geqslant M$. Hence (recall that $F^n(x_1) \neq F^m(x_1)$ for all $n \neq m$)

$$F^n(x_1) \in \bigcup_{i=1}^{s} \bigcup_{j=1}^{s} (\gamma_i \cap (F^k)^{-1}(\gamma_j)).$$

It implies that for each $k \geqslant 0$, there exist a sequence of numbers $\{n_i^k\}_{i\geqslant 0}$ ($n_i^k \geqslant M$) and two curves γ_1^k, γ_2^k among γ_i such that $F^{n_i^k}(x_1) \in \gamma_1^k \cap (F^k)^{-1}(\gamma_2^k)$, for all $i \geqslant 0$. That is, the intersection of the two curves γ_1^k and $F^{-1}(\gamma_2^k)$ have infinitely many points. Since γ_1^k is irreducible, it follows that $\gamma_1^k \subset (F^k)^{-1}(\gamma_2^k)$.

For all k, γ_1^k and γ_2^k are irreducible components of $\overline{S_F}$ then there are $l < s$ such that $\gamma_1^l = \gamma_1^s$ and $\gamma_2^l = \gamma_2^s$. We have $\gamma_1^l \subset (F^l)^{-1}(\gamma_2^l)$ so

$$F^l(\gamma_1^l) = F^l(\gamma_1^s) \subset \gamma_2^l = \gamma_2^s.$$

Similarly, $\gamma_1^s \subset (F^s)^{-1}(\gamma_2^s)$ hence $F^l(\gamma_1^s) \subset (F^{s-l})^{-1}(\gamma_2^s)$. Thus

$$F^l(\gamma_1^s) \subset \gamma_2^s \cap (F^{s-l})^{-1}(\gamma_2^s).$$

On the other hand, $F^{n_i^s}(x_1) \in \gamma_1^s$ for all $i \geq 0$, then $F^{n_i^s+l}(x_1) \in F^l(\gamma_1^s)$. So, two curves γ_2^s and $(F^{s-l})^{-1}(\gamma_2^s)$ both contain infinitely many points $F^{n_i^s+l}(x_1)$ for all $i \geq 0$, while γ_2^s is again irreducible then $\gamma_2^s \subset (F^{s-l})^{-1}(\gamma_2^s)$. Put $G = F^{s-l}$, then $G(\gamma) \subset \gamma$, where $\gamma := \gamma_2^s$ is an irreducible component of S_F.

One claims that $\#G^{-1}(F^n(x_1)) = 1$ for all $n \geq M + s - l - 1$. Since, if $x \in G^{-1}(F^n(x_1))$, then $F^{s-l}(x) = F^n(x_1)$, this means $F^{s-l-1}(x), F^{n-1}(x_1) \in F^{-1}(F^n(x_1))$, due to ii), it implies that $F^{s-l-1}(x) = F^{n-1}(x_1)$. Continue this process, finally, we obtain that $x = F^{n-s+l}(x_1)$. Put $y_i = F^{n_i^s+s}(x_1)$, then for all i large enough, $y_i \in G(\gamma) \subset \gamma$ and $G^{-1}(y_i) \in \gamma$.

Next, we will prove that $G(\gamma) = \gamma$. Indeed, consider the restriction map $G_|\gamma : \gamma \to \gamma$. It follows from Proposition 2.4 and the previous explaination that there exists a finite subset $W \subset \gamma$ such that the number of points in the preimage $(G_|\gamma)^{-1}(y)$ is one (modulo 2) for all $y \in \gamma \setminus W$. It means that $G(\gamma) = \gamma \setminus B$ for some finite subset (possibly empty) B of γ. If γ is bounded, then it is a compact set, the image $G(\gamma)$ is also a compact set, hence $G(\gamma) = \gamma$. The case when γ is unbounded is handled as follows.

For an unbounded curve C in $\mathbb{R}^2$, if we denote by B_R the open ball centered at the origin with radius R, then once $R \geq R_0$ with $R_0 > 0$ is large enough, the complement $C \setminus B_R$ has finite number of unbounded connected components, each of which is diffeomorphic to $[0, \infty)$, the number of components does not depend on R. We call those connected components the branches at infinity of the curve C.

Since $\gamma \setminus G(\gamma)$ is a finite set, for $R > 0$ large enough, $\gamma \setminus B_R$ consists of finite branches at infinity $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, all of which are subsets of $G(\gamma)$. We see that for each $i = 1, \ldots, m$, the inverse $G^{-1}(\Gamma_i)$ is a closed semialgebraic subsets of the curve γ and it is unbounded (otherwise, $\Gamma_i = G(G^{-1}(\Gamma_i))$ is bounded), hence, there is one branch at infinity Δ_i of γ such that $\Gamma_i \subset G(\Delta_i)$ (we may choose R bigger if needed). That means, each branch at infinity of γ is the image of some branch via G. Because the number of branches is finite, therefore G maps each branch at infinity of γ to a branch at infinity of γ.

Now, fix $R > 0$ large enough and consider the inverse

$$K := (G_{|\gamma})^{-1}(\overline{B}_R) = G^{-1}(\overline{B}_R) \cap \gamma,$$

where $\overline{B}_R$ is the closed ball centered at the origin with radius R. It implies from the previous observation that K is a bounded subset of γ and is also closed. So it is a compact set. Therefore, we obtain that

$$G(\gamma) = G(K) \cup (\bigcup_{i=1}^{m} \Gamma_i)$$

is a closed subset of γ. Hence B is empty. In other words, $G(\gamma) = \gamma$. It follows that for all $n > 0$ we have $G^n(\gamma) = \gamma$, then for a fixed $i > 0$ large enough:

$$y_i = F^{n_i^s+s}(x_1) \in \gamma = G^n(\gamma) = F^{(s-l)n}(\gamma).$$

When n is big enough, we get $(s-l)n > n_i^s + s$ implying that

$$F^{n_i^s+s}(x_1) \in F^{n_i^s+s+1}(\mathbb{R}^2).$$

This contradicts to i). The proof is complete.

□

For a polynomial mapping $F = (p, q) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, the set of critical values of F is a semialgebraic set and by the Sard's Theorem, it has measure zero. Hence, for generic $(a, b) \in \mathbb{R}^2$ (i.e. outside some algebraic curve), the preimage $F^{-1}(a, b)$ is either an empty set or a set consisting of finitely many points. Let $C_F \subset \mathbb{R}^2$ be the union of the set of critical values and the asymptotic variety of F, then C_F divides the plane into some connected components. On each such component R, for all $y \in R$ the preimage $F^{-1}(y)$ has the same number of points.

Theorem 3.3. *Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a polynomial mapping with even degree such that the complement of the image is a finite set. Then, there exists a natural number N such that the image $F^k(\mathbb{R}^2)$ of F^k are equal for all $k \geqslant N$.*

Proof. It follows from the hypothesis that there is an algebraic curve $S \subset \mathbb{R}^2$ such that for all point $x \notin S$, the Euler characteristic of the fiber $F^{-1}(x)$ is even. We will choose S such that it contains the curve C_F in the previous explanation, then the fiber $F^{-1}(x)$ is a finite set, so it contains an even number of points. Because there are at most finitely many points whose preimage is empty, if $x \notin S$, the set $F^{-1}(x)$ is nonempty and contains even number of points, this means it has more than one point.

We repeat the argument as in the proof of Theorem 1.1, replacing the asymptotic variety S_F by this curve S. More precisely, if there does not exist natural number N such that $F^N(\mathbb{R}^2) = F^{N+1}(\mathbb{R}^2)$, then there is some point $x_1 \in \mathbb{R}^2$ such that the following two conditions hold

i) $F^n(x_1) \notin F^{n+1}(\mathbb{R}^2)$, for all $n \geqslant 0$;
ii) $\#F^{-1}(F^n(x_1)) = 1$, for all $n \geqslant M$, where M is some non-negative integer.

It implies from ii) that $F^n(x_1) \in S$, for all $n \geqslant M$. Now, the proof is complete by following all argument as in the proof of Theorem 1.1. □

The followings are some examples where the assumptions of Theorem 1.1 hold.

Example 3.4. $F_1(x, y) = (x^2y - x - y, (x^2y - x + y)(xy - 1/2)) : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$.

We have the following remarks on this mapping:

1) $J(F_1) = (x^2y - x + y)^2 + 4(xy - 1/2)^2$, $J(F_1) \neq 0$ at all points except two points $A = (1, 1/2)$ and $B = (-1, -1/2)$. Thus the map has two critical points $F_1(A) = (-1, 0)$ and $F_1(B) = (1, 0)$.

2) Fibers the map: $F_1^{-1}(a, b), (a, b) \in \mathbb{R}^2$. Let $(x, y) \in F_1^{-1}(a, b)$, this means

$$\begin{cases} x^2y - x - y = a, \\ (x^2y - x + y)(xy - 1/2) = b. \end{cases} \tag{3.2}$$

There are two cases:

Case 1: $a \neq \pm 1$. From the first equation in 3.2 we get $y = \frac{a+x}{x^2-1}$, substituting to the other equation, one obtains that

$$(a - 2b)x^4 + 2(a^2 + 1)x^3 + (6a + 4b)x^2 + 2(a^2 + 1)x + (a - 2b) = 0. \tag{3.3}$$

Then, one can check that if $a - 2b \neq 0$, the equation 3.3 has two solutions, which implying that the fiber $F_1^{-1}(a, b)$ is a set consisting of two points. If $a - 2b = 0$, the fiber $F_1^{-1}(a, b)$ consists of one point.

Case 2: $a = 1$ or $a = -1$. By computation, one can also check that each of the points $(1, 0), (-1, 0), (1, \frac{1}{2}), (-1, -\frac{1}{2})$ has exactly one point in the preimage and other points have two points in the preimage.

Thus the map is surjective and the asymptotic variety of this map is the line $\{u - 2v = 0\}$. The map has degree zero.

Example 3.5. (Pinchuk map, see [12, 4, 2]) Put

$$t = xy - 1, h = t(tx + 1), f = (tx + 1)^2(h + 1/x)$$

and

$$u(f, h) = -170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - (75/4)h^4.$$

The Pinchuk map is $(p, q) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, where $p = f + h$ and $q = -t^2 - 6th(h + 1) + u(f, h)$.

The image of the Pinchuk map is $\mathbb{R}^2 \setminus \{(0,0), (-1, -163/4)\}$, it has degree zero and all points outside the asymptotic variety has two preimage.

Remark 3.6. In Theorem 1.1 and Theorem 3.3, we can not remove the assumption that the complement of the image of the map has only finitely many points. Indeed, consider the following mapping.

$$F(x, y) = (x^2 + 1, y) : \mathbb{R}^2 \to \mathbb{R}^2.$$

Then the image of the iterated map F^k is never stable.

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Conflict of Interest Statement

The author has no conflict of interest to declare.

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