Tamed-adaptive Euler-Maruyama approximation for SDEs with superlinearly growing and piecewise continuous drift, superlinearly growing and locally Hölder continuous diffusion

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May 15, 2023

Abstract

In this paper, we consider stochastic differential equations whose drift coefficient is superlinearly growing and piece-wise continuous, and whose diffusion coefficient is superlinearly growing and locally Hölder continuous. We first prove the existence and uniqueness of the solution to such stochastic differential equations and then propose a tamed-adaptive Euler-Maruyama approximation scheme. We study the rate of convergence in the L^1-norm of the scheme in both finite and infinite time intervals.

Keywords: Adaptive Euler-Maruyama; Discontinuous drift; Locally Hölder continuous diffusion; Stochastic differential equations; Tamed Euler-Maruyama; Uniform in time approximation.
Mathematics Subject Classification: 60H35, 60H10

1 Introduction

Stochastic differential equations (SDEs) are used to model a variety of random dynamical phenomena; therefore, they play a significant role in many fields of science and industry. In many cases, the solution to many SDEs cannot be computed explicitly and has to be approximated by a numerical scheme.

In this paper, we consider the numerical approximation for stochastic processes $X = (X_t)_{t\in[0,+\infty)}$ defined by the following stochastic differential equation (SDE),

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t, \\ X_0 = x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $W = (W_t)_{t\in[0,+\infty)}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual condition.

For SDEs with Lipschitz continuous coefficients, it is well-known that the explicit Euler-Maruyama approximation scheme converges at the strong rate of order $\frac{1}{2}$ (see [11, 18]). The numerical approximation for SDEs with irregular coefficients has recently been considered extensively. The divergence of the classical Euler-Maruyama scheme when applying for SDEs with super-linear growth coefficients has been pointed out in [9]. Thereafter, many modified Euler-Maruyama schemes have been introduced for SDEs with super-linear growth coefficients, such as the tamed Euler-Maruyama scheme (see [7, 9, 27, 28]), the truncated Euler-Maruyama scheme (see [16]), the implicit Euler-Maruyama scheme (see [17]), and the adaptive scheme (see [4]). The convergence of the Euler-Maruyama scheme for SDEs with Hölder continuous diffusion coefficients was first studied in [5] by using the Yamada-Watanabe approximation technique. This technique has been developed in [26, 25] to study the strong convergence of the tamed Euler-Maruyama scheme for SDEs with super-linear growth coefficients.

There are several approaches to study numerical approximation for SDEs with discontinuous drift coefficient. The first approach is to transform an SDE with discontinuous drift into a new SDE with Lipschitz

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continuous coefficients. Using this transformation technique, Leobacher and Szölgyenyi developed an approximation method for SDEs with piecewise Lipschitz continuous drift and global Lipschitz continuous diffusion (see [14, 15]). Thereafter, the transformation technique was applied to study a quasi-Milstein scheme for a smaller family of SDEs, i.e. those with coefficients having piecewise continuous Lipschitz derivatives (see [20, 21]). The second approach is to use a Gaussian bound for the density of the Euler-Maruyama approximated solution, which was proven in [13]. Using this Gaussian bound, in [26], Ngo and Taguchi showed that for SDEs the Euler-Maruyama approximation with uniformly elliptic, bounded, $(1/2+\alpha)$-Hölder diffusion coefficient and one-sided Lipschitz, bounded variation drift coefficient, the Euler-Maruyama strongly converges in L^1-norm at the order of $\alpha \in (0, 1/2]$. In [25], by introducing a drift removal transformation technique, they can get rid of the one-sided Lipschitz continuous condition of drift for one-dimensional SDEs (see also [1]). The third approach is to use adaptive schemes (see [31, 22]). The basic idea is that the step size of the scheme will be adjusted to be smaller near the discontinuous points of the drift coefficient. The fourth approach is to use the regularisation of the noise. In [3], by exploiting the regularising effect of the noise, the authors showed that the rate of strong convergence of the Euler-Maruyama is arbitrarily close to 1/2 when applying for a larger class of non-degenerated SDEs with irregular drift. The fifth approach is to use the stochastic sewing lemma. In [12, 2], they studied a tamed Euler-Maruyama scheme of strong order 1/2 for a class of SDEs with an integrable drift coefficient and elliptic regular diffusion coefficient. The approximation for SDEs with superlinearly growing coefficients, locally Lipschitz diffusion, and piece-wise continuous drift, has been studied very recently in [19, 6].

So far, all the numerical studies for SDEs with discontinuous drift have only considered the approximation in a finite time interval. For infinite time intervals, Fang and Giles (see [4]) developed an adaptive Euler-Maruyama scheme for a class of SDEs with a polynomial growth Lipschitz continuous drift, and bounded, globally Lipschitz continuous diffusion. Using the method of Lyapunov functions, they proved that the proposed scheme converges in L^p-norm at the rate of order 1/2. To extend this result for a larger class of SDEs, i.e. those with locally Lipschitz continuous and one-sided Lipschitz drift and locally $(\alpha + 1/2)$-Hölder continuous diffusion coefficients, Kieu et. al. (see [10]) established a tamed-adaptive Euler-Maruyama approximation scheme, utilized the Yamada-Watanabe approximation to obtain upper bounds for some p^{th} moments of the approximated solution, and then showed that the scheme converges in L^1-norm at the rate of order $\alpha \in (0, 1/2]$.

In this paper, we are interested in the uniformly in time numerical approximation for the SDEs of the form (1), where b is superlinearly growing and piecewise locally Lipschitz continuous, and σ is superlinerly growing and locally $(\alpha + 1/2)$-Hölder continuous. By combining and improving the tamed and adaptive models in the literature, we introduce a tamed-adaptive Euler-Maruyama approximation scheme that converges in both finite and infinite time intervals. The scheme is proved to converge in L^1-norm at the rate of order α, and, specifically, under some condition on the growths of b and σ, the scheme is proved to converge in infinite time intervals. The proof is based on an enhancement from the techniques introduced by Kieu et. al. [10], and Yaroslavtseva [31], and an introduction of a new function φ to control the discontinuity of drift coefficient, which is the most novel technical contribution of this work. To the best of our knowledge, this is the first approximation scheme for SDEs with superlinearly growing and piecewise continuous drift, superlinearly growing, and locally Hölder continuous diffusion.

The rest of this paper is structured as follows. In Section 2, we first introduce a list of assumptions on the coefficients b and σ, and state theorems on the existence and uniqueness of solution for some classes of SDEs with superlinearly growing and piecewise continuous drift, superlinearly growing and locally Hölder continuous diffusion. Next, we introduce the tamed-adaptive Euler-Maruyama approximation scheme and state our main results on the rate of convergence of this scheme in both finite and infinite time intervals. All the proofs are deferred to Section 3. In Section 4, we present a numerical experiment to illustrate the performance of the new scheme for various SDEs.

2 Tame-adaptive Euler-Maruyama approximation

2.1 Existence and uniqueness of solution

We first introduce some assumptions on the coefficients of equation (1).

(A1) There exist constants $\gamma, \eta \in \mathbb{R}$ and $p_0 \in [2, +\infty)$ such that

$$xb(x) + \frac{p_0 - 1}{2}|\sigma(x)|^2 \leq \gamma x^2 + \eta, \text{ for any } x \in \mathbb{R}.$$

(A2) There exists a constant L_1 such that

$$(x-y)(b(x)-b(y)) \leq L_1(x-y)^2, \text{ for any } x,y \in \mathbb{R}.$$

(A3) There exist a sequence $\xi_1 < \xi_2 < \ldots < \xi_k$, some constants $L_2 > 0$ and $l \geq 1$ such that for any $(x,y) \in \mathcal{S} := \cup_{i=1}^{k-1}(\xi_i, \xi_{i+1})^2 \cup (-\infty, \xi_1)^2 \cup (\xi_k, +\infty)^2$,

$$|b(x)-b(y)| \leq L_2(1+|x|^l+|y|^l)|x-y|.$$

(A4) There exist constants $m \geq 1, L_3 \geq 0$ and $\alpha \in [0, \frac{1}{2}]$ such that

$$|\sigma(x)-\sigma(y)| \leq L_3(1+|x|^m+|y|^m)|x-y|^{\alpha+1/2}, \text{ for any } x,y \in \mathbb{R}.$$

(A5) The function σ satisfies $\sigma(\xi_i) > 0$ for all $i = 1, \ldots, k$, where (ξ_i) is the sequence defined in (A3).

(A6) There exist constants $L_4, h, \xi' > 0$ such that

$$\sup_{|x-y|\leq h, |x| \geq \xi'} \frac{\sigma(x)}{\sigma(y)} \leq L_4.$$

(A'2) There exists a constant L_1 such that

$$(x-y)(b(x)-b(y)) \leq L_1(x-y)^2,$$

for any $(x,y) \in (-\infty, \xi_1)^2 \cup (\xi_k, +\infty)^2$, where ξ_1, ξ_k are defined in Assumption (A3).

(A'5) There exists a positive constant κ such that $\sigma(x) \geq \kappa$ for any $x \in \mathbb{R}$.

Remark 2.1. It can be seen that

- Assumption (A3) implies that there exists a constant $L_2' > 0$ such that

$$|b(x)-b(y)| \leq L_2' + L_2'(1+|x|^l+|y|^l)|x-y|, \text{ for any } x,y \in \mathbb{R}.$$

- Assumption (A'5) implies Assumption (A5); Assumption (A2) implies Assumption (A'2).

Theorem 2.2. *Suppose that Assumptions (A1),(A'2), (A3),(A4),(A'5) hold, and $p_0 \geq (4l+4) \vee (4+4\alpha+4m)$. Moreover, if either (A6) hold or (A'2) hold for $L_2 < 0$, then the equation* (1) *has a unique strong solution.*

Theorem 2.3. *Suppose that (A1)–(A5) hold for $p_0 \geq (4l+4) \vee (4m+4\alpha+4)$. Then the equation* (1) *has a unique strong solution.*

The existence and uniqueness of solution to one dimensional SDEs with super-linear growth coefficients, discontinuous drift and degenerate locally Lipschitz continuous diffusion coefficient have been studied recently in [19, 6]. The novel of our Theorems 2.2 and 2.3 is that they can be applied for SDE with locally Hölder continuous coefficient. Our approach is based on a localization technique (see [24]).

2.2 Tamed-adaptive Euler-Maruyama approximation scheme

Throughout this paper, we always assume that Assumptions (A1), (A3) and (A4) hold. Let $\Xi = \{\xi_1, \ldots, \xi_k\}$, $d(x, \Xi) = \min_{1 \leq i \leq k} |x - \xi_i|$, and $\Xi^\varepsilon = \{x \in \mathbb{R} : d(x, \Xi) \leq \varepsilon\}$ for any $\varepsilon > 0$. Since σ is continuous and $\sigma(\xi_i) \neq 0$, there exist $\mu, \nu > 0$ depending only on σ such that $\inf_{|x-\xi_i| \leq \mu} \sigma(x) \geq \nu$. Let $\varepsilon_0 = \mu \wedge \min_{1 \leq i \leq k-1}(\xi_{i+1} - \xi_i)$.

Let Δ_0 be a positive constant satisfying

$$\Delta \log^4(1/\Delta) < \sqrt{\Delta} \log^2(1/\Delta) < \frac{1}{2}\varepsilon_0,$$

for all $\Delta \in (0, \Delta_0)$. For each $\Delta \in (0, \Delta_0)$, we define the functions σ_Δ and h_Δ by

$$\sigma_\Delta(x) := \frac{\sigma(x)}{1 + \Delta^{1/2}|\sigma(x)|},$$

and

$$h_\Delta(x) = \begin{cases} \dfrac{\Delta}{[1+|b(x)|+|\sigma(x)|+|x|^l]^2} & \text{if} \quad x \in (\Xi^{\varepsilon_1})^c, \\ \dfrac{[d(x,\Xi)]^2}{\log^4(1/\Delta)[1+|b(x)|+|\sigma(x)|+|x|^l]^2} & \text{if} \quad x \in \Xi^{\varepsilon_1} \setminus \Xi^{\varepsilon_2}, \\ \dfrac{\Delta^2 \log^4(1/\Delta)}{[1+|b(x)|+|\sigma(x)|+|x|^l]^2} & \text{if} \quad x \in \Xi^{\varepsilon_2}, \end{cases} \tag{2}$$

where $\varepsilon_1 := \sqrt{\Delta}\log^2(1/\Delta)$ and $\varepsilon_2 := \Delta \log^4(1/\Delta)$.

The tamed-adaptive Euler-Maruyama scheme is defined as follows. Let $Y_0 = x_0, t_0 = 0$, and for each $i \geq 0$,

$$\begin{cases} t_{i+1} = t_i + h_\Delta(Y_{t_i}), \\ Y_t = Y_{t_i} + b(Y_{t_i})(t - t_i) + \sigma_\Delta(Y(t_i))(W_t - W_{t_i}), \quad t_i < t \leq t_{i+1}. \end{cases} \tag{3}$$

Note that if Assumptions (A1), (A3) and (A4) hold, then by following the argument in the proof of Proposition 2.1 in [10], it can be shown that the scheme is well-defined, i.e. $\lim_{i\to\infty} t_i = +\infty$, a.s.

For each $t \geq 0$, we define $\underline{t} := \max\{t_i, t_i \leq t\}$, which is a stopping time, and $\overline{Y}_t := Y_{\underline{t}}$.

Let $[\frac{p_0}{2}]$ denote the integer part of $\frac{p_0}{2}$. The following result shows the convergence in L^1-norm of the tamed-adaptive Euler-Maruyama scheme.

Theorem 2.4. *Let Assumptions (A1)–(A5) hold and $[\frac{p_0}{2}] \geq (l+1) \vee (1+2\alpha+2m)$. Then there exists a positive constant C which does not depend on Δ such that*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|Y_t - X_t|\right] \leq \begin{cases} C\Delta^\alpha & \text{if } 0 < \alpha \leq \dfrac{1}{2}, \\ \dfrac{C}{\log \frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{4}$$

Moreover, if γ and L_1 are negative, then the constant C does not depend on T either.

Assumption (A2) is in fact quite restrictive since it excludes some very simple functions, such as $b(x) = 1_{(0,\infty)}(x)$. In the following, we will study the convergence of the tamed-adaptive scheme (3) under Assumption (A'2), which only requires that b is one-sided Lipschitz continuous outside the interval (ξ_1, ξ_k). However, we need to assume that σ satisfies Assumption (A'5), which is stronger than Assumption (A5).

Theorem 2.5. *Suppose that Assumptions (A1),(A'2), (A3),(A4) and (A'5) hold, and $[\frac{p_0}{2}] \geq (l+1) \vee (1+2\alpha+2m)$.*

a) If (A6) holds, then there exists a positive constant C which does not depend on Δ such that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|Y_t - X_t|\right] \leq \begin{cases} C\Delta^\alpha & \text{if } 0 < \alpha \leq \dfrac{1}{2}, \\ \dfrac{C}{\log \frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{5}$$

b) If (A'2) holds for some $L_1 < 0$, then the estimate (5) *also holds. Moreover, if we suppose further that $\gamma < 0$, then the constant C does not depend on T either.*

Finally, we consider the complexity of the scheme (3). For any $T > 0$, let N_T be the number of time-steps required by a path approximation on $[0, T]$. More precisely, we write

$$N_T = 1 + \sum_{k=1}^{\infty} 1_{\{t_k < T\}}.$$

Theorem 2.6. *Suppose that Assumptions (A1),(A3),(A4) and (A5) hold, and $[\frac{p_0}{2}] \geq (l+1)\vee(1+2\alpha+2m)$. Then there exists a constant $C > 0$ which does not depend on Δ such that*

$$\mathbb{E}[N_T] \leq \begin{cases} CT\Delta^{\alpha/2-7/4} & \text{if} \quad [\frac{p_0}{2}] \leq \frac{3}{2\alpha+1}(1+2\alpha+2m), \\ CT\Delta^{-1} & \text{if} \quad [\frac{p_0}{2}] > \frac{3}{2\alpha+1}(1+2\alpha+2m). \end{cases}$$

Moreover, if $\gamma < 0$, then the constant C does not depend on T either.

Remark 2.7. If $[\frac{p_0}{2}] > (l+1) \vee \frac{3(1+2\alpha+2m)}{1+2\alpha}$, then the number of time discretizations on the interval $[0, T]$ of the scheme (3) is proportional to the one of the classical Euler-Maruyama scheme.

3 Proofs

3.1 Yamada-Watanabe approximation

We recall the Yamada-Watanabe approximation ([30]). For each $\delta > 1$ and $\varepsilon > 0$, there exists a continuous function $\psi_{\delta\varepsilon} : \mathbb{R} \to \mathbb{R}^+$ with $\text{supp}\psi_{\delta\varepsilon} \subset [\varepsilon/\delta; \varepsilon]$ such that $\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta\varepsilon}(z)dz = 1$ and $0 \le \psi_{\delta\varepsilon}(z) \le \dfrac{2}{z \log \delta}, \ z > 0$. Define $\phi_{\delta\varepsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta\varepsilon}(z)dzdy, \ x \in \mathbb{R}$. It is easy to verify that $\phi_{\delta\varepsilon}$ has the following useful properties: for any $x \in \mathbb{R}\backslash\{0\}$,

(YW1) $\phi'_{\delta\varepsilon}(x) = \dfrac{x}{|x|}\phi'_{\delta\varepsilon}(|x|)$,

(YW2) $0 \le |\phi'_{\delta\varepsilon}(x)| \le 1$,

(YW3) $|x| \le \varepsilon + \phi_{\delta\varepsilon}(x)$,

(YW4) $\dfrac{\phi'_{\delta\varepsilon}(|x|)}{|x|} \le \dfrac{\delta}{\varepsilon}$,

(YW5) $\phi''_{\delta\varepsilon}(|x|) = \psi_{\delta\varepsilon}(|x|) \le \dfrac{2}{|x| \log \delta} 1_{[\frac{\varepsilon}{\delta};\varepsilon]}(|x|) \le \dfrac{2\delta}{\varepsilon \log \delta}$.

3.2 Some moment estimates

The following result has been proven in [10].

Proposition 3.1. *Suppose that the coefficients b and σ satisfy Assumption (A1), and σ is bounded on every compact subset of $\mathbb{R}$. Then, for any $p \in [0, p_0]$,*

$$\mathbb{E}[|X_t|^p] \le \left| x_0^2 e^{2\gamma t} + \frac{\eta}{\gamma}(e^{2\gamma t} - 1) \right|^{p/2}.$$

Proposition 3.2. *Suppose that the coefficients b and σ satisfy Assumptions (A1), (A3) and (A4). Then for any positive number $k \le [p_0/2]$, there exists a positive constant $K = K(x_0, k, \eta, \gamma, L_1, L_2, L_3)$ which does not depend on Δ, such that*

$$\mathbb{E}\left[|Y_t|^{2k}\right] \le \begin{cases} Ke^{2k\gamma t} & \text{if } \gamma > 0, \\ K(1+t)^k & \text{if } \gamma = 0, \\ K & \text{if } \gamma < 0. \end{cases} \tag{6}$$

Proof. Thanks to Hölder's inequality, it is sufficient to show (6) for $0 < k \le p_0/2$. From the definition of the scheme, we have

$$\max\left\{|\overline{Y}_s b(\overline{Y}_s)(s - \underline{s})|, \ b^2(\overline{Y}_s)(s - \underline{s})^2, \ \mathbb{E}\left[\sigma_\Delta^2(\overline{Y}_s)(W_s - W_{\underline{s}})^2 \middle| \mathcal{F}_{\underline{s}}\right]\right\} \le C\Delta.$$

Then the proof follows directly from the argument of Theorem 2.4 in [10]. □

Lemma 3.3. *Suppose that the coefficients b and σ satisfy all conditions of Proposition 3.2. Then for any $p > 0$, there exists a positive constant C_p depending only on p such that*

$$\sup_{t \ge 0} \mathbb{E}\left[|Y_t - \overline{Y}_t|^p\right] \le C_p \Delta^{p/2}.$$

Proof. From (3),

$$\begin{aligned} |Y_t - \overline{Y}_t|^p &= \left|b(\overline{Y}_t)(t - \underline{t}) + \sigma_\Delta(\overline{Y}_t)(W_t - W_{\underline{t}})\right|^p \\ &\le 2^{p-1}\left(\left|b(\overline{Y}_t)(t - \underline{t})\right|^p + \left|\sigma_\Delta(\overline{Y}_t)(W_t - W_{\underline{t}})\right|^p\right) \\ &\le 2^{p-1}\left(\left|b(\overline{Y}_t)\right|^p \left|h_\Delta(\overline{Y}_t)\right|^p + \left|\sigma_\Delta(\overline{Y}_t)\right|^p \left|W_t - W_{\underline{t}}\right|^p\right). \end{aligned}$$

By (2), we have $\left|b(\overline{Y}_t)h_\Delta(\overline{Y}_t)\right| \le \dfrac{\Delta}{4}$, and $\left|\sigma_\Delta(\overline{Y}_t)\left|h_\Delta(\overline{Y}_t)\right|^{1/2}\right| \le \Delta^{1/2}$, which implies the desired result. □

Lemmas 3.4 and 3.5 are modified versions of Lemmas 6 and 7 in [31] for the stochastic differential equations with Hölder continuous diffusion coefficient.

Lemma 3.4. *Suppose that Assumptions (A1),(A3),(A4) and (A5) hold for* $2[\frac{p_0}{2}] \geq (l+1) \vee (4m+4\alpha+2)$. *Let* $f : [0,+\infty) \to [0,+\infty)$ *be a Borel measurable function. Let* $n = \frac{[\frac{p_0}{2}]}{1+2\alpha+2m}$ *and* $q = \frac{4n-3}{4n}$. *Then there exists a constant* c *which does not depend on* Δ *such that for any* $h \geq 1, \varepsilon < \varepsilon_0, \Delta < \Delta_0$,

$$\mathbb{E}\Big[\int_t^{t+h} f(d(Y_s,\Xi))1_{\Xi^\varepsilon}(Y_s)ds\Big] \leq \begin{cases} ch\displaystyle\int_0^\varepsilon f(x)dx + ch\Delta^{\frac{1}{4}+\frac{\alpha}{2}} \sup_{x\in[0,\varepsilon]} f(x), & \text{if } [\frac{p_0}{2}] \leq \frac{3(1+2\alpha+2m)}{2\alpha+1}, \\ ch\displaystyle\int_0^\varepsilon f(x)dx + ch \sup_{x\in[0,\varepsilon]} f(x)\big(\varepsilon^{q+\frac{\alpha}{2}+\frac{1}{4}} + \Delta\big) & \text{if } [\frac{p_0}{2}] > \frac{3(1+2\alpha+2m)}{2\alpha+1}. \end{cases}$$

Moreover, if $\gamma < 0$, *then the constant* c *does not depend on* t.

Proof. It is enough to show that for all $i = 1,\dots,k$, there exists $c \in (0,+\infty)$ such that for all $\Delta < \Delta_0, \varepsilon < \varepsilon_0, h \geq 1$,

$$\mathbb{E}\Big[\int_t^{t+h} f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)ds\Big] \leq \begin{cases} ch\displaystyle\int_0^\varepsilon f(x)dx + ch\Delta^{\frac{1}{4}+\frac{\alpha}{2}} \sup_{x\in[0,\varepsilon]} f(x), & \text{if } [\frac{p_0}{2}] \leq \frac{3(1+2\alpha+2m)}{2\alpha+1}, \\ ch\displaystyle\int_0^\varepsilon f(x)dx + ch \sup_{x\in[0,\varepsilon]} f(x)\big(\varepsilon^{q+\frac{\alpha}{2}+\frac{1}{4}} + \Delta\big) & \text{if } [\frac{p_0}{2}] > \frac{3(1+2\alpha+2m)}{2\alpha+1}. \end{cases}$$

For each $a \in \mathbb{R}$, let $L^a(Y) = (L^a_t(Y))_{t\in[0,+\infty)}$ be the local time of Y at the point a. From (3), by Tanaka's formula, for any $h \geq 1$, we have

$$|Y_t - a| = |x_0 - a| + \int_0^t sgn(Y_s - a)b(\overline{Y}_s)ds + \int_0^t sgn(Y_s - a)\sigma_\Delta(\overline{Y}_s)dW_s + L^a_t(Y).$$

Hence

$$\begin{aligned} |L^a_{t+h}(Y) - L^a_t(Y)| \leq& |Y_{t+h} - Y_t| + \left|\int_t^{t+h} sgn(Y_s - a)b(\overline{Y}_s)ds\right| + \left|\int_t^{t+h} sgn(Y_s - a)\sigma_\Delta(\overline{Y}_s)dW_s\right| \\ \leq& \left|\int_t^{t+h} b(\overline{Y}_s)ds\right| + \left|\int_t^{t+h} \sigma_\Delta(\overline{Y}_s)dW_s\right| \\ &+ \left|\int_t^{t+h} sgn(Y_s - a)b(\overline{Y}_s)ds\right| + \left|\int_t^{t+h} sgn(Y_s - a)\sigma_\Delta(\overline{Y}_s)dW_s\right|. \end{aligned}$$

For the rest of the proof, we denote by $K_1, K_2, \dots$ some constants that do not depend on Δ, h or a. Moreover, when $\gamma < 0$, these constants do not depend on t either. By taking expectations on both sides of the above inequality and using Doob's inequality, we get

$$\begin{aligned} \mathbb{E}\Big[|L^a_{t+h}(Y) - L^a_t(Y)|\Big] \leq& 2\int_t^{t+h} \mathbb{E}[|b(\overline{Y}_s)|]ds + 2\left|\int_t^{t+h} \mathbb{E}\Big[\sigma^2_\Delta(\overline{Y}_s)\Big]ds\right|^{1/2} \\ \leq& K_1\int_t^{t+h}\Big(\mathbb{E}\Big[(1+|\overline{Y}_s)|^l)|\overline{Y}_s|\Big] + 1\Big)ds \\ &+ K_1\left|\int_t^{t+h}\Big(\mathbb{E}\Big[\big(1+|\overline{Y}_s|^m\big)^2|\overline{Y}_s|^{2\alpha+1}\Big] + 1\Big)ds\right|^{1/2}, \end{aligned}$$

where the last estimate is derived from Assumptions (A3), (A4), Remark 2.1, and the fact that $\sigma^2_\Delta(x) \leq \sigma^2(x)$. Thanks to Proposition 3.2,

$$\sup_{s\in[0,t]} \mathbb{E}\Big[|\overline{Y}_s|^{l+1}\Big] + \sup_{s\in[0,t)} \mathbb{E}\Big[|\overline{Y}_s|^{2m+2\alpha+1}\Big] \leq K_2.$$

This implies that

$$\mathbb{E}\Big[|L^a_{t+h}(Y) - L^a_t(Y)|\Big] \leq K_3\sqrt{h} + K_3 h \leq 2K_3 h. \tag{7}$$

On the other hand, by using the occupation time formula, for all $\varepsilon < \varepsilon_0$ and $\Delta < \Delta_0$,

$$\mathbb{E}\Big[\int_0^t f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)\sigma_\Delta^2(\overline{Y}_s)ds\Big] = \int_{\mathbb{R}} f(|a-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(a)\mathbb{E}\big[L_t^a(Y)\big]da.$$

Hence, it follows from (7) that

$$\begin{aligned}\mathbb{E}\Big[\int_t^{t+h} f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)\sigma_\Delta^2(\overline{Y}_s)ds\Big] =& \int_{\mathbb{R}} f(|a-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(a)\Big(\mathbb{E}\big[L_{t+h}^a(Y)\big]-E\big[L_t^a(Y)\big]\Big)da \\ \le& K_4 h\int_0^\varepsilon f(x)dx. \end{aligned} \tag{8}$$

Next, it follows from Assumption (A4), Lemma 3.3, Proposition 3.2, $2[\frac{p_0}{2}] \ge (4mn+4n\alpha+2n)$ and the Cauchy-Schwarz inequality that

$$\begin{aligned}&\Big(\mathbb{E}\Big[\,\big|\sigma^2(Y_s)-\sigma^2(\overline{Y}_s)\big|^{2n}\,\Big]\Big)^{\frac{1}{2n}} \\ &\le L_3^2\Big(\mathbb{E}\Big[\big(1+|Y_s|^m+|\overline{Y}_s|^m\big)^{2n}|Y_s-\overline{Y}_s|^{(\alpha+\frac12)2n}\big(2\sigma(0)+|Y_s|^{m+\alpha+\frac12}+|Y_s|^{\frac12+\alpha}+|\overline{Y}_s|^{m+\alpha+\frac12}+|\overline{Y}_s|^{\frac12+\alpha}\big)^{2n}\Big]\Big)^{\frac{1}{2n}} \\ &\le K_5\Delta^{\frac14+\frac{\alpha}{2}}.\end{aligned}$$

Therefore,

$$\Big(\mathbb{E}\Big[|\sigma^2(Y_s)-\sigma^2(\overline{Y}_s)|^{\frac{4n}{3}}\Big]\Big)^{\frac{3}{4n}} \le \Big(\mathbb{E}\Big[|\sigma^2(Y_s)-\sigma^2(\overline{Y}_s)|^{2n}\Big]\Big)^{\frac{1}{2n}} \le K_5\Delta^{\frac14+\frac{\alpha}{2}}. \tag{9}$$

Note that $|\sigma_\Delta^2(x)-\sigma^2(x)| \le 2|\sigma(x)|^3\Delta^{\frac12}$. Thus

$$\begin{aligned}&\Big(\mathbb{E}\big[\,\big|\sigma_\Delta^2(\overline{Y}_s)-\sigma^2(\overline{Y}_s)\big|^{\frac{4n}{3}}\,\big]\Big)^{\frac{3}{4n}} = \Big(\mathbb{E}\Big[\Big(\frac{2\Delta^{1/2}|\sigma^3(\overline{Y}_s)|+\Delta\sigma^4(\overline{Y}_s)}{\big[1+\Delta^{1/2}|\sigma(\overline{Y}_s)|\big]^2}\Big)^{\frac{4n}{3}}\Big]\Big)^{\frac{3}{4n}} \\ &\le K_6\Delta^{1/2}\Big(\mathbb{E}\Big[|\sigma^{4n}(\overline{Y}_s)|\Big]\Big)^{\frac{3}{4n}} \le K_7\Delta^{1/2}\Big(\mathbb{E}\Big[\sigma^{4n}(0)+L_3^{4n}|\overline{Y}_s|^{4n(\alpha+\frac12)}\Big(1+|\overline{Y}_s^m|\Big)^{4n}\Big]\Big)^{\frac{3}{4n}} \\ &\le K_8\Delta^{1/2}.\end{aligned} \tag{10}$$

From (8),(9) and (10), for all $\varepsilon \le \varepsilon_0$, we have

$$\begin{aligned}&\mathbb{E}\Big[\int_t^{t+h} f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)ds\Big] \\ \le& \frac{1}{\nu^2}\mathbb{E}\Big[\int_t^{t+h} f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)\sigma^2(Y_s)ds\Big] \\ \le& \frac{1}{\nu^2}\mathbb{E}\Big[\int_t^{t+h} f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)\sigma_\Delta^2(\overline{Y}_s)ds\Big] \\ &+\frac{1}{\nu^2}\mathbb{E}\Big[\int_t^{t+h} f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)\big[\sigma^2(\overline{Y}_s)-\sigma_\Delta^2(\overline{Y}_s)\big]ds\Big] \\ &+\frac{1}{\nu^2}\mathbb{E}\Big[\int_t^{t+h} f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)\big[\sigma^2(Y_s)-\sigma^2(\overline{Y}_s)\big]ds\Big] \\ \le& \frac{K_4h}{\nu^2}\int_0^\varepsilon f(x)dx+\Big(\frac{K_8h^{\frac{3}{4n}}\Delta^{1/2}}{\nu^2}+\frac{K_5\Delta^{\frac14+\frac{\alpha}{2}}h^{\frac{3}{4n}}}{\nu^2}\Big)\sup_{x\in[0,\varepsilon]}f(x)\Big(\int_t^{t+h}\mathbb{P}(Y_s\in[\xi_i-\varepsilon,\xi_i+\varepsilon])ds\Big)^{\frac{4n-3}{4n}} \\ \le& K_9h\int_0^\varepsilon f(x)dx+K_{10}h^{\frac{3}{4n}}\Delta^{\frac14+\frac{\alpha}{2}}\sup_{x\in[0,\varepsilon]}f(x)\Big(\int_t^{t+h}\mathbb{P}(Y_s\in[\xi_i-\varepsilon,\xi_i+\varepsilon])ds\Big)^{\frac{4n-3}{4n}}\end{aligned} \tag{11}$$

where the fourth estimate is obtained by using Hölder inequality .

If $\frac14+\frac{\alpha}{2}+q<1$, then $\frac{\alpha}{2}+\frac14-\frac{3}{4n}<0$. In this case, we get the desired result from (11) since $\mathbb{P}(Y_s \in [\xi_i-\varepsilon,\xi_i+\varepsilon]) \le 1$.

It remains to consider the case when $\frac{1}{4}+\frac{\alpha}{2}+q\geq 1$. By choosing $f=1$ in (11), we get

$$\begin{aligned}&\int_t^{t+h}\mathbb{P}(Y_s\in[\xi_i-\varepsilon,\xi_i+\varepsilon])ds\\ &\leq K_9h\varepsilon+K_{10}h^{\frac{3}{4n}}\Delta^{\frac{1}{4}+\frac{\alpha}{2}}\Big(\int_t^{t+h}\mathbb{P}(Y_s\in[\xi_i-\varepsilon,\xi_i+\varepsilon])ds\Big)^{\frac{4n-3}{4n}}\\ &\leq K_9h\varepsilon+K_{10}h\Delta^{\frac{1}{4}+\frac{\alpha}{2}}.\end{aligned}$$

By applying the above estimate to (11) again, we get

$$\begin{aligned}&\mathbb{E}\Big[\int_t^{t+h}f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)ds\Big]\\ \leq &K_9h\int_0^\varepsilon f(x)dx+K_{10}h^{1-q}\sup_{x\in[0,\varepsilon]}f(x)\Delta^{\frac{1}{4}+\frac{\alpha}{2}}\Big(K_9h\varepsilon+K_{10}h\Delta^{\frac{1}{4}+\frac{\alpha}{2}}\Big)^q\end{aligned}$$

By Hölder's inequality and the fact that $\frac{1}{4}+\frac{\alpha}{2}+q\geq 1$, we have

$$\begin{aligned}\varepsilon^q\Delta^{\frac{1}{4}+\frac{\alpha}{2}}&\leq\frac{q}{q+\frac{\alpha}{2}+\frac{1}{4}}\varepsilon^{q+\frac{\alpha}{2}+\frac{1}{4}}+\frac{\frac{\alpha}{2}+\frac{1}{4}}{q+\frac{\alpha}{2}+\frac{1}{4}}\Delta^{q+\frac{\alpha}{2}+\frac{1}{4}}\\ &\leq\frac{q}{q+\frac{\alpha}{2}+\frac{1}{4}}\varepsilon^{q+\frac{\alpha}{2}+\frac{1}{4}}+\frac{\frac{\alpha}{2}+\frac{1}{4}}{q+\frac{\alpha}{2}+\frac{1}{4}}\Delta.\end{aligned}$$

Therefore,

$$\mathbb{E}\Big[\int_t^{t+h}f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)ds\Big]\leq\quad K_9h\int_0^\varepsilon f(x)dx+hK_{11}\sup_{x\in[0,\varepsilon]}f(x)\Big(\varepsilon^{q+\frac{\alpha}{2}+\frac{1}{4}}+\Delta^{(\frac{1}{4}+\frac{\alpha}{2})(1+q)}+\Delta\Big).\tag{12}$$

We will prove by induction that for any $k\in\mathbb{N}$, there exists a constant c_k such that

$$\mathbb{E}\Big[\int_t^{t+h}f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)ds\Big]\leq K_9h\int_0^\varepsilon f(x)dx+hc_k\sup_{x\in[0,\varepsilon]}f(x)\Big(\varepsilon^{q+\frac{\alpha}{2}+\frac{1}{4}}+\Delta^{(\frac{1}{4}+\frac{\alpha}{2})(1+q+\ldots+q^k)}+\Delta\Big).\tag{13}$$

Indeed, the estimate (12) verifies (13) for $k=1$. Assume that (13) holds for $k=m>1$. Substituting $f=1$ in (13) for $k=m$, we have

$$\int_t^{t+h}\mathbb{P}(Y_s\in[\xi_i-\varepsilon,\xi_i+\varepsilon])ds\leq K_9h\varepsilon+hc_m\big(\varepsilon+\Delta+\Delta^{(\frac{1}{2}+\frac{\alpha}{4})(1+q+\ldots+q^m)}\big).$$

Hence, from (11) and by using Young's inequality, we obtain

$$\begin{aligned}&\mathbb{E}\Big[\int_t^{t+h}f(|Y_s-\xi_i|)1_{[\xi_i-\varepsilon,\xi_i+\varepsilon]}(Y_s)ds\Big]\\ &\leq K_9h\int_0^\varepsilon f(x)dx+K_{10}h^{1-q}\Delta^{\frac{1}{4}+\frac{\alpha}{2}}\sup_{x\in[0,\varepsilon]}f(x)\Big(\int_t^{t+h}\mathbb{P}(Y_s\in[\xi_i-\varepsilon,\xi_i+\varepsilon])ds\Big)^q\\ &\leq K_9h\int_0^\varepsilon f(x)dx+K_{10}h\Delta^{\frac{1}{4}+\frac{\alpha}{2}}\sup_{x\in[0,\varepsilon]}f(x)\Big[(K_{12}+c_m)\varepsilon+c_m\big(\Delta^{(\frac{1}{4}+\frac{\alpha}{2})(1+q+\ldots+q^m)}+\Delta\big)\Big]^q\\ &\leq K_9h\int_0^\varepsilon f(x)dx+hc_{m+1}\sup_{x\in[0,\varepsilon]}f(x)\big(\varepsilon^{q+\frac{\alpha}{2}+\frac{1}{4}}+\Delta^{(\frac{1}{4}+\frac{\alpha}{2})(1+q+\ldots+q^{m+1})}+\Delta\big),\end{aligned}$$

which implies that (13) is true for $k=m+1$. Therefore, by induction, (13) holds for any positive integer k. The desired result follows from the fact that $1+q+q^2+\ldots=\frac{4n}{3}>p$. □

Lemma 3.5. *For any $\beta>0$, there exist positive constants d_1,d_2 and d_3 such that for all $\Delta\leq\Delta_0$,*

i) $\sup_{t>0}\mathbb{P}\Big(|Y_t-Y_{\underline{t}}|\geq\beta\varepsilon_1, Y_{\underline{t}}\in(\Xi^{\varepsilon_1})^c\Big)\leq d_1\Delta^{\beta\log(1/\Delta)}$, *ii)* $\sup_{t>0} \mathbb{P}\Big(|Y_t - Y_{\underline{t}}| \geq \beta d(Y_{\underline{t}}, \Xi), Y_{\underline{t}} \in \Xi^{\varepsilon_1} \setminus \Xi^{\varepsilon_2}\Big) \leq d_2 \Delta^{\beta \log(1/\Delta)}$,

iii) $\sup_{t>0} \mathbb{P}\Big(|Y_t - Y_{\underline{t}}| \geq \beta \varepsilon_2, Y_{\underline{t}} \in \Xi^{\varepsilon_2}\Big) \leq d_3 \Delta^{\beta \log(1/\Delta)}$.

The proof of this Lemma is similar to the one of Lemma 7 in [31], so it shall not be provided here.

Lemma 3.6. *For each $t > 0$, there exists a positive constant C which does not depend on Δ such that for all $h \geq 1, \Delta \leq \Delta_0$,*

$$\mathbb{E}\Big[\int_t^{t+h} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) ds\Big] \leq Ch\Delta^{\frac{1}{4}+\frac{\alpha}{2}}.$$

If $\gamma < 0$, then the constant C does not depend on t either.

Proof. We write

$$\begin{aligned}
&\mathbb{E}\Big[\int_t^{t+h} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) ds\Big] \\
&= \mathbb{E}\Big[\int_t^{t+h} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) 1_{(\Xi^{\varepsilon_1})^c}(Y_{\underline{s}}) ds\Big] + \mathbb{E}\Big[\int_t^{t+h} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) 1_{\Xi^{\varepsilon_1} \setminus \Xi^{\varepsilon_2}}(Y_{\underline{s}}) ds\Big] + \mathbb{E}\Big[\int_t^{t+h} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) 1_{\Xi^{\varepsilon_2}}(Y_{\underline{s}}) ds\Big] \\
&\leq \int_t^{t+h} \mathbb{P}\Big(|Y_s - Y_{\underline{s}}| \geq \varepsilon_1, Y_{\underline{s}} \in (\Xi^{\varepsilon_1})^c\Big) ds + \int_t^{t+h} \mathbb{P}\Big(|Y_s - Y_{\underline{s}}| \geq d(Y_{\underline{s}}, \Xi), Y_{\underline{s}} \in \Xi^{\varepsilon_1} \setminus \Xi^{\varepsilon_2}\Big) ds \\
&\quad + \int_t^{t+h} \mathbb{P}\big(Y_{\underline{s}} \in \Xi^{\varepsilon_2}\big) ds.
\end{aligned}$$

By Lemma 3.5, the first and second terms of the last expression are bounded by $d_1 h\Delta$ and $d_2 h\Delta$, respectively. For the last term, it can be bounded by $H_1 + H_2$, where

$$H_1 = \int_t^{t+h} \mathbb{P}\big(|Y_s - Y_{\underline{s}}| \geq \varepsilon_2, Y_{\underline{s}} \in \Xi^{\varepsilon_2}\big) ds, \quad H_2 = \int_t^{t+h} \mathbb{P}\big(Y_s \in \Xi^{2\varepsilon_2}\big) ds.$$

Note that $H_1 \leq d_3\sqrt{h}\Delta$. By using Lemma 3.4 with $f = 1, \varepsilon = \varepsilon_2$, we have $H_2 \leq 4c\sqrt{h}\varepsilon_2 + ch\Delta^{\frac{1}{4}+\frac{\alpha}{2}}$, where c is a constant not depending on Δ. Moreover, if $\gamma < 0$, then c does not depend on t either. This concludes the proof. ☐

Lemma 3.7. *For each $t > 0$ and $L_0 \in \mathbb{R}$, there exists a positive constant K which does not depend on Δ such that for all $\Delta < \Delta_0$, it holds that*

$$\mathbb{E}\Big[\int_0^t e^{-L_0 s} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) ds\Big] \leq K e^{-L_0 t} \Delta^{\frac{1}{4}+\frac{\alpha}{2}}. \tag{14}$$

When both γ and L_0 are negative, the constant K does not depend on t either.

Proof. It is sufficient to prove (14) for $t \in \mathbb{N}$. For C being the constant in the statement of Lemma 3.6, we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t e^{-L_0 s} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) ds\Big] = \sum_{i=0}^{t-1} \mathbb{E}\Big[\int_i^{i+1} e^{-L_0 s} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) ds\Big] \\
&\leq e^{|L_0|} \sum_{i=0}^{t-1} e^{-L_0(i+1)} \mathbb{E}\Big[\int_i^{i+1} 1_{\mathbb{S}}(Y_s, \overline{Y}_s) ds\Big] \leq C e^{|L_0|} \sum_{i=0}^{t-1} e^{-L_0(i+1)} \Delta^{\frac{1}{4}+\frac{\alpha}{2}} = \quad C e^{|L_0|} \frac{e^{-L_0(t+1)} - e^{-L_1}}{e^{-L_0} - 1} \Delta^{\frac{1}{4}+\frac{\alpha}{2}},
\end{aligned}$$

which implies the desired result. ☐

3.3 Proof of Theorem 2.4

Put $Z_t := X_t - Y_t$. Applying Property (YW3) and Itô's formula to $e^{-L_1 t}\phi_{\delta\varepsilon}(Y_t)$ gives

$$e^{-L_1 t}|Z_t| \leq e^{-L_1 t}\varepsilon + e^{-L_1 t}\phi_{\delta\varepsilon}(Z_t)$$

$$=e^{-L_1 t}\varepsilon+\int_0^t e^{-L_1 s}\left[-L_1\phi_{\delta\varepsilon}(Z_s)+\phi'_{\delta\varepsilon}(Z_s)\left(b(X_s)-b(\overline{Y}_s)\right)+\frac{1}{2}\phi''_{\delta\varepsilon}(Z_s)\left|\sigma(X_s)-\sigma_\Delta(\overline{Y}_s)\right|^2\right]ds$$
$$+\int_0^t e^{-L_1 s}\phi'_{\delta\varepsilon}(Z_s)\left(\sigma(X_s)-\sigma_\Delta(\overline{Y}_s)\right)dW_s. \tag{15}$$

Set $J_1(s)=\phi'_{\delta\varepsilon}(Z_s)\left(b(X_s)-b(\overline{Y}_s)\right)$ and $J_2(s)=\frac{1}{2}\phi''_{\delta\varepsilon}(Z_s)\left|\sigma(X_s)-\sigma_\Delta(\overline{Y}_s)\right|^2$. Firstly, we write

$$J_2(s)=\frac{1}{2}\phi''_{\delta\varepsilon}(Z_s)\left|\sigma(X_s)-\sigma(Y_s)+\sigma(Y_s)-\sigma(\overline{Y}_s)+\sigma(\overline{Y}_s)-\sigma_\Delta(\overline{Y}_s)\right|^2.$$

Using Property (YW5) and Assumption (A4), we have

$$\begin{aligned}J_2(s)&\le\frac{3}{|Z_s|\log\delta}1_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Z_s|)\left(|\sigma(X_s)-\sigma(Y_s)|^2+\left|\sigma(Y_s)-\sigma(\overline{Y}_s)\right|^2+\left|\sigma(\overline{Y}_s)-\sigma_\Delta(\overline{Y}_s)\right|^2\right)\\
&\le\frac{3}{|Z_s|\log\delta}1_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Z_s|)\Big(L_3^2\left(1+|X_s|^m+|Y_s|^m\right)^2|X_s-Y_s|^{1+2\alpha}+\\
&\quad+L_3^2\left(1+|Y_s|^m+|\overline{Y}_s|^m\right)^2|Y_s-\overline{Y}_s|^{1+2\alpha}+\Delta\left|\sigma(\overline{Y}_s)\right|^4\Big)\\
&\le\frac{3}{|Z_s|\log\delta}1_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Z_s|)\Big(3L_3^2\left(1+|X_s|^{2m}+|Y_s|^{2m}\right)|X_s-Y_s|^{1+2\alpha}+\\
&\quad+3L_3^2\left(1+|Y_s|^{2m}+|\overline{Y}_s|^{2m}\right)|Y_s-\overline{Y}_s|^{1+2\alpha}+\Delta\left|\sigma(\overline{Y}_s)\right|^4\Big)\\
&\le\frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1+|X_s|^{2m}+|Y_s|^{2m}\right)+\frac{9L_3^2\delta}{\varepsilon\log\delta}\left(1+|Y_s|^{2m}+|\overline{Y}_s|^{2m}\right)|Y_s-\overline{Y}_s|^{2\alpha+1}\\
&\quad+\frac{3\delta\Delta|\sigma(\overline{Y}_s)|^4}{\varepsilon\log\delta}.\end{aligned}$$

Using the fact that $\left(1+|Y_s|^{2m}+|\overline{Y}_s|^{2m}\right)|Y_s-\overline{Y}_s|^{2\alpha+1}\le\left(1+|Y_s|^{2m}+|\overline{Y}_s|^{2m}\right)^2\Delta^{\frac{1}{2}+\alpha}+|Y_s-\overline{Y}_s|^{4\alpha+2}\Delta^{-\frac{1}{2}-\alpha}$, we have

$$\begin{aligned}J_2(s)&\le\frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1+|X_s|^{2m}+|Y_s|^{2m}\right)+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\left(1+|Y_s|^{2m}+|\overline{Y}_s|^{2m}\right)^2\Delta^{1/2+\alpha}\\
&\quad+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-\frac{1}{2}-\alpha}|Y_s-\overline{Y}_s|^{4\alpha+2}+\frac{C_1\delta\Delta\left(|\overline{Y}_s|^{2+4\alpha+4m}+1\right)}{\varepsilon\log\delta}\\
&\le\frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1+|X_s|^{2m}+|Y_s|^{2m}\right)+\frac{27L_3^2\delta}{2\varepsilon\log\delta}\left(1+|Y_s|^{4m}+|\overline{Y}_s|^{4m}\right)\Delta^{1/2+\alpha}\\
&\quad+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-\frac{1}{2}-\alpha}|Y_s-\overline{Y}_s|^{2+4\alpha}+\frac{C_1\delta\Delta\left(|\overline{Y}_s|^{2+4\alpha+4m}+1\right)}{\varepsilon\log\delta},\end{aligned}\tag{16}$$

for some constant $C_1>0$. Secondly, we write

$$J_1(s)=\phi'_{\delta\varepsilon}(Z_s)\left(b(X_s)-b(Y_s)\right)+\phi'_{\delta\varepsilon}(Z_s)\left(b(Y_s)-b(\overline{Y}_s)\right).$$

Thanks to Properties (YW1), (YW2) and Assumptions (A2), (A3), we have

$$\begin{aligned}J_1(s)\le&\frac{\phi'_{\delta\varepsilon}(|Z_s|)}{|Z_s|}Z_s\left(b(X_s)-b(Y_s)\right)+\left|\phi'_{\delta\varepsilon}(Z_s)\left(b(Y_s)-b(\overline{Y}_s)\right)\right|\\
\le&L_1\phi'_{\delta\varepsilon}(|Z_s|)|Z_s|+|b(Y_s)-b(\overline{Y}_s)|1_{\mathbb{S}}(Y_s,\overline{Y}_s)+|b(Y_s)-b(\overline{Y}_s)|1_{\mathbb{S}^c}(Y_s,\overline{Y}_s)\\
\le&L_1\phi'_{\delta\varepsilon}(|Z_s|)|Z_s|+|b(Y_s)-b(\overline{Y}_s)|1_{\mathbb{S}}(Y_s,\overline{Y}_s)+L_2(1+|Y_s|^l+|\overline{Y}_s|^l)|Y_s-\overline{Y}_s|.\end{aligned}$$

It follows from Remark 2.1 that

$$\begin{aligned}J_1(s)&\le L_1\phi'_{\delta\varepsilon}(|Z_s|)|Z_s|+C_21_{\mathbb{S}}(Y_s,\overline{Y}_s)+C_2(1+|Y_s|^l+|\overline{Y}_s|^l)|Y_s-\overline{Y}_s| \qquad (17)\\
&\le L_1\phi'_{\delta\varepsilon}(|Z_s|)|Z_s|+C_21_{\mathbb{S}}(Y_s,\overline{Y}_s)+\frac{3}{2}C_2\Delta^{1/2}\left(1+|Y_s|^{2l}+|\overline{Y}_s|^{2l}\right)+\frac{1}{2}C_2\Delta^{-1/2}|Y_s-\overline{Y}_s|^2,\end{aligned}$$

where the constant $C_2>0$ depends on $L_2, n, \xi_i, b(\xi_i+), b(\xi_i-)$.

From (15),(16), and the property $-L_1\phi_{\delta\varepsilon}(x)+L_1\phi'_{\delta\varepsilon}(|x|)|x| \leq \max\{L_1\varepsilon;0\}$,

$$\begin{aligned}
&\mathbb{E}\left[e^{-L_1t}|Z_t|\right]\\
&\leq e^{-L_1t}\varepsilon+\int_0^t e^{-L_1s}\Bigg[\max\{L_1\varepsilon;0\}+\frac{3}{2}C_2\Delta^{1/2}\left(1+\mathbb{E}\left[|Y_s|^{2l}\right]+\mathbb{E}\left[|\overline{Y}_s|^{2l}\right]\right)\\
&+\frac{1}{2}C_2\Delta^{-1/2}\mathbb{E}\left[|Y_s-\overline{Y}_s|^2\right]+C_2 1_S(Y_s,\overline{Y}_s)+\frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1+\mathbb{E}\left[|X_s|^{2m}\right]+\mathbb{E}\left[|Y_s|^{2m}\right]\right)\\
&+\frac{27L_3^2\delta}{2\varepsilon\log\delta}\left(1+\mathbb{E}\left[|Y_s|^{4m}\right]+\mathbb{E}\left[|\overline{Y}_s|^{4m}\right]\right)\Delta^{1/2+\alpha}+\frac{9L_3^2\delta}{2\varepsilon\log\delta}\Delta^{-1/2-\alpha}\mathbb{E}\left[|Y_s-\overline{Y}_s|^{2+4\alpha}\right]\\
&+\frac{C_1\delta\Delta\left(\mathbb{E}\left[|\overline{Y}_s|^{2+4\alpha+4m}\right]+1\right)}{\varepsilon\log\delta}\Bigg]ds.
\end{aligned}$$

Thanks to the condition $p_0\geq(2l+2)\vee(2+4\alpha+4m)$, Proposition 3.2, Proposition 3.1, and Lemma 3.3, there exists a constant $C>0$, which does not depend on Δ, such that for any $0\leq t\leq T$, it holds that

$$\mathbb{E}\left[e^{-L_1t}|Z_t|\right]\leq e^{-L_1t}\varepsilon+C\left[\varepsilon+\Delta^{\frac{1}{2}}+\Delta+\Delta^{\frac{1}{4}+\frac{\alpha}{2}}+\frac{\varepsilon^{2\alpha}}{\log\delta}+\frac{\delta\Delta^{1/2+\alpha}}{\varepsilon\log\delta}+\frac{\delta\Delta}{\varepsilon\log\delta}\right]\int_0^t e^{-L_1s}ds.$$

If $\alpha\in(0,\frac{1}{2}]$, by choosing $\varepsilon=\Delta^{\frac{1}{2}}$, $\delta=2$, we obtain $\sup_{t\leq T}\mathbb{E}\left[|Z_t|\right]\leq C\Delta^{\alpha}$.

If $\alpha=0$, by choosing $\varepsilon=\Delta^{\frac{1}{4}}$, $\delta=\Delta^{-\frac{1}{4}}$, we obtain $\sup_{t\leq T}\mathbb{E}\left[|Z_t|\right]\leq\frac{C}{\log\frac{1}{\Delta}}$.

Moreover, if $\gamma, L_1<0$, the constant C does not depend on T. We conclude the proof of Theorem 2.4.

3.4 Proof of Theorem 2.5

3.4.1 Control drift function

In [25], the authors used the function φ, which is a solution to the equation $b\varphi'+\frac{1}{2}\sigma^2\varphi''=0$ to handle the discontinuity of the drift coefficient b when it is no longer one-sided Lipschitz. Our assumptions on the boundedness of b and σ are not as strict as those in [25]. Moreover, we want to show the convergence of the approximation scheme on the whole interval $(0,+\infty)$. Therefore, we will modify the approach in [25] by introducing a new function φ defined as follows.

First, we consider the following properties on the drift coefficient b.

(P_b1): $b(y)\geq0$ for all $y>\xi_k$.

(P_b2): $b(y)\leq0$ for all $y<\xi_1$.

If Property (P_b1) does not hold, we choose $\xi_{k+1}>\xi_k$ such that $b(\xi_{k+1})<0$. Otherwise, we choose $\xi_{k+1}=\xi_k+1$.

If Property (P_b2) does not hold, we choose $\xi_0<\xi_1$ such that $b(\xi_0)>0$. Otherwise, we choose $\xi_0=\xi_1-1$.

Note that if $L_1<0$ then it follows from Assumption (A'2) that neither Property (P_b1) nor Property (P_b2) holds, which implies that $b(\xi_0)>0>b(\xi_{k+1})$. Also, note that b is continuous at ξ_0 and ξ_{k+1}.

Next, we define a function $\varphi\in C^1(\mathbb{R})$ as follows:

- For $y\in[\xi_0,\xi_{k+1}]$,

$$\varphi(y)=b(\xi_0)+\int_{\xi_0}^y\exp\left(\int_{\xi_0}^x\frac{-2b(t)}{\sigma^2(t)}\,dt\right)\left[\int_{\xi_0}^x\exp\left(\int_{\xi_0}^t\frac{2b(s)}{\sigma^2(s)}\,ds\right)\frac{2R(t)}{\sigma^2(t)}\,dt+K\right]dx,$$

where the constant K is chosen such that

$$K>2\int_{\xi_0}^{\xi_{k+1}}\left|\exp\left(\int_{\xi_0}^t\frac{2b(s)}{\sigma^2(s)}\,ds\right)\frac{2R(t)}{\sigma^2(t)}\right|dt+2\exp\left(\int_{\xi_0}^{\xi_{k+1}}\frac{|2b(s)|}{\sigma^2(s)}\,ds\right)+2,$$

and

$$R(x)=b(\xi_0)+\frac{(x-\xi_0)(b(\xi_{k+1})-b(\xi_0))}{\xi_{k+1}-\xi_0},$$

for any $x\in(\xi_0,\xi_{k+1})$.

- For $y \in (-\infty, \xi_0)$,

$$\varphi(y) = \begin{cases} \varphi(\xi_0) - \int_y^{\xi_0} \left(1 + [\varphi'(\xi_0) - 1] \exp\left(\int_{\xi_0}^x \frac{-2b(s)}{\sigma^2(s)} ds\right)\right) dx & \text{if } (P_b2) \text{ holds,} \\ b(\xi_0) + (y - \xi_0)\varphi'(\xi_0) & \text{otherwise.} \end{cases}$$

- For $y \in (\xi_{k+1}, +\infty)$, then

$$\varphi(y) = \begin{cases} \varphi(\xi_{k+1}) + \int_{\xi_{k+1}}^y \left(1 + [\varphi'(\xi_{k+1}) - 1] \exp\left(\int_{\xi_{k+1}}^x \frac{-2b(s)}{\sigma^2(s)} ds\right)\right) dx & \text{if } (P_b1) \text{ holds,} \\ \varphi(\xi_{k+1}) + (x - \xi_{k+1})\varphi'(\xi_{k+1}) & \text{otherwise,} \end{cases}$$

We show some useful properties of the function φ.

Lemma 3.8. *Under the assumption of Theorem 2.5, the function φ satisfies*

(P1) $\varphi \in C^1(\mathbb{R})$,

(P2) There exists a positive constant H such that $1 \le \varphi'(x) \le H$ for any $x \in \mathbb{R}$,

(P3) φ'' exists and is bounded on $\mathbb{R}\backslash\Xi$,

(P4) $\varphi'(x)b(x) + \frac{1}{2}\varphi''(x)\sigma^2(x) = \Psi(x)$ for any $x \in \mathbb{R}\backslash\Xi$, where

$$\Psi(x) = \begin{cases} R(x) & \text{if } \ x \in (\xi_0, \xi_{k+1})\backslash\Xi, \\ \varphi'(\xi_0)b(x) & \text{if } x \in (-\infty, \xi_0] \quad \text{and } (P_2b) \text{ does not hold,} \\ b(x) & \text{if } x \in (-\infty, \xi_0] \quad \text{and } (P_2b) \text{ hold,} \\ \varphi'(\xi_{k+1})b(x) & \text{if } x \in [\xi_{k+1}, \infty) \text{ and } (P_1b) \text{ does not hold,} \\ b(x) & \text{if } x \in [\xi_{k+1}, \infty) \text{ and } (P_1b) \text{ holds.} \end{cases} \tag{18}$$

Proof. One can proves Properties (P1),(P2) and (P4) easily. To verify Property (P3), we will show that $\sup_{x>\xi_{k+1}} |\varphi''(x)| < +\infty$. The proof for the fact that $\sup_{x<\xi_0} |\varphi''(x)| < +\infty$ is similar.

Case 1: Suppose (A'2) holds for some $L_1 < 0$ then Property (P_b1) does not hold, the result follows straightforward from the fact that $\varphi''(x) = 0$ for any $x \ge \xi_{k+1}$.

Case 2: Suppose that both (A6) and (P_b1) hold. Then for any $x > \max\{\xi' + h, \xi_{k+1} + h\}$

$$\begin{aligned} |\varphi''(x)| &= [\varphi'(\xi_{k+1}) - 1]\frac{2b(x)}{\sigma^2(x)} \exp\left(\int_{\xi_{k+1}}^x \frac{-2b(s)}{\sigma^2(s)} ds\right) \\ &\le [\varphi'(\xi_{k+1}) - 1]\frac{2b(x)}{\sigma^2(x)} \exp\left(\int_{x-h}^x \frac{-2b(s)}{\sigma^2(s)} ds\right). \end{aligned}$$

By the mean value theorem, there exists $\xi \in [x - h, x]$ such that

$$\begin{aligned} |\varphi''(x)| \le & [\varphi'(\xi_{k+1}) - 1]\frac{2b(x)}{\sigma^2(x)} \exp\left(h\frac{-2b(\xi)}{\sigma^2(\xi)}\right) \\ = & [\varphi'(\xi_{k+1}) - 1]\frac{2b(\xi)}{\sigma^2(\xi)} \exp\left(h\frac{-2b(\xi)}{\sigma^2(\xi)}\right) \\ & + [\varphi'(\xi_{k+1}) - 1] \exp\left(h\frac{-2b(\xi)}{\sigma^2(\xi)}\right) \frac{2b(\xi)}{\sigma^2(\xi)}\left(\frac{\sigma^2(\xi)}{\sigma^2(x)} - 1\right) \\ & + [\varphi'(\xi_{k+1}) - 1] \exp\left(h\frac{-2b(\xi)}{\sigma^2(\xi)}\right) \frac{(b(x) - b(\xi))(x - \xi)}{\sigma^2(\xi)(x - \xi)}. \end{aligned}$$

Note that $\sup_{x>0} xe^{-hx} < +\infty$. Using this fact and Assumptions (A2), (A5), and (A6), we obtain the desired result. ☐

We can further see that the function Ψ defined in (18) is one-sided Lipschitz continuous, i.e. there exists a constant l_Ψ such that

$$(x-y)(\Psi(x)-\Psi(y)) \le l_\Psi (x-y)^2, \text{ for all } x,y\in\mathbb{R}.$$

We denote $\bar{\mathbb{S}} := \cup_{i=0}^k (\xi_i,\xi_{i+1})^2 \cup (-\infty,\xi_0)^2 \cup (\xi_{k+1},+\infty)^2$, and

$$L_\Psi = \begin{cases} \frac{l_\Psi}{H} & \text{if } l_\Psi < 0, \\ l_\Psi & \text{if } l_\Psi \ge 0. \end{cases} \tag{19}$$

3.4.2 Proof of Theorem 2.5

First, we have $|Z_t| = |X_t - Y_t| \le |\varphi(X_t)-\varphi(Y_t)| \le \varepsilon + \phi_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))$. By using Itô's formula and Property (P4), we have

$$e^{-L_\Psi t}\phi_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t)) = J_1+J_2+J_3+J_4+J_5+J_6, \tag{20}$$

where

$$\begin{aligned}
J_1 &= \int_0^T -L_\Psi e^{-L_\Psi t}\phi_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\,dt,\\
J_2 &= \int_0^T e^{-L_\Psi t}\phi'_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\,(\Psi(X_t)-\Psi(Y_t))\;dt,\\
J_3 &= \int_0^T e^{-L_\Psi t}\phi'_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\varphi'(Y_t)(b(Y_{\underline{t}})-b(Y_t))\right]\;dt,\\
J_4 &= \int_0^T e^{-L_\Psi t}\phi'_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\frac{1}{2}\sigma_\Delta^2(Y_{\underline{t}})\varphi''(Y_t)-\frac{1}{2}\sigma^2(Y_t)\varphi''(Y_t)\right]\;dt,\\
J_5 &= \int_0^T \frac{1}{2}e^{-L_\Psi t}\phi''_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\varphi'(X_t)\sigma(X_t)-\varphi'(Y_t)\sigma_\Delta(Y_{\underline{t}})\right]^2\;dt,\\
J_6 &= \int_0^T e^{-L_\Psi t}\phi'_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\varphi'(X_t)b(X_t)-\varphi'(Y_t)b(Y_{\underline{t}})\right]\;dW_t.
\end{aligned}$$

We now give a bound for each term on the left-hand side of (20). First, by Property (YW1), we have

$$\begin{aligned}
\phi'_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\,(\Psi(X_t)-\Psi(Y_t)) &= \frac{\phi'_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))}{\varphi(X_t)-\varphi(Y_t)}\,(\varphi(X_t)-\varphi(Y_t))\,(\Psi(X_t)-\Psi(Y_t))\\
&= \frac{\phi'_{\delta\varepsilon}(|\varphi(X_t)-\varphi(Y_t)|)}{|\varphi(X_t)-\varphi(Y_t)|}\,\frac{\varphi(X_t)-\varphi(Y_t)}{X_t-Y_t}(X_t-Y_t)\,(\Psi(X_t)-\Psi(Y_t))\\
&\le \frac{\phi'_{\delta\varepsilon}(|\varphi(X_t)-\varphi(Y_t)|)}{|\varphi(X_t)-\varphi(Y_t)|}\,\frac{\varphi(X_t)-\varphi(Y_t)}{X_t-Y_t}l_\Psi(X_t-Y_t)^2\\
&= l_\Psi\frac{\phi'_{\delta\varepsilon}(|\varphi(X_t)-\varphi(Y_t)|)}{|\varphi(X_t)-\varphi(Y_t)|}\,(\varphi(X_t)-\varphi(Y_t))\,(X_t-Y_t).
\end{aligned}$$

Using the definition of L_Ψ in (19) and Property (P2), we get

$$\begin{aligned}
\phi'_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\,(\Psi(X_t)-\Psi(Y_t)) &\le L_\Psi\frac{\phi'_{\delta\varepsilon}(|\varphi(X_t)-\varphi(Y_t)|)}{|\varphi(X_t)-\varphi(Y_t)|}\,(\varphi(X_t)-\varphi(Y_t))^2\\
&= L_\Psi\phi'_{\delta\varepsilon}(|\varphi(X_t)-\varphi(Y_t)|)|\varphi(X_t)-\varphi(Y_t)|.
\end{aligned}$$

Since $-L_\Psi\phi_{\delta\varepsilon}(x)+L_\Psi|x|\phi'_{\delta\varepsilon}(|x|)\le\max\{L_\Psi\varepsilon,0\}$, we get

$$\begin{aligned}
J_1+J_2 &= \int_0^T\left[-L_\Psi e^{-L_\Psi t}\phi_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))+e^{-L_\Psi t}\phi'_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))\,(\Psi(X_t)-\Psi(Y_t))\right]\,dt\\
&\le \int_0^T \max\{L_\Psi\varepsilon,0\}e^{-L_\Psi t}dt.
\end{aligned}$$

For the rest of the proof, we denote by $K_1, K_2, ...$ some constants that do not depend on Δ. Moreover, when $\gamma<0$, these constants do not depend on t either.

Using Properties (YW2),(P2), Assumption (A3), and Remark 2.1, we get

$$
\begin{aligned}
&\left|\phi'_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))\varphi'(Y_t)(b(Y_{\underline{t}})-b(Y_t))\right| \le H\left|b(Y_{\underline{t}})-b(Y_t)\right| \\
&= H\left|b(Y_{\underline{t}})-b(Y_t)\right| 1_{\bar{\mathbb{S}}}(Y_t,Y_{\underline{t}}) + H\left|b(Y_{\underline{t}})-b(Y_t)\right| 1_{\mathbb{R}^2\setminus\bar{\mathbb{S}}}(Y_t,Y_{\underline{t}}) \\
&\le K_1\left(1_{\bar{\mathbb{S}}}(Y_t,Y_{\underline{t}}) + (1+|Y_t|^l+|Y_{\underline{t}}|^l)|Y_t-Y_{\underline{t}}|\right).
\end{aligned}
$$

Then, Proposition 3.2, Lemma 3.3 and Lemma 3.7 give

$$
\mathbb{E}[J_3] \le K_1\int_0^T e^{-L_\Psi t}\mathbb{E}\left(1_{\bar{\mathbb{S}}}(Y_t,Y_{\underline{t}}) + (1+|Y_t|^l+|Y_{\underline{t}}|^l)|Y_t-Y_{\underline{t}}|\right)dt \le K_2\int_0^T e^{-L_\Psi t}\Delta^{\frac{1}{4}+\frac{\alpha}{2}}\,dt.
$$

For J_4, we first note that

$$
\begin{aligned}
&\left|\phi'_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\sigma^2_\Delta(Y_{\underline{t}})\varphi''(Y_t)-\sigma^2(Y_t)\varphi''(Y_t)\right]\right| \\
&\le K_3\left[(1+|Y_t|^{2m+\alpha+1/2}+|Y_{\underline{t}}|^{2m+\alpha+1/2})|Y_t-Y_{\underline{t}}|^{1/4+\alpha/2} + (1+|Y_{\underline{t}}|^{2m+2\alpha+1})\Delta^{1/2}\right];
\end{aligned}
$$

hence, by Proposition 3.2 and Lemma 3.3

$$
\begin{aligned}
\mathbb{E}[J_4] \le& \int_0^T \mathbb{E}\left[\left|e^{-L_\Psi t}\phi'_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left(\frac{1}{2}\sigma^2_\Delta(Y_{\underline{t}})\varphi''(Y_t)-\frac{1}{2}\sigma^2(Y_t)\varphi''(Y_t)\right)\right|\right]dt \\
\le& K_4\int_0^T e^{-L_\Psi t}\mathbb{E}\left[(1+|Y_t|^{2m+\alpha+1/2}+|Y_{\underline{t}}|^{2m+\alpha+1/2})|Y_t-Y_{\underline{t}}|^{1/2+\alpha} + (1+|Y_{\underline{t}}|^{2m+2\alpha+1})\Delta^{1/2}\right]dt \\
\le& K_5\int_0^T e^{-L_\Psi t}(\Delta^{1/4+\alpha/2}+\Delta^{1/2})\,dt.
\end{aligned}
$$

For J_5, using the estimate $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, we get

$$
\begin{aligned}
&\int_0^T e^{-L_\Psi t}\phi''_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\varphi'(X_t)\sigma(X_t)-\varphi'(Y_t)\sigma_\Delta(Y_{\underline{t}})\right]^2 dt \\
&\le 4\int_0^T e^{-L_\Psi t}\phi''_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\varphi'(X_t)\right]^2\left[\sigma(X_t)-\sigma(Y_t)\right]^2 dt \\
&+4\int_0^T e^{-L_\Psi t}\phi''_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\varphi'(Y_t)\right]^2\left[\sigma(Y_t)-\sigma(Y_{\underline{t}})\right]^2 dt \\
&+4\int_0^T e^{-L_\Psi t}\phi''_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))\left[\varphi'(Y_t)\right]^2\left[\sigma_\Delta(Y_{\underline{t}}-\sigma(Y_{\underline{t}})\right]^2 dt \\
&+4\int_0^T e^{-L_\Psi t}\phi''_{\delta_\varepsilon}(\varphi(X_t)-\varphi(Y_t))\sigma^2(Y_t)\left[\varphi'(X_t)-\varphi'(Y_t)\right]^2 dt \\
&:= 4(J_{5,1}+J_{5,2}+J_{5,3}+J_{5,4}).
\end{aligned}
$$

Using Properties (YW5), (P2), Assumption (A4), Proposition 3.2 and Proposition 3.1, we get

$$
\begin{aligned}
\mathbb{E}[J_{5,1}] \le& H^2\int_0^T e^{-L_\Psi t}\mathbb{E}\left[\phi''_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))|\varphi(X_t)-\varphi(Y_t)|\frac{\left[\sigma(X_t)-\sigma(Y_t)\right]^2}{|\varphi(X_t)-\varphi(Y_t)|}\right]dt \\
\le& K_6\int_0^T e^{-L_\Psi t}\mathbb{E}\left[\frac{2}{\log\delta}1_{\{|\varphi(X_t)-\varphi(Y_t)|\le\varepsilon\}}\frac{|X_t-Y_t|^{1+2\alpha}(1+|X_t|^{2m}+|Y_t|^{2m})}{|\varphi(X_t)-\varphi(Y_t)|}\right]dt \\
\le& K_7\int_0^T e^{-L_\Psi t}\frac{\varepsilon^{2\alpha}}{\log\delta}\mathbb{E}\left[(1+|X_t|^{2m}+|Y_t|^{2m})\right]dt \\
\le& K_8\int_0^T e^{-L_\Psi t}\frac{\varepsilon^{2\alpha}}{\log\delta}\,dt,
\end{aligned}
$$

and

$$
\mathbb{E}[J_{5,4}] = \int_0^T e^{-L_\Psi t}\mathbb{E}\left[\phi''_{\delta\varepsilon}(\varphi(X_t)-\varphi(Y_t))|\varphi(X_t)-\varphi(Y_t)|\sigma^2(Y_t)\frac{|\varphi'(X_t)-\varphi'(Y_t)|^2}{|\varphi(X_t)-\varphi(Y_t)|^2}|\varphi(X_t)-\varphi(Y_t)|\right]dt
$$

$$\leq K_9 \int_0^T e^{-L_\Psi t} \mathbb{E}\left[\frac{2}{\log\delta} 1_{\{|\varphi(X_t)-\varphi(Y_t)|\leq\varepsilon\}} |\varphi(X_t) - \varphi(Y_t)|\sigma^2(Y_t)\right] dt$$
$$\leq K_{10} \int_0^T e^{-L_\Psi t} \frac{2}{\log\delta} \varepsilon \mathbb{E}\left[\sigma^2(Y_t)\right] dt$$
$$\leq K_{11} \int_0^T e^{-L_\Psi t} \frac{2\varepsilon}{\log\delta} dt.$$

Using Properties (YW5), (P2), Assumption (A4), Proposition 3.2, and Lemma 3.3, we get

$$\mathbb{E}[J_{5,2}] \leq K_{12} \int_0^T e^{-L_\Psi t} \mathbb{E}\left[\left(\sigma(Y_t) - \sigma(Y_{\underline{t}})\right)^2 \frac{2\delta}{\varepsilon\log\delta}\right] dt \leq K_{13} \int_0^T e^{-L_\Psi t} \Delta^{\alpha+1/2} \frac{2\delta}{\varepsilon\log\delta} dt.$$

Using similar estimates as in (10), we get

$$\mathbb{E}[J_{5,3}] \leq K_{14} \int_0^T e^{-L_\Psi t} \mathbb{E}\left[\frac{2\delta}{\varepsilon\log\delta} \left(\sigma_\Delta(Y_{\underline{t}}) - \sigma(Y_{\underline{t}})\right)^2\right] dt \leq K_{15} \int_0^T e^{-L_\Psi t} \frac{2\delta}{\varepsilon\log\delta} \Delta\, dt.$$

Finally, $\mathbb{E}[J_6] = 0$. To sum up,

$$\begin{aligned}\mathbb{E}[e^{-L_\Psi T}|Z_T|] &\leq \mathbb{E}[J_1 + J_2 + J_3 + J_4 + J_5 + J_6] + e^{-L_\Psi T}\varepsilon \\ &\leq K_{16} \int_0^T e^{-L_\Psi t} \left[\varepsilon + \Delta^{1/2} + \Delta^{\frac{1}{4}+\frac{\alpha}{2}} + \frac{\varepsilon^{2\alpha}}{\log\delta} + \frac{\delta\Delta^{1/2+\alpha}}{\varepsilon\log\delta} + \frac{\delta\Delta}{\varepsilon\log\delta} + \frac{2\varepsilon}{\log\delta}\right] dt.\end{aligned}$$

If $\alpha \in (0, \frac{1}{2}]$, then by choosing $\varepsilon = \Delta^{1/2}$ and $\delta = 2$, we obtain

$$\mathbb{E}[|Z_T|] \leq C \frac{e^{L_\Psi T} - 1}{L_\Psi} \Delta^\alpha. \tag{21}$$

If $\alpha = 0$, then by choosing $\varepsilon = \Delta^{1/4}$ and $\delta = \Delta^{-1/4}$, we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\left[|Z_t|\right] \leq C \frac{e^{L_\Psi T} - 1}{L_\Psi} \frac{1}{\log\frac{1}{\Delta}}. \tag{22}$$

Recall that the constants C in (21) and (22) do not depend on T when L_1, γ and L_Ψ are negative. This implies the desired result.

3.5 Proof of Theorem 2.6

Proof. We first note that

$$N_T = 1 + \sum_{k=1}^\infty 1_{\{t_k<T\}} = 1 + \sum_{k=1}^\infty 1_{\{t_k<T\}} \int_{t_{k-1}}^{t_k} \frac{1}{h_\Delta(\overline{Y}_s)} ds = 1 + \int_0^T \frac{1}{h_\Delta(\overline{Y}_s)} ds.$$

We write $\mathbb{E}\Big[\int_0^T \frac{1}{h_\Delta(\overline{Y}_s)} ds\Big] = I_1 + I_2 + I_3$, where $I_1 = \mathbb{E}\Big[\int_0^T \frac{1}{h_\Delta(\overline{Y}_s)} 1_{(\Xi^{\varepsilon_1})^c}(Y_{\underline{s}}) ds\Big]$, $I_2 = \mathbb{E}\Big[\int_0^T \frac{1}{h_\Delta(\overline{Y}_s)} 1_{\Xi^{\varepsilon_1}\setminus\Xi^{\varepsilon_2}}(Y_{\underline{s}}) ds\Big]$, $I_3 = \mathbb{E}\Big[\int_0^T \frac{1}{h_\Delta(\overline{Y}_s)} 1_{\Xi^{\varepsilon_2}}(Y_{\underline{s}}) ds\Big]$.

For the rest of the proof, we denote by $K_1, K_2, ...$ some constants that do not depend on Δ. Moreover, when $\gamma < 0$, these constants do not depend on t either. It follows from (2) and Proposition 3.2 that

$$I_1 \leq \mathbb{E}\Big[\int_0^T \frac{\left[1 + |b(Y_{\underline{s}})| + |\sigma(Y_{\underline{s}})| + |Y_{\underline{s}}|^l\right]^2}{\Delta} ds\Big] \leq K_1 T \Delta^{-1}.$$

Thanks to Assumptions (A3) and (A4), $\sup_{y\in\Xi^{\varepsilon_2}} \left[1 + |b(y)| + |\sigma(y)| + |y|^l\right]^2 < +\infty$. It follows from (2) that

$$I_3 = \mathbb{E}\Big[\int_0^T \frac{\left[1 + b(Y_{\underline{s}}) + \sigma(Y_{\underline{s}}) + |Y_{\underline{s}}|^l\right]^2}{\Delta^2 \log^4(1/\Delta)} 1_{\Xi^{\varepsilon_2}}(Y_{\underline{s}}) ds\Big] \leq \frac{K_2}{\Delta^2 \log^4(1/\Delta)} \int_0^T \mathbb{P}\Big(Y_{\underline{s}} \in \Xi^{\varepsilon_2}\Big) ds. \tag{23}$$

Similarly, we have

$$I_2 = \mathbb{E}\Big[\int_0^T \frac{\log^4(1/\Delta)\big[1+|b(Y_{\underline{s}})|+|\sigma(Y_{\underline{s}})|+|Y_{\underline{s}}|^l\big]^2}{\big[d(Y_{\underline{s}},\Xi)\big]^2} 1_{\Xi^{\varepsilon_1}\setminus\Xi^{\varepsilon_2}}(Y_{\underline{s}})ds\Big]$$
$$\leq K_3\log^4(1/\Delta)\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_{\underline{s}},\Xi)\big]^2} 1_{\Xi^{\varepsilon_1}\setminus\Xi^{\varepsilon_2}}(Y_{\underline{s}})ds\Big] = K_3\log^4(1/\Delta)(I_{21}+I_{22}),$$

where

$$I_{2,1} = \mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_{\underline{s}},\Xi)\big]^2} 1_{\Xi^{\varepsilon_1}\setminus\Xi^{\varepsilon_2}}(Y_{\underline{s}})1_{\{|Y_s-Y_{\underline{s}}|\geq\frac{1}{2}d(Y_{\underline{s}},\Xi)\}}ds\Big],$$
$$I_{2,2} = \mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_{\underline{s}},\Xi)\big]^2} 1_{\Xi^{\varepsilon_1}\setminus\Xi^{\varepsilon_2}}(Y_{\underline{s}})1_{\{|Y_s-Y_{\underline{s}}|<\frac{1}{2}d(Y_{\underline{s}},\Xi)\}}ds\Big].$$

By applying Lemma 3.5.ii, for all $\Delta < \Delta_0$ we obtain

$$I_{2,1} \leq \frac{1}{\varepsilon_2^2}\int_0^T \mathbb{P}\Big(|Y_s - Y_{\underline{s}}| \geq \frac{1}{2}d(Y_{\underline{s}},\Xi), Y_{\underline{s}}\in\Xi^{\varepsilon_1}\setminus\Xi^{\varepsilon_2}\Big)ds \leq K_4\frac{T}{\Delta^2\log^8(1/\Delta)}\Delta^{\frac{1}{2}\log(1/\Delta)}.$$

Note that $|Y_s - Y_{\underline{s}}| < \frac{1}{2}d(Y_{\underline{s}},\Xi)$ implies $\frac{1}{2}d(Y_{\underline{s}},\Xi) \leq d(Y_s,\Xi) \leq \frac{3}{2}d(Y_{\underline{s}},\Xi)$. Hence, if $\varepsilon_2 \leq d(Y_{\underline{s}},\Xi) \leq \varepsilon_1$, then $\frac{1}{2}\varepsilon_2 \leq d(Y_s,\Xi) \leq \frac{3}{2}\varepsilon_1$. Hence

$$I_{2,2} \leq \frac{9}{4}\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\frac{3}{2}\varepsilon_1}\setminus\Xi^{\frac{1}{2}\varepsilon_2}}(Y_s)ds\Big].$$

Now we will consider two cases.

Case 1: $[\frac{p_0}{2}] > \frac{3}{2\alpha+1}(1+2\alpha+2m)$.

We can choose n such that $n > \frac{3}{2\alpha+1}$ and $[\frac{p_0}{2}] > \frac{3}{2\alpha+1}(1+2\alpha+2m)n$, and then choose $p \in (\frac{4}{2\alpha+1}, \frac{4n}{3})$. Let $q = \frac{4n-3}{4n}$. Define the sequence a_i as follow: $a_0 = \frac{1}{2}$, and $a_{i+1} = \frac{1}{2}(a_i(q+\frac{\alpha}{2}+\frac{1}{4})+1)$ for $i \geq 0$. Let $\varepsilon_{a_i} := \Delta^{a_i}\log^{4a_i}(1/\Delta)$. Since $q+\frac{\alpha}{2}+\frac{1}{4} > 1$, we have $\lim_{i\to\infty} a_i > 1$. Thus, there exists a positive integer h such that $a_h \leq 1 < a_{h+1}$. Then we write

$$I_{2,2} \leq \frac{9}{4}\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\frac{3}{2}\varepsilon_1}\setminus\Xi^{\varepsilon_{a_1}}}(Y_s)ds\Big] + \frac{9}{4}\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\varepsilon_{a_h}}\setminus\Xi^{\frac{1}{2}\varepsilon_2}}(Y_s)ds\Big]$$
$$+\frac{9}{4}\sum_{i=1}^{h-1}\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\varepsilon_{a_i}}\setminus\Xi^{\varepsilon_{a_{i+1}}}}(Y_s)ds\Big]$$
$$\leq \frac{9}{4}\sum_{i=0}^{h-1}\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\frac{3}{2}\varepsilon_{a_i}}\setminus\Xi^{\frac{1}{2}\varepsilon_{a_{i+1}}}}(Y_s)ds\Big] + \frac{9}{4}\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\frac{3}{2}\varepsilon_{a_h}}\setminus\Xi^{\frac{1}{2}\varepsilon_2}}(Y_s)ds\Big].$$

Applying Lemma 3.4 with $f(x) = \frac{1}{(\max\{\frac{1}{2}\varepsilon_{a_{i+1}},x\})^2}$, $\varepsilon = \frac{3}{2}\varepsilon_{a_i}$, we have that for each $i = 0,\dots,h-1$,

$$\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\frac{3}{2}\varepsilon_{a_i}}\setminus\Xi^{\frac{1}{2}\varepsilon_{a_{i+1}}}}(Y_s)ds\Big] \leq \mathbb{E}\Big[\int_0^T \frac{1}{\max\big\{\frac{1}{2}\varepsilon_{a_{i+1}}, d(Y_s,\Xi)\big\}^2} 1_{\Xi^{\frac{3}{2}\varepsilon_{a_i}}\setminus\Xi^{\frac{1}{2}\varepsilon_{a_{i+1}}}}(Y_s)ds\Big]$$
$$\leq K_5T\int_0^{\frac{3}{2}\varepsilon_{a_i}} \frac{1}{(\max\{\frac{1}{2}\varepsilon_{a_{i+1}},x\})^2}dx + K_5T\frac{4}{\varepsilon_{a_{i+1}}^2}\Big(\big(\frac{3}{2}\varepsilon_{a_i}\big)^{q+\frac{\alpha}{2}+\frac{1}{4}} + \Delta + \Delta^{(\frac{1}{4}+\frac{\alpha}{2})p}\Big)$$
$$\leq K_6T(\Delta^{-1}+\Delta^{(\frac{1}{4}+\frac{\alpha}{2})p-2})\frac{1}{\log^4(1/\Delta)}$$
$$\leq 2K_6T\Delta^{-1}\frac{1}{\log^4(1/\Delta)},$$

where the last estimate follows from the fact that $(\frac{1}{4}+\frac{\alpha}{2})p > 1$.
Similarly, applying Lemma 3.4 with $f(x) = \frac{1}{(\max\{\frac{1}{2}\varepsilon_2, x\})^2}$, $\varepsilon = \frac{3}{2}\varepsilon_{a_h}$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\frac{3}{2}\varepsilon_{a_h}} \setminus \Xi^{\frac{1}{2}\varepsilon_2}}(Y_s)ds\Big] \le \mathbb{E}\Big[\int_0^T \frac{1}{\max\big\{\frac{1}{2}\varepsilon_2, d(Y_s,\Xi)\big\}^2} 1_{\Xi^{\frac{3}{2}\varepsilon_{a_h}} \setminus \Xi^{\frac{1}{2}\varepsilon_2}}(Y_s)ds\Big] \\
&\le K_7 T \int_0^{\frac{3}{2}\varepsilon_{a_h}} \frac{1}{(\max\{\frac{1}{2}\varepsilon_2, x\})^2}dx + K_7 T \frac{4}{\varepsilon_2^2}\Big(\big(\frac{3}{2}\varepsilon_{a_h}\big)^{q+\frac{\alpha}{2}+\frac{1}{4}} + \Delta + \Delta^{(\frac{1}{4}+\frac{\alpha}{2})p}\Big) \\
&\le K_8 T(\Delta^{-1} + \Delta^{(\frac{1}{4}+\frac{\alpha}{2})p-2})\frac{1}{\log^4(1/\Delta)} \\
&\le K_9 T \Delta^{-1}\frac{1}{\log^4(1/\Delta)}.
\end{aligned}$$

To estimate I_3, applying Lemma 3.4 with $f(x) = 1$ and $\varepsilon = \varepsilon_2$, it follows from (23) that

$$I_3 \le \frac{K_{10}T}{\Delta^2\log^4(1/\Delta)}\Big(\varepsilon_2 + (\varepsilon_2 + \Delta + \Delta^{(\frac{1}{4}+\frac{\alpha}{2})p})\Big) \le K_{11}\frac{T}{\Delta\log^4(1/\Delta)}.$$

Case 2: $[\frac{p_0}{2}] \le \frac{3}{2\alpha+1}(1+2\alpha+2m)$.

Applying Lemma 3.4 with $f(x) = \frac{1}{(\max\{\frac{3}{2}\varepsilon_2, x\})^2}$, $\varepsilon = \frac{3}{2}\varepsilon_1$, for all $\Delta < \Delta_0$, we have

$$\begin{aligned}
I_{2,2} &\le \frac{9}{4}\mathbb{E}\Big[\int_0^T \frac{1}{\big[d(Y_s,\Xi)\big]^2} 1_{\Xi^{\frac{3}{2}\varepsilon_1}}(Y_{\underline{s}})ds\Big] \le K_{12}T\int_0^{\frac{3}{2}\varepsilon_1}\frac{1}{(\max\{\frac{3}{2}\varepsilon_2, x\})^2}dx + K_{12}T\Delta^{\frac{1}{4}+\frac{\alpha}{2}}\frac{1}{\varepsilon_2^2} \\
&\le K_{13}T\frac{1}{\log^4(1/\Delta)}\Delta^{\alpha/2-7/4}.
\end{aligned}$$

It follows from Lemma 3.4 with $f(x) = 1$ and $\varepsilon = \varepsilon_2$, and (23) that

$$I_3 \le K_{14}\frac{T}{\Delta\log^4(1/\Delta)}(\Delta^{\frac{1}{4}+\frac{\alpha}{2}} + \Delta^2\log^4(1/\Delta)) \le K_{15}T\Delta^{\alpha/2-7/4}.$$

When $\gamma < 0$, it follows from the uniform boundedness of the moment of Y that the constants $(K_i)_{1\le i\le 15}$ does not depend on T. We conclude the proof. □

3.6 Proofs of Theorem 2.2 and Theorem 2.3

The following lemma is needed for the proofs of Theorem 2.2 and Theorem 2.3.

Lemma 3.9. *Suppose that Assumptions (A1), (A2) and (A4) hold for some $m \ge 0$. Moreover, there exists a constant $C > 0$ such that for any solution X of the equation* (1)*, we have*

$$\sup_{t\in[0,T]}\mathbb{E}[|X_t|^{2m\vee 1}] \le C.$$

Then, the equation (1) *has at most one strong solution.*

Proof. Assume that (X') is another solution of the equation (1), we will show that $\mathbb{E}[|X_t - X'_t|] = 0$ for all $t \in [0,T]$, which implies the uniqueness of the solution.
By applying Itô's formula for $\phi_{\delta\varepsilon}(X_t - X'_t)$ and using Property (YW3), we have

$$\begin{aligned}
|X_t - X'_t| \le \varepsilon &+ \int_0^t \phi'_{\delta\varepsilon}(X_s - X'_s)\,[b(X_s) - b(X'_s)]\,ds \\
&+ \frac{1}{2}\int_0^t \phi''_{\delta\varepsilon}(X_s - X'_s)\,[\sigma(X_s) - \sigma(X'_s)]^2\,ds \\
&+ \int_0^t \phi'_{\delta\varepsilon}(X_s - X'_s)\,[\sigma(X_s) - \sigma(X'_s)]\,dW_s. \qquad (24)
\end{aligned}$$

Using Assumption (A2) and Properties (YW1), (YW2), we get

$$\begin{aligned}\int_0^t \phi'_{\delta\varepsilon}(X_s - X'_s)[b(X_s) - b(X'_s)]ds &= \int_0^t \frac{\phi'_{\delta\varepsilon}(X_s - X'_s)}{X_s - X'_s}(X_s - X'_s)[b(X_s) - b(X'_s)]ds \\ &\le \int_0^t \frac{\phi'_{\delta\varepsilon}(|X_s - X'_s|)}{|X_s - X'_s|} L_1 (X_s - X'_s)^2 ds \\ &\le \int_0^t |L_1||X_s - X'_s|ds.\end{aligned} \tag{25}$$

Using Assumption (A4), Property (YW5), and Proposition 3.1, we have

$$\mathbb{E}\Big[\int_0^t \phi''_{\delta\varepsilon}(X_s - X'_s)\,[\sigma(X_s) - \sigma(X'_s)]^2\,ds\Big] \le \frac{2\varepsilon^{2\alpha}}{\log\delta}\mathbb{E}\left[\int_0^t 3L_3 + 3L_3|X_s|^{2m} + 3L_3|X'_s|^{2m}ds\right] \le \frac{K_1\varepsilon^{2\alpha}}{\log\delta},$$

where K_1 depend on neither δ nor ε. Together with (24) and (25), this implies

$$\mathbb{E}[|X_t - X'_t|] \le \varepsilon + |L_1|\int_0^t \mathbb{E}[|X_s - X'_s|]ds + \frac{K_1\varepsilon^{2\alpha}}{\log\delta}.$$

By letting $\varepsilon \to 0, \delta \to \infty$, we get

$$\mathbb{E}[|X_t - X'_t|] \le |L_1|\int_0^t \mathbb{E}[|X_s - X'_s|]ds.$$

By Gronwall's inequality, we induce that $\mathbb{E}\left[|X_t - X'_t|\right] = 0$. The proof is complete. □

Proof of Theorem 2.2

We will use the localization technique as in [24]. For each $N > 1$, set

$$b_N(x) = \begin{cases} b(x) & \text{if } |x| \le N, \\ b\Big(\frac{Nx}{|x|}\Big)\big(N + 1 - |x|\big) & \text{if } N < |x| < N+1, \\ 0 & \text{if } |x| \ge N+1,\end{cases}$$

and

$$\sigma_N(x) = \begin{cases} \sigma(x) & \text{if } |x| \le N, \\ \Big(\sigma\Big(\frac{Nx}{|x|}\Big) - 1\Big)(N + 1 - |x|) + 1 & \text{if } N < |x| < N+1, \\ 1 & \text{if } |x| \ge N+1.\end{cases}$$

It can verify that b_N is bounded, and σ_N is Hölder continuous and uniformly elliptic. Then the equation

$$X_N(t) = x_0 + \int_0^t b_N(X_s)ds + \int_0^t \sigma_N(X_s)dW_s$$

has a unique strong solution. Moreover, we can verify that $xb_N(x) + \frac{p_0-1}{2}|\sigma_N(x)^2| \le \gamma'|x|^2 + \eta'$ for some constants γ', η' depending only on γ, η. Hence, it follows from Lemma 3.1 in [LT19] that if $p_0 \ge (l+4) \vee (2m + 2\alpha + 4)$ and $2 \le p \le (p_0 - (l \vee (2m + 2\alpha)))/2$, then

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|X_N(t)|^p\Big] \le C(x, T, l, p, \gamma, \eta, m).$$

Using the argument in the proof of Theorem 3.1.i in [24], we can show that when $N \to \infty$, X_N will converge in probability to a process X which satisfies the equation (1).

It remains to prove the uniqueness of the solution. Let φ be the function defined in Theorem 2.5. Note that from the properties (P1)–(P3), φ^{-1} exists and has bounded and Lipschitz continuous derivative. Let $\overline{b} := \Psi \circ \varphi^{-1}$ and $\overline{\sigma} := (\varphi'.\sigma) \circ \varphi^{-1}$. It can be verified that $\overline{b}$ and $\overline{\sigma}$ satisfy the following properties: for any $x, y \in \mathbb{R}$,

$$(x - y)(\overline{b}(x) - \overline{b}(y)) \le C(x - y)^2,$$

and

$$|\overline{\sigma}(x) - \overline{\sigma}(y)| \le C|x-y|^{\alpha+1/2}(1+|x|^{m+1}+|y|^{m+1})$$

for some positive constant C. Let $U_t := \varphi(X_t)$. Using Itô's formula and Property (P4), we obtain

$$dU_t = \overline{b}(U_t)dt + \overline{\sigma}(U_t)dW_t. \tag{26}$$

It follows from Lemma 3.1 and Property (P2) that

$$\mathbb{E}[|U_t|^{2m+2}] \le C_1 + C_1\mathbb{E}[|X_t|^{2m+2}] < +\infty.$$

Then it follows from Lemma 3.9 that there exists at most one solution to equation (26). Since φ is strictly increasing, we obtain the uniqueness for solution of equation (1). This concludes the proof of Theorem 2.2.

Proof of Theorem 2.3

For the proof of Theorem 2.3, we need the following result.

Lemma 3.10. *Suppose that Assumptions (A1)–(A5) hold for $m = l = \alpha = 0$, and b is bounded by $\|b\|_\infty < +\infty$. Then the equation* (1) *has a unique strong solution.*

Proof. For each $N > \frac{1}{\varepsilon_0}$, we define

$$b_N(x) = \begin{cases} b(x) \quad \text{if} \quad d(x,\Xi) > \frac{1}{N}, \\ b(\xi_i - \frac{1}{N}) + \frac{\left(x-\xi_i+\frac{1}{N}\right)\left(b(\xi_i+\frac{1}{N})-b(\xi_i-\frac{1}{N})\right)}{2/N} \quad \text{if} \quad d(x,\xi_i) \le \frac{1}{N} \text{ for some } i \in \{1,\dots,k\}. \end{cases} \tag{27}$$

One can verify that b_N is locally Lipschitz continuous, and there exists a constant K not depending on N such that

$$\sup_{x\in\Xi^{\varepsilon_0}} \Big(|b(x)-b_N(x)| + |b_M(x)-b_N(x)|\Big) \le K, \text{ for any } M, N > \frac{1}{\varepsilon_0}, \tag{28}$$

and

$$(x-y)(b_N(x)-b_N(y)) \le |L_1|(x-y)^2, \text{ for any } x, y \in \mathbb{R}, N > \frac{1}{\varepsilon_0}. \tag{29}$$

From (28) and Assumption (A3), we have

$$|b_N(x) - b_N(y)| \le L_2' + 2K + 3L_2'|x-y|. \tag{30}$$

Also, from Assumption (A1) and (28),

$$xb_N(x) + \frac{p_0-1}{2}\sigma^2(x) \le K\max\{|\xi_k|, |\xi_1|\} + \eta + \gamma x^2. \tag{31}$$

By using Theorem 3.1 in [24], the following SDE

$$X_N(t) = x_0 + \int_0^t b_N(X_N(s))ds + \int_0^t \sigma(X_N(s))dW_s$$

has a unique strong solution.
For any $N > M > \frac{1}{\varepsilon_0}$ and $u \le T$, we have $\phi_{\delta\varepsilon}(X_M(u) - X_N(u)) = J_1 + \frac{1}{2}J_2 + J_3$, where

$$\begin{aligned} J_1 &= \int_0^u \phi'_{\delta\varepsilon}(X_M(t) - X_N(t))(b_M(X_M(t)) - b_N(X_N(t)))dt, \\ J_2 &= \int_0^u \phi''_{\delta\varepsilon}(X_M(t) - X_N(t))[\sigma(X_M(t)) - \sigma(X_N(t))]^2 dt, \\ J_3 &= \int_0^u \phi'_{\delta\varepsilon}(X_M(t) - X_N(t))[\sigma(X_M(t)) - \sigma(X_N(t))]dW_t. \end{aligned}$$

First, by using Assumption (A4), the Cauchy-Schwarz inequality, and Property (YW5), we have

$$\mathbb{E}[J_2] \le \int_0^u 9L_3^2\mathbb{E}\left[\frac{2}{|X_M(t)-X_N(t)|\log\delta}1_{\{|X_M(t)-X_N(t)|\le\varepsilon\}}|X_M(t)-X_N(t)|\right]dt \le \frac{18L_3^2T}{\log\delta}.$$

Next, we write $J_1 = J_{1,1} + J_{1,2}$, where

$$J_{1,1} = \int_0^u \phi'_{\delta\varepsilon}(X_M(t) - X_N(t))(b_M(X_M(t)) - b_M(X_N(t)))dt,$$
$$J_{1,2} = \int_0^u \phi'_{\delta\varepsilon}(X_M(t) - X_N(t))(b_M(X_N(t)) - b_N(X_N(t)))dt.$$

Using (29) and Properties (YW1) and (YW2),

$$\begin{aligned}J_{1,1} &= \int_0^u \frac{\phi'_{\delta\varepsilon}(X_M(t) - X_N(t)}{X_M(t) - X_N(t))}(X_M(t) - X_N(t))(b_M(X_M(t)) - b_M(X_N(t)))dt \\ &\leq \int_0^u \frac{\phi'_{\delta\varepsilon}(|X_M(t) - X_N(t)|)}{|X_M(t) - X_N(t))|}|L_1|(X_M(t) - X_N(t))^2 dt \leq |L_1| \int_0^u |X_M(t) - X_N(t)|dt.\end{aligned}$$

Note that for $N > M$, $b_N(x) = b_M(x)$ when $d(x, \Xi) > \frac{1}{M}$. This implies that

$$\mathbb{E}[J_{1,2}] \leq \int_0^u 2K\mathbb{E}\big(1_{\{X_N(t) \in \Xi^{1/M}\}}\big)dt = \sum_{i=1}^k \int_0^u 2K\mathbb{E}\big[1_{\{\xi_i - \frac{1}{M} \leq X_N(t) \leq \xi_i + \frac{1}{M}\}}\big]dt.$$

We will prove that there exists a constant c not depending on M, N such that for all $M, N \geq \frac{1}{\varepsilon_0}$,

$$\int_0^u \mathbb{E}\big[1_{\{\xi_i - \frac{1}{M} \leq X_N(t) \leq \xi_i + \frac{1}{M}\}}\big]dt \leq \frac{c}{M}. \tag{32}$$

Indeed, thanks to Tanaka's formula, for all $a \in \mathbb{R}$ we have

$$|X_N(u) - a| = |x_0 - a| + \int_0^u sgn(X_N(t) - a)b_N(X_N(t))ds + \int_0^u sgn(X_N(t) - a)\sigma(X_N(t))dW_t + L_u^a(X_N).$$

Hence

$$|L_u^a(X_N)| \leq |X_N(u) - x_0| + \left|\int_0^u sgn(X_N(t) - a)b_N(X_N(t))dt\right| + \left|\int_0^u sgn(X_N(t) - a)\sigma(X_N(t))dW_t\right|. \tag{33}$$

By taking expectation on both sides of the above estimate, using Doob's inequality, Proposition 3.1, Assumptions (A3), (A4), the estimates (31) and (30), there exists a constant c_1 that does not depend on N and a such that

$$\begin{aligned}\mathbb{E}[|L_u^a(X_N)|] \leq& \mathbb{E}\left[\left|\int_0^u b_N(X_N(t))dt\right|\right] + \mathbb{E}\left[\left|\int_0^u \sigma(X_N(t))dW_t\right|\right] \\ &+ \mathbb{E}\left[\left|\int_0^u sgn(X_N(t) - a)b_N(X_N(t))dt\right|\right] + \mathbb{E}\left[\left|\int_0^u sgn(X_N(t) - a)\sigma(X_N(t))dW_t\right|\right] \\ \leq& 2\int_0^u \mathbb{E}\left[|b_N(X_N(t))|\right]dt + 2\left[\left|\int_0^u \mathbb{E}\left[\sigma^2(X_N(t))\right]dt\right|\right]^{1/2} \\ \leq& 2\int_0^T \mathbb{E}\left[|b_N(X_N(t))|\right]dt + 2\left[\left|\int_0^T \mathbb{E}\left[\sigma^2(X_N(t))\right]dt\right|\right]^{1/2} \leq c_1.\end{aligned}$$

By using the occupation time formula, we obtain

$$\begin{aligned}\mathbb{E}\Big[\int_0^u 1_{[\xi_i - \frac{1}{M}, \xi_i + \frac{1}{M}]}(X_N(t))dt\Big] &\leq \frac{1}{\nu^2}\mathbb{E}\Big[\int_0^u 1_{[\xi_i - \frac{1}{M}, \xi_i + \frac{1}{M}]}(X_N(t))\sigma^2(X_N(t))dt\Big] \\ &= \frac{1}{\nu^2}\int_{-\infty}^{+\infty} 1_{[\xi_i - \frac{1}{M}, \xi_i + \frac{1}{M}]}(a)\mathbb{E}[L_u^a(X_N)]da \leq \frac{c_1}{\nu^2}\frac{2}{M} = \frac{c}{M},\end{aligned}$$

where $c = \frac{2c_1}{\nu^2}$. Note that $\mathbb{E}[J_3] = 0$. In summary, by the Property (YW3),

$$\mathbb{E}\big[|X_M(u) - X_N(u)|\big] \leq \varepsilon + \mathbb{E}\big[\phi_{\delta\varepsilon}(X_M(u) - X_N(u))\big]$$

$$\leq \varepsilon + \frac{2kKc}{M} + \frac{18L_3^2 T}{\log \delta} + |L_1| \int_0^u \mathbb{E}\big[|X_M(t) - X_N(t)|\big] dt.$$

By letting $\varepsilon \to 0$ and $\delta \to \infty$, we get

$$\mathbb{E}\big[|X_M(u) - X_N(u)|\big] \leq \frac{2kKc}{M} + |L_1| \int_0^u \mathbb{E}\big[|X_M(t) - X_N(t)|\big] dt.$$

Using Gronwall's inequality, we have

$$\mathbb{E}[|X_M(u) - X_N(u)|] \leq \frac{2kKc}{M} e^{|L_1|u} \text{ for all } N > M > \frac{1}{\varepsilon_0}.$$

Then $X_M(u)$ is a Cauchy sequence in $L^1(\Omega)$, and it converges to a random variable $X(u)$. By letting $N \to +\infty$, it follows from Fatou's lemma that

$$\mathbb{E}[|X(u) - X_M(u)|] \leq \frac{2kKc}{M} e^{|L_1|u}, \text{ for all } u \leq T.$$

From here, by using Assumption (A4), Hölder's inequality and Young's inequality, we have

$$\mathbb{E}\left[\left(\int_0^u \sigma(X_N(t))dW_t - \int_0^u \sigma(X(t))dW_t\right)^2\right] = \mathbb{E}\left[\int_0^u \left(\sigma(X_N(t) - \sigma(X(t))\right)^2 dt\right]$$
$$\leq 9L_3^2 \mathbb{E}\left[\int_0^u |X_N(t) - X(t)| dt\right] \leq \frac{18L_3^2 kKcu}{N} e^{|L_1|u}. \tag{34}$$

Next, from (32), we obtain

$$\mathbb{E}\left[\int_0^u |b(X_N(t)) - b_N(X_N(t)| dt\right] \leq K \sum_{i=1}^k \mathbb{E}\Big[\int_0^u 1_{[\xi_i - \frac{1}{N}, \xi_i + \frac{1}{N}]}(X_N(t)) dt\Big] \leq \frac{Kkc}{N}.$$

For $N > M$, we write $\mathbb{E}\left[\int_0^u |b(X_t) - b(X_N(t))| dt\right] = I_1 + I_2$, where

$$I_1 = \mathbb{E}\left[\int_0^u |b(X_t) - b(X_N(t))| 1_{\{X_t \in \Xi^{1/2M}\}} dt\right],$$
$$I_2 = \mathbb{E}\left[\int_0^u |b(X_t) - b(X_N(t))| 1_{\{X_t \notin \Xi^{1/2M}\}} dt\right].$$

From (32), by letting $N \to \infty$,

$$\mathbb{E}\Big[\int_0^u 1_{[\xi_i - \frac{1}{2M}, \xi_i + \frac{1}{2M}]}(X(t)) dt\Big] \leq \limsup_{N \to \infty} \mathbb{E}\Big[\int_0^u 1_{(\xi_i - \frac{1}{M}, \xi_i + \frac{1}{M})}(X_N(t)) dt\Big] \leq \frac{c}{M},$$

implying that

$$\mathbb{E}\Big[\int_0^u 1_{\{X_t \in \Xi^{1/2M}\}} dt\Big] \leq \frac{ck}{M}. \tag{35}$$

By using the boundedness of b and (35), we have

$$I_1 \leq 2\|b\|_\infty \int_0^u \mathbb{E}[1_{\{X_t \in \Xi^{1/2M}\}}] \leq 2\|b\|_\infty \frac{ck}{M}.$$

To estimate I_2, we write

$$I_2 = \mathbb{E}\left[\int_0^u |b(X_t) - b(X_N(t))| 1_{\{X_t \notin \Xi^{1/2M}\}} 1_{\{(X_t, X_N(t)) \in S\}} dt\right]$$
$$+ \mathbb{E}\left[\int_0^u |b(X_t) - b(X_N(t))| 1_{\{X_t \notin \Xi^{1/2M}\}} 1_{\{(X_t, X_N(t)) \notin S\}} dt\right]$$

$$\leq 3L_2 \int_0^u \mathbb{E}[|X(t) - X_N(t)|]dt + 2\|b\|_\infty \int_0^u \mathbb{E}[|1_{\{|X_t - X_N(t)|>1/2M\}}]dt$$
$$\leq 3L_2 \int_0^u \mathbb{E}[|X(t) - X_N(t)|]dt + 4\|b\|_\infty M \int_0^u \mathbb{E}[|X_t - X_N(t)|]dt$$
$$\leq (3L_2 + 4M\|b\|_\infty)\frac{2kKc}{N}e^{|L_1|u}.$$

By choosing $N = M^2 \to \infty$, we have

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^u |b(X(t)) - b_N(X_N(t)|dt\right] = 0.$$

Together with (34), this concludes the existence of solution to equation (1). The uniqueness of solution is obtained by using Lemma 3.9. □

Now we are ready to prove Theorem 2.3. For any $N > \max\{|\xi_1|, |\xi_k|\}$, let's denote

$$b_N(x) = \begin{cases} b(x) & \text{if } |x| \leq N, \\ b\Big(\frac{Nx}{|x|}\Big)\big(N+1-|x|\big) & \text{if } N < |x| < N+1, \\ 0 & \text{if } |x| \geq N+1, \end{cases}$$

and

$$\sigma_N(x) = \begin{cases} \sigma(x) & \text{if } |x| \leq N, \\ \sigma\Big(\frac{Nx}{|x|}\Big)(N+1-|x|) & \text{if } N < |x| < N+1, \\ 0 & \text{if } |x| \geq N+1. \end{cases}$$

One can check that b_N and σ_N satisfy all conditions of Lemma 3.10, with the note that a compactly supported $(\alpha + 1/2)$-Hölder continuous function is also a 1/2-Hölder continuous function. Then the equation

$$\begin{cases} dX_N(t) = b_N(X_t)dt + \sigma_N(X_t)dW_t \\ X_0^N = x_0 \in \mathbb{R} \end{cases},$$

has a unique strong solution. Moreover, we can verify that $xb_N(x) + \frac{p_0-1}{2}|\sigma_N(x)^2| \leq \gamma'|x|^2 + \eta'$ for some constant γ', η' which depend only on γ, η. Hence, it follows from Lemma 3.1 in [23] that if $p_0 \geq (l \vee m) + 4$ and $2 \leq p \leq (p_0 - (l \vee m))/2$, then

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^N|^p\Big] \leq C(x, T, l, p, \gamma, \eta, m).$$

Again, by following the argument in the proof of Theorem 3.1.i in [24], we can show that X_N will converge in probability to a process X which satisfies equation (1).

The uniqueness of solution is obtained by using Lemma 3.9. We conclude the proof.

4 Numerical analysis

In this section, we conduct the proposed algorithm for several SDEs to support our theory. Recall that if X is the unique strong solution of the SDE (1) and Y^Δ is our approximation scheme corresponding to the step-size parameter Δ, then under the conditions stated in Theorem 2.4 and Theorem 2.5, for $\alpha \in (0, 1/2]$, we have

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|Y_t^\Delta - X_t|\right] \leq C\Delta^\alpha$$

for some $C \in (0, \infty)$ and for all $\Delta \in (0, \Delta_0)$. Hence, by using the triangle inequality, we obtain

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[|Y_t^{2^{-(k+1)}\Delta_0} - Y_t^{2^{-k}\Delta_0}|\Big] \leq C2^{-k\alpha}$$

for some positive constant C.

If we denote $r_k := |Y_t^{2^{-(k+1)}\Delta_0} - Y_t^{2^{-k}\Delta_0}|$, then we can use $\widetilde{r}_k := \frac{1}{N_k}\sum_{n=1}^{N_k} r_{k,n}$ as an approximation for $\mathbb{E}[r_k]$, in which $r_{k,n}$ is the n^{th} numerical path of r_k for all $n = 1, 2, ..., N_k$. The main issue here is that the simulations of the fine and coarse paths in the formula of r_k have to share the same driving Brownian motion during the overlap time interval. To do this, we adapt a simulating method from Fang and Giles (see [4]).

To determine the convergence rate α for each numerical example, observe that since

$$\widetilde{r}_k \leq c2^{-k\alpha}$$

when the approximation $\widetilde{r}_k$ is relatively good, i.e. when N_k is sufficiently large, we also have

$$\log \widetilde{r}_k \leq \log c - \alpha(k \log 2).$$

Therefore, we can estimate the rate α by computing the slope of the regression line for the pairs $(-k\log 2, \log \widetilde{r}_k)$, i.e. it should be no more than an empirical rate $\tilde{\alpha}$ which satisfies $-\log \widetilde{r}_k = -\log c + \tilde{\alpha}(k \log 2) + o(1)$.

In Theorem 2.6, we have provided the rate of the computation cost ζ satisfying $\mathbb{E}[N_T] \leq CT\Delta^{\zeta}$ for our scheme. This theoretical rate can also be estimated by an empirical rate $\tilde{\zeta}$ in the same way as above.

The two SDEs that we consider in Table 1 satisfy the conditions in Theorem 2.4, while those in Table 2 satisfy the conditions provided in Theorem 2.5. For each example, we compute $\widetilde{r}_k$ for $k = 1, 2, 3, 4, 5$ with $\Delta_0 = 1.8 \cdot 10^{-4}$. The number of samples is $N_k = 10^3$ for $k = 1, 2, 3, 4$. The results for the case $T = 1$ are summarized in Table 3.

Example	$b(x)$	$\sigma(x)$	x_0	p_0	l	m	α	γ	L_1
1	$\begin{cases} -x - x^3 & \text{if } x \geq 0, \\ 1 - x - x^3 & \text{if } x < 0. \end{cases}$	$(1+x)(1+x^{2/3})1_{[-1,\infty)}(x)$	0	20	2	1	$\frac{1}{6}$	-1	-1
2	$\begin{cases} -1 + x - x^3 & \text{if } x \geq 0, \\ x - x^3 & \text{if } x < 0. \end{cases}$	$(1+x)(1+x^{2/3})1_{[-1,\infty)}(x)$	0	20	2	1	$\frac{1}{6}$	1	1

Table 1: Examples of SDEs satisfying the conditions in Theorem 2.4.

Example	$b(x)$	$\sigma(x)$	x_0	p_0	l	m	α	γ	L_2
3	$\begin{cases} 1 + x - x^3 & \text{if } x > 2, \\ x^2 + 1 & \text{if } 0 \leq x \leq 2, \\ x - x^3 & \text{if } x < 0. \end{cases}$	$1 + \sqrt{\frac{x^4 + x^{4/3}}{14}}$	0.2	26	2	1	$\frac{1}{6}$	-1	-1
4	$\begin{cases} 1 + x - x^{2/3} & \text{if } x > 2, \\ x^2 + 1 & \text{if } 0 \leq x \leq 2, \\ x & \text{if } x < 0. \end{cases}$	$1 + x^{2/3}$	0	20	1	$\frac{4}{3}$	$\frac{1}{6}$	1	1

Table 2: Examples of SDEs satisfying the conditions in Theorem 2.5.

Example	$\tilde{\alpha}$	95% CI for $\tilde{\alpha}$	$\tilde{\nu}$	95% CI for $\tilde{\zeta}$
1	0.557	$[0.422, 0.692]$	-1.131	$[-1.286, -0.977]$
2	0.534	$[0.376, 0.692]$	-1.139	$[-1.279, -0.999]$
3	0.601	$[0.463, 0.739]$	-1.252	$[-1.375, -1.129]$
3	0.608	$[0.485, 0.731]$	-1.094	$[-1.137, -1.052]$

Table 3: Estimation for the rates α and ζ in Example 1,2, 3 and 4 for $T = 1$.

In these examples, for $T = 1$, the empirical rate of convergence is larger than the theoretical rate 1/6, and the empirical rate of computation cost is almost the same as the theoretical rate -1. Hence, the numerical results support the theoretical findings.

Example	$T=1$	$T=5$
1	-4.390	-4.184
3	-1.925	-1.878

Table 4: The intercepts of the regression line for the pairs $(-k\log 2, \log \widetilde{r}_k)$ in Example 1 and 3.

In addition, since $\gamma < 0$ in Example 1 and 3, the constant C in the estimates (4) and (5) does not depend on T. Hence, the intercept of the regression line for the pairs $(-k\log 2, \log \widetilde{r}_k)$ should not change drastically when the time T is adjusted. This can be verified when we compare the values of this intercept in the case $T = 1$ and $T = 5$ for both examples. The results are shown in Table 4.

It has been shown that when $\gamma < 0$, the constant C in Theorem 2.6 does not depend on T either. To demonstrate this, we note that the intercept of the regression line in the estimation of ζ is $\log C + \log T$. The values of this intercept for Example 1 in the case $T = 1$ and $T = 5$ are 7.082 and 7.675, respectively. We can see that $7.645 - 7.082 \approx \log 5$, which suggests that the constant C does not depend on T.

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