LIPSCHITZ-CONTINUITY OF TIME CONSTANT IN GENERALIZED FIRST-PASSAGE PERCOLATION

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Abstract. In this article, we consider a generalized First-passage percolation model, where each edge in $\mathbb{Z}^d$ is independently assigned an infinite weight with probability $1-p$, and a random finite weight otherwise. The existence and positivity of the time constant have been established in [CT16]. Recently, using sophisticated multi-scale renormalizations, Cerf and Dembin [CD22] proved that the time constant of chemical distance in super-critical percolation is Lipschitz continuous. In this work, we propose a different approach leveraging lattice animal theory and a simple one-step renormalization with the aid of Russo's formula, to show the Lipschitz continuity of the time constant in generalized First-passage percolation.

1. Introduction

1.1. **Model and main results.** First-passage percolation (FPP), which was introduced by Hammersley and Welsh in the 1960s, serves as a prototype for models of random growth or infection models. Let $d \geq 2$ and $(\mathbb{Z}^d, \mathcal{E}(\mathbb{Z}^d))$ represent the d-dimensional integer lattice, where the edge set $\mathcal{E}(\mathbb{Z}^d)$ consists of pairs of nearest neighbours in $\mathbb{Z}^d$. To each edge $e \in \mathcal{E}(\mathbb{Z}^d)$, we assign a random variable ω_e with values in $[0,\infty)$, assuming that the family $(\omega_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ is independent and identically distributed. The random variable ω_e can be interpreted as the time needed for the infection to cross the edge e. We define a random pseudo-metric T: for any pair of vertices $x, y \in \mathbb{Z}^d$, $\mathrm{T}(x,y)$ is the shortest time to go from x to y. The main object of FPP is to know how the infection grows in the lattice, or equivalently how is the asymptotic behavior of the passage time $\mathrm{T}(0,x)$ as $||x||_\infty$ tends to infinity. There has been a great and consistent interest of mathematicians for more than sixty years to answer this question, see, for instance, [ADH17] and references therein. While most studies focus on the case of finite edge weight, i.e. ω_e takes a value in $[0,\infty)$, recently there have been several results on the behavior of generalized models allowing the infinite value, see e.g. [GM04, CT16]. The emergence of infinite weight can explain the situation that some edges in the lattice are not available for the spread of infection.

In this paper, we consider a generalized FPP that is mixed from the Bernoulli percolation and classical FPP. More precisely, given F a distribution supported on $[0,\infty)$, and $p \in [0,1]$, we define a new distribution F_p by

$$F_p := pF + (1-p)\delta_\infty.$$

Let $\tau := (\tau_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ be a family of edge-weights with the same distribution F_p, interpreted as the time to pass each edge in $\mathbb{Z}^d$. The usual first passage time $\mathrm{T}_p(x,y)$ on $\mathbb{Z}^d$ for $x, y \in \mathbb{Z}^d$ is defined by

$$\mathrm{T}_p(x,y) := \inf_{\gamma:x\to y} \mathrm{T}_p(\gamma) := \inf_{\gamma:x\to y} \sum_{e\in\gamma} \tau_e,$$

where the infimum is taken over all paths from x to y in $\mathbb{Z}^d$. We impose the following constraint on p and F:

$$p > p_c(d) > F(0), \tag{1.1}$$

where $p_c(d)$ is the critical parameter of Bernoulli percolation on $\mathbb{Z}^d$. The condition $p > p_c(d)$ guarantees the unique infinite cluster composed of finite weight edges, while the assumption $F(0) < p_c(d)$ rules out the possibility of having an infinite cluster with zero weight. Since the passage time $\mathrm{T}_p(x,y)$ may take the infinite value (when x and y are not connected by a path of finite weight edges), we consider a modification as follows. Let $\mathcal{C}_p$ denote the unique infinite cluster of edges with finite weights. Given points $x, y \in \mathbb{R}^d$, we define the regularized passage time as

$$\widetilde{\mathrm{T}}_p(x,y) := \mathrm{T}_p([x]_p, [y]_p),$$

where $[x]_p$ denotes the d_1-closest point to x in $\mathcal{C}_p$ with a deterministic rule breaking ties. Traditionally, the main object of interest in generalized FPP is the asymptotic behavior of $\widetilde{\mathrm{T}}_p$. Particularly, the weak law of large numbers was obtained in [GM90, CT16]: there exists a constant $\mu_p \in [0,\infty)$ such that

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{T}}_p(0, n\mathbf{e}_1)}{n} = \mu_p \qquad \text{in probability,} \tag{1.2}$$

where $\mathbf{e}_1$ is the first unit vector in $\mathbb{R}^d$. Moreover, Garet and Marchand [GM04, Remark 1] proved if $\mathbb{E}[\tau^{2+\delta}\mathbf{1}_{\tau<\infty}] < \infty$ with some $\delta > 0$, then the convergence in (1.2) holds true almost surely and in L_1. Our first result is the strong law of large numbers for the regularized first passage time assuming solely the finiteness of first moment of $\tau\mathbf{1}_{\tau<\infty}$. We prove it in Appendix C. [1]

Theorem 1.1. (SLLN of the regularized passage time) *If $p > p_c(d)$ and $\mathbb{E}[\tau\mathbf{1}_{\tau<\infty}] < \infty$, then*

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{T}}_p(0, n\mathbf{e}_1)}{n} = \mu_p \qquad a.s.\ and\ in\ L_1.$$

The continuity and regularity of the time constant of First-passage percolation and chemical distance in super-critical percolation have been subjects of investigation since the 1980s. The continuity has been explored in works [Cox80, CK81, GMPT17], while the regularity has been addressed in [Dem21, CD22, KT22].[2] In particular, Cerf and Dembin [CD22] have established the Lipschitz continuity of the time constant for the chemical distance, i.e., $F = \delta_1$. Going further, the authors also claim a quantitative estimate of difference of time constants for two distributions (that includes Theorem 1.2 below), though they do not give detailed proof.

In this context, we present our main result as follows:

Theorem 1.2. (Lipschitz continuity) *For all $p_0 > p_c(d)$, there exists a constant $C = C(p_0) > 0$ such that for all p, q in the interval $[p_0, 1]$,*

$$|\mu_p - \mu_q| \le C|p - q|.$$

Notably, the time constant can be expressed as the limit of a truncated passage time defined below, which implies that the moment condition on weight is not necessary for this theorem.

1.2. Outline of the proof.

The proofs in [CD22] utilizes a sophisticated multi-scale renormalization technique. However, in our paper, we propose an alternative approach that employs lattice animal theory combined with a straightforward one-step renormalization process. Let us explain the outline of the proof here.

Let $M := M_n := (\log n)^3$ and $K := K_n := n^2$. We denote by $\mathrm{T}_M^{\Lambda_K}(x, y)$ the first passage time between x and y associated with the truncated weights $(\tau_e^M)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ using only paths inside Λ_K, where $\tau_e^M := \tau_e \wedge M$. Then the proof of Theorem 1.2 is decomposed into two steps:

Step 1 (Time constant as the limit of truncated passage time): We aim to show

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\mathrm{T}_M^{\Lambda_K}(0, n\mathbf{e}_1)\right]}{n} = \mu_p. \tag{1.3}$$

The proof goes as follows. Let λ be a large positive constant and $q := \mathbb{P}(\tau_e \le \lambda)$, see Apendix B.1 for the choice of λ. We consider the percolation of q-open edges consisting of $\{e \in \mathcal{E}(\mathbb{Z}^d) : \ \tau_e \le \lambda\}$ and use similar notations, such as $\mathcal{C}_q$ and $[x]_q$, for this percolation. Note that $\mathcal{C}_q \subset \mathcal{C}_p$, and the vertices in $\mathcal{C}_q$ can be connected to each other along paths whose weights are at most λ. According to [GMPT17, Lemma 2.11], we have

$$\lim_{n\to\infty} \frac{\mathrm{T}_p([0]_q, [n\mathbf{e}_1]_q)}{n} = \mu_p \quad \text{a.s. and in } L_1.$$

In this step, we further aim to show

$$\mathbb{E}\left[\left|\mathrm{T}_p([0]_q, [n\mathbf{e}_1]_q) - \mathrm{T}_M^{\Lambda_K}(0, n\mathbf{e}_1)\right|\right] = \mathcal{O}(\lambda M).$$

To prove this estimate, we introduce the notation of **effective radius** $(R_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ in Section 2.2. Roughly speaking, given an edge e belonging to a geodesic of the truncated passage time, R_e measures the effect when flipping the state of e. Under the event that $\{R_e \le (\log n)^{5/2}\, \forall e \in [-n^2, n^2]^d\}$ which occurs with overwhelming probability, we show that $\left|\mathrm{T}_p([0]_q, [n\mathbf{e}_1]_q) - \mathrm{T}_M^{\Lambda_K}(0, n\mathbf{e}_1)\right| = \mathcal{O}(\lambda M)$. In particular, we have (1.3). We refer to Section 3.2 for the details.

Step 2 (Linear bound via Russo's formula): Let $\mathrm{T}_{M,\pm,e}^{\Lambda_K}(0, n\mathbf{e}_1)$ be the first passage time when the weight of the edge e is set to M for + and 0 for −, respectively. We take γ to be a geodesic of $\mathrm{T}_M^{\Lambda_K}(0, n\mathbf{e}_1)$, and we

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[1] Although the proof of Theorem 1.1 is based on classical Kingman's sub-additive ergodic theorem and is quite simple, we could not find any reference for it.

[2] Note that in [KT22], a distribution defined as $F_p = p\delta_0 + (1-p)\delta_1$ was considered, and explicit bounds for the Lipschitz constants were obtained.

define $\Delta_e \mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1) := \mathrm{T}_{M,+,e}^{\Lambda_K}(0,n\mathbf{e}_1) - \mathrm{T}_{M,-,e}^{\Lambda_K}(0,n\mathbf{e}_1)$. We aim to show

$$\left|\frac{\mathrm{d}\mathbb{E}\left[\mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)\right]}{\mathrm{d}p}\right| \le \mathbb{E}\left[\sum_{e\in\gamma}\Delta_e \mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)\right] \le \mathcal{O}(1)\mathbb{E}\left[\sum_{e\in\gamma} R_e\right] \le \mathcal{O}(n). \tag{1.4}$$

The first inequality follows from a standard application of Russo's formula. The second inequality simply follows from the construction of effective radius appearing above. The proof of the last inequality in (1.4) uses properties of effective radius, i.e., a local dependence and a good probability decay, and lattice animal theory.

The effective radius along with the utilization of lattice animal theory is proved to be robust in estimating the effect of flipping edge in percolation. In fact, we can use these ingredients to establish the sub-diffusive concentration of chemical distance in Bernoulli percolation as in [CN23].

1.3. **Notation.**

We summarize some notation frequently used throughout the paper.

- *Integer interval.* We define $[a] := [1,a]\cap\mathbb{Z}$ for all $a \ge 1$.
- *Box and its boundary.* For every $x\in\mathbb{Z}^d$ and $t>0$, we define $\Lambda_t(x) := x + [-t,t]^d$ the box with center x and radius t. For simplicity, we write $\Lambda_t := \Lambda_t(0)$. We define the boundary of $\Lambda_t(x)$ as $\partial\Lambda_t(x) := \Lambda_t(x)\setminus\Lambda_{t-1}(x)$.
- *Edge set.* Given a set $A\subset\mathbb{Z}^d$, we denote by $\mathcal{E}(A)$ the set of edges both of whose endpoints belong to A.
- *Set distance.* For $X, Y\subset\mathbb{Z}^d$, we consider several kinds of distance between X and Y as

$$\mathrm{d}_\star(X,Y) := \min\{\|x-y\|_\star : x\in X, y\in Y\}, \qquad \star\in\{1,2,\infty\}.$$

- *Path* and *open path.* We say that a sequence $\gamma = (v_0,\dots,v_n)$ is a **path** if $|v_i - v_{i-1}|_1 = 1$ and $v_i\neq v_j$ for all $i\neq j\in[n]$. Given $A\subset\mathbb{Z}^d$, let $\mathcal{P}(A)$ denote the set of all paths inside A. Given a Bernoulli percolation on $\mathbb{Z}^d$ with parameter p, we say that a path is p**-open** if all of its edges are open. An **open cluster** is a maximal connected component in the percolation. An open cluster $\mathcal{C}$ is called a q-crossing in Λ if in each direction there is an open path in $\mathcal{C}$ connecting the two opposite faces of Λ. In that case, we write q-crossing cluster $\mathcal{C}\subset\Lambda$.
- *Geodesic* and *truncated passage time* : Let T be the first passage time associated with weights $(\omega_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$. Given $x,y\in\mathbb{Z}^d$, a path γ between x and y is termed a **geodesic** of T if its passage time matches $\mathrm{T}(x,y)$, i.e. $\mathrm{T}(\gamma) := \sum_{e\in\gamma}\omega_e = \mathrm{T}(x,y)$. Given $H>0$ and $A\subset\mathbb{Z}^d$, we define the **truncated passage time**, denoted by T_H^A, as the first passage time associated with the truncated weights $(\omega_e\wedge H)_{e\in\mathcal{E}(Z^d)}$ using only paths inside A. When $A=\mathbb{Z}^d$, we write $\mathrm{T}_H := \mathrm{T}_H^{\mathbb{Z}^d}$.

1.4. **Organization.**

The paper is organized as follows. In Section 2, we introduce the main ingredients of proof including Russo's formula, effective radius, and lattice animal theory. In Section 3, we prove Step 1 and Step 2 using the elements prepared in Section 2. In the Appendix, we prove the strong law of large numbers of the passage time (Theorem 1.1), Russo's formula and properties of effective radius.

2. MAIN INGREDIENTS OF PROOF

In this section, we introduce three main elements in proving the Lipschitz continuity. The first result is Russo's type formula (Lemma 2.1). The second result considers the effects of resampling an edge (Propositions 2.4 and 2.5), and the third result provides an upper bound on the total cost of resampling along a random path using the lattice animal theory (Corollary 2.8). Although they have been already investigated in previous research, e.g., [CN19] and [CN23], we provide the proofs of these results in Appendix for the completeness of the paper.

2.1. **Russo's type formula.**

Let $L\in\mathbb{R}_+\cup\{\infty\}$. Let ν be a random variable with the distribution G supported in $[0,L]$. For $p\in(p_c(d),1)$, we define the distribution G_p on $[0,L]$ by

$$G_p := pG + (1-p)\delta_L,$$

where δ_L stands for the Dirac delta distribution at L.

Lemma 2.1. *Let E be a finite set, $\xi = (\xi_e)_{e\in E}$ i.i.d. random variables with the common distribution G_p, and $X : [0,L]^E\to\mathbb{R}$ be a function. Suppose that $\xi^{+,e}$ and ξ^e are obtained from ξ by replacing ξ_e with L and with ν respectively, where ν is an independent random variable with distribution G. Then, we have*

$$\frac{\mathrm{d}\mathbb{E}[X(\xi)]}{\mathrm{d}p} = \sum_{e\in E}(\mathbb{E}[X(\xi^e)] - \mathbb{E}[X(\xi^{+,e})]).$$

2.2. **The effect of resampling edges.** As we will see in the next section, using Russo's type formula (Lemma 2.1), the problem of Lipschitz continuity of time constant can be reduced to controlling the effect of resampling the edges along the geodesics. Given an edge e, we introduce the **effective radius** R_e, which measures the change of chemical distance when flipping the state of e from open to closed.

Given a coupling of Bernoulli percolation models for parameters p, a path γ is called p-open if all of its edges are open in the corresponding percolation with parameter p. We define the set of p-open paths in $A \subset \mathbb{Z}^d$ by

$$\mathbb{O}_p(A) := \{\gamma \in \mathcal{P}(A) : \ \gamma \text{ is } p\text{-open}\}.$$

For $A, B, U \subset \mathbb{Z}^d$, we define the **chemical distance**

$$\mathrm{D}_p^U(A, B) := \inf\{|\gamma| : x \in A,\ y \in B,\ \gamma \text{ is a } p\text{-open path from } x \text{ to } y \text{ inside } U\}.$$

When $U = \mathbb{Z}^d$, we simply write D_p for $\mathrm{D}_p^{\mathbb{Z}^d}$. Given $p \in [0, 1]$ and $\lambda \in \mathbb{R}$, we define

$$q := q(p, \lambda) := \mathbb{P}(\tau_e \le \lambda) = pF([0, \lambda]). \tag{2.1}$$

Let δ_0 be a sufficiently small positive constant as in Lemma B.1 below. Given $p_0 \in (p_c(d), 1]$, we define $q_0 := \frac{p_0 + p_c(d)}{2}$ and take $\lambda = \lambda(p_0, F)$ sufficiently large such that $F([0, \lambda]) \ge \max\left\{\frac{q_0}{p_0}, 1 - \delta_0\right\}$, which implies

$$q_0 \le q \le p \le q + \delta_0 \ \forall\, p \in [p_0, 1]. \tag{2.2}$$

We say that an edge e is q-open or p-open if $\tau_e \le \lambda$ or $\tau_e < \infty$, respectively. We call q-percolation and p-percolation the associated percolation models. Let $\mathcal{C}_q$ and $\mathcal{C}_p$ be the corresponding infinite clusters. We will see in Appendix B.1 that the condition (2.2) assures that a large cluster in $\mathcal{C}_q$ and a long path in $\mathcal{C}_p$ would intersect with high probability. Given an edge $e \in \mathcal{E}(\mathbb{Z}^d)$, we fix a rule to write $e = (x_e, y_e)$ so that $\|x_e\|_1 < \|y_e\|_1$. For $N \ge 1$, and $e = (x_e, y_e)$, we define $\Lambda_N(e) := \Lambda_N(x_e)$, and an annulus

$$\mathrm{A}_N(e) := \Lambda_{3N}(e) \setminus \Lambda_N(e). \tag{2.3}$$

We say that γ is a crossing path of $\mathrm{A}_N(e)$ if γ is a path inside $\mathrm{A}_N(e)$ that joins $\partial\Lambda_N(e)$ and $\partial\Lambda_{3N}(e)$. Let $\mathscr{C}(\mathrm{A}_N(e))$ be the collection of all crossing paths of $\mathrm{A}_N(e)$. Given $H > 0$ and $A \subset \mathbb{Z}^d$, recall that T_H^A is the first passage time associated with the truncated weights $(\tau_e \wedge H)_{e \in \mathcal{E}(Z^d)}$ using only paths inside A. For $u, v \in A$, we define the set of geodesics of $\mathrm{T}_H^A(u, v)$ as

$$\mathbb{G}_H(u, v; A) := \{\gamma = (u, \dots, v) \in \mathcal{P}(A) : \mathrm{T}_H(\gamma) = \mathrm{T}_H^A(u, v)\}.$$

We also define

$$\mathbb{G}_H(A) := \bigcup_{u, v \in A} \mathbb{G}_H(u, v; A).$$

If $A = \mathbb{Z}^d$, we simply write $\mathbb{G}_H(x, y)$ for $\mathbb{G}_H(x, y; \mathbb{Z}^d)$ and write $\mathbb{G}_H$ for $\mathbb{G}_H(\mathbb{Z}^d)$.

Remark 2.2. *Given $B \subset A \subset \mathbb{Z}^d$ and $H > 0$, if $\gamma \in \mathbb{G}_H(A)$ and π is a sub-path of γ such that $\pi \subset B$, then $\pi \in \mathbb{G}_H(B)$. We note that $\mathbb{G}_H(A)$ is measurable with respect to the weights of edges inside A.*

Let C_* be a positive constant. For each $e \in \mathcal{E}(\mathbb{Z}^d)$, we define the q-**effective radius** of e as

$$R_e := R_e(C_*, H) := \inf\left\{N \ge 3 : \forall \gamma_1, \gamma_2 \in \mathbb{G}_H(\Lambda_{C_* N}(e)) \cap \mathscr{C}(\mathrm{A}_N(e)),\ \mathrm{D}_q^{\mathrm{A}_N(e)}(\gamma_1, \gamma_2) \le C_* N\right\}.$$

Remark 2.3. *By the definition of effective radius and Remark 2.2, for all $e \in \mathcal{E}(\mathbb{Z}^d)$ and $t \ge 1$ the event $\{R_e = t\}$ depends solely on the states of edges within the box $\Lambda_{C_* t}(e)$.*

The followings give a large deviation estimate for effective radii and build a bypass along with a geodesic. The proofs are postponed until Appendix since they are standard in percolation theory.

Proposition 2.4. [3] *Let $p_0 \in (p_c(d), 1]$. There exist $C_* \ge 3$, $\lambda > 0$ and $c \in (0, 1)$ depending on p_0 such that for all $p \in [p_0, 1]$ and $H > 0$,*

$$\mathbb{P}(R_e \ge t) \le c^{-1} \exp(-c\sqrt{t}) \qquad \forall e \in \mathcal{E}(\mathbb{Z}^d), \quad \forall\, t \in [cH^2].$$

We fix C_* and λ as in Proposition 2.4, and set $q = q(p, \lambda)$ throughout the paper.

Proposition 2.5. *Let $x, y \in \mathbb{Z}^d$ and $\gamma \in \mathbb{G}_H(x, y)$ be a geodesic of $\mathrm{T}_H(x, y)$. Suppose that $e \in \gamma$ is an edge satisfying $x, y \notin \Lambda_{3R_e}(e)$. Then there exists another path η_e between x and y such that:*

(a) $\eta_e \cap \Lambda_{R_e - 1}(e) = \emptyset$ and $\eta_e \setminus \gamma$ consists only of q-open edges;
(b) $|\eta_e \setminus \gamma| \le C_* R_e$.

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[3] A stronger (exponential) bound for Proposition 2.4 is obtained in [CN23, Section 3], though the present estimate is sufficient for our current purpose.

2.3. **Lattice animals of dependent weight.** To manage the cumulative cost of edge resampling, we aim to estimate the sum of effective radii along a random path. While these effective radii are not mutually independent, their interdependence is relatively local (Remark 2.3). We utilize lattice animal theory to provide an upper bound for the sum of these radii. We first revisit a result that controls the total weight of paths in dependent environments in [CN23] using the theory of greedy lattice animals.

Let $\mathcal{P}_L$ be the set of all paths γ inside Λ_L of length at most L. For all $\gamma \in \mathcal{P}_L$, we define

$$\Gamma(\gamma) := \sum_{e\in\gamma} I_{e,N}, \quad \Gamma_{L,N} := \max_{\gamma\in\mathcal{P}_L} \Gamma(\gamma).$$

Lemma 2.6. [CN19, Lemma 2.6] *Given $N, A \in \mathbb{N}$, suppose that $(I_{e,N})_{e\in\mathcal{E}(\mathbb{Z}^d)}$ is a collection of Bernoulli random variables satisfying that for all $e \in \mathcal{E}(\mathbb{Z}^d)$, the variable $I_{e,N}$ is independent of all the random variables $(I_{e',N})_{e'\notin\mathcal{E}(\Lambda_{AN}(e))}$. Then there exists a positive constant C depending on A, d such that for all $L \in \mathbb{N}$,*

$$\mathbb{E}[\Gamma_{L,N}] \le CLN^d q_N^{1/d}, \quad \textit{where} \quad q_N := \sup_{e\in\mathcal{E}(\mathbb{Z}^d)} \mathbb{E}[I_{e,N}].$$

Proof. We give a simplified proof here. Given $A \in \mathbb{N}$, let us consider a decomposition $\mathcal{E}(\mathbb{Z}^d) = \bigcup_{i=1}^{(2dAN)^d} E_i$ such that E_i's are disjoint, and for each E_i, $\mathrm{d}_\infty(\{x,y\},\{x',y'\}) \ge 2A$ for all $e = (x,y) \ne e' = (x',y') \in E_i$ (see [CN19, Lemma 2.6] for a concrete example). This implies that $(I_{e,N})_{e\in E_i}$ are independent from each other. Let $(\bar{I}_{e,N})_{e\in\mathcal{E}(\mathbb{Z}^d)}$ be i.i.d. Bernoulli random variables where to each e, the distribution of $\bar{I}_{e,N}$ is the same as that of $I_{e,N}$. Fix $L \in \mathbb{N}$, and observe that

$$\mathbb{E}[\Gamma_{L,N}] \le \sum_{i=1}^{(2dAN)^d} \mathbb{E}\left[\max_{\gamma\in\mathcal{P}_L} \sum_{e\in\gamma\cap E_i} I_{e,N}\right] = \sum_{i=1}^{(2dAN)^d} \mathbb{E}\left[\max_{\gamma\in\mathcal{P}_L} \sum_{e\in\gamma\cap E_i} \bar{I}_{e,N}\right] \le \sum_{i=1}^{(2dAN)^d} \mathbb{E}\left[\max_{\gamma\in\mathcal{P}_L} \sum_{e\in\gamma} \bar{I}_{e,N}\right].$$

By Peierls's argument, e.g., [DHS15, Lemma 6.8], $\mathbb{E}\left[\max_{\gamma\in\mathcal{P}_L}\sum_{e\in\gamma}\bar{I}_{e,N}\right] \le \mathcal{O}(Lq_N^{1/d})$, which yields the claim. □

The following result controls the total weight of an arbitrary random path.

Lemma 2.7. *Let $A > 0$ and $(X_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ be a family of non-negative random variables such that for all $e \in \mathcal{E}(\mathbb{Z}^d)$ and $N \in \mathbb{N}$,*

$$\textit{the event } \{N-1 \le X_e < N\} \textit{ is independent of } (X_{e'})_{e'\in\mathcal{E}(\mathbb{Z}^d\setminus\Lambda_{AN}(e))}. \tag{2.4}$$

We define $q_N := \sup_{e\in\mathcal{E}(\mathbb{Z}^d)} \mathbb{P}(N-1 \le X_e < N)$. Let $f : [0,\infty) \to [0,\infty)$ be a function satisfying

$$B := \sum_{N=1}^{\infty} f_*(N)^2 N^d q_N^{1/d} < \infty, \quad \textit{where} \quad f_*(N) := \sup_{N-1\le x<N} f(x). \tag{2.5}$$

Then there exists $C = C(A,B) > 0$ such that for all random paths γ starting from 0 in the same probability space of $(X_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$, and $L \in \mathbb{N}$,

$$\mathbb{E}\left[\sum_{e\in\gamma} f(X_e)\right] \le CL + C\sum_{\ell\ge L} \ell(\mathbb{P}(|\gamma| = \ell))^{1/2}.$$

Proof. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{e\in\gamma} f(X_e)\right] &= \mathbb{E}\left[\sum_{e\in\gamma} f(X_e)\mathbf{1}(|\gamma| < L)\right] + \mathbb{E}\left[\sum_{e\in\gamma} f(X_e)\mathbf{1}_{|\gamma|\ge L}\right] \\
&\le \mathbb{E}\left[\max_{\gamma\in\mathcal{P}_L}\sum_{e\in\gamma} f(X_e)\right] + \sum_{\ell=L}^{\infty}\mathbb{E}\left[\sum_{e\in\gamma} f(X_e)\mathbf{1}_{|\gamma|=\ell}\right] \\
&\le \left(\mathbb{E}\left[\left(\max_{\gamma\in\mathcal{P}_L}\sum_{e\in\gamma} f(X_e)\right)^2\right]\right)^{1/2} + \sum_{\ell=L}^{\infty}\left(\mathbb{E}\left[\left(\max_{\gamma\in\mathcal{P}_\ell}\sum_{e\in\gamma} f(X_e)\right)^2\right]\right)^{1/2} (\mathbb{P}[|\gamma| = \ell])^{1/2}.
\end{aligned} \tag{2.6}$$

Let $m \ge L$. By Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left(\max_{\gamma\in\mathcal{P}_m}\sum_{e\in\gamma} f(X_e)\right)^2\right] \le \mathbb{E}\left[\max_{\gamma\in\mathcal{P}_m}|\gamma|\sum_{e\in\gamma} f^2(X_e)\right] \le m\mathbb{E}\left[\max_{\gamma\in\mathcal{P}_m}\sum_{e\in\gamma} f^2(X_e)\right]. \tag{2.7}$$

Let $I_{e,N} := \mathbf{1}_{N-1\le X_e<N}$. We have

$$\sum_{e\in\gamma} f^2(X_e) = \sum_{e\in\gamma}\sum_{N\ge 1} f^2(X_e) I_{e,N} \le \sum_{N\ge 1} f_*^2(N)\sum_{e\in\gamma} I_{e,N}.$$

Let $\Gamma_{m,N} := \max_{\gamma\in\mathcal{P}_m}\sum_{e\in\gamma} I_{e,N}$. Therefore,

$$\mathbb{E}\left[\max_{\gamma\in\mathcal{P}_m}\sum_{e\in\gamma} f^2(X_e)\right] \le \mathbb{E}\left[\sum_{N\ge 1} f_*^2(N)\max_{\gamma\in\mathcal{P}_m}\sum_{e\in\gamma} I_{e,N}\right] = \sum_{N\ge 1} f_*^2(N)\mathbb{E}\left[\Gamma_{m,N}\right]. \tag{2.8}$$

By Lemma 2.6 with (2.4), for all $N \ge 1$, $\mathbb{E}[\Gamma_{m,N}] = \mathcal{O}(m)N^d q_N^{1/d}$. Combined with (2.8), this yields

$$\mathbb{E}\left[\max_{\gamma\in\mathcal{P}_m}\sum_{e\in\gamma} f^2(X_e)\right] = \mathcal{O}(m)\sum_{N\ge 1} f_*^2(N)N^d q_N^{1/d} = \mathcal{O}(m),$$

by the assumption of f. Finally, combining this with (2.6) and (2.7), we derive the claim. □

Applying Lemma 2.7 with $X_e = R_e\mathbf{1}_{R_e\le M}$, $A = 2C_*$, and $f(x) = x$, since the conditions (2.4) and (2.5) follow from Remark 2.3 and Proposition 2.4 respectively, we have the following:

Corollary 2.8. *For any $C > 0$, there exists C' such that the following holds. For all $L \in \mathbb{N}$ and a random path γ starting from 0 satisfying $\mathbb{P}(|\gamma| = \ell) \le \ell^{-5}$ for all $\ell \ge CL$, we have*

$$\mathbb{E}\left[\sum_{e\in\gamma} R_e\mathbf{1}_{R_e\le M}\right] \le C'L.$$

3. LIPSCHITZ CONTINUITY OF THE TIME CONSTANT: PROOF OF THEOREM 1.2

In this section, we shall apply the results of effective radius to the truncated passage time $\mathrm{T}_M^{\Lambda_K}$. Recall λ from Section 2 and $q = pF([0,\lambda]) \ge q_0$ with $q_0 = \frac{p_0+p_c(d)}{2} > p_c(d)$.

3.1. Length of geodesics.

We recall some estimates on the sizes of holes and chemical distances.

Lemma 3.1. [Pis96, Theorem 2] *There exists $c = c(q_0) \in (0,1)$ such that for all $t \ge 1$,*

$$\mathbb{P}\left(\Lambda_t \cap \mathcal{C}_q = \emptyset\right) \le \mathbb{P}\left(\Lambda_t \cap \mathcal{C}_{q_0} = \emptyset\right) \le c^{-1}\exp(-ct^{d-1}). \tag{3.1}$$

Consequently, for all $x \in \mathbb{Z}^d$ and $t > 0$,

$$\mathbb{P}(\|x - [x]_q\|_\infty \ge t) \le c^{-1}\exp(-ct^{d-1}). \tag{3.2}$$

Lemma 3.2. [AP96, (4.49)] *There exists $\rho = \rho(q_0) \ge 1$ such that for all $x \in \mathbb{Z}^d$ and all $t \ge \rho\|x\|_\infty$,*

$$\max\{\mathbb{P}(\mathrm{D}_q(0,x) \in [t,\infty)),\, \mathbb{P}(\mathrm{D}_q([0]_q,[x]_q) \ge t)\} \le \rho\exp(-t/\rho). \tag{3.3}$$

Accordingly, it is natural to expect $\mathrm{T}_p([0]_q,[n\mathbf{e}_1]_q)/n$ is close to $\mathrm{T}_p([0]_p,[n\mathbf{e}_1]_p)/n$. In fact, it was shown in [GMPT17, Lemma 2.11] that for all $p > p_c(d)$,

$$\mu_p = \lim_{n\to\infty}\frac{\mathrm{T}_p([0]_q,[n\mathbf{e}_1]_q)}{n} \quad \text{a.s. and in } L_1. \tag{3.4}$$

Next, we cite a result on the length of a geodesic in First-passage percolation.

Lemma 3.3. [Kes86, Proposition 5.8] *Assume that G, the edge weight distribution in generalized First-passage percolation, satisfies $G(0) < p_c(d)$. Then there exists $c = c(G) \in (0,1)$ such that for all $\ell \in \mathbb{N}$,*

$$\mathbb{P}\left(\exists\,\gamma \in \mathcal{P}_*(0):\ |\gamma| \ge \ell,\ \mathrm{T}(\gamma) \le c\ell\right) \le \exp\left(-c\ell\right), \tag{3.5}$$

where $\mathcal{P}_*(0)$ is the set of all paths starting at 0.

The following result gives large deviation estimates of the length of geodesics.

Lemma 3.4. *Recall that $M = (\log n)^3$, $K = n^2$. Let $p_0 > p_c(d)$. There exists $C_1 = C_1(F,p_0) > 0$ such that for all $p \in [p_0,1]$, $\ell \ge C_1 n$ and $x \in \Lambda_{2n}(0)$, we have*

$$\max\{\mathbb{P}(\exists\,\gamma \in \mathbb{G}_M(0,x;\Lambda_K) : |\gamma| \ge \ell),\ \mathbb{P}(\exists\,\gamma \in \mathbb{G}_M(0,x) : |\gamma| \ge \ell)\} \le C_1\exp(-\ell/(C_1 M)).$$

Proof. We first claim that there exists $C = C(q_0) > 0$, for all $q \geq q_0$, and $\ell \geq n$ and $x \in \Lambda_{2n}$,

$$\mathbb{P}(\mathcal{E}_q^c) \leq \exp(-\ell/(CM)), \text{ with } \mathcal{E}_q := \{\exists\, u \in \Lambda_{\ell/M}(0) \cap \mathcal{C}_q, \exists\, v \in \Lambda_{\ell/M}(x) \cap \mathcal{C}_q : \mathrm{D}_q(u,v) \leq C\ell\}. \tag{3.6}$$

Since $q \mapsto \mathbb{P}(\mathcal{E}_q^c)$ is non-increasing, it suffices to show (3.6) with q_0. By Lemma 3.1, $\mathbb{P}(\Lambda_{\ell/M}(z) \cap \mathcal{C}_{q_0} = \emptyset) \leq e^{-\ell/(C'M)}$ for $z \in \{0, x\}$ with some $C' = C'(q_0) > 0$. Moreover, by Lemma 3.2, there exists $C'' = C''(q_0) > 0$,

$$\mathbb{P}(\exists\, u \in \Lambda_{\ell/M}(0) \cap \mathcal{C}_{q_0}, \exists\, v \in \Lambda_{\ell/M}(x) \cap \mathcal{C}_{q_0} :\ \mathrm{D}_{q_0}(u,v) > C''\ell) \leq \exp(-\ell/C''),$$

which yields (3.6). Let $q := \mathbb{P}(\tau_e \leq \lambda)$. On the event $\mathcal{E}_q$, there exist $u \in \Lambda_{\ell/M}(0) \cap \mathcal{C}_q$ and $v \in \Lambda_{\ell/M}(x) \cap \mathcal{C}_q$ such that $\mathrm{D}_q^{\Lambda_K}(u,v) \leq C\ell$. Hence, if $\ell \leq 4dnM$, then since $\ell \leq 4dnM = o(K)$, one has $\mathrm{D}_q^{\Lambda_K}(u,v) = \mathrm{D}_q(u,v)$. Thus,

$$\mathrm{T}_M^{\Lambda_K}(u,v) \leq \lambda \mathrm{D}_q^{\Lambda_K}(u,v) = \lambda \mathrm{D}_q(u,v) \leq C\lambda\ell.$$

Therefore, if $\ell \leq 4dnM$ and $\mathcal{E}_q$ occurs, for n large enough, then

$$\mathrm{T}_M^{\Lambda_K}(0,x) \leq \mathrm{T}_M^{\Lambda_K}(0,u) + \mathrm{T}_M^{\Lambda_K}(u,v) + \mathrm{T}_M^{\Lambda_K}(v,x) \leq 2d\ell + C\lambda\ell = (2d + C\lambda)\ell. \tag{3.7}$$

If $\ell > 4dnM$, then we have the same bound since $\mathrm{T}_M^{\Lambda_K}(0,x) \leq 2dM|x|_\infty \leq 4dnM$. Let $C_1 := \frac{2d + C\lambda}{c}$ with $c = c(F_1^1) \in (0,1)$ as in Lemma 3.3. We write $\mathbb{P}_G$ for the probability measure of First-passage percolation with weight distribution G. By (3.7), we get for all $\ell \geq n$,

$$\begin{aligned} \mathbb{P}(\exists\, \gamma \in \mathbb{G}_M(0,x;\Lambda_K) : |\gamma| \geq C_1\ell, \mathcal{E}_q) &\leq \mathbb{P}(\exists\, \gamma \in \mathbb{G}_M(0,x;\Lambda_K) : |\gamma| \geq C_1\ell, \mathrm{T}_M(\gamma) \leq (2d + C\lambda)\ell) \\ &\leq \mathbb{P}_{F_p^M}(\exists\, \gamma \in \mathcal{P}_*(0);\ |\gamma| \geq C_1\ell, \mathrm{T}(\gamma) \leq cC_1\ell). \end{aligned}$$

Also, we have the same bound for $\mathbb{G}_M(0,x)$ instead of $\mathbb{G}_M(0,x;\Lambda_K)$. Since F_p^M stochastically dominates F_1^1 for n large enough and $F_1^1(0) = F(0) < p_c(d)$, the right-hand side is bounded from above by

$$\mathbb{P}_{F_1^1}(\exists\, \gamma \in \mathcal{P}_*(0) :\ |\gamma| \geq C_1\ell, \mathrm{T}(\gamma) \leq cC_1\ell) \leq \exp(-cC_1\ell).$$

Combining this with (3.6), the result follows with $\max\{C, C_1\}$ in place of C_1. □

3.2. **Comparison of $\mathrm{T}_p([0]_q, [n\mathbf{e}_1]_q)$ and $\mathrm{T}_M^{\Lambda_K}(0, n\mathbf{e}_1)$.**

Proposition 3.5. *For all $p \in [p_0, 1]$, we have*

$$\mathbb{E}\left[\left|\mathrm{T}_p([0]_q, [n\mathbf{e}_1]_q) - \mathrm{T}_M^{\Lambda_K}(0, n\mathbf{e}_1)\right|\right] = \mathcal{O}(M). \tag{3.8}$$

Note that (1.3) follows by combining (3.4) and (3.8). The proof of (3.8) is divided into

$$\mathbb{E}\left[|\mathrm{T}_p([0]_q, [n\mathbf{e}_1]_q) - \mathrm{T}_M([0]_q, [n\mathbf{e}_1]_q)|\right] = \mathcal{O}(M), \tag{3.9}$$

$$\mathbb{E}\left[\left|\mathrm{T}_M([0]_q, [n\mathbf{e}_1]_q) - \mathrm{T}_M^{\Lambda_K}(0, n\mathbf{e}_1)\right|\right] = \mathcal{O}(M). \tag{3.10}$$

Proof of (3.9). Recall that $q = pF([0,\lambda]) \leq p$ and an edge e is q-open if and only if $\tau_e \leq \lambda$. Thus,

$$\max\{\mathrm{T}_p([0]_q, [n\mathbf{e}_1]_q), \mathrm{T}_M([0]_q, [n\mathbf{e}_1]_q)\} \leq \lambda \mathrm{D}_q([0]_q, [n\mathbf{e}_1]_q). \tag{3.11}$$

Let γ_M be a geodesic of $\mathrm{T}_M([0]_q, [n\mathbf{e}_1]_q)$. Define

$$\mathcal{E}_n := \mathcal{E}_n^{(1)} \cap \mathcal{E}_n^{(2)} := \{\max\{\|0 - [0]_q\|_\infty, \|n\mathbf{e}_1 - [n\mathbf{e}_1]_q\|_\infty\} \leq M\} \cap \{\forall e \in \gamma_M,\ R_e \leq (\log n)^{5/2}\},$$

Let C_1 be a positive constant as in Lemma 3.4. Note that

$$(\mathcal{E}_n^{(2)})^c \cap \mathcal{E}_n^{(1)} \cap \{|\gamma_M| \leq C_1 n\} \subset \{\exists\, e \in \mathcal{E}(\Lambda_{2C_1 n}) :\ R_e \geq (\log n)^{5/2}\}.$$

Thus, we have

$$\mathbb{P}(\mathcal{E}_n^c) \leq 2\mathbb{P}(\|0 - [0]_q\|_\infty > M) + \mathbb{P}(\mathcal{E}_n^{(1)};\ |\gamma_M| > C_1 n) + \mathbb{P}(\exists\, e \in \mathcal{E}(\Lambda_{2C_1 n}) :\ R_e \geq (\log n)^{5/2}). \tag{3.12}$$

By Lemma 3.1, there exists a positive constant c, such that

$$\mathbb{P}(\|0 - [0]_q\|_\infty > M) \leq \exp(-cM^{d-1}).$$

Using Lemma 3.4, we have

$$\mathbb{P}(\mathcal{E}_n^{(1)};\ |\gamma_M| > C_1 n) \leq C_1 (2n)^{2d} \exp(-n/(C_1 M)). \tag{3.13}$$

Finally, Proposition 2.4 yields

$$\mathbb{P}(\exists\, e \in \mathcal{E}(\Lambda_{2C_1 n}) :\ R_e \geq (\log n)^{5/2}) \leq C_2 n^d \exp(-(\log n)^{5/4}/C_2),$$

with some positive constant C_2. Putting things together, we have, with some positive constant $C > 0$,

$$\mathbb{P}(\mathcal{E}_n^c) \leq C \exp(-(\log n)^{5/4}/C). \tag{3.14}$$

We next prove that on the event $\mathcal{E}_n$,

$$\tau_e < M \text{ and } e \text{ is } p\text{-open}, \quad \forall\, e \in \gamma_M \setminus \mathcal{E}(\Lambda_{2M}(0) \cup \Lambda_{2M}(n\mathbf{e}_1)). \tag{3.15}$$

Assume $\mathcal{E}_n$ and $e \in \gamma_M \setminus \mathcal{E}(\Lambda_{2M}(0) \cup \Lambda_{2M}(n\mathbf{e}_1))$. If $[0]_q \in \Lambda_{3R_e}(e)$, then one has $\mathrm{d}_\infty(0,e) \leq \mathrm{d}_\infty(0,[0]_q) + \mathrm{d}_\infty([0]_q,e) \leq M + 3R_e < 2M-1$, which contradicts $e \in \gamma_M \setminus \mathcal{E}(\Lambda_{2M}(0) \cup \Lambda_{2M}(n\mathbf{e}_1))$. Thus, we have $[0]_q \notin \Lambda_{3R_e}(e)$. Similarly, we have $[n\mathbf{e}_1]_q \notin \Lambda_{3R_e}(e)$. Applying Proposition 2.5 to $\gamma = \gamma_M \in \mathbb{G}_M$, we obtain a path η_e from $[0]_q$ to $[n\mathbf{e}_1]_q$ such that e' is q-open for all $e' \in \eta_e \setminus \gamma_M$, i.e., $\tau_e \leq \lambda$, $|\eta_e \setminus \gamma_M| \leq C_* R_e$, and $e \notin \eta_e$. Thus,

$$\begin{aligned}
\mathrm{T}_M(\gamma_M) \leq \mathrm{T}_M(\eta_e) &= \mathrm{T}_M(\eta_e \cap \gamma_M) + \mathrm{T}_M(\eta_e \setminus \gamma_M)\\
&\leq \mathrm{T}_M(\gamma_M) - \tau_e^M + \mathrm{T}_M(\eta_e \setminus \gamma_M) \leq \mathrm{T}_M(\gamma_M) - \tau_e^M + C_*\lambda R_e,
\end{aligned}$$

which yields $\tau_e^M \leq C_*\lambda R_e < M$. Thus $\tau_e < M$, and e is p-open.

Using Lemma B.1 and Lemma 3.2, there exist $C = C(q_0) > 0$ such that for all $x \in \mathbb{Z}^d$ and $N \in \mathbb{N}$,

$$\mathbb{P}(\mathcal{E}'_N(x)) \leq C\exp(-N/C), \tag{3.16}$$

where

$$\mathcal{E}'_N(x) := \{\exists\, \eta \in \mathbb{O}_p(\Lambda_{3N}(x)) :\ \mathrm{Diam}(\eta) \geq 3N/2, \eta \cap \mathcal{C}_q = \emptyset\} \cup \{\exists\, u,v \in \Lambda_{3N}(x) : \mathrm{D}_q(u,v) \in [CN,\infty)\}.$$

Here, we remark that q has been chosen appropriately to apply Lemma B.1, see (2.2) and Appendix B.1. Suppose that $\mathcal{E}_n^* := \mathcal{E}_n \cap \mathcal{E}'_{2M}(0)^c \cap \mathcal{E}'_{2M}(n\mathbf{e}_1)^c$ occurs. On the event $\mathcal{E}_n^*$, γ_M crosses the annuli $\mathrm{A}_{2M}(0)$ and $\mathrm{A}_{2M}(n\mathbf{e}_1)$. Hence, by (3.15), we find two vertices $u \in \gamma_M \cap \mathrm{A}_{2M}(0) \cap \mathcal{C}_q$ and $v \in \gamma_M \cap \mathrm{A}_{2M}(n\mathbf{e}_1) \cap \mathcal{C}_q$, such that $\mathrm{D}_q([0]_q,u), \mathrm{D}_q([n\mathbf{e}_1]_q,v) \leq 2CM$, and $\mathrm{T}_M(u,v) = \mathrm{T}_p(u,v)$. Hence, we have

$$\mathrm{T}_p([0]_q,[n\mathbf{e}_1]_q) \leq \mathrm{T}_p([0]_q,u) + \mathrm{T}_p(u,v) + \mathrm{T}_p(v,[n\mathbf{e}_1]_q) \leq 4C\lambda M + \mathrm{T}_M([0]_q,[n\mathbf{e}_1]_q).$$

Combining this with $\mathrm{T}_M([0]_q,[n\mathbf{e}_1]_q) \leq \mathrm{T}_p([0]_q,[n\mathbf{e}_1]_q)$, we arrive at

$$|\mathrm{T}_p([0]_q,[n\mathbf{e}_1]_q) - \mathrm{T}_M([0]_q,[n\mathbf{e}_1]_q)|\mathbf{1}_{\mathcal{E}_n^*} \leq 4C\lambda M. \tag{3.17}$$

By (3.14) and (3.16), we have $\mathbb{P}((\mathcal{E}_n^*)^c) \leq C\exp(-(\log n)^{5/4}/(4C))$. By (3.11) and Lemma 3.2, we have

$$\begin{aligned}
\mathbb{E}\left[|\mathrm{T}_p([0]_q,[n\mathbf{e}_1]_q) - \mathrm{T}_M([0]_q,[n\mathbf{e}_1]_q)|\, 1_{(\mathcal{E}_n^*)^c}\right] &\leq 2\lambda\mathbb{E}\left[\mathrm{D}_q([0]_q,[n\mathbf{e}_1]_q)1_{(\mathcal{E}_n^*)^c}\right]\\
&\leq 2\lambda\left(\mathbb{E}\left[\mathrm{D}_q^2([0]_q,[n\mathbf{e}_1]_q)\right]\right)^{1/2}(\mathbb{P}((\mathcal{E}_n^*)^c))^{1/2},
\end{aligned}$$

which converges to 0 as $n \to \infty$. Combining the last two displays, we obtain (3.9). □

Proof of (3.10). We have

$$\mathbb{E}\left[\left|\mathrm{T}_M([0]_q,[n\mathbf{e}_1]_q) - \mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)\right|\right] \leq \mathbb{E}[|\mathrm{T}_M([0]_q,[n\mathbf{e}_1]_q) - \mathrm{T}_M(0,n\mathbf{e}_1)|] + \mathbb{E}\left[\left|\mathrm{T}_M(0,n\mathbf{e}_1) - \mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)\right|\right].$$

By the triangular inequality, the translation invariance, and (3.2), the first term is bounded from above by

$$\mathbb{E}[\mathrm{T}_M(0,[0]_q)] + \mathbb{E}[\mathrm{T}_M(n\mathbf{e}_1,[n\mathbf{e}_1]_q)] \leq 2dM\mathbb{E}[\mathrm{d}_\infty(0,[0]_q)] = \mathcal{O}(M).$$

We now estimate the last term. Let γ_M be a geodesic of $\mathrm{T}_M(0,n\mathbf{e}_1)$. If $|\gamma_M| < n^2 = K$, then $\mathrm{T}_M(0,n\mathbf{e}_1) = \mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)$. Therefore, since $|\mathrm{T}_M(0,n\mathbf{e}_1) - \mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)| \leq Mn$, by Lemma 3.4, we have

$$\begin{aligned}
\mathbb{E}\left[\left|\mathrm{T}_M(0,n\mathbf{e}_1) - \mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)\right|\right] &= \mathbb{E}\left[\left|\mathrm{T}_M(0,n\mathbf{e}_1) - \mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)\right| \mathbf{1}_{|\gamma_M| \geq n^2}\right]\\
&\leq Mn\mathbb{P}(|\gamma_M| \geq n^2) \leq CMn\exp(-n^2/(CM)),
\end{aligned} \tag{3.18}$$

with some $C = C(F,p_0) > 0$. This yields (3.10). □

3.3. **Bound on the derivative of first passage time.**

Proposition 3.6. *There exists a positive constant $C = C(p_0)$ such that for all $p \in [p_0,1)$,*

$$\left|\frac{\mathrm{d}\mathbb{E}\left[\mathrm{T}_M^{\Lambda_K}(0,n\mathbf{e}_1)\right]}{\mathrm{d}p}\right| \leq Cn.$$

Proof. Let $\Delta_e \mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1) := \mathrm{T}^{\Lambda_K}_{M,+,e}(0,n\mathbf{e}_1) - \mathrm{T}^{\Lambda_K}_{M,-,e}(0,n\mathbf{e}_1)$, where $\mathrm{T}^{\Lambda_K}_{M,\pm,e}(0,n\mathbf{e}_1)$ is the first passage time when the weight of the edge e is set to M for $+$ and 0 for $-$. Let γ be a geodesic of $\mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1)$. Since $\Delta_e \mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1) = 0$ for all $e \notin \gamma$ and $\mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1)$ is an increasing function of weights $(\tau^M_e)_{e\in\Lambda_K}$, applying Lemma 2.1 with $L = M, \xi = \tau^M, E = \mathcal{E}(\Lambda_K), X = \mathrm{T}^M_{\Lambda_K}(0,n\mathbf{e}_1)$, we have

$$\left| \frac{\mathrm{d}\mathbb{E}\left[\mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1)\right]}{\mathrm{d}p} \right| \leq \mathbb{E}\left[\sum_{e\in\mathcal{E}(\Lambda_K)} \Delta_e \mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1)\right] = \mathbb{E}\left[\sum_{e\in\gamma} \Delta_e \mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1)\right]. \tag{3.19}$$

We give an upper bound for (3.19). Let $(R_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$and C_* be as in Proposition 2.4. We fix $e\in\gamma$ and define $\mathcal{U}_e := \{0, n\mathbf{e}_1 \notin \Lambda_{3R_e}(e)\}$. Notice that if $|\gamma| < n^2$ then γ is a geodesic of $\mathrm{T}_M(0,n\mathbf{e}_1)$. Hence, on $\mathcal{U}_e \cap \{|\gamma| < n^2\}$, by Proposition 2.5, there exists a path η_e from 0 to $n\mathbf{e}_1$ satisfying $\eta_e \setminus \gamma$ consisting of edges with weights at most λ and $|\eta_e\setminus\gamma| \leq C_* R_e$. Thus, on $\{R_e \leq M\} \cap \mathcal{U}_e \cap \{|\gamma| < n^2\}$, one has the bound

$$\Delta_e \mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1) = \mathrm{T}^{\Lambda_K}_{M,+,e}(0,n\mathbf{e}_1) - \mathrm{T}^{\Lambda_K}_{M,-,e}(0,n\mathbf{e}_1) \leq \lambda|\eta_e\setminus\gamma| \leq C_*\lambda R_e.$$

Otherwise, we use a trivial bound $\Delta_e \mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1) \leq M$. We note that the event $\{R_e \leq M\} \cap \mathcal{U}_e^c$ implies $\mathrm{d}_\infty(0,e) \wedge \mathrm{d}_\infty(n\mathbf{e}_1,e) \leq 3M$. Therefore,

$$\begin{aligned}\sum_{e\in\gamma} \Delta_e \mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1) &\leq C_*\lambda\sum_{e\in\gamma} R_e \mathbf{1}_{R_e\leq M} + M\sum_{e\in\gamma}\mathbf{1}_{\mathrm{d}_\infty(0,e)\wedge\mathrm{d}_\infty(n\mathbf{e}_1,e)\leq 3M} + M\sum_{e\in\gamma}\mathbf{1}_{R_e>M} + M|\gamma|\mathbf{1}_{|\gamma|\geq n^2}\\ &\leq C_*\lambda\sum_{e\in\gamma} R_e \mathbf{1}_{R_e\leq M} + 4dM(6M+1)^d + M\sum_{e\in\gamma}\mathbf{1}_{R_e>M} + M|\gamma|\mathbf{1}_{|\gamma|\geq n^2}. \end{aligned} \tag{3.20}$$

On the other hand, thanks to Lemma 3.4, for all $\ell \geq Cn$ with n large enough,

$$\mathbb{P}(|\gamma|\geq\ell) \leq \exp(-\ell/(CM)) \leq \ell^{-5}, \tag{3.21}$$

where $C = C(p_0)$ is a positive constant. Therefore, using Corollary 2.8 with $L = n$,

$$\mathbb{E}\left[\sum_{e\in\gamma} R_e\mathbf{1}_{R_e\leq M}\right] \leq C'n, \tag{3.22}$$

with some $C' > 0$. In addition, by using (3.21), Proposition 2.4 and $M = (\log n)^3$,

$$\begin{aligned}\mathbb{E}\left[\sum_{e\in\gamma}\mathbf{1}_{R_e>M}\right] &\leq \mathbb{E}\left[\sum_{e\in\gamma}\mathbf{1}_{R_e>M};\ |\gamma|\leq Cn\right] + \mathbb{E}\left[|\gamma|;\ |\gamma|\geq Cn\right]\\ &\leq n^{2d}\,\mathbb{P}(\exists\, e\in\mathcal{E}(\Lambda_{Cn}) : R_e > M) + \sum_{\ell\geq Cn}\ell\,\mathbb{P}(|\gamma|=\ell) = \mathcal{O}(1),\end{aligned}$$

and $\mathbb{E}[M|\gamma|\mathbf{1}_{|\gamma|\geq n^2}] = \mathcal{O}(1)$. Combined with (3.19), (3.20) and (3.22), this yields the desired result. □

3.4. Proof of Theorem 1.2.

We write $\mathbb{E}_u$ to emphasize that the considering parameter is u. By Proposition 3.5 and (3.4), and Proposition 3.6, there exists a positive constant $C = C(p_0)$ such that for all $p_1, p_2 \in [p_0, 1]$,

$$\begin{aligned}|\mu_{p_2} - \mu_{p_1}| &= \lim_{n\to\infty}\frac{1}{n}\left|\mathbb{E}_{p_2}\left[\mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1)\right] - \mathbb{E}_{p_1}\left[\mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1)\right]\right|\\ &= \lim_{n\to\infty}\frac{1}{n}\left|\int_{p_1}^{p_2}\frac{\mathrm{d}\mathbb{E}_u\left[\mathrm{T}^{\Lambda_K}_M(0,n\mathbf{e}_1)\right]}{\mathrm{d}u}\mathrm{d}u\right| \leq C|p_2-p_1|.\end{aligned}$$

□

Appendix A. Russo's formula: Proof of Lemma 2.1

Proof. We enumerate $E = \{e_1, e_2, \ldots, e_n\}$. For all vector $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in [0,1)^n$, let $\xi^{\mathbf{p}} = (\xi^{\mathbf{p}}_{e_i})_{i\in[n]}$ be a collection of independent random variables with the distributions $(G_{p_i})_{i\in[n]}$. Let $(U_i)_{i=1}^n$ be i.i.d. random variables uniformly distributed on $[0,1]$ and $s = (s_{e_i})_{i=1}^n$ i.i.d. random variables taking values on $[0,L]$ with the same distribution of ν, which are independent from (U_i). Let us define $\omega^{\mathbf{p}} = (\omega^{\mathbf{p}}_{e_i})_{i=1}^n$ by

$$\omega^{\mathbf{p}}_{e_i} := \mathbf{1}(U_i \leq p_i)s_{e_i} + \mathbf{1}(U_i > p_i)L. \tag{A.1}$$

It is clear that $\omega^{\mathbf{p}}$ has the law as $\xi^{\mathbf{p}}$. Given $i\in[n]$, we consider $\widehat{\omega}^{\mathbf{p}}_{e_i}$ so that $\omega^{\mathbf{p}} = (\widehat{\omega}^{\mathbf{p}}_{e_i}, \omega^{\mathbf{p}}_{e_i})$ to emphasize i is the considering coordinate. Let $\mathbf{e}_i$ be the i^{th} unit vector in $\mathbb{R}^n$. If $U_i \notin (p_i, p_i+\varepsilon]$, then $X(\omega^{\mathbf{p}+\varepsilon\mathbf{e}_i}) = X(\omega^{\mathbf{p}})$.

Otherwise, $X(\omega^{\mathbf{p}+\varepsilon\mathbf{e}_i}) = X(\widehat{\omega}^{\mathbf{p}}_{e_i}, s_{e_i})$ and $X(\omega^{\mathbf{p}}) = X(\widehat{\omega}^{\mathbf{p}}_{e_i}, L)$. Therefore, by the independence of $(U_i)_{i=1}^n$ and $(s_{e_i})_{i=1}^n$, defining $f(\mathbf{p}) := \mathbb{E}[X(\xi^{\mathbf{p}})]$,

$$f(\mathbf{p}+\varepsilon\mathbf{e}_i) - f(\mathbf{p}) = \mathbb{E}\big[(X(\widehat{\omega}^{\mathbf{p}}_{e_i}, s_{e_i}) - X(\widehat{\omega}^{\mathbf{p}}_{e_i}, L))\mathbf{1}(U_i \in (p_i, p_i+\varepsilon])\big] = \varepsilon(\mathbb{E}[X(\widehat{\omega}^{\mathbf{p}}_{e_i}, s_{e_i})] - \mathbb{E}[X(\widehat{\omega}^{\mathbf{p}}_{e_i}, L)]).$$

Let $\xi^{\mathbf{p},i}$ and $\xi^{\mathbf{p},+,i}$ be the configurations obtained from $\xi^{\mathbf{p}}$ by replacing $\xi^{\mathbf{p},+,i}_{e_i}$ with s_{e_i} and with L respectively. Therefore, we have

$$\frac{\partial f(\mathbf{p})}{\partial p_i} = \lim_{\varepsilon\to 0} \frac{f(\mathbf{p}+\varepsilon\mathbf{e}_i) - f(\mathbf{p})}{\varepsilon} = \mathbb{E}[X(\xi^{\mathbf{p},i})] - \mathbb{E}[X(\xi^{\mathbf{p},+,i})].$$

Combining this with the chain rule, $\dfrac{\mathrm{d}\mathbb{E}[X]}{\mathrm{d}p} = \sum_{i=1}^n \dfrac{\partial f(\mathbf{p})}{\partial p_i}\Big|_{p_1=\ldots=p_n=p}$, we get the desired result. □

APPENDIX B. EFFECT OF RESAMPLING: PROOF OF PROPOSITIONS 2.4 AND 2.5

For $m, N \in \mathbb{N}$, let $\mathcal{B}_N(m)$ denote the set of all boxes of side length m in Λ_N.

B.1. The choice of λ and good box.

Given $p_c(d) < q \le p \le 1$ and $m, N \in \mathbb{N}$, we define

$$A_{p,q,m,N} := \{\exists\, q\text{-crossing cluster } \mathcal{C} \subset \Lambda_N,\ \exists\, \gamma \in \mathbb{O}_p(\Lambda_N) : \mathrm{Diam}(\gamma) \ge m/2,\ \gamma \cap \mathcal{C} = \emptyset\}.$$

Lemma B.1. *For all $p_0 > p_c(d)$, there exist δ_0, $C > 0$ depending on p_0, such that for all $p \in [p_0, 1]$, $q \in [p-\delta_0, p]$, and $N \in \mathbb{N}$, $(\log N)^2 \le m \le N$,*

$$\mathbb{P}(A_{p,q,m,N}) \le C\exp(-m/C).$$

Proof. First, we consider the case $m = N$. For simplicity, we write $A_{p,q,N}$ for $A_{p,q,N,N}$. Let $q_0 := (p_0 + p_c(d))/2$. By [Gri89, Lemma 7.104], for all $k, N \in \mathbb{N}$, $q \ge q_0$, we have

$$\mathbb{P}(\exists\,\text{two } q\text{-open clusters } \mathcal{C}_1, \mathcal{C}_2 \subset \Lambda_N : \mathrm{Diam}(\mathcal{C}_1), \mathrm{Diam}(\mathcal{C}_2) \ge k, \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset) \le C\exp(-k/C), \tag{B.1}$$

with C a positive constant depending on q_0.[4] Consequently, $\mathbb{P}(A_{q,q,N}) \le C\exp(-N/C)$.

Moreover, by standard use of Russo's formula, e.g., [GMPT17, (3.4)], we have

$$\mathbb{P}(A_{p,q,N}) \le \mathbb{P}(A_{q,q,N})\exp(N\log(1+(p-q)/p)).$$

Combining the last two displays, as long as q is close enough to p, we have the claim.

Next, we consider a general m. Using [Gri89, Theorem 7.68] and the assumption $(\log N)^2 \le m \le N$,

$$\mathbb{P}(\exists\, q\text{-crossing cluster} \subset \Lambda_N,\ \exists\, q\text{-crossing cluster} \subset \Lambda \text{ for all } \Lambda \in \mathcal{B}_N(m)) \le C\exp(-m/C),$$

with $C = C(p_0)$ a positive constant. It follows from this estimate and (B.1) that

$$\mathbb{P}(\mathcal{E}_{q,m,N}) \ge 1 - C\exp(-m/C), \tag{B.2}$$

where C is a positive constant depending on p_0 and

$$\mathcal{E}_{q,m,N} := \{\exists\, q\text{-crossing cluster } \mathcal{C} \subset \Lambda_N \text{ that contains a } q\text{-crossing cluster in } \Lambda \text{ for all } \Lambda \in \mathcal{B}_N(m)\}.$$

Remark that if the event $A^c_{q,q,N}$ occurs, then there is at most one q-crossing cluster in Λ_N. Notice further that given a path γ with $\mathrm{Diam}(\gamma) \ge m$ in Λ_N, we can find $\Lambda \in \mathcal{B}_N(m)$ such that Λ contains a sub-path of γ with diameter at least $m/2$. Therefore, if $A_{p,q,m,N} \cap A^c_{q,q,N} \cap \mathcal{E}_{q,m,N}$ occurs, then there exists a box $\Lambda \in \mathcal{B}_N(m)$, a p-open path $\gamma' \in \mathbb{O}_p(\Lambda)$ with $\mathrm{Diam}(\gamma') \ge m/2$ and a q-crossing cluster $\mathcal{C}' \subset \Lambda$ such that $\gamma' \cap \mathcal{C}' = \emptyset$. Hence, using the claim for $A_{p,q,m}$ and $(\log N)^2 \le m \le N$,

$$\mathbb{P}(A_{p,q,m,N} \cap A^c_{q,q,N} \cap \mathcal{E}_{q,m,N}) \le |\mathcal{B}_N(m)|\mathbb{P}(A_{p,q,m}) \le C\exp(-m/C),$$

with C a positive constant. Combining all together gives the desired result. □

With a positive constant δ_0 as in Lemma B.1, the constant $\lambda = \lambda(\delta_0, p_0, F)$ is then defined as in (2.2).

Lemma B.2. *There exist $C = C(p_0) \ge 3$ such that for all $t \ge C$, $H > 0$ and $N \in [H^2/C]$,*

$$\mathbb{P}(\exists\, q\text{-crossing cluster } \mathcal{C} \subset \Lambda_N,\ \exists\, \pi \in \mathcal{P}(\Lambda_N) \cap \mathbb{G}_H(\Lambda_{tN}) : \mathrm{Diam}(\pi) \ge N/2,\ \pi \cap \mathcal{C} = \emptyset) \le C\exp(-\sqrt{N}/C).$$

---

[4] Though [Gri89, Lemma 7.104] is only stated in $d \ge 3$, the result also holds for planar percolation by standard arguments.

Proof. Using Lemma B.1, there exists $C_1 = C_1(q_0) > 0$ such that for all $N \geq 1$,

$$\mathbb{P}(\exists\, q\text{-crossing cluster } \mathcal{C} \subset \Lambda_N,\ \exists\, \pi \in \mathbb{O}_p(\Lambda_N) : \mathrm{Diam}(\pi) \geq \sqrt{N},\ \pi \cap \mathcal{C} = \emptyset) \leq C_1 \exp(-\sqrt{N}/C_1).$$

Hence, the result follows if there exists $C_2 = C_2(q_0) > 0$ such that for all $N \leq M^2/C_2$,

$$\begin{aligned} &\mathbb{P}(\forall\, \pi \in \mathcal{P}(\Lambda_N) \cap \mathbb{G}_H(\Lambda_{tN}) \text{ with } \mathrm{Diam}(\pi) \geq N/2,\ \exists\, \eta \subset \pi : \eta \in \mathbb{O}_p(\Lambda_N) \text{ and } \mathrm{Diam}(\eta) \geq \sqrt{N}) \\ &\quad \geq 1 - C_2 \exp(-\sqrt{N}/C_2). \end{aligned} \tag{B.3}$$

Let $\mathrm{cl}_p(\pi)$ denote the set of p-closed edges of π. Observe that if $\mathrm{Diam}(\pi) \geq N/2$ and $|\mathrm{cl}_p(\pi)| \leq \sqrt{N}/2$, then π contains a p-open sub-path, say η, with $\mathrm{Diam}(\eta) \geq \sqrt{N}$. Moreover, if $\pi = (x, \dots, y) \in \mathbb{G}_H(x, y; \Lambda_{tN})$ satisfies $|\mathrm{cl}_p(\pi)| \geq \sqrt{N}/2$, then $\mathrm{T}_H^{\Lambda_{tN}}(x, y) = \mathrm{T}_H(\pi) \geq \sqrt{N}H/2$. Hence, it suffices to show

$$\mathbb{P}(\exists\, x, y \in \Lambda_N : \mathrm{T}_H^{\Lambda_{tN}}(x, y) \geq \sqrt{N}H/2) \leq C_2 \exp(-\sqrt{N}/C_2), \tag{B.4}$$

with some $C_2 = C_2(q_0) > 0$. By Lemmas 3.1 and 3.2, there exists $C_3 = C_3(q_0) > 0$ such that

$$\begin{aligned} &\mathbb{P}(\mathcal{A}_N) \leq C_3 \exp(-\sqrt{N}/C_3), && \mathcal{A}_N := \{\exists\, x \in \Lambda_N : \mathrm{d}_1(x, [x]_q) \geq \sqrt{N}/8\}, \\ &\mathbb{P}(\mathcal{B}_N) \leq C_3 \exp(-N/C_3), && \mathcal{B}_N := \{\exists\, u, v \in \Lambda_{2N} \cap \mathcal{C}_q : \mathrm{D}_q(u, v) \geq C_3 N\}. \end{aligned}$$

Given $x, y \in \Lambda_N$, let η_x (resp. η_y) be a shortest path in $\mathbb{Z}^d$-lattice from x to $[x]_q$ (resp. from y to $[y]_q$), and $\eta_{x,y}$ a geodesic of $\mathrm{D}_q([x]_q, [y]_q)$. Construct a path from x to y by $\eta := \eta_x \cup \eta_{x,y} \cup \eta_y$. On the event, $\mathcal{A}_N^c \cap \mathcal{B}_N^c$, for all $t \geq 2C_3$ and $x, y \in \Lambda_N$, since $\eta \in \mathcal{P}(\Lambda_{2C_3 N})$,

$$\mathrm{T}_H^{\Lambda_{tN}}(x, y) \leq \mathrm{T}_H(\eta_x) + \mathrm{T}_H(\eta_{x,y}) + \mathrm{T}_H(\eta_y) \leq H[\mathrm{d}_1(x, [x]_q) + \mathrm{d}_1(y, [y]_q)] + \lambda \mathrm{D}_q([x]_q, [y]_q) < \sqrt{N}H/2,$$

provided that $N \leq H^2/(8C_3\lambda)^2$. Hence, (B.4) follows. □

Recall $\mathrm{A}_N(e) = \Lambda_{3N}(e) \setminus \Lambda_N(e)$. Fix ρ and $C(p_0)$ as in Lemma 3.2, B.2, and set

$$N_\rho := \lfloor N/8\rho^2 \rfloor, \quad C_* := C(p_0) + (48\rho^2)^d. \tag{B.5}$$

Definition B.3. *For each $e \in \mathcal{E}(\mathbb{Z}^d)$, we say that the box $\Lambda_{3N}(e)$ is q-**good** if the following hold:*

(i) There exists a q-crossing cluster $\mathcal{C}$ in Λ_{3N} that contains a crossing cluster in Λ for all $\Lambda \in \mathcal{B}_{3N}(N_\rho)$,

(ii) For all $x, y \in \mathrm{A}_N(e)$ with $\mathrm{d}_\infty(\{x, y\}, \partial \mathrm{A}_N(e)) \geq N/2$ and $\mathrm{d}_\infty(x, y) \leq 2N_\rho$, if $\mathrm{D}_q(x, y) < \infty$, then $\mathrm{D}_q^{\mathrm{A}_N(e)}(x, y) = \mathrm{D}_q(x, y) \leq 4\rho N_\rho$.

(iii) If $\pi \in \mathcal{P}(\Lambda_{3N}(e)) \cap \mathbb{G}_H(\Lambda_{C_* N}(e))$ satisfies $\mathrm{Diam}(\pi) \geq N_\rho$, *then $\pi \cap \mathcal{C} \neq \emptyset$.*

Lemma B.4. *There exists $C = C(p_0) > 0$ such that for all $q \geq q_0$, $H > 0$ and $N \in [H^2/C]$*

$$\mathbb{P}(\Lambda_{3N}(e) \text{ is } q\text{-}\textbf{\textit{good}}) \geq 1 - C \exp(-\sqrt{N}/C).$$

Proof. Using (B.2), there exists a positive constant $C = C(p_0)$, such that

$$\mathbb{P}(\Lambda_{3N} \text{ does not satisfies (i)}) \leq \mathbb{P}(\mathcal{E}^c_{q, N_\rho, 3N}) \leq C \exp(-N/C).$$

Observe that if $\mathrm{A}_N(e)$ does not satisfy (ii), then there exist $x, y \in \mathrm{A}_N(e)$ such that $\mathrm{d}_\infty(\{x, y\}, \partial \mathrm{A}_N(e)) \geq N/2$, $\mathrm{d}_\infty(x, y) \leq 2N_\rho$, $\mathrm{D}_q(x, y) \in [4\rho N_\rho, \infty)$. Hence, thanks to the union bound and Lemma 3.2, there exists a positive constant $C = C(p_0, \rho) > 16\rho^2$ such that

$$\mathbb{P}(\Lambda_{3N} \text{ does not satisfy (ii)}) \leq C |\mathrm{A}_N(e)|^2 \exp(-N_\rho/C) \leq C \exp(-N/(C^2)). \tag{B.6}$$

Suppose now that $\mathrm{A}_N(e)$ satisfies (i) but not (iii). Then there exist $\pi \in \mathcal{P}(\Lambda_{3N}(e)) \cap \mathbb{G}_H(\Lambda_{C_* N}(e))$ and a q-crossing cluster $\mathcal{C} \subset \Lambda_{3N}$ such that $\mathrm{Diam}(\pi) \geq N_\rho$, and $\mathcal{C}$ crosses all $\Lambda \in \mathcal{B}_{3N}(N_\rho)$, and $\pi \cap \mathcal{C} = \emptyset$. Note that there exists a vertex $x \in \Lambda_{3N}$ and a sub-path $\pi' \in \mathcal{P}(\Lambda_{N_\rho/2}(x)) \cap \mathbb{G}_H(\Lambda_{C_* N_\rho/2}(x))$ of π such that $\mathrm{Diam}(\pi') \geq N_\rho/2$ and $\pi' \cap \mathcal{C} = \emptyset$. Thus, by Lemma B.2, there exists $C = C(p_0, \rho) > 0$ such that

$$\begin{aligned} &\mathbb{P}(\Lambda_{3N} \text{ satisfies (i) but not (iii)}) \\ &\leq \mathbb{P}\begin{pmatrix} \exists\, x \in \Lambda_{3N},\ \exists\, q\text{-crossing cluster } \mathcal{C}' \subset \Lambda_{N_\rho/2}(x), \exists\, \pi' \in \mathcal{P}(\Lambda_{N_\rho/2}(x)) \cap \mathbb{G}_H(\Lambda_{C_* N_\rho/2}(x)) : \\ \mathrm{Diam}(\pi') \geq N_\rho/2,\ \pi' \cap \mathcal{C}' = \emptyset \end{pmatrix} \\ &\leq C N^d \exp(-\sqrt{N}/C). \end{aligned}$$

Putting things together, we have the claim. □

B.2. **Proof of Proposition 2.4.** Recall ρ, N_ρ, and C_* from Lemma 3.1 and (B.5). Let

$$\mathcal{V}_N(e) := \{\forall\, \gamma_1, \gamma_2 \in \mathbb{G}_H(\Lambda_{C_* N}(e)) \cap \mathscr{C}(\mathrm{A}_N(e)),\ \mathrm{D}_q^{\mathrm{A}_N(e)}(\gamma_1, \gamma_2) \le C_* N\}.$$

Fix $e \in \mathcal{E}(\mathbb{Z}^d)$. By the definition of R_e and Lemma B.4, the result follows from

$$\{\Lambda_{3N}(e) \text{ is } q\textbf{-good}\} \subset \mathcal{V}_N(e). \tag{B.7}$$

To this end, we assume that $\Lambda_{3N}(e)$ is q-**good**. Let $\gamma_1, \gamma_2 \in \mathbb{G}_H(\Lambda_{C_* N}(e)) \cap \mathscr{C}(\mathrm{A}_N(e))$. For each $j \in \{1,2\}$, there exists a connected path $\pi_j \subset \gamma_j \cap \left\{\Lambda_{2N+\frac{N_\rho}{2}}(e) \setminus \Lambda_{2N-\frac{N_\rho}{2}}(e)\right\}$ satisfying

$$\forall j \in \{1,2\}, \quad \pi_j \in \mathcal{P}(\Lambda_{3N}(e)) \cap \mathbb{G}_H(\Lambda_{C_* N}(e)), \quad \mathrm{diam}(\pi_j) \ge N_\rho, \quad \mathrm{d}_\infty(\pi_j, \partial \mathrm{A}_N(e)) \ge 3N/4.$$

Then by Definition B.3 (iii), we have $\pi_1 \cap \mathcal{C} \neq \emptyset$ and $\pi_2 \cap \mathcal{C} \neq \emptyset$, with $\mathcal{C}$ the cluster crossing all sub-boxes of side-length N_ρ of Λ_{3N}. Therefore, there exist $u, v \in \mathrm{A}_N(e)$ such that $u \in \pi_1 \cap \mathcal{C}$, $v \in \pi_2 \cap \mathcal{C}$, and $\mathrm{d}_\infty(\{u,v\}, \partial \mathrm{A}_N(e)) \ge 3N/4$. Moreover, since $\mathcal{C}$ contains a crossing cluster in Λ for all $\Lambda \in \mathcal{B}_{3N}(N_\rho)$, we find a sequence of vertices $(x_i)_{i=0}^h \subset \mathcal{C}$ with $h \le (6N/N_\rho)^d = (48\rho^2)^d$ such that

$$x_0 = u,\ x_h = v; \quad \mathrm{d}_\infty(x_i, \partial \mathrm{A}_N(e)) \ge N/2 \quad \forall\, i \in [h-1]; \qquad \mathrm{d}_\infty(x_{i-1}, x_i) \le 2N_\rho \quad \forall\, i \in [h].$$

Remark further that $\mathrm{D}_q(x_{i-1}, x_i) < \infty$, as $(x_i)_{i=0}^h \subset \mathcal{C}$. Hence, it follows from Definition B.3 (ii) that $\mathrm{D}_q^{\mathrm{A}_n(e)}(x_{i-1}, x_i) \le 4\rho N_\rho$. Therefore, $\mathcal{V}_N(e)$ holds since

$$\mathrm{D}_q^{\mathrm{A}_N(e)}(\gamma_1, \gamma_2) \le \sum_{i=1}^h \mathrm{D}_q^{\mathrm{A}_N(e)}(x_{i-1}, x_i) \le (6N/N_\rho)^d (4\rho N_\rho) \le C_* N.$$

□

B.3. **Proof of Proposition 2.5.** Assume that $\gamma = (x_i)_{i=1}^\ell \in \mathbb{G}_H$ is a path between x and y with $x, y \in \mathbb{Z}^d$. If $e \in \gamma$ and $x, y \notin \Lambda_{3R_e}(e)$, then γ crosses the annulus $\mathrm{A}_{R_e}(e)$ at least twice. The first and last sub-path of γ crossing A are defined by $\gamma_1 = (x_{i_-}, \dots, x_{i_+})$ and $\gamma_2 = (x_{o_-}, \dots, x_{o_+})$, where

$$\begin{aligned} i_+ &:= \min\{i \ge 1 : x_i \in \partial \Lambda_N\}, \quad i_- := \max\{i \le i_+ : x_i \in \partial \Lambda_{3N}\}, \\ o_- &:= \max\{i \ge 1 : x_i \in \partial \Lambda_N\}, \quad o_+ := \min\{i \ge o_- : x_i \in \partial \Lambda_{3N}\}. \end{aligned}$$

We have $\gamma_1, \gamma_2 \in \mathbb{G}_H$ and $\gamma_1, \gamma_2 \subset \mathrm{A}_{R_e}(e) \subset \Lambda_{C_* R_e}(e)$, which implies $\gamma_1, \gamma_2 \in \mathscr{C}(\mathrm{A}_{R_e}(e)) \cap \mathbb{G}_H(\Lambda_{C_* R_e}(e))$. By definition of R_e, $\mathrm{D}_q^{\mathrm{A}_{R_e}(e)}(\gamma_1, \gamma_2) \le C_* R_e$. Let $\widetilde{\eta}_e$ be a geodesic of $\mathrm{D}_q^{\mathrm{A}_{R_e}(e)}(\gamma_1, \gamma_2)$. Then it is a q-open path $\widetilde{\eta}_e$ such that $|\widetilde{\eta}_e| = \mathrm{D}_q^{\mathrm{A}_{R_e}(e)}(\gamma_1, \gamma_2) \le C_* R_e$. For $u, v \in \gamma$, we write $\gamma_{u,v}$ for the sub-path of γ from u to v. Let z_1 and z_2 be points where the path $\widetilde{\eta}_e$ intersects with γ_1 and γ_2, respectively. We define

$$\eta_e := \gamma_{x, z_1} \cup \widetilde{\eta}_e \cup \gamma_{z_2, y}.$$

Notice that $|\eta_e \setminus \eta| = |\widetilde{\eta}_e| \le C_* R_e$. Furthermore, since γ_1 and γ_2 are first and last sub-path of γ crossing $\mathrm{A}_{R_e}(e)$, one has $\gamma_{x, z_1} \cap \Lambda_{R_e - 1}(e) = \emptyset$ and $\gamma_{z_2, y} \cap \Lambda_{R_e - 1}(e) = \emptyset$. In addition, $\widetilde{\eta}_e \cap \Lambda_{R_e - 1}(e) = \emptyset$ since $\widetilde{\eta}_e \subset \mathrm{A}_{R_e}(e)$. Hence, $\eta_e \cap \Lambda_{R_e - 1}(e) = \emptyset$. Hence, η_e is a desired path. □

APPENDIX C. THE STRONG CONVERGENCE TO TIME CONSTANT: PROOF OF THEOREM 1.1

Theorem 1.1 directly follows from Kingman's sub-additive ergodic theorem, e.g., [ADH17, Theorem 2.2], assuming the following integrability of passage time recalling that $\widetilde{\mathrm{T}}_p(x,y) := \mathrm{T}_p([x]_p, [y]_p)$.

Lemma C.1. *If* $\mathbb{E}[\tau \mathbf{1}_{\tau < \infty}] < \infty$ *and* $p > p_c(d)$*, then* $\mathbb{E}[\mathrm{T}_p([0]_p, [\mathbf{e}_1]_p)] < \infty$.

Proof. Define $X := \inf\{m : \mathrm{D}_p^{\Lambda_m}([0]_p, [\mathbf{e}_1]_p) < \infty\}$. If $X = k$, then $[0]_p$ and $[\mathbf{e}_1]_p$ are connected in Λ_k, and thus $\widetilde{\mathrm{T}}_p(0, \mathbf{e}_1) \le \sum_{e \in \Lambda_k} \tau_e \mathbf{1}_{\tau_e < \infty}$. Let $\mathcal{E}_k := \{X \ge k\} = \{\mathrm{D}_p^{\Lambda_{k-1}}([0]_p, [\mathbf{e}_1]_p) = \infty\}$. Hence,

$$\mathbb{E}[\mathrm{T}_p([0]_p, [\mathbf{e}_1]_p)] \le \sum_{k=1}^\infty \mathbb{E}\left[\sum_{e \in \Lambda_k} \tau_e \mathbf{1}_{\tau_e < \infty} \mathbf{1}_{X = k}\right] \le \sum_{k=1}^\infty \mathbb{E}\left[\sum_{e \in \Lambda_k} \tau_e \mathbf{1}_{\tau_e < \infty} \mathbf{1}_{\mathcal{E}_k}\right]. \tag{C.1}$$

Since the event $\mathcal{E}_k$ is measurable with $(\mathbf{1}_{\tau_e < \infty})_{e \in \mathcal{E}(\mathbb{Z}^d)}$, we have

$$\mathbb{E}\left[\tau_e \mathbf{1}_{\tau_e < \infty} \mathbf{1}_{\mathcal{E}_k}\right] = \mathbb{E}\left[\tau_e \mathbf{1}_{\tau_e < \infty} \mathbb{E}\left[\mathbf{1}_{\mathcal{E}_k} \mathbf{1}_{\tau_e < \infty} \mid \tau_e\right]\right] \le \mathbb{E}\left(\tau_e \mathbf{1}_{\tau_e < \infty}\right) \mathbb{P}(\mathcal{E}_k) / \mathbb{P}(\tau_e < \infty).$$

By Lemma 3.1 and Lemma 3.2, there exists a positive constant c, such that

$$\mathbb{P}(\mathcal{E}_k) \le \mathbb{P}(\{[0]_p, [\mathbf{e}_1]_p\} \not\subset \Lambda_{ck}) + \mathbb{P}(\exists\, u, v \in \Lambda_{ck} : \mathrm{D}_p(u,v) \in (k/2, \infty)) \le c^{-1} \exp(-ck).$$

Combining this with (C.1) yields that

$$\mathbb{E}[\mathrm{T}_p([0]_p, [\mathbf{e}_1]_p)] \leq \sum_{k=1}^{\infty} (2k+1)^d (pc)^{-1} \exp(-ck) \mathbb{E}\left[\tau_e \mathbf{1}_{\tau_e < \infty}\right] < \infty.$$

□

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