MEAN-FIELD SPIN MODELS – FLUCTUATION OF THE MAGNETIZATION AND MAXIMUM LIKELIHOOD ESTIMATOR

VAN HAO CAN * ADRIAN RÖLLIN †

Abstract

Consider the mean-field spin models where the Gibbs measure of each configuration depends only on its magnetization. Based on the Stein and Laplace methods, we give a new and short proof for the scaling limit theorems with convergence rate for the magnetization in a perturbed model. As an application, we derive the scaling limit theorems for the maximum likelihood estimator in linear models.

1 INTRODUCTION

The Ising model was originally proposed for the purpose to study the properties of ferromagnetic materials, but it has become since a prototype spin model on general graphs, see Ellis (1985); Hofstad (2021+); Niss (2005, 2009). Recently, it has also become a model for describing the pairwise interactions in networks, see e.g. Contucci and Giardina (2013); Geman and Graffigne (1986); Green and Richardson (2002) for its application in social networks, computer vision, and biology. However, in some situations, pairwise interaction is not enough to express the dependence of spins in networks, which motivated the study of higher-order Ising models, where multi-atom interactions are allowed; see for example Heringa, Blote and Hooglan (1989); Suzuki (1972); Yamashiro, Ohkuwa, Nishimori and Lidar (2019). As far as we know, one of the first rigorous results in this direction is due to Mukherjee, Son and Bhattacharya (2021), who considered the p-spin Curie-Weiss model for $p \geqslant 2$, given by the Gibbs measure

$$\mu_n(\omega) \propto \exp\left(\frac{\beta}{n^{p-1}} \sum_{1 \leqslant i_1, \ldots, i_p \leqslant n} \omega_{i_1} \ldots \omega_{i_p} + h \sum_{i=1}^{n} \omega_i\right), \quad \omega \in \Omega_n = \{1, -1\}^n, \tag{1.1}$$

where $\beta > 0$ denotes the inverse temperature and where $h \in \mathbb{R}$ denotes the external field; here and below, for any measure μ, the notation $\mu(\omega) \propto f(w)$ means that the value of $\mu(\omega)$ is proportional to $f(\omega)$ up to a normalising constant that only depends on the model parameters. In (1.1), all possible p-tuples in the complete graph of size n contribute to the Hamiltonian, and

*Institute of Mathematics, Vietnam Academy of Science and Technology. Email: cvhao@math.ac.vn.

†Department of Statistics and Data Science, Vietnam National University of Singapore. Email: adrian.roellin@nus.edu.sg.

hence to the corresponding Gibbs measure, and it can therefore be rewritten as

$$\mu_n(\omega) \propto \exp\big(n\big(\beta\bar{\omega}^p + h\bar{\omega}\big)\big), \quad \bar{\omega} = \frac{\omega_1 + \ldots + \omega_n}{n}, \tag{1.2}$$

and using this compact form and the Laplace method, Mukherjee, Son and Bhattacharya (2021) and Mukherjee, Son and Bhattacharya (2021+) investigated the fluctuation of the magnetization $\bar{\omega}$, as well as the maximum likelihood estimators for the parameters β and h.

In this article, we go further and study the fluctuation of the magnetisation the case where the interaction can be expressed by a general (smooth enough) function of $\bar{\omega}$ instead of just being a polynomial of $\bar{\omega}$ as in (1.2). Consider the generalized linear model

$$\mu_n(\omega) \propto \exp\Big(n\big(\beta_1 f_1(\bar{\omega}_+) + \ldots + \beta_l f_l(\bar{\omega}_+)\big)\Big), \quad \bar{\omega}_+ = \frac{\big|\{i : \omega_i = 1\}\big|}{n}, \tag{1.3}$$

where $f_1, \ldots, f_l$ are smooth functions and $\beta_1, \ldots, \beta_l$ are real-valued model parameters. Note that $\bar{\omega}_+ = (\bar{\omega}+1)/2$, and so studying $\bar{\omega}$ and $\bar{\omega}_+$ is equivalent. Particularly, if we choose $l = 2$ and $f_1(a) = 2a-1$, $\beta_1 = h$, $f_2(a) = (2a-1)^p$, and $\beta_2 = \beta$, we obtain the p-spin Curie-Weiss model in (1.2).

Denoting $X_n = |\{i : \omega_i = 1\}| = n\bar{\omega}_+$, the linear model (1.3) can be characterized by the simpler model

$$\mathbb{P}[X_n = k] \propto \exp\big(nF(k/n)\big)\binom{n}{k}, \quad 0 \leqslant k \leqslant n,$$

where

$$F(a) = \beta_1 f_1(a) + \ldots + \beta_l f_l(a), \qquad a \in [0,1].$$

Observe further that

$$\frac{1}{n}\log\binom{n}{k} \approx I(k/n), \quad I(a) = -a\log a + (a-1)\log(1-a),$$

here, I is the entropy function. Combining Stein's method for normal approximation and Laplace's method, we derive a complete description of the fluctuation of $\bar{\omega}_+$ (and thus, $\bar{\omega}$). It turns out that the order of the fluctuation depends on the order of regularity of the maximizers of a function $A : [0,1] \to \mathbb{R}$ given as

$$A(a) = F(a) + I(a). \tag{1.4}$$

For instance, if A has a unique maximizer at, say, $a_* \in (0,1)$ and if this maximizer is $2m$-regular, that is, if $A^{(k)}(a_*) = 0$ for $1 \leqslant k \leqslant 2m-1$ and if $A^{(2m)}(a_*) < 0$, then our general result implies that $\bar{\omega}_+$ concentrates around a_* and the order of concentration is $n^{-1/(2m)}$.

The second question we address in this article is the construction of suitable estimators of the model parameters. The maximum likelihood estimators (MLEs) in the p-spin Curie-Weiss model was studied by Comets and Gidas (1991) for $p = 2$ and by Mukherjee, Son and Bhattacharya (2021+) for $p \geqslant 3$, and for Markov random fields on lattices by Comets (1992); Pickard (1987). The maximum pseudo likelihood estimation problem of the

Ising model on general graphs has been discussed by Chatterjee (2007) and Ghosal and Mukherjee (2020). We refer to Mukherjee, Son and Bhattacharya (2021+) and the references therein for further discussion on the history and development of the problem.

In this article, we follow the usual approach to construct the MLE for each parameter β_i using only one sample ω. In fact, we can construct a consistent estimator $\hat{\beta}_{i,n}$ of β_i using only the quantity $\bar{\omega}_+$; see more in Section 4. Apart from consistency, we can also show that, after suitable scaling, $\hat{\beta}_{i,n} - \beta_i$ converges to a non-degenerate random variable. A standard approach to study the fluctuation and scaling limits of $\hat{\beta}_{i,n}$ is to prove limit theorems for a perturbed model of (1.3); see for example Comets and Gidas (1991) and Mukherjee, Son and Bhattacharya (2021+) for p-spin Curie-Weiss models.

In the general setting, we consider the perturbed model

$$\mathbb{P}[X_n = k] \propto \exp\Big(nA_n(k/n) + n^{1/(2m)}B_n(k/n)\Big), \quad 0 \leqslant k \leqslant n, \tag{1.5}$$

where $A_n, B_n : \{0, \frac{1}{n}, \dots, 1\} \to \mathbb{R}$; here, A_n is the main term driving the model and B_n is the perturbation. We assume in addition that A_n and B_n are well approximated by smooth functions $A, B : [0,1] \to \mathbb{R}$ and $2m$ is the regularity order of the maximizers of A. Particularly, for the linear model (1.3), the knowledge of the fluctuation of X_n with A given by (1.4) and B suitably chosen would lead to the scaling limit of estimators $\hat{\beta}_{1,n}, \dots, \hat{\beta}_{l,n}$ of the linear model (1.3). We refer to Section 4 for detailed proofs.

The usual strategy to investigate the Gibbs measure of the form (1.3) (or the more general form (1.5)) is using Laplace's method to prove the concentration and scaling limit of magnetization around maximizers of $A(a)$. This approach usually requires many tedious and difficult computations of exponential functionals. Our main innovation in the study of the perturbed model (1.5) is exploiting Stein's method to avoid some of these complicated computations. Moreover, as a additional bonus of using Stein's method, we also obtain the rate of convergence in our limit theorems. We refer to Section 2 for more details.

We briefly summarize the main findings of this paper.

- ▷ In Theorems 2.1–2.3, combining Stein's and Laplace's methods, we give a short proof for scaling limit theorems of the magnetization (or for X_n) with convergence rate in Wasserstein distance. This distance measures the difference of random variables over the space of test functions having bounded first derivative.

- ▷ In Theorem 4.1, applying the limit theorems of magnetization in pertubed models, we show the scaling limits of maximum likelihood estimators of the linear model form (1.3).

1.1 Notation

For any random variables X and Y, we consider the Kolmogorov and Wasserstein probability metrics, defined as

$$d_{\mathrm{K}}(X,Y) = \sup_{t\in\mathbb{R}} |\,\mathbb{P}[X \leqslant t] - \mathbb{P}[Y \leqslant t]|,$$

$$d_{\mathrm{W}}(X,Y) = \sup_{\|h'\|\leqslant 1} |\mathbb{E}h(X) - \mathbb{E}h(Y)|.$$

For $a > 0$, we denote by $N^{+}(0,a)$ (resp. $N^{-}(0,a)$) the positive (resp. negative) half-normal distribution, that is the distribution of $|N(0,a)|$ (resp. $-|N(0,a)|$). Let X be a random variable with density $p(x)$. We write $p(x) \propto f(x)$ if $p(x)$ is proportional to $f(x)$ up to a normalizing constant, and in such a case, we also write $X \propto f(x)$ if X has distribution with density given by $p(x)$. Let f and g be two real functions. We write $f = \mathrm{O}(g)$ if there exists a universal constant $C > 0$ such that $f(x) \leqslant Cg(x)$ for all x in the domain of f and g. We also write $f = g + \mathrm{O}(h)$ when $|f - g| = \mathrm{O}(|h|)$, and write $f = \exp(g + \mathrm{O}(h))$ if $|\log f - g| = \mathrm{O}(|h|)$. In some cases, we write $f = \mathrm{O}_{\delta}(g)$ to emphasize that the constant C may depend on δ.

2 THE MAGNETIZATION IN PERTURBED MODELS

Let $A_n, B_n : \{0, 1/n, \dots, 1\} \to \mathbb{R}$ and $m_* \in \mathbb{N}$. We consider the integer-valued random variable X_n defined by the model

$$\mathbb{P}[X_n = k] = \frac{1}{Z_n} \exp\bigl(H_n(k/n)\bigr), \quad 0 \leqslant k \leqslant n,$$

where

$$H_n(k/n) = nA_n(k/n) + n\sigma_{*,n}B_n(k/n), \quad \sigma_{*,n} = n^{-1+1/(2m_*)},$$
$$Z_n = \sum_{k=0}^{n} \exp\bigl(H_n(k/n)\bigr).$$

In what follows, we will make use of various technical assumptions. Let ε_*, δ_*, and C_* be positive constants, let $(a_j, m_j)_{j\in J}$ be a finite collection of pairs with $a_j \in (0,1)$ and $m_j \in \mathbb{N}$ for $j \in J$, and let $A, B : [0,1] \to \mathbb{R}$ be functions such that $A \in C^{2m_*+1}([0,1])$ and $B \in C^{2}([0,1])$. Consider the following assumptions:

(A1) $(a_j)_{j\in J}$ are all the maximizers of A, and $\max_{j\in J} m_j = m_*$. We have $A'(a_j) = \ldots = A^{(2m_j-1)}(a_j) = 0$ and $\max_{|x-a_j|\leqslant\delta_*} A^{(2m_j)}(x) < 0$ for all $j \in J$. The intervals $(a_j - \delta_*, a_j + \delta_*)$, $j \in J$, are disjoint and contained in $(0,1)$.

(A2) For n large enough and for all k for which $|k/n - a_j| \geqslant \delta_*$ for all $j \in J$, we have

$$A_n(k/n) \leqslant \max_{x\in[0,1]} A(x) - \varepsilon_*, \quad |B_n(k/n)| \leqslant C_*.$$

(A3) For n large enough and for all k and ℓ for which there is $j \in J$ such that $|k/n - a_j| < \delta_*$ and $|\ell/n - a_j| < \delta_*$, we have

$$|A_n(k/n) - A(k/n)| + |B_n(k/n) - B(k/n)| \leqslant C_*/n,$$

and

$$\bigl|[A_n(k/n) - A_n(\ell/n)] - [A(k/n) - A(\ell/n)]\bigr| \leqslant C_*|k-\ell|/n^2,$$
$$\bigl|[B_n(k/n) - B_n(\ell/n)] - [B(k/n) - B(\ell/n)]\bigr| \leqslant C_*|k-\ell|/n^2.$$

(A4) For n large enough and for $k_j = [na_j]$, $j \in J$, we have

$$\sup_{i,j\in J_2} \big|[A_n(k_i/n) - A_n(k_j/n)]\big| \leqslant \frac{C_*}{n^2\sigma_{*,n}},$$

where

$$J_1 = \{j \in J : B(a_j) = \max_{k\in J} B(a_k)\}, \quad J_2 = \{j \in J_1 : m_j = \max_{k\in J_1} m_k\}.$$

Theorem 2.1 (Weak law of large numbers). *Under Assumptions* (A1)–(A4), *we have*

$$\frac{X_n}{n} \xrightarrow{\mathscr{L}} \sum_{j\in J_2} p_j \delta_{a_j}, \tag{2.1}$$

where for $j \in J_2$,

$$p_j = \frac{q_j}{\sum_{k\in J_2} q_k}, \qquad q_j = \int_{\mathbb{R}} \exp(c_j x^{2m_j} + b_j x)dx,$$

with

$$c_j = \frac{A^{(2m_j)}(a_j)}{(2m_j)!}, \qquad b_j = B'(a_j)\,\mathrm{I}[m_j = m_*]. \tag{2.2}$$

Theorem 2.2 (Concentration). *Assume* (A1)–(A3), *and let* $\delta \in (0, \delta_*)$. *There exist a positive constants* c *such that*

$$\mathbb{P}[|X_n/n - a_j| > \delta \text{ for all } j \in J] \leqslant \exp(-cn) \tag{2.3}$$

and

$$\mathbb{P}[|X_n/n - a_j| > \delta \text{ for all } j \in J_1] \leqslant \exp(-cn\sigma_{*,n}). \tag{2.4}$$

Moreover, for any $j_2 \in J_2$, *there exists a constant* C *such that if* $J_1 \neq J_2$,

$$\mathbb{P}[|X_n/n - a_j| > \delta \text{ for all } j \in J_2] \leqslant C \max_{j_1\in J_1\setminus J_2} n^{1/(2m_{j_1})-1/(2m_{j_2})}, \tag{2.5}$$

and, for any $j \in J_2$,

$$\mathbb{P}[|X_n/n - a_j| \leqslant \delta_*] = p_j + \mathrm{O}(\tau_{*,n}) + \mathrm{O}\bigg(\max_{k\in J_1\setminus J_2} n^{1/(2m_k)-1/(2m_j)}\bigg), \tag{2.6}$$

where

$$\tau_{*,n} = \frac{(\log n)^{2m_*+1}}{n^{1/(2m_*)}} + n^{1/(2m_*)-1/(2m_{j_2})} \log n\, \mathrm{I}[m_{j_2} \neq m_*].$$

Theorem 2.3 (Distributional limit theorem). *Under Assumptions* (A1)–(A3), *we have for all* $j \in J$ *and* $l \in \mathbb{N}$ *that*

$$\mathbb{E}\big\{|X_n/n - a_j|^l \,\big|\, |X_n - na_j| \leqslant n\delta_*\big\} = \mathrm{O}\big(n^{-l/(2m_j)}\big);$$

and for all $j \in J$ *that*

$$\begin{aligned} &d_{\mathrm{W}}\big(\mathscr{L}\big(n^{1/(2m_j)}(X_n/n - a_j)\,\big|\,|X_n - na_j| \leqslant n\delta_*\big), \mathscr{L}(Y_j)\big) \\ &\qquad = \mathrm{O}\big(n^{-1/(2m_j)}\big) + \mathrm{O}\big(n^{1/(2m_*)-1/(2m_j)}\,\mathrm{I}[m_j \neq m_*]\big), \end{aligned}$$

where $Y_j \propto \exp(c_j x^{2m_j} + b_j x)$ *with* c_j *and* b_j *given as in* (2.2).

Remark 2.4. In Theorem 2.1, the Condition (A4) is not needed when A has a unique maximizer. In fact, Condition (A4) is only required in (3.7) to prove (2.1), where we compare the Gibbs measure around the maximizers.

To simplify notation, we will drop the dependence on n in what follows and write X, W, σ and τ instead of X_n, W_n, σ_n and τ_n, and introduce some notation

$$\sigma_j = n^{1/(2m_j)-1}; \qquad J_* = \{j \in J : \sigma_j = \sigma_*\} = \{j \in J : m_j = m_*\}.$$

In order to prove Theorems 2.1, 2.2 and 2.3, the following result is key.

Proposition 3.1. *Assume* (A1)–(A3)*, and let* $\delta \in (0, \delta_*]$*. Then for all* $j \in J$*, we have*

$$\begin{aligned} Z_{n,j}(\delta) &:= \sum_{|k/n - a_j| \leqslant \delta} \exp(H_n(k/n)) \\ &= (q_j + \mathrm{O}_\delta(\tau_j))\sigma_j^{-1} \exp(nA_n(k_j/n) + n\sigma_* B(a_j)), \end{aligned}$$

where $k_j = [na_j]$ *and* τ_j*,* q_j*,* c_j *and* b_j *are given in Theorem 2.1.*

The proof of Proposition 3.1 is based on Laplace's method and will be presented at the end of this section.

3.1 Concentration and weak law of large numbers

Proof of Theorems 2.1 and 2.2. We start by proving the concentration inequalities. We first show that for any $\delta \in (0, \delta_*)$, one has

$$\mathbb{P}\big[|X/n - a_j| > \delta \text{ for all } j \in J\big] \leqslant \exp(-cn), \tag{3.1}$$

where $c = c(\delta) > 0$ is a constant. Let k be an integer such that $|k - na_j| \geqslant \delta n$ for all $j \in J$. We claim that there exist $i \in J$ and $c > 0$, such that

$$A_n(k/n) - A_n(k_i/n) \leqslant -c, \tag{3.2}$$

where recall that $k_i = [na_i]$. Indeed, if $|k - na_j| \geqslant \delta_* n$ for all $j \in J$ then let i be an arbitrary element of J and using by (A2) and (A3), we have

$$\begin{aligned} &A_n(k/n) - A_n(k_i/n) \\ &\quad = A_n(k/n) - A(a_i) + A(a_i) - A(k_i/n) + A(k_i/n) - A_n(k_i/n) \\ &\quad \leqslant -\varepsilon_* + \mathrm{O}(|k_i/n - a_i|) \leqslant -2\varepsilon_*/3, \end{aligned}$$

where we have used $|k_i/n - a_i| \leqslant 1/n$. Otherwise, suppose that $|k - na_i| \leqslant \delta_* n$ for some $i \in J$. Then

$$\begin{aligned} &A_n(k/n) - A_n(k_i/n) \\ &\quad = A(k/n) - A(k_i/n) + \mathrm{O}(1/n) = A(k/n) - A(a_i) + \mathrm{O}(1/n) \\ &\quad \leqslant \sup_{x: |x - a_i| \leqslant \delta_*} A''(x)\delta_*^2/2 + \mathrm{O}(1/n) \leqslant -c, \end{aligned}$$

where $c = c(\delta_*) > 0$. Here, for the first two equations, we used (A3) and $|k_i/n - a_i| \leqslant 1/n$, for the remaining inequalities, we used Taylor expansion and (A1). The proof of (3.2) is complete.

Next, note that by (A2), $|B_n(k/n)| \leqslant C_*$ when $|k/n - a_j| \geqslant \delta_*$ for all $j \in J$, and by (A3) for k such that $|k/n - a_j| \leqslant \delta_*$ for some $j \in J$ one has $|B_n(k/n)| \leqslant |B(k/n)| + C_*/n \leqslant 2\max_{x\in[0,1]} |B(x)|$. Therefore,

$$\max_{0\leqslant k\leqslant n} |B_n(k/n)| = \mathrm{O}(1). \tag{3.3}$$

Combining (3.2) and (3.3) yields that for all n sufficiently large

$$\begin{aligned} &H_n(k/n) - H_n(k_i/n) \\ &\quad = n[A_n(k/n) - A_n(k_i/n)] + n\sigma_*[B_n(k/n) - B_n(k_i/n)] \\ &\quad \leqslant -cn + \mathrm{O}(n\sigma_*) \leqslant -cn/2, \end{aligned} \tag{3.4}$$

and thus

$$\mathbb{P}[X = k] \leqslant \exp(-cn/2)\,\mathbb{P}[X = k_i] \leqslant \exp(-cn/4),$$

and (3.1) is proved by using the union bound.

By Proposition 3.1, for any fixed $\delta \in (0, \delta_*)$, for all $j \in J$ and n sufficiently large

$$\begin{aligned} Z_{n,j}(\delta) &:= \sum_{|k/n-a_j|\leqslant\delta} \exp(H_n(k/n)) \\ &= (q_j + \mathrm{O}_\delta(\tau_j))\sigma_j^{-1} \exp(nA_n(k_j/n) + n\sigma_* B(a_j)), \end{aligned} \tag{3.5}$$

where $k_j = [na_j]$ and

$$\tau_j = \frac{(\log n)^{2m_*+1}}{n\sigma_*} + \frac{\sigma_* \log n}{\sigma_j}\,\mathrm{I}[j \in J \setminus J_*],$$

and

$$q_j = \int_{\mathbb{R}} \exp(c_j x^{2m_j} + b_j x)dx,$$

with c_j, b_j as in (2.2).

Note that $nA_n(k_j/n) = nA(a_j) + \mathrm{O}(1) = n\max_{x\in[0,1]} A(x) + \mathrm{O}(1)$ by (A3). Therefore, the leading terms of $(Z_{n,j})_{j\in J}$ are the ones at which the sequence $(B(a_j))_{j\in J}$ attains the maximum. Recall that

$$J_1 = \{j \in J : B(a_j) = \max_{k\in J} B(a_k)\}.$$

Let $\delta \in (0, \delta_*)$ be any fixed constant. By the above, (3.1) and (3.5) yield that, if $J_1 \neq J$,

$$\begin{aligned} &\mathbb{P}\big[|X/n - a_j| > \delta \text{ for all } j \in J_1\big] \\ &\quad \leqslant \exp(-cn) + \frac{\sum_{j\in J\setminus J_1} Z_{n,j}}{\sum_{j\in J} Z_{n,j}} \\ &\quad \leqslant \exp(-cn) + \mathrm{O}_\delta(1) \sum_{j\in J\setminus J_1} \frac{\sigma_{j_1}}{\sigma_j} \exp(n\sigma_*(B(a_j) - B(a_{j_1}))) \\ &\quad \leqslant \exp(-c_1 n\sigma_*), \end{aligned}$$

where c and c_1 are positive constants depending on δ, and j_1 is an element of J_1. Similarly, if $J_2 \neq J_1$,

$$\mathbb{P}\big[|X/n - a_j| > \delta \text{ for all } j \in J_2\big] \leqslant \mathrm{O}_\delta(1) \max_{j_1 \in J_1 \setminus J_2} \sigma_{j_2}/\sigma_{j_1}, \qquad (3.6)$$

with j_2 an element of J_2. The two above inequalities and (3.1) yields the concentration estimates in (2.3), (2.4) and (2.5).

We now prove the weak law of large numbers (2.1) and the estimate (2.6). By (A4) for all $i, j \in J_2$

$$|nA_n(k_i/n) - nA_n(k_j/n)| = \mathrm{O}(1/n\sigma_*). \qquad (3.7)$$

Hence, it follows from (3.5) that for any $\delta \in (0, \delta_*)$, and for all $j \in J_2$

$$\frac{Z_{n,j}(\delta)}{\sum_{k \in J_2} Z_{n,k}(\delta)} = p_j + \mathrm{O}_\delta(\tau_*), \qquad (3.8)$$

where

$$p_j = \frac{q_j}{\sum_{k \in J_2} q_k},$$

and

$$\tau_* = \tau_{j_2} = \frac{(\log n)^{2m_*+1}}{n\sigma_*} + \frac{\sigma_* \log n}{\sigma_{j_2}} \mathrm{I}[J_2 \neq J_*],$$

with j_2 an element of J_2 (note here that $\sigma_{j_2} = \sigma_{j_2'}$ for all $j_2, j_2' \in J_2$). Combining (3.8) and (3.6), we have

$$X/n \xrightarrow{\mathscr{L}} \sum_{j \in J_2} p_j \, \delta_{a_j},$$

and for all $j \in J_2$

$$\mathbb{P}[|X/n - a_j| \leqslant \delta_*] = p_j + \mathrm{O}(\tau_*) + \mathrm{O}(1) \max_{j_1 \in J_1 \setminus J_2} \sigma_{j_2}/\sigma_{j_1}.$$

The proof of (2.1) and (2.6) is complete. □

3.2 Stein's method

We first state and derive what is needed to implement Stein's method for target distributions of the form $p(y) \propto \exp(cy^{2m} + by)$. The following result is a consequence of the general approach of Chaterjee and Shao (2011).

Lemma 3.2. *Let m be a positive integer, and let Y be a random variable with density function $p(y) \propto \exp(cy^{2m} + by)$ with $c < 0$ and $b \in \mathbb{R}$. Then there exists a positive constant $K = K(c, b, m)$ such that for any random variable W,*

$$d_{\mathrm{W}}(W, Y) \leqslant \sup_{f \in C_K^2(\mathbb{R})} \Big| \mathbb{E}\Big\{ f'(W) + \tfrac{p'(W)}{p(W)} f(W) \Big\} \Big|,$$

where

$$C_K^2(\mathbb{R}) = \big\{ f \in C^2(\mathbb{R}) : \|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty \leqslant K \big\},$$

with $C^2(\mathbb{R})$ the space of twice differentiable functions and $\|g\|_\infty = \sup_{x \in \mathbb{R}} |g(x)|$.

Proof. Let h be Lipschitz continuous and consider the Stein equation

$$f'(w) + p'(w)f(w)/p(w) = h(w) - \mathbb{E}h(Y). \tag{3.9}$$

Chaterjee and Shao (2011, Lemma 4.1) showed that the solution f_h of the functional equation (3.9) belongs to $C^2(\mathbb{R})$ and satisfies

$$\|f\|_\infty \vee \|f'\|_\infty \vee \|f''\|_\infty \leqslant (1 + d_1)(1 + d_2)(1 + d_3)\|h'\|_\infty,$$

where

$$d_1 = \sup_{x\in\mathbb{R}} \frac{\min\{P(x), 1 - P(x)\}}{p(x)}, \quad d_2 = \sup_{x\in\mathbb{R}} \frac{\min\{P(x), 1 - P(x)\}p'(x)}{p^2(x)},$$

and $d_3 = \sup_{x\in\mathbb{R}} Q(x)$, with $P(x) = \int_{-\infty}^{x} p(t)dt$ and

$$Q(x) = \frac{1 + |(p'/p)'(x)|}{p(x)} \min\big\{\mathbb{E}\{Y\,\mathrm{I}[Y \leqslant x]\} + \mathbb{E}|Y|P(x),$$
$$\mathbb{E}\{Y\,\mathrm{I}[Y > x]\} + \mathbb{E}|Y|(1 - P(x))\big\}.$$

We now show that d_3 is a finite constant depending only on c, b and m. The proof for d_1 and d_2 is similar but simpler, hence omitted. It is clear that

$$d_3 = \max\Big\{\sup_{x\leqslant -C} Q(x), \sup_{|x|\leqslant C} Q(x), \sup_{x\geqslant C} Q(x)\Big\}, \quad C = 1 + \frac{4 + |b|}{m|c|}. \tag{3.10}$$

First, consider $x \geqslant C$; since $(p'/p)'(x) = 2m(2m - 1)cx^{2m-2}$ and $\mathbb{E}Y < \infty$,

$$Q(x) \leqslant C_1 \frac{x^{2m-2}\int_x^\infty yp(y)dy}{p(x)} = C_1 \frac{x^{2m-2}\int_x^\infty yq(y)dy}{q(x)}, \tag{3.11}$$

with $C_1 = C_1(c, b, m)$ a finite constant and $q(x) = \exp(cx^{2m} + bx)$. Using integration by parts and the fact that $q'(y) = q(y)(2mcy^{2m-1} + b) < 0$ for $y \geqslant x \geqslant C$,

$$\begin{aligned}\int_x^\infty yq(y)dy &= \int_x^\infty \frac{y}{2mcy^{2m-1} + b} d(q(y)) \leqslant \int_x^\infty \frac{y^{2-2m}}{mc} d(q(y)) \\ &= \frac{x^{2-2m}q(x)}{m|c|} + \int_x^\infty \frac{y^{1-2m}(2 - 2m)}{mc} q(y)dy \\ &\leqslant \frac{x^{2-2m}q(x)}{m|c|} + \frac{1}{2}\int_x^\infty yq(y)dy,\end{aligned}$$

and hence

$$\int_x^\infty yq(y)dy \leqslant \frac{2x^{2-2m}q(x)}{m|c|}.$$

Combining this with (3.11) we have $\sup_{x\geqslant C} Q(x) \leqslant 2C_1/(m|c|)$. The same inequality holds for $\sup_{x\leqslant -C} Q(x)$. Since Q is continuous, it also follows that $\sup_{|x|\leqslant C} Q(x) < \infty$. Hence, by (3.10), we have $d_3 < \infty$.

Finally, considering (3.9) with w replaced by W and taking expectation, the claim easily follows. □

Lemma 3.3. *(i) Let W, Y and Z be random variables such that $|W - Y| \leqslant |Z|$ almost surely. Then*

$$d_{\mathrm{K}}(W,Y) \leqslant \inf_{\delta>0}\Big(\sup_{s\in\mathbb{R}}\mathbb{P}[s < Y \leqslant s+\delta] + \mathbb{P}[|Z| \geqslant \delta]\Big).$$

(ii) Let Y be a random variable satisfying

$$M_Y := \sup_{\delta>0}\sup_{s\in\mathbb{R}}\frac{1}{\delta}\,\mathbb{P}[s \leqslant Y \leqslant s+\delta] < \infty.$$

Then there exists a positive constant $C = C(M_Y)$, such that for all random variable W,

$$d_{\mathrm{K}}(W,Y) \leqslant C\, d_{\mathrm{W}}(W,Y)^{1/2}.$$

Proof. Since $Y - |Z| \leqslant W \leqslant Y + |Z|$, we have for all $s \in \mathbb{R}$ and $\delta > 0$

$$\mathbb{P}[Y \leqslant s-\delta] - \mathbb{P}[|Z| \geqslant \delta] \leqslant \mathbb{P}[W \leqslant s] \leqslant \mathbb{P}[Y \leqslant s+\delta] + \mathbb{P}[|Z| \geqslant \delta].$$

Subtracting $\mathbb{P}[Y \leqslant s]$ everywhere and taking supremum over s, (i) now easily follows. Item (ii) is proved by Ross (2011, Proposition 1.2). □

3.3 Distributional limit theorem

Proof of Theorem 2.3. We shall prove that for all $j \in J$ and $l \in \mathbb{N}$,

$$\mathbb{E}\big\{|X/n - a_j|^l \,\big|\, |X - na_j| \leqslant n\delta_*\big\} = \mathrm{O}\big(1/(n\sigma_j)^l\big), \tag{3.12}$$

and for $j \in J$

$$\begin{aligned} &d_{\mathrm{W}}\big(\mathscr{L}(W_j \,|\, |X - na_j| \leqslant n\delta_*), \mathscr{L}(Y_j)\big) \\ &\quad = \mathrm{O}(1/(n\sigma_j)) + \mathrm{O}(\sigma_*/\sigma_j\, \mathrm{I}[j \in J \setminus J_*]), \end{aligned} \tag{3.13}$$

where

$$W_j = \sigma_j(X - na_j), \qquad Y_j \propto \mathbf{p}_j \propto \exp(c_j x^{2m_j} + b_j x),$$

with c_j and b_j given as in (2.2). Let $\tilde{X}_j$ be a random variable having the conditional distribution of X given $|X - na_j| \leqslant n\delta_*$; that is,

$$\mathbb{P}[\tilde{X}_j = k] = \frac{\exp(H_n(k/n))}{Z_{n,j}}, \qquad \ell_j \leqslant k \leqslant L_j, \tag{3.14}$$

where

$$\ell_j = \lceil n(a_j - \delta_*)\rceil, \quad L_j = [n(a_j + \delta_*)], \quad Z_{n,j} = Z_{n,j}(\delta_*).$$

Then

$$\mathbb{E}\big\{|X/n - a_j|^l \,\big|\, |X - na_j| \leqslant n\delta_*\big\} = \mathbb{E}\big\{|\tilde{X}/n - a_j|^l\big\}, \tag{3.15}$$

and from Lemma 3.2, we have

$$\begin{aligned} &d_{\mathrm{W}}\big(\mathscr{L}(W_j \,|\, |X - na_j| \leqslant n\delta_*), \mathscr{L}(Y_j)\big) \\ &\quad\leqslant \sup_{f\in C_K^2(\mathbb{R})}\Big|\mathbb{E}\big\{f'(W_j) + \tfrac{\mathbf{p}_j'(W_j)}{\mathbf{p}_j(W_j)}f(W_j) \,\big|\, |X - na_j| \leqslant n\delta_*\big\}\Big| \\ &\quad= \sup_{f\in C_K^2(\mathbb{R})}\Big|\mathbb{E}\big\{f'(\tilde{W}_j) + \tfrac{\mathbf{p}_j'(\tilde{W}_j)}{\mathbf{p}_j(\tilde{W}_j)}f(\tilde{W}_j)\big\}\Big| \end{aligned} \tag{3.16}$$

where $K = K(c_j, b_j, m_j)$ is a finite constant, and

$$\tilde{W}_j = \sigma_j(\tilde{X} - na_j).$$

Given $f \in C_K^2(\mathbb{R})$, we define the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = f(\sigma_j(x - na_j)).$$

For any bounded function $h : \mathbb{R} \to \mathbb{R}$ and $\delta > 0$, let $\Delta_\delta h(x) = h(x+\delta) - h(x)$; we have

$$\Delta_1 g(\tilde{X}_j) = \Delta_{\sigma_j} f(\tilde{W}_j), \qquad g(\tilde{X}_j) = f(\tilde{W}_j). \tag{3.17}$$

For $x = (\ell_j - 1)/n, \ell_j/n, \ldots, (L_j - 1)/n$, let

$$D_n(x) = \Delta_{1/n} H_n(x) = n\Delta_{1/n} A_n(x) + n\sigma_* \Delta_{1/n} B_n(x),$$

and also let $D_n((\ell_j - 1)/n) = 0$. Note that by (3.14), for $\ell_j - 1 \leqslant k \leqslant L_j - 1$,

$$\frac{\mathbb{P}[\tilde{X}_j = k+1]}{\mathbb{P}[\tilde{X}_j = k]} = \exp\bigl(D_{1/n}(k/n)\bigr).$$

Hence, (3.17) and straightforward calculations now yield

$$\mathbb{E}\Delta_{\sigma_j} f(\tilde{W}_j) = \mathbb{E}\Delta_1 g(\tilde{X}_j) = \mathbb{E}\bigl\{g(\tilde{X}_j)\bigl[\exp\bigl(-D_n(\tfrac{\tilde{X}_j - 1}{n})\bigr) - 1\bigr]\bigr\} + r_1, \tag{3.18}$$

where

$$r_1 = \frac{1}{Z_{n,j}}\bigl[-g(L_j + 1)\exp\bigl(H_n(L_j/n)\bigr) + g(\ell_j)\exp\bigl(H_n(\ell_j/n)\bigr)\bigr].$$

By (3.4), we have

$$\max\{H_n(L_j/n), H_n(\ell_j/n)\} \leqslant H_n(k_j/n) - cn,$$

for some $c > 0$. Moreover,

$$|H_n(k_j/n) - nA_n(k_j/n) - n\sigma_* B(a_j)| = n\sigma_* |B_n(k_j/n) - B(a_j)| = \mathrm{O}(n\sigma_*).$$

Therefore,

$$\max\{H_n(L_j/n), H_n(\ell_j/n)\} \leqslant nA_n(k_j/n) + n\sigma_* B(a_j) - cn/2.$$

Combining this estimate with (3.5), we obtain

$$r_1 \leqslant \|f\|_\infty \exp(-cn/4). \tag{3.19}$$

Moreover, by Taylor's expansion,

$$\left|\frac{1}{\sigma_j}\Delta_{\sigma_j} f(\tilde{W}_j) - f'(\tilde{W}_j)\right| \leqslant \sigma_j \|f''\|_\infty. \tag{3.20}$$

It follows from (3.18), (3.19) and (3.20) that

$$\left|\mathbb{E}\left\{f'(\tilde{W}_j) - \frac{1}{\sigma_j}\Bigl(\exp\bigl(-D_n(\tfrac{\tilde{X}_j - 1}{n})\bigr) - 1\Bigr)f(\tilde{W}_j)\right\}\right|$$

$$\leqslant \|f\|_\infty \exp(-cn/4) + \sigma_j \|f''\|_\infty. \quad (3.21)$$

We now estimate the error when replacing $\sigma_j^{-1}(\exp(-D_n(\frac{X-1}{n}))-1)$ by $\mathbf{p}_j'(W_j)/\mathbf{p}_j(W_j)$ in (3.21). For $|k-k_j| \leqslant \delta_* n$, using (A3) and Taylor's expansion we have

$$\begin{aligned} A_n(k/n) &- A_n((k-1)/n) - n^{-1}A'(k/n) \\ &= [A_n(k/n) - A_n((k-1)/n)] - [A(k/n) - A((k-1)/n)] \\ &\quad + A(k/n) - A((k-1)/n) - n^{-1}A'(k/n) = \mathrm{O}(n^{-2}). \end{aligned}$$

Thus

$$n\Delta_{1/n}A_n((k-1)/n) = A'(k/n) + \mathrm{O}(n^{-1}).$$

Similarly,

$$n\Delta_{1/n}B_n((k-1)/n) = B'(k/n) + \mathrm{O}(n^{-1}).$$

Therefore,

$$|D_n((k-1)/n) - [A'(k/n) + \sigma_* B'(k/n)]| = \mathrm{O}(n^{-1}). \quad (3.22)$$

Furthermore, $|e^u - e^v| = e^v|e^{u-v}-1| \leqslant 2e^v|u-v|$ when $|u-v|$ is sufficiently small. Hence, by using (3.22) we have for all n large enough

$$\begin{aligned} &\big|\exp(-D_n((k-1)/n) - \exp(-A'(k/n) - \sigma_* B'(k/n))\big| \\ &\quad \leqslant 2 \max_{|x-a_j|\leqslant \delta_*} \exp(|A'(x)| + \sigma_*|B'(x)|) \\ &\qquad \times |D_n((k-1)/n) - [A'(k/n) + \sigma_* B'(k/n)]| = \mathrm{O}(n^{-1}). \end{aligned} \quad (3.23)$$

Moreover, by applying Taylor's expansion to the function $e^{-A'(x)-\sigma_* B'(x)}$ around $x = a_j$ and noting that $A^{(k)}(a_j) = 0$ for all $1 \leqslant k \leqslant 2m_j - 1$,

$$\begin{aligned} \exp(-A'(k/n) - \sigma_* B'(k/n)) = 1 - \frac{A^{(2m_j)}(a_j)}{(2m_j-1)!}(k/n - a_j)^{2m_j-1} - \sigma_* B'(a_j) \\ + \mathrm{O}\big((k/n-a_j)^{2m_j} + \sigma_*|k/n - a_j|\big). \end{aligned}$$

Note further that

$$\begin{aligned} &\frac{A^{(2m_j)}(a_j)}{(2m_j-1)!}(\tilde{X}_j/n - a_j)^{2m_j-1} + \sigma_* B'(a_j) \\ &\quad = 2m_j c_j (\tilde{X}_j/n - a_j)^{2m_j-1} + \sigma_* B'(a_j) \\ &\quad = \sigma_j(2m_j c_j \tilde{W}_j^{2m_j-1} + b_j) - \sigma_j b_j + \sigma_* B'(a_j) \\ &\quad = \sigma_j \frac{\mathbf{p}_j'(\tilde{W}_j)}{\mathbf{p}_j(\tilde{W}_j)} + \mathrm{O}(\sigma_*\, \mathrm{I}[j \in J \setminus J_*]), \end{aligned}$$

since $\mathbf{p}_j'(w)/\mathbf{p}_j(w) = 2m_j c_j w^{2m_j-1} + b_j$, and

$$\tilde{W}_j = \sigma_j(\tilde{X}_j - na_j) = \sigma_j^{-1/(2m_j-1)}(\tilde{X}_j/n - a_j),$$

and

$$|\sigma_j b_j - \sigma_* B'(a_j)| = \begin{cases} 0 & \text{if } j \in J_* \\ |\sigma_* B'(a_j)| = \mathrm{O}(\sigma_*) & \text{if } j \in J \setminus J_*. \end{cases}$$

Therefore,

$$\begin{aligned}\exp\big(-A'(\tilde X_j/n) - \sigma_* B'(\tilde X_j/n)\big) - 1 \\ = -\sigma_j \frac{\mathbf{p}_j'(\tilde W_j)}{\mathbf{p}_j(\tilde W_j)} + \mathrm{O}(\sigma_* \,\mathrm{I}[j \in J\setminus J_*]) \\ + \mathrm{O}\big((\tilde X_j/n - a_j)^{2m_j} + \sigma_* |\tilde X_j/n - a_j|\big).\end{aligned} \tag{3.24}$$

It follows from (3.23) and (3.24), and the fact that $\sigma_* \leqslant \sigma_j$ that

$$\begin{aligned}\mathbb{E}\Big\{\Big|\sigma_j^{-1}\big(\exp(-D_n(\tfrac{\tilde X_j - 1}{n})) - 1\big) f(\tilde W_j) + \tfrac{\mathbf{p}_j'(\tilde W_j)}{\mathbf{p}_j(\tilde W_j)} f(\tilde W_j)\Big|\Big\} \\ \leqslant C\|f\|_\infty \,\mathbb{E}\Big\{\big(\sigma_j^{-1}(\tilde X_j/n - a_j)^{2m_j} + |\tilde X_j/n - a_j|\big)\Big\} \\ + C\sigma_j^{-1}\sigma_* \,\mathrm{I}[j \in J\setminus J_*],\end{aligned} \tag{3.25}$$

where C is a positive constant. In order to estimate the above term, we analyse $\mathbb{P}[\tilde X_j = k]$. By Proposition 3.1, if $|k/n - a_j| \leqslant \delta_*$, we have

$$\begin{aligned}\mathbb{P}[\tilde X_j = k] &= \frac{\mathbb{P}[X_j = k]}{Z_{n,j}(\delta_*)} \\ &= \mathrm{O}(1)\sigma_j \exp(n(A_n(k/n) + \sigma_* B_n(k/n) - A_n(k_j/n) - \sigma_* B(a_j))).\end{aligned} \tag{3.26}$$

Now,

$$A_n(k/n) = A(k/n) + \mathrm{O}(1/n) \leqslant A(a_j) + \alpha_j (k/n - a_j)^{2m_j} + \mathrm{O}(1/n),$$

where for the first equation, we used (A3), and for the second one, we used Taylor expansion and (A1) and as well as the fact that

$$\alpha_j := \max_{|x - a_j| \leqslant \delta_*} \frac{A^{(2m_j)}(x)}{(2m_j)!} < 0.$$

Furthermore,

$$|A(a_j) - A_n(k_j/n)| \leqslant |A(a_j) - A(k_j/n)| + |A_n(k_j/n) - A(k_j/n)| = \mathrm{O}(1/n).$$

Therefore,

$$A_n(k/n) \leqslant A_n(k_j/n) + \alpha_j (k/n - a_j)^{2m_j} + \mathrm{O}(1/n).$$

By (A3),

$$|B_n(k/n) - B(a_j)| \leqslant |B_n(k/n) - B(k/n)| + |B(k/n) - B(a_j)| = \mathrm{O}(|k/n - a_j|).$$

Using the last two display equations, (3.26) and $\sigma_j \leqslant \sigma_*$, we have

$$\mathbb{P}[\tilde X_j = k] \leqslant C\sigma_j \exp\big(\alpha_j n (k/n - a_j)^{2m_j} + C n \sigma_j |k/n - a_j|\big)$$

for some finite constant C. Next, by using $\sigma_j^{2m_j} = n^{1-2m_j}$ and integral approximations, we have for all $l \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\{|\tilde{X}_j/n - a_j|^l\} \\
&\quad \leqslant C\sigma_j \sum_{k:|k/n-a_j|\leqslant\delta_*} |k/n - a_j|^l \exp\Big(\alpha_j n(k/n - a_j)^{2m_j} + Cn\sigma_j|k/n - a_j|\Big) \\
&\quad = \mathrm{O}(\sigma_j) \int_{-n\delta_*}^{n\delta_*} (|x|/n)^l \exp(\alpha_j(x\sigma_j)^{2m_j} + C|x\sigma_j|)dx \\
&\quad = \mathrm{O}((n\sigma_j)^{-l}) \int_{-n\sigma_j\delta_*}^{n\sigma_j\delta_*} |y|^l \exp(\alpha_j y^{2m_j} + C|y|)dy = \mathrm{O}((n\sigma_j)^{-l}),
\end{aligned}$$

since $\alpha_j < 0$. This estimate and (3.15) implies (3.12). In particular, we have

$$\begin{aligned}
&\mathbb{E}\Big\{\sigma_j^{-1}(\tilde{X}_j/n - a_j)^{2m_j} + |\tilde{X}_j/n - a_j|\Big\} \\
&\quad = \mathrm{O}(\sigma_j^{-1}(n\sigma_j)^{-2m_j}) + \mathrm{O}((n\sigma_j)^{-1}) = \mathrm{O}\big((n\sigma_j)^{-1}\big),
\end{aligned}$$

where we used that $\sigma_j^{2m_j} = n^{1-2m_j}$. Therefore, by (3.25),

$$\mathbb{E}\Big\{\Big|\sigma_j^{-1}\big(\exp(-D_n(\tfrac{\tilde{X}_j-1}{n})) - 1\big)f(\tilde{W}_j) + \tfrac{\mathbf{p}_j'(\tilde{W}_j)}{\mathbf{p}_j(\tilde{W}_j)}f(\tilde{W}_j)\Big|\Big\} = \mathrm{O}\big(\|f\|_\infty/(n\sigma_j)\big).$$

Combining the above inequality with (3.21) we yield that for all $K > 0$

$$\begin{aligned}
&\sup_{f\in C_K^2(\mathbb{R})}\Big|\mathbb{E}\Big\{f'(\tilde{W}_j) + \tfrac{\mathbf{p}_j'(\tilde{W}_j)}{\mathbf{p}_j(\tilde{W}_j)}f(\tilde{W}_j)\Big\}\Big| \\
&\quad = \mathrm{O}(K/(n\sigma_j)) + \mathrm{O}(\sigma_*) + \mathrm{O}(\sigma_j/\sigma_*\,\mathrm{I}[j \in J \setminus J_*]) \\
&\quad = \mathrm{O}(K/(n\sigma_j)) + \mathrm{O}(\sigma_j/\sigma_*\,\mathrm{I}[j \in J \setminus J_*]).
\end{aligned}$$

Then the desired estimate (3.13) follows from this bound and (3.16). □

3.4 Free energy

Proof of Proposition 3.1. Fix a constant $\delta \in (0, \delta_*]$. We aim to approximate

$$Z_{n,j}(\delta) := \sum_{|k/n-a_j|\leqslant\delta} \exp(H_n(k/n)).$$

Let $\varepsilon \in (0, \delta)$ be a suitably small constant chosen later (see (3.29)). For $n\varepsilon \leqslant |k - na_j| \leqslant n\delta$, by (A3)

$$\begin{aligned}
&A_n(k/n) - A_n(k_j/n) = A(k/n) - A(k_j/n) + \mathrm{O}(|k - k_j|/n^2) \\
&\quad \leqslant \max_{\varepsilon\leqslant|x-a_j|\leqslant\delta}(A(x) - A(a_j)) + \mathrm{O}(|k_j/n - a_j|) + \mathrm{O}(|k - k_j|/n^2) \leqslant -\eta
\end{aligned}$$

with $\eta = \eta(\varepsilon) > 0$, since a_j is the unique maximizer of the smooth function A in $[a_j - \delta_*, a_j + \delta_*]$. Therefore, since B_n is uniformly bounded by (3.3),

$$\begin{aligned}
&H_n(k/n) - H_n(k_j/n) \\
&\quad = n[A_n(k/n) - A_n(k_j/n)] + n\sigma_*[B_n(k/n) - B_n(k_j/n)] \leqslant -\eta n/2.
\end{aligned}$$

Thus

$$\sum_{k=0}^{n} \frac{\exp(H_n(k/n))}{\exp(H_n(k_j/n))} \mathrm{I}[\varepsilon \leqslant |k/n - a_j| \leqslant \delta] \leqslant n \exp(-\eta n/2). \tag{3.27}$$

Next, we consider $\sigma_j^{-1} \log n \leqslant |k - na_j| \leqslant n\varepsilon$. By (A3) for all $|k/n - a_j| \leqslant \delta_*$

$$A_n(k/n) - A_n(k_j/n) = A(k/n) - A(k_j/n) + \mathrm{O}(|k - k_j|/n^2).$$

Moreover, using Taylor expansion around a_j with $A^{(m)}(a_j) = 0$ for $1 \leqslant m \leqslant 2m_j - 1$, we have

$$\begin{aligned} A(k/n) - A(k_j/n) &= A(k/n) - A(a_j) + A(a_j) - A(k_j/n) \\ &= c_j(k/n - a_j)^{2m_j} + \mathrm{O}(|k/n - a_j|^{2m_j+1}) + \mathrm{O}(n^{-2}), \end{aligned}$$

where we recall that $c_j = A^{(2m_j)}(a_j)/(2m_j)!$ and $|k_j/n - a_j|^2 \leqslant n^{-2}$. It follows from the last two estimates that for all $|k - na_j| \leqslant n\delta_*$

$$\begin{aligned} &A_n(k/n) - A_n(k_j/n) \\ &\quad = c_j(k/n - a_j)^{2m_j} + \mathrm{O}(|k/n - a_j|^{2m_j+1}) + \mathrm{O}(|k - k_j|/n^2) + \mathrm{O}(n^{-2}). \end{aligned} \tag{3.28}$$

In particular, there exists a constant $C_1 = C_1(a_j, c_j, A) > 0$ such that

$$A_n(k/n) - A_n(k_j/n) \leqslant c_j(k/n - a_j)^{2m_j} + C_1|k/n - a_j|^{2m_j+1} + C_1/n.$$

By taking

$$\varepsilon = |c_j|/(2C_1), \tag{3.29}$$

we yield that for $|k/n - a_j| \leqslant \varepsilon$,

$$A_n(k/n) - A_n(k_j/n) \leqslant c_j(k/n - a_j)^{2m_j}/2 + C_1/n, \tag{3.30}$$

by noting that $c_j < 0$. On the other hand for all $|k/n - a_j| \leqslant \delta_*$, by (A3)

$$n\sigma_*[B_n(k/n) - B_n(k_j/n)] = n\sigma_*[B(k/n) - B(k_j/n)] + \mathrm{O}(\sigma_*|k - k_j|/n).$$

Moreover,

$$\begin{aligned} B(k/n) - B(k_j/n) &= B(k/n) - B(a_j) + B(a_j) - B(k_j/n) \\ &= B'(a_j)(k/n - a_j) + \mathrm{O}(|k/n - a_j|^2) + \mathrm{O}(n^{-1}). \end{aligned}$$

Thus for all $|k/n - a_j| \leqslant \delta_*$,

$$\begin{aligned} &n\sigma_*[B_n(k/n) - B_n(k_j/n)] \\ &\quad = \sigma_*(k - na_j)(B'(a_j) + \mathrm{O}(|k/n - a_j|)) + \mathrm{O}(\sigma_*). \end{aligned} \tag{3.31}$$

Hence, using (3.30) and (3.31) and $\sigma_* \leqslant \sigma_j$, and noting that $\sigma_j^{2m_j} = n^{1-2m_j}$,

$$H_n(k/n) - H_n(k_j/n) \leqslant \frac{c_j}{2}(\sigma_j(k - na_j))^{2m_j} + C\sigma_j|k - na_j| + C,$$

with C some positive constant. Therefore,

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\exp(H_n(k/n))}{\exp(H_n(k_j/n))} \mathrm{I}[(\log n)/\sigma_j \leqslant |k - na_j| \leqslant n\varepsilon] \\
&\quad \leqslant \sum_{|k-na_j| \geqslant (\log n)/\sigma_j} \exp\Big(\frac{c_j}{2}(\sigma_j(k-na_j))^{2m_j} + C\sigma_j|k-na_j| + C\Big) \\
&\quad = \mathrm{O}(1) \int_{|x| \geqslant (\log n)/\sigma_j} \exp\Big(\frac{c_j}{2}(\sigma_j x)^{2m_j} + C|\sigma_j x| + C\Big) dx = \mathrm{O}(1/n).
\end{aligned} \tag{3.32}$$

Here, in the last inequality we have used $\int_{|y| \geqslant \log n} \exp(c_j y^{2m_j} + Cy + C) dy = \mathrm{O}(n^{-2})$ since $c_j < 0$ and $m_j \geqslant 1$. It follows from (3.27) and (3.32) that

$$\begin{aligned}
Z_{n,j}(\delta) &= \big(1 + \mathrm{O}(1/n)\big) \sum_{\substack{|k-na_j| \\ \leqslant (\log n)/\sigma_j}} \exp(H_n(k/n)) \\
&= \big(1 + \mathrm{O}(1/n)\big) \exp(H_n(k_j/n)) \sum_{\substack{|k-na_j| \\ \leqslant (\log n)/\sigma_j}} \frac{\exp(H_n(k/n))}{\exp(H_n(k_j/n))} \\
&= \big(1 + \mathrm{O}(\sigma_*)\big) \exp(nA_n(k_j/n) + n\sigma_* B(a_j)) \\
&\qquad \times \sum_{\substack{|k-na_j| \\ \leqslant (\log n)/\sigma_j}} \frac{\exp(H_n(k/n))}{\exp(H_n(k_j/n))},
\end{aligned} \tag{3.33}$$

where for the last equation we used (A3) to derive that

$$\big|nA_n(k_j/n) + n\sigma_* B(a_j) - H_n(k_j/n)\big| = \big|n\sigma_*\big(B_n(k_j/n) - B(a_j)\big)\big| = \mathrm{O}(\sigma_*).$$

By (3.31), if $|k - na_j| \leqslant \sigma_j^{-1} \log n$ then

$$\begin{aligned}
&n\sigma_*[B_n(k/n) - B_n(k_j/n)] \\
&\quad = B'(a_j)\sigma_*(k - na_j) + \mathrm{O}((\log n)^2 \sigma_*/n\sigma_j^2) + \mathrm{O}(\sigma_*) \\
&\quad = b_j \sigma_j (k - na_j) + \mathrm{O}(\sigma_*(\log n)/\sigma_j \, \mathrm{I}[\sigma_j \neq \sigma_*]) \\
&\qquad + \mathrm{O}((\log n)^2 \sigma_*/n\sigma_j^2) + \mathrm{O}(\sigma_*),
\end{aligned}$$

since $b_j = B'(a_j)\,\mathrm{I}[\sigma_j = \sigma_*]$. Similarly, by (3.28) for $|k - na_j| \leqslant \sigma_j^{-1} \log n$,

$$\begin{aligned}
n[A_n(k/n) - A_n(k_j/n)] &= c_j n(k/n - a_j)^{2m_j} + \mathrm{O}((\log n)^{2m_j+1}/n\sigma_j) \\
&= c_j(\sigma_j(k - na_j))^{2m_j} + \mathrm{O}((\log n)^{2m_j+1}/n\sigma_j).
\end{aligned}$$

Therefore,

$$H_n(k/n) - H_n(k_j/n) = c_j(\sigma_j(k - na_j))^{2m_j} + b_j\sigma_j(k - na_j) + \mathrm{O}(\tau_j), \tag{3.34}$$

where

$$\tau_j = \frac{(\log n)^{2m_j+1}}{n\sigma_j} + \frac{\sigma_* \log n}{\sigma_j} \mathrm{I}[\sigma_j \neq \sigma_*].$$

We now compute

$$\begin{aligned}\sum_{\substack{|k-na_j|\\ \leqslant(\log n)/\sigma_j}} &\exp\big(c_j(\sigma_j(k-na_j))^{2m_j}+b_j\sigma_j(k-na_j)\big)\\ &=\sum_{i\in\Gamma_n}\exp\big(c_j(i\sigma_j)^{2m_j}+b_j(i\sigma_j)\big),\end{aligned}\tag{3.35}$$

where $\Gamma_n=\{k-na_j : k\in\mathbb{Z}, |k-na_j|\leqslant(\log n)/\sigma_j\}$. Denote by $h(x)=\exp\big(c_jx^{2m_j}+b_jx\big)$. Then for all $i\in\Gamma_n$, by Taylor expansion

$$\Big|h(i\sigma_j)-\sigma_j^{-1}\int_{i\sigma_j}^{(i+1)\sigma_j}h(x)dx\Big|\leqslant\sigma_j\sup_{i\sigma_j\leqslant x\leqslant(i+1)\sigma_j}|h'(x)|.$$

Hence,

$$\begin{aligned}\Big|&\sum_{i\in\Gamma_n}h(i\sigma_j)-\sigma_j^{-1}\int_{\mathbb{R}}h(x)dx\Big|\\ &\leqslant\sigma_j\sum_{i\in\Gamma_n}\sup_{i\sigma_j\leqslant x\leqslant(i+1)\sigma_j}|h'(x)|+\int_{|x|\geqslant\log n}h(x)dx.\end{aligned}\tag{3.36}$$

Since $h'(x)=\exp(c_jx^{2m_j}+b_jx)(2m_jc_jx^{2m_j-1}+b_j)$ with $c_j<0$, we can find a positive constant $C=C(c_j,m_j,b_j)$, such that if $|y|\geqslant C$ then

$$\sup_{x\in\mathbb{R}}|h'(x)|\leqslant C,\qquad\sup_{y\leqslant x\leqslant y+1}|h'(x)|\leqslant\exp(-c_jy^{2m_j}/2).$$

Therefore, we have

$$\begin{aligned}\sum_{i\in\Gamma_n}\sup_{i\sigma_j\leqslant x\leqslant(i+1)\sigma_j}|h'(x)|&\leqslant 2C^2/\sigma_j+\sum_{i\in\Gamma_n}\exp(-c_j(i\sigma_j)^{2m_j}/2)\\ &\leqslant \mathrm{O}(1/\sigma_j)+\int_{|x|\leqslant(\log n)/\sigma_j}\exp(-c_j(x\sigma_j)^{2m_j}/2)dx\\ &=\mathrm{O}(1/\sigma_j),\end{aligned}$$

which together with (3.36) yields that

$$\begin{aligned}\sum_{i\in\Gamma_n}h(i\sigma_j)&=\sigma_j^{-1}\int_{\mathbb{R}}h(x)dx+\mathrm{O}(1)+\int_{|x|\geqslant\log n}h(x)dx\\ &=\sigma_j^{-1}q_j+\mathrm{O}(1),\end{aligned}$$

since $q_j=\int_{\mathbb{R}}h(x)dx$. Combining this with (3.34) and (3.35) we obtain that

$$\sum_{\substack{|k-na_j|\\ \leqslant(\log n)/\sigma_j}}\frac{\exp(H_n(k/n))}{\exp(H_n(k_j/n))}=(1+\mathrm{O}(\tau_j))\sigma_j^{-1}q_j+\mathrm{O}(1)=(1+\mathrm{O}(\tau_j))\sigma_j^{-1}q_j,$$

since $\tau_j\geqslant(\log n)^{2m_j+1}/(n\sigma_j)\geqslant\sigma_j$. We finally deduce (3.5) from the above estimate and (3.33). □

4 MAXIMUM LIKELIHOOD ESTIMATOR OF LINEAR MODELS

We first recall the generalized linear model (1.3) given as

$$\mu_n(\omega) = \frac{1}{Z_n}\exp(H_n(\omega)), \qquad \omega \in \Omega_n = \{+1,-1\}^n,$$

where

$$Z_n = \sum_{\omega\in\Omega_n} \exp(H_n(\omega)),$$

and

$$H_n(\omega) = n\big(\beta_1 f_1(\bar{\omega}_+) + \ldots + \beta_l f_l(\bar{\omega}_+)\big), \quad \bar{\omega}_+ = \frac{\big|\{i : \omega_i = 1\}\big|}{n}.$$

Since we construct the estimator for each parameter β_i considering the others $(\beta_j)_{j\neq i}$ to be known, for simplicity we rewrite

$$H_n(\omega) = n\big(\beta f(\bar{\omega}_+) + g(\bar{\omega}_+)\big), \tag{4.1}$$

where $f, g : [0,1] \to \mathbb{R}$ are non-constant smooth enough and known functions. Our aim is to estimate the parameter β. In order to build the MLE of β, we compute the log-likelihood function of the model as

$$L_n(\beta,\omega) = \frac{1}{n}\log\mu_n(\omega) = \beta f(\bar{\omega}_+) + g(\bar{\omega}_+) - \varphi_n(\beta)$$

with

$$\varphi_n(\beta) = \frac{1}{n}\log Z_n.$$

Then the MLE of β, denoted by $\hat{\beta}_n$, is a solution of

$$0 = \partial_\beta L_n = f(\bar{\omega}_+) - u(\beta),$$

where

$$u(\beta) = \partial_\beta\varphi_n = \mathbb{E}_\beta f(\bar{\omega}_+)$$

with $\mathbb{E}_\beta$ the Gibbs expectation with respect to μ_n for given β. Note that

$$\partial_\beta u = \mathbb{E}_\beta f(\bar{\omega}_+)^2 - \mathbb{E}_\beta\{f(\bar{\omega}_+)\}^2 > 0$$

since f is non-constant. Therefore, u is strictly increasing in β, and thus

$$\hat{\beta}_n = u^{-1}(f(\bar{\omega}_+)). \tag{4.2}$$

Before stating the main result of this section, recall the entropy function $I : [0,1] \to \mathbb{R}$ defined as $I(a) = -a\log a + (a-1)\log(1-a)$ for $a \in [0,1]$ with the convention that $0\cdot\log 0 = 0$.

Theorem 4.1. *Consider the maximum likelihood estimator $\hat{\beta}_n$ as in (4.2) of the linear model having Hamiltonian given by (4.1) with $f, g \in C^{2m_*+1}([0,1])$ and $m_* \in \mathbb{N}$. Suppose that the function $A : [0,1] \to \mathbb{R}$ given as $A(a) = \beta f(a) + g(a) + I(a)$ has finite maximizers, denoted by $(a_j)_{j\in J}$, satisfying* *that* $A^{(k)}(a_j) = 0$ *for all* $1 \leqslant k \leqslant 2m_j - 1$ *and* $A^{(2m_j)}(a_j) < 0$ *for all* $j \in J$, *with* $(m_j)_{j\in J} \subset \mathrm{N}$ *and* $m_* = \max_{j\in J} m_j$. *Define*

$$J_1^+ = \{j \in J : f(a_j) = \max_{k\in J} f(a_k)\}, \quad J_2^+ = \{j \in J_1^+ : m_j = \max_{k\in J_1^+} m_k\},$$
$$J_1^- = \{j \in J : f(a_j) = \min_{k\in J} f(a_k)\}, \quad J_2^- = \{j \in J_1^- : m_j = \max_{k\in J_1^-} m_k\}.$$

Assume that $(J_2^- \cup J_2^+) \subset J_* := \{j \in J : m_j = m_*\}$, *and assume that there exist* $j \in J_2^-$ *and* $k \in J_2^+$ *such that*

$$f'(a_j)f'(a_k) \neq 0. \tag{4.3}$$

Then
$$(\hat\beta_n - \beta)n^{1-1/(2m_*)} \xrightarrow{\mathscr{L}} U,$$
where the distribution of U *is given as in* (4.22)–(4.24).

Proof. For simplicity we omit the subscript n in all involved terms. Let

$$X = n\bar\omega_+, \qquad \sigma_j = n^{1/(2m_j)-1} \text{ for } j \in J, \qquad \sigma_* = n^{1/(2m_*)-1}.$$

For $\gamma \in \mathbb{R}$, we call $\mathbb{P}_\gamma$ the Gibbs measure at parameter γ and $\mathbb{E}_\gamma$ the corresponding expectation. With $X = n\bar\omega_+$, we have for $0 \leqslant k \leqslant n$ that

$$\mathbb{P}_\beta[X = k] \propto \exp\big(n\big(\beta f(k/n) + g(k/n)\big)\big)\binom{n}{k} = \exp(nA_n(k/n)),$$

where $A_n : \{0, 1/n, \ldots, 1\} \to \mathbb{R}$ is defined as

$$A_n(k/n) = \beta f(k/n) + g(k/n) + \frac{1}{n}\log\binom{n}{k}.$$

Recall that $A(a) = \beta f(a) + g(a) + I(a)$ and $\frac{1}{n}\log\binom{n}{k}$ is well approximated by $I(k/n)$. Let $B \in C^2([0,1])$ and define $B_n : \{0, 1/n, \ldots, 1\} \to \mathbb{R}$ as $B_n(k/n) = B(k/n)$ for $0 \leqslant k \leqslant n$. Then it is straightforward to check that there exist ε_*, δ_* and C_* such that (A1)–(A4) hold. For any $j \in J$, we define the event
$$\mathcal{A}_j = \{|X/n - a_j| \leqslant \delta_*\},$$
and for $t \in \mathbb{R}$ define the random variable

$$Y_j(t) \propto \exp(c_j x^{2m_j} + t b_j x),$$

where
$$c_j = \frac{A^{(2m_j)}(a_j)}{(2m_j)!}, \qquad b_j = B'(a_j)\,\mathrm{I}[j \in J_*].$$

Fix $t < 0$, by the definition of $\hat\beta$ and the monotonicity of u we have

$$\begin{aligned}&\mathbb{P}_\beta[(\hat\beta - \beta)/\sigma_* \leqslant t] \\ &\quad= \mathbb{P}_\beta\big[u^{-1}(f(X/n)) \leqslant \beta + t\sigma_*\big] = \mathbb{P}_\beta[f(X/n) \leqslant u(\beta + t\sigma_*)].\end{aligned}$$

Part 1. We start by estimating $u(\beta + t\sigma_*)$. Note that $u(\beta + t\sigma_*) = \mathbb{E}_{\beta+t\sigma_*} f(X/n)$, and in the application of Theorem 2.1, the measure $\mathbb{P}_{\beta+t\sigma_*}$ corresponds to the case $B = tf$. Hence, with $t < 0$, we have $J_1 \equiv J_1^-$ and $J_2 \equiv J_2^-$. Thus by Theorem 2.1,

$$X/n \overset{\mathbb{P}_{\beta+t\sigma_*}}{\longrightarrow} \sum_{j\in J_2^-} p_j^-(t)\delta_{a_j},$$

where for $j, i \in J_2^-$, we have

$$p_j^-(t) = \frac{q_j(t)}{\sum_{i\in J_2^-} q_i(t)}, \qquad q_i(t) = \int_{\mathbb{R}} \exp(c_i x^{2m_i} + tb_i x)dx.$$

Note that $b_j = B'(a_j)\,\mathrm{I}[j \in J_*] = B'(a_j)$ for $j \in J_2^-$, since we assume that $J_2^- \subset J_*$. This assumption also yields that $\sigma_{j_2} = \sigma_*$ for all $j_2 \in J_2^-$. Therefore, using Theorem 2.2, we have

$$\mathbb{P}_{\beta+t\sigma_*}[\mathcal{A}_j] = p_j^-(t) + \mathrm{O}(\tau_* + \tau_*^-) \quad \text{for all } j \in J_2^-, \tag{4.4}$$

$$\mathbb{P}_{\beta+t\sigma_*}[\mathcal{A}_j] = \mathrm{O}(\sigma_*/\sigma_j) \quad \text{for all } j \in J_1^- \setminus J_2^-, \tag{4.5}$$

$$\mathbb{P}_{\beta+t\sigma_*}\big[\cap_{j\in J_1^-} \mathcal{A}_j^c\big] \leqslant \exp(-cn\sigma_*), \tag{4.6}$$

where c is a positive constant and

$$\tau_* = (\log n)^{2m_*+1}/(n\sigma_*), \quad \tau_*^- = \max_{j\in J_1^-\setminus J_2^-} \sigma_*/\sigma_j.$$

In addition, Theorem 2.3 yields that for any $j \in J$,

$$\mathbb{E}_{\beta+t\sigma_j}\big\{(X/n - a_j)^2\big|\mathcal{A}_j\big\} = \mathrm{O}(1/(n\sigma_j)^2), \tag{4.7}$$

and

$$\begin{aligned} &d_{\mathrm{W}}\big(\mathscr{L}_{\mathbb{P}_{\beta+t\sigma_*}}(\sigma_j(X - na_j)|\mathcal{A}_j), \mathscr{L}(Y_j(t))\big) \\ &\quad = \mathrm{O}(1/(n\sigma_j)) + \mathrm{O}(\sigma_*/\sigma_j\, \mathrm{I}[j \notin J_*]). \end{aligned} \tag{4.8}$$

We remark that here and below the notation O depends on $\|B\|_\infty = |t|\|f\|_\infty$ and $\|A\|_\infty$. Let $\lambda_- = \min_{j\in J} f(a_j)$. Then $\lambda_- = f(a_j)$ for all $j \in J_1^-$, and therefore

$$\begin{aligned} u(\beta + t\sigma_*) - \lambda_- &= \mathbb{E}_{\beta+t\sigma_*}\big\{f(X/n) - \lambda_-\big\} \\ &= \sum_{j\in J_1^-} \mathbb{E}_{\beta+t\sigma_*}\big\{f(X/n) - f(a_j)\big|\mathcal{A}_j\big\}\mathbb{P}_{\beta+t\sigma_*}[\mathcal{A}_j] \\ &\qquad + \mathbb{E}_{\beta+t\sigma_*}\big\{(f(X/n) - \lambda_-)\,\mathrm{I}\big[\cap_{j\in J_1^-} \mathcal{A}_j^c\big]\big\}. \end{aligned} \tag{4.9}$$

For $j \in J$, by Taylor's expansion,

$$\begin{aligned} &\mathbb{E}_{\beta+t\sigma_*}\{f(X/n) - f(a_j)|\mathcal{A}_j\} \\ &\quad = \mathbb{E}_{\beta+t\sigma_*}\big\{f'(a_j)\sigma_k(X - na_j)\big|\mathcal{A}_j\big\}/(n\sigma_j) + \mathrm{O}(1)\mathbb{E}_{\beta+t\sigma_*}\big\{(X/n - a_j)^2\big|\mathcal{A}_j\big\}. \end{aligned}$$

In addition, by (4.8),

$$\mathbb{E}_{\beta+t\sigma_*}\{f'(a_j)\sigma_j(X-na_j)\big|\mathcal{A}_j\}$$
$$= f'(a_j)\mathbb{E}Y_j(t) + \mathrm{O}(1/(n\sigma_j)) + \mathrm{O}(\sigma_*/\sigma_j\,\mathrm{I}[j\notin J_*]).$$

The last two estimates and (4.7) yields that

$$\begin{aligned}&\mathbb{E}_{\beta+t\sigma_*}\{f(X/n)-f(a_j)|\mathcal{A}_j\}\\ &\quad= f'(a_j)\mathbb{E}Y_j(t)/(n\sigma_j) + \mathrm{O}(1/(n\sigma_j)^2) + \mathrm{O}(\sigma_*/n\sigma_j^2\,\mathrm{I}[j\notin J_*]).\end{aligned} \tag{4.10}$$

Combining this with (4.4) and the fact that $\sigma_j=\sigma_*$ for all $j\in J_2^-$, and $J_2^-\subset J_*$, we obtain that

$$\sum_{j\in J_2^-}\mathbb{E}_{\beta+t\sigma_*}\{f(X/n)-f(a_j)|\mathcal{A}_j\}\mathbb{P}_{\beta+t\sigma_*}[\mathcal{A}_j]$$
$$= (n\sigma_*)^{-1}\sum_{k\in J_2^-} f'(a_j)\mathbb{E}Y_j(t)p_j^-(t) + \mathrm{O}((\tau_*+\tau_*^-)/n\sigma_*).$$

Using (4.5) and (4.10), we have

$$\sum_{j\in J_1^-\setminus J_2^-}\mathbb{E}_{\beta+t\sigma_*}\{f(X/n)-f(a_j)|\mathcal{A}_j\}\mathbb{P}_{\beta+t\sigma_*}[\mathcal{A}_j]$$
$$= \mathrm{O}(1)\sum_{j\in J_1^-\setminus J_2^-}\sigma_*/n\sigma_j^2 = \mathrm{O}(\tau_*^-/n\sigma_*),$$

and by (4.6)

$$\mathbb{E}_{\beta+t\sigma_*}\big\{(f(X/n)-\lambda_-)\,\mathrm{I}\big[\cap_{j\in J_1^-}\mathcal{A}_j^c\big]\big\} \leqslant \exp(-cn\sigma_*/2).$$

It follows from the last three display equations and (4.9) that

$$n\sigma_*(u(\beta+t\sigma_*)-\lambda_-) = e_-(t) + \mathrm{O}(\tau_*+\tau_*^-), \tag{4.11}$$

where

$$e_-(t) = \sum_{j\in J_2^-} f'(a_j)\mathbb{E}Y_j(t)p_j^-(t). \tag{4.12}$$

Note that

$$e_-(t) = \frac{\sum_{j\in J_2^-}\int_{\mathbb{R}} f'(a_j)x\exp(c_jx^{2m_j}+tf'(a_j)x)dx}{\sum_{j\in J_2^-}\int_{\mathbb{R}}\exp(c_jx^{2m_j}+tf'(a_j)x)dx}.$$

Moreover, if $f'(a_j)\neq 0$ by changing variable $y=tf'(a_j)x$,

$$\int_{\mathbb{R}} f'(a_j)x\exp(c_jx^{2m_j}+tf'(a_j)x)dx$$
$$= \frac{\mathrm{sgn}(tf'(a_j))}{t^2f'(a_j)}\int_{\mathbb{R}} y\exp\big(c_jy^{2m_j}/(tf'(a_j))^{2m_j}+y\big)dy < 0,$$

since $t<0$ and $\int_{\mathbb{R}} y\exp(cy^{2m}+y)dy>0$ for all $c<0$ and $m\in\mathbb{N}$. In addition, by the assumption (4.3) there exists $j\in J_2^-$ such that $f'(a_j)\neq 0$. Thus by the two above display equations, we have

$$e_-(t)\in(-\infty,0)$$

is a negative and finite constant.

Part 2. We proceed to compute $\mathbb{P}_\beta[f(X/n) \leqslant u(\beta + t\sigma_*)]$. In the application of Theorem 2.1, the measure $\mathbb{P}_\beta$ corresponds to the case $B \equiv 0$, or $J_1 = J$ and $J_2 = J_*$. Hence, by Theorem 2.1, we have

$$X/n \xrightarrow{\mathbb{P}_\beta} \sum_{j \in J_*} p_j \delta_{a_j},$$

where for i and j in J,

$$p_j = \frac{q_j}{\sum_{i \in J_*} q_i}, \qquad q_i = \int_{\mathbb{R}} \exp(c_i x^{2m_i}) dx.$$

Moreover, by Theorem 2.2,

$$\mathbb{P}_\beta[\mathcal{A}_j] = p_j + \mathrm{O}(\tau_* + \tau_*') \quad \text{for all } j \in J_*, \qquad \mathbb{P}_\beta\big[\cap_{j \in J_*} \mathcal{A}_j^c\big] = \mathrm{O}(\tau_*'), \tag{4.13}$$

where $\tau_*' = \max_{j \in J \setminus J_*} \sigma_*/\sigma_j$. By Theorem 2.3,

$$\mathbb{E}_\beta\big\{(X/n - a_j)^2 \big| \mathcal{A}_j\big\} = \mathrm{O}\big((n\sigma_j)^{-2}\big), \tag{4.14}$$

and

$$d_{\mathrm{W}}\big(\mathscr{L}_{\mathbb{P}_\beta}(\sigma_j(X - na_j)|\mathcal{A}_j), \mathscr{L}(Y_j)\big) = \mathrm{O}\big((n\sigma_j)^{-1}\big), \tag{4.15}$$

where $Y_j = Y_j(0) \propto \exp(c_j x^{2m_j})$. It follows from (4.13) that

$$\begin{aligned} &\mathbb{P}_\beta[f(X/n) \leqslant u(\beta + t\sigma_*)] \\ &\quad = \sum_{j \in J_*} \mathbb{P}_\beta[f(X/n) \leqslant u(\beta + t\sigma_*)|\mathcal{A}_j]p_j + \mathrm{O}(\tau_* + \tau_*'). \end{aligned} \tag{4.16}$$

By Lemma 3.3 (ii) and (4.15)

$$\begin{aligned} d_{\mathrm{K}}(\mathscr{L}_{\mathbb{P}_\beta}(\sigma_j(X - na_j)|\mathcal{A}_j), \mathscr{L}(Y_j)) &\leqslant d_{\mathrm{W}}(\mathscr{L}_{\mathbb{P}_\beta}(\sigma_j(X - na_j)|\mathcal{A}_j), \mathscr{L}(Y_j))^{1/2} \\ &= \mathrm{O}((n\sigma_j)^{-1/2}). \end{aligned} \tag{4.17}$$

In particular, for all $\delta > 0$

$$\begin{aligned} &\sup_{s \in \mathbb{R}} \mathbb{P}_\beta\big[s \leqslant f'(a_j)\sigma_j(X - na_j) \leqslant s + \delta \big| A_j\big] \\ &\quad \leqslant \sup_{s \in \mathbb{R}} \mathbb{P}_\beta\big[s \leqslant f'(a_j)Y_j \leqslant s + \delta\big] + \mathrm{O}((n\sigma_j)^{-1/2}) = \mathrm{O}(\delta) + \mathrm{O}((n\sigma_j)^{-1/2}), \end{aligned}$$

since Y_j has the bounded density. Using the inequality that $|f(x) - f(a) - f'(a)(x - a)| \leqslant \|f\|_\infty (x - a)^2/2$ and Lemma 3.3(i), and the above estimate, we have

$$\begin{aligned} &d_{\mathrm{K}}\big(\mathscr{L}_{\mathbb{P}_\beta}[n\sigma_j(f(X/n) - f(a_j))|\mathcal{A}_j], \mathscr{L}(f'(a_j)\sigma_j(X - na_j)\big|\mathcal{A}_j)\big) \\ &\quad \leqslant \inf_{\delta > 0} \Big(\sup_{s \in \mathbb{R}} \mathbb{P}_\beta[s \leqslant f'(a_j)\sigma_j(X - na_j) \leqslant s + \delta|\mathcal{A}_j] \\ &\qquad\qquad + \mathbb{P}_\beta[\|f''\|_\infty (n\sigma_j(X/n - a_j)^2) \geqslant 2\delta|\mathcal{A}_j]\Big) \\ &\quad = \mathrm{O}(1) \inf_{\delta > 0} \big\{\delta + \mathbb{P}_\beta\big[\|f''\|_\infty (n\sigma_j(X/n - a_j)^2) \geqslant 2\delta\big|\mathcal{A}_j\big]\big\} + \mathrm{O}\big((n\sigma_j)^{-1/2}\big). \end{aligned}$$

Moreover, by Markov's inequality and (4.14)

$$\begin{aligned}&\mathbb{P}_\beta\big[\|f''\|_\infty(n\sigma_j(X/n-a_j)^2)\geqslant 2\delta\big|\mathcal{A}_j\big]\\&\qquad=\mathrm{O}(1)\,\mathbb{E}\big\{n\sigma_j(X/n-a_j)^2\big|\mathcal{A}_j\big\}/\delta=\mathrm{O}\big((\delta n\sigma_j)^{-1}\big).\end{aligned}$$

Combining the two above estimates and taking $\delta=(n\sigma_j)^{-1/2}$, we obtain

$$\begin{aligned}&d_{\mathrm{K}}\big(\mathscr{L}_{\mathbb{P}_\beta}(n\sigma_j(f(X/n)-f(a_j))|\mathcal{A}_j),\mathscr{L}(f'(a_j)\sigma_j(X-na_j)|\mathcal{A}_j)\big)\\&\qquad\qquad=\mathrm{O}((n\sigma_j)^{-1/2}),\end{aligned}$$

which together with (4.17) implies that for all $j\in J$

$$d_{\mathrm{K}}\big(\mathscr{L}_{\mathbb{P}_\beta}(n\sigma_j(f(X/n)-f(a_j))|\mathcal{A}_j),\mathscr{L}(f'(a_j)Y_j)\big)=\mathrm{O}((n\sigma_j)^{-1/2}).\tag{4.18}$$

If $j\in J_*\setminus J_1^-$ then by the definition of J_1^-, we have $f(a_j)>\lambda_-$. Hence, by (4.11),

$$u(\beta+t\sigma_*)=\lambda_-+o(1)\leqslant(f(a_j)+\lambda_-)/2.$$

Thus

$$\begin{aligned}&\mathbb{P}_\beta[f(X/n)\leqslant u(\beta+t\sigma_*)|\mathcal{A}_j]\\&\quad\leqslant\mathbb{P}_\beta[f(X/n)\leqslant(\lambda_-+f(a_j))/2|\mathcal{A}_j]\\&\quad=\mathbb{P}_\beta[n\sigma_j(f(X/n)-f(a_j))\leqslant n\sigma_j(\lambda_--f(a_j))/2|\mathcal{A}_j]\\&\quad\leqslant d_{\mathrm{K}}\big(\mathscr{L}_{\mathbb{P}_\beta}(n\sigma_j(f(X/n)-f(a_j))|\mathcal{A}_j),\mathscr{L}(f'(a_j)Y_j)\big)\\&\qquad+\mathbb{P}[f'(a_j)Y_j\leqslant n\sigma_j(\lambda_--f(a_j))/2]=\mathrm{O}((n\sigma_j)^{-1/2}),\end{aligned}$$

by using (4.18) and the following estimate

$$\mathbb{P}[f'(a_j)Y_j\leqslant n\sigma_j(\lambda_--f(a_j))/4]\leqslant\exp(-c(n\sigma_j)^2),$$

for some $c>0$, since $Y_j\propto\exp(c_jx^{2m_j}+b_jx)$ with $c_j<0$, and $\lambda_-<f(a_j)$.

Next, assume that $j\in J_*\cap J_1^-$. Then $\sigma_j=\sigma_*$ and $f(a_j)=\lambda_-$. Therefore,

$$\begin{aligned}&\mathbb{P}_\beta[f(X/n)\leqslant u(\beta+t\sigma_*)|\mathcal{A}_j]\\&\quad=\mathbb{P}_\beta\big[n\sigma_j(f(X/n)-f(a_j))\leqslant n\sigma_*(u(\beta+t\sigma_*)-\lambda_-)\big|\mathcal{A}_j\big].\end{aligned}$$

Combining this with (4.18) yields that

$$\begin{aligned}&\mathbb{P}_\beta\big[f(X/n)\leqslant u(\beta+t\sigma_*)\big|\mathcal{A}_j\big]\\&\quad=\mathbb{P}\big[f'(a_j)Y_j\leqslant n\sigma_*(u(\beta+t\sigma_*)-\lambda_-)\big]+\mathrm{O}(1/(n\sigma_*)^{1/2}).\end{aligned}\tag{4.19}$$

Recall that by (4.11)

$$n\sigma_*(u(\beta+t\sigma_*)-\lambda_-)=e_-(t)+\mathrm{O}(\tau_*+\tau_*^-),$$

where $e_-(t)\in(-\infty,0)$ is given in (4.12). Hence, if $f'(a_j)=0$ then

$$\mathbb{P}[f'(a_j)Y_j\leqslant n\sigma_*(u(\beta+t\sigma_*)-\lambda_-)]=0.\tag{4.20}$$

If $f'(a_j) \neq 0$, since Y_j has the symmetric law with bounded density,

$$\begin{aligned}
&\mathbb{P}[f'(a_j)Y_j \leqslant n\sigma_*(u(\beta + t\sigma_*) - \lambda_-)] \\
&\quad = \mathbb{P}\big[Y_j \leqslant n\sigma_*(u(\beta + t\sigma_*) - \lambda_-)/f'(a_j)\big] \\
&\quad = \mathbb{P}\big[Y_j \leqslant e_-(t)/f'(a_j) + \mathrm{O}(\tau_* + \tau_*^-)\big] \\
&\quad = \mathbb{P}\big[Y_j \leqslant e_-(t)/f'(a_j)\big] + \mathrm{O}(\tau_* + \tau_*^-).
\end{aligned}$$

Combining this with (4.19), we obtain that if $f'(a_j) \neq 0$ then

$$\begin{aligned}
&\mathbb{P}_\beta[f(X/n) \leqslant u(\beta + t\sigma_*)|\mathcal{A}_j] \\
&\quad = \mathbb{P}\big[Y_j \leqslant e_-(t)/f'(a_j)\big] + \mathrm{O}(\tau_* + \tau_*^-) + \mathrm{O}(1/(n\sigma_*)^{1/2}).
\end{aligned} \tag{4.21}$$

Part 3. We now combine the results from Parts 1 and 2. Using (4.12), (4.16), (4.20) and (4.21) we have for any fixed negative real number t,

$$\begin{aligned}
&\mathbb{P}[(\hat{\beta} - \beta)/\sigma_* \leqslant t] \\
&\quad = \mathbb{P}_\beta[f(X/n) \leqslant u(\beta + t\sigma_*)] \\
&\quad = \sum_{j \in J_*} \mathbb{P}_\beta[f(X/n) \leqslant u(\beta + t\sigma_*)|\mathcal{A}_j]p_j + \mathrm{O}(\tau_* + \tau_*') \\
&\quad = \sum_{j \in J_2^-} \mathbb{P}\left[Y_j \leqslant \sum_{k \in J_2^-} \frac{f'(a_k)}{f'(a_j)} p_k^-(t)\mathbb{E}Y_k(t)\right] p_j\, \mathrm{I}[f'(a_j) \neq 0] \\
&\qquad + \mathrm{O}((n\sigma_*)^{-1/2}) + \mathrm{O}(\tau_* + \tau_*').
\end{aligned}$$

Note here that $\tau_*^- \leqslant \tau_*'$. Similarly, for $t > 0$

$$\begin{aligned}
&\mathbb{P}[(\hat{\beta} - \beta)/\sigma_* > t] \\
&\quad = \sum_{j \in J_2^+} \mathbb{P}\left[Y_j > \sum_{k \in J_2^+} \frac{f'(a_k)}{f'(a_j)} p_k^+(t)\mathbb{E}Y_k(t)\right] p_j\, \mathrm{I}[f'(a_j) \neq 0] \\
&\qquad + \mathrm{O}((n\sigma_*)^{-1/2}) + \mathrm{O}(\tau_* + \tau_*'),
\end{aligned}$$

where for $k \in J_2^+$

$$p_k^+(t) = \frac{q_k(t)}{\sum_{i \in J_2^+} q_i(t)}, \qquad q_i(t) = \int_{\mathbb{R}} \exp(c_i x^{2m_i} + tb_i x)dx.$$

We recall that the term O depends on t, $\|f\|_\infty$ and $\|g\|_\infty$. Hence, for any fixed real number $t \neq 0$, there is a positive constant $C = C(t)$, such that for all n sufficiently large

$$|\mathbb{P}[(\hat{\beta} - \beta)/\sigma_* \leqslant t] - \mathbb{P}[U \leqslant t]| \leqslant C[(n\sigma_*)^{-1/2} + \theta_- + \theta_+] = \mathrm{o}(1),$$

where U has the distribution as

$$\begin{aligned}
&\mathbb{P}[U \leqslant t] \\
&\quad = \sum_{j \in J_2^-} p_j\, \mathrm{I}[f'(a_j) \neq 0]\mathbb{P}\left[Y_j \leqslant \sum_{k \in J_2^-} \frac{f'(a_k)}{f'(a_j)} p_k^-(t)\mathbb{E}Y_k(t)\right], \quad t < 0,
\end{aligned} \tag{4.22}$$

$$\mathbb{P}[U > t] = \sum_{j\in J_2^+} p_j \, \mathrm{I}[f'(a_j) \neq 0]\mathbb{P}\Big[Y_j > \sum_{k\in J_2^+} \frac{f'(a_k)}{f'(a_j)} p_k^+(t)\mathbb{E}Y_k(t)\Big], \quad t > 0, \tag{4.23}$$

$$\mathbb{P}[U = 0] = 1 - \frac{1}{2} \sum_{j\in(J_2^+ \cup J_2^-)} p_j \, \mathrm{I}[f'(a_j) \neq 0]. \tag{4.24}$$

Note that the value $\mathbb{P}[U = 0] = 1 - \mathbb{P}[U < 0] - \mathbb{P}[U > 0]$ is obtained as follows. Letting $t \to 0^+$ and $t \to 0^-$ in the formulas of $\mathbb{P}[U \leqslant t]$ and $\mathbb{P}[U > t]$, since $\mathbb{E}Y_k(0) = 0$ and $\mathbb{P}[Y_j \leqslant 0] = 1/2$, we have

$$\mathbb{P}[U < 0] = \frac{1}{2} \sum_{j\in J_2^-} p_j \, \mathrm{I}[f'(a_j) \neq 0], \qquad \mathbb{P}[U > 0] = \frac{1}{2} \sum_{j\in J_2^+} p_j \, \mathrm{I}[f'(a_j) \neq 0].$$

We finally conclude that

$$(\hat{\beta} - \beta)/\sigma_* \xrightarrow{\mathscr{L}} U,$$

and finish the proof of Theorem 4.2. □

Remark 4.2. We consider some special cases. If $|J| = 1$ then $J_2^+ = J_2^- = J$, and we denote by a_* the unique maximizer and assume that $f'(a_*) \neq 0$. In this case, the distribution of U is as follows. For all $t \in \mathbb{R}$,

$$\mathbb{P}[U \leqslant t] = \mathbb{P}[Y \leqslant \mathbb{E}Y(t)],$$

where, by denoting m_* the order of regularity of a_*,

$$Y = Y(0), \qquad Y(t) \propto \exp(c_* x^{2m_*} + tf'(a_*)x), \quad c_* = \frac{A^{2m_*}(a_*)}{(2m_*)!} < 0.$$

Note that if $m_* = 1$ then $Y(t) \sim N\big(tf'(a_*)/2|c_*|, 1/2|c_*|\big)$, and we can compute

$$U = N\big(0, 2|c_*|/f'(a_*)^2\big).$$

Next, consider the case all the maximizers have the same order of regularity, i.e. $m_j = m_*$ for all $j \in J$. Then $J_2^- = J_1^- = J_- = \{j \in J : f(a_j) = \min_{k\in J} f(a_k)\}$, and $J_2^+ = J_1^+ = J_+ = \{j \in J : f(a_j) = \max_{k\in J} f(a_k)\}$, and we assume that there exist $j \in J_-$ and $k \in J_+$ such that $f'(a_j)f'(a_k) \neq 0$. The law of U is given as in (4.22)–(4.24) when replacing J_2^- and J_2^+ by J_- and J_+.
Finally, we consider the case $m_j = 1$ for all $j \in J$, and

$$\begin{aligned} c_j = c_k = c_-, &\quad f'(a_j) = f'(a_k) = d_- \quad \text{for all } k, j \in J_-, \\ c_j = c_k = c_+, &\quad f'(a_j) = f'(a_k) = d_+ \quad \text{for all } k, j \in J_+. \end{aligned}$$

Then for $j \in J_-$ and $t < 0$, we have $p_j^-(t) = 1/|J_-|$, and $Y_j(t) \sim N(\frac{td_-}{2|c_-|}, \frac{1}{2|c_-|})$. Therefore, for $t \in \mathbb{R}_-$,

$$\mathbb{P}[U \leqslant t] = p_- \, \mathbb{P}\Big[N\Big(0, \tfrac{1}{2|c_-|}\Big) \leqslant \tfrac{td_-}{2|c_-|}\Big] = p_- \, \mathbb{P}\Big[N\Big(0, \tfrac{2|c_-|}{d_-^2}\Big) \leqslant t\Big],$$

where

$$p_- = \sum_{j\in J_-} p_j \, \mathrm{I}[f'(a_j) \neq 0].$$

Similarly for $t \in \mathbb{R}_+$,

$$\mathbb{P}[U > t] = \mathbb{P}\Big[N\Big(0, \tfrac{2|c_+|}{d_+^2}\Big) > t\Big]p_+, \qquad p_+ = \sum_{j\in J_+} p_j \, \mathrm{I}[f'(a_j) \neq 0].$$

Thus

$$U = \tfrac{p_-}{2} N^-\Big(0, \tfrac{2|c_-|}{d_-^2}\Big) + \tfrac{p_+}{2} N^+\Big(0, \tfrac{2|c_+|}{d_+^2}\Big) + \Big(1 - \tfrac{p_- + p_+}{2}\Big)\delta_0,$$

where recall that $N^-(0, \sigma^2)$ (resp. $N^+(0, \sigma^2)$) is negative (resp. positive) half-normal distribution.

5 SOME EXAMPLES

In this section, we apply Theorems 2.1, 2.3 and 4.1 to the p-spin Curie-Weiss model and the annealed Ising model on random regular graphs. We say that a maximizer a_* of a smooth function A is $2m$-regular (with $m \in \mathbb{N}$) if $A^{(k)}(a_*) = 0$ for $k = 1, \ldots, 2m-1$ and $A^{(2m)}(a_*) < 0$.

5.1 p-spin Curie-Weiss model

Let $2 \leqslant p \in \mathbb{N}$, we consider the p-spin Curie-Weiss model with Hamiltonian

$$H_n(\omega) = \frac{\beta}{n^{p-1}} \sum_{1\leqslant i_1,\ldots,i_p\leqslant n} \omega_{i_1}\ldots\omega_{i_p} + h\sum_{i=1}^n \omega_i = n f_{\beta,h}(\bar{\omega}_+),$$

with

$$f_{\beta,h}(a) = \beta(2a-1)^p + h(2a-1), \qquad a \in [0,1]. \tag{5.1}$$

We now study the maximizers of

$$A(a) = f_{\beta,h}(a) + I(a), \quad a \in [0,1].$$

Mukherjee, Son and Bhattacharya (2021) have fully characterized the maximizers of the function A by showing that the parameter space $(\beta, h) \in \mathbb{R}_+ \times \mathbb{R}$ is partitioned into disjoint regions:

(i) regular region $R_1 = \{(\beta, h) : A$ has an unique maximizer $a_* \in (0,1)\}$ (in this case a_* is 2-regular);

(ii) p-critical curve $R_2 = \{(\beta, h) : A$ has multiple maximizers in $(0,1)\}$ (in this case all the maximizers are 2-regular);

(iii) p-special points $R_3 = \{(\beta, h) : A$ has an unique maximizer $a_* \in (0,1)$, $A''(a_*) = 0\}$ (in this case a_* is 4-regular).

We refer the reader to Appendix B of Mukherjee, Son and Bhattacharya (2021) for a complete picture of the partition (R_1, R_2, R_3).

Now, given the additional parameters $(\bar{\beta}, \bar{h})$, Mukherjee, Son and Bhattacharya (2021+) considered the perturbed Hamiltonians

$$H_n^r(\omega) = n f_{\beta,h}(\bar{\omega}_+) + \sqrt{n} B(\bar{\omega}_+)$$
$$H_n^s(\omega) = n f_{\beta,h}(\bar{\omega}_+) + n^{1/4} B(\bar{\omega}_+),$$

where

$$B(a) = f_{\bar{\beta},\bar{h}}(a), \quad a \in [0,1],$$

with $f_{\bar{\beta},\bar{h}}$ defined as in (5.1). Denoting the corresponding Gibbs measures by μ_n^r and μ_n^s and using Theorems 2.1 and 2.3, we obtain the following result.

Theorem 5.1. *Consider the magnetization $M_n = \sum_{i=1}^n \omega_i$ under the perturbed measures μ_n^r and μ_n^s. Corresponding to the cases (i)–(iii) we have the following.*

(I) If $(\beta, h) \in R_1$ then

$$d_{\mathrm{W}}\Big(\mathscr{L}_{\mu_n^r}(W_n), N\Big(\tfrac{2B'(a_*)}{|A''(a_*)|}, \tfrac{4}{|A''(a_*)|}\Big)\Big) = \mathrm{O}(n^{-1/2}),$$

where

$$W_n = \frac{M_n - n(2a_* - 1)}{\sqrt{n}}.$$

(II) If $(\beta, h) \in R_2$ then A has multiple maximizers, say $0 < a_1 < a_2 < \ldots < a_k < 1$. Let $\delta_* > 0$ be a constant such that the intervals $((a_i - \delta_*, a_i + \delta_*))_{i=1}^k$ are disjoint. Then under μ_n^r,

$$M_n/n \xrightarrow{\mathscr{L}} \sum_{i=1}^k p_i \delta_{2a_i - 1},$$

with $(p_i)_{i=1}^k$ being explicit constants. Moreover, for $1 \leqslant i \leqslant k$,

$$d_{\mathrm{W}}\Big(\mathscr{L}_{\mu_n^r}(W_{n,i} | \bar{\omega}_+ \in (a_i - \delta_*, a_i + \delta_*)), N\Big(\tfrac{2B'(a_i)}{|A''(a_i)|}, \tfrac{4}{|A''(a_i)|}\Big)\Big) = \mathrm{O}(n^{-1/2}),$$

where

$$W_{n,i} = \frac{M_n - n(2a_i - 1)}{\sqrt{n}}.$$

(III) If $(\beta, h) \in R_3$, then under μ_n^s,

$$d_{\mathrm{W}}(W_n, Y) = \mathrm{O}(n^{-1/4}), \qquad W_n = \frac{M_n - n(2a_* - 1)}{n^{3/4}},$$

where

$$Y \propto \exp\Big(\frac{c_* x^4}{16} + \frac{b_* x}{2}\Big), \quad c_* = \frac{A^{(4)}(a_*)}{24}, \quad b_* = B'(a_*).$$

Remark that here we transfer our results for X_n to M_n via the relation $M_n = 2X_n - n$. The above theorem covers Theorem 2.1 of Mukherjee, Son and Bhattacharya (2021) (the main result in this paper) and Theorem 3.1 of Mukherjee, Son and Bhattacharya (2021+) (the key result leading to the maximum likelihood estimators).

Now we aim to apply Theorem 4.1 to find the scaling limits of MLEs. First, we have to check the non-degeneracy condition in (4.3). Observe that this condition is always true for the parameter h, since the corresponding function $f_h(a) = 2a - 1$ is not degenerated at any $a \in [0, 1]$. However, that condition for β does not hold when $\beta \leqslant \tilde{\beta}_p$ and $h = 0$, where $\tilde{\beta}_p = \sup\{\beta \geqslant 0 : \sup_{a \in [0,1]} A(a) = 0\}$. In fact, in this case $a = 1/2$ is a maximizer of A that belongs to the set J_-, and the corresponding function $f_\beta(a) = (2a-1)^p$ is degenerated at this point. In summary, we have the following.

Theorem 5.2. *Consider the maximum likelihood estimators of the p-spin Curie-Weiss model denoted by $\hat{\beta}_n$ and $\hat{h}_n$.*

(Ia) *If $(\beta, h) \in R_1$, then*

$$\sqrt{n}(\hat{h}_n - h) \xrightarrow{\mathscr{L}} N(0, \sigma_h),$$

with σ_h a positive constant.

(Ib) *If $(\beta, h) \in R_1 \setminus \{(\beta, 0) : \beta \leqslant \tilde{\beta}_p\}$, then*

$$\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\mathscr{L}} N(0, \sigma_\beta),$$

with σ_β a positive constant.

(IIa) *If $(\beta, h) \in R_2$, then*

$$\sqrt{n}(\hat{h}_n - h) \xrightarrow{\mathscr{L}} U_h,$$

where

$$U_h = p_h^- N^-(0, \sigma_h^-) + p_h^+ N^+(0, \sigma_h^+) + (1 - p_h^- - p_h^+)\delta_0,$$

with $p_h^\pm, \sigma_h^\pm$ positive constants.

(IIb) *If $(\beta, h) \in R_2 \setminus \{(\tilde{\beta}_p, 0)\}$, then*

$$\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\mathscr{L}} U_\beta,$$

where

$$U_\beta = p_\beta^- N^-(0, \sigma_\beta^-) + p_\beta^+ N^+(0, \sigma_\beta^+) + (1 - p_\beta^- - p_\beta^+)\delta_0,$$

with $p_\beta^\pm, \sigma_\beta^\pm$ positive constants.

(III) *If $(\beta, h) \in R_3$, then*

$$n^{3/4}(\hat{\beta}_n - \beta) \xrightarrow{\mathscr{L}} Z_\beta, \qquad n^{3/4}(\hat{h}_n - h) \xrightarrow{\mathscr{L}} Z_h,$$

where for $\gamma \in \{\beta, h\}$ *the random variable* Z_γ *has the distribution*

$$\mathbb{P}[Z_\gamma \leqslant t] = \mathbb{P}[Y_\gamma(0) \leqslant \mathbb{E}Y_\gamma(t)],$$

where

$$Y_\gamma(t) \propto \exp\big(c_* y^4 + t f'_\gamma(a_*) y\big),$$

with $c_* = A^{(4)}(a_*)/24$, *and* $f_\beta(a) = (2a-1)^p$ *and* $f_h(a) = (2a-1)$.

Note that in (IIb), all the points $(\beta, 0)$ with $\beta < \tilde{\beta}_p$ are not in R_2 (in fact, these points are in R_1). The above result covers Theorems 2.2–2.7 of Mukherjee, Son and Bhattacharya (2021+), except for the estimator $\hat{\beta}_n$ when $h = 0$ and $\beta \leqslant \tilde{\beta}_p$, which is corresponding to the results (2.19), (2.22) and (2.26) in this paper.

Remark 5.3. A natural extension of the homogeneous p-spin Curie-Weiss model is the mixed spin model with Hamiltonian given as

$$H_n(\omega) = n f_{\boldsymbol{p},\boldsymbol{\beta}}(\bar{\omega}_+),$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k$ and $\boldsymbol{p} = (p_1, \ldots, p_k) \in \mathbb{N}^k$ and

$$f_{\boldsymbol{p},\boldsymbol{\beta}}(a) = \sum_{i=1}^{k} \beta_i (2a-1)^{p_i}.$$

This Hamiltonian satisfies the conditions (A1)–(A4). Hence, we can apply our theorems to this model. The remaining task is to analyze the maximizers of $A(a)$ and check the non-degeneracy condition of $f_{\beta_i}(a) = (2a-1)^{p_i}$ at these points. This problem is non-trivial, hence left for future research.

5.2 Annealed Ising model on random regular graphs

Let $G_n = (V_n, E_n)$ be the random regular graph of degree $d \geqslant 3$ with n vertices $V_n = \{v_1, \ldots, v_n\}$. The Gibbs measure of annealed Ising model is defined as follows. For $\omega \in \{1, -1\}^n$,

$$\mu_n(\omega) \propto \mathbb{E}\{\exp(H_n(\omega))\}, \qquad H_n(\omega) = \beta \sum_{(v_i, v_j) \in E_n} \omega_i \omega_j + h \sum_{i=1}^{n} \omega_i,$$

where expectation is taken over the space of random regular graphs with respect to a uniform distribution. Can (2019, Eq. (3.2) and Lemma 2.1) proved that if $\bar{\omega}_+ = k/n$ then

$$\mu_n(\omega) \propto \exp(2hk) g(\beta, dk, dn),$$

where $\{g(\beta, m, l)\}_{m \leqslant l}$ satisfies that

$$\big|l^{-1} \log g(\beta, m, l) - g_\beta(m/l)\big| = \mathrm{O}(1/l), \tag{5.2}$$
$$\Big|\big(l^{-1} \log g(\beta, m, l) - g_\beta(m/l)\big) - \big(l^{-1} \log g(\beta, k, l) - g_\beta(k/l)\big)\Big| = \mathrm{O}\big(|k-m|/l^2\big),$$

with

$$g_\beta(a) = \int_0^{a\wedge(1-a)} \frac{e^{-2\beta}(1-2s) + \sqrt{1+(e^{-4\beta}-1)(1-2s)^2}}{2(1-s)} ds.$$

Therefore, with $X_n = n\bar{\omega}_+$, we have

$$\mu_n(X_n = k) \propto \exp(nA_n(k/n))$$

with

$$A_n\big(k/n\big) = 2hk/n + \frac{1}{n}\log g(\beta, dk, dn) + \frac{1}{n}\log\binom{n}{k}.$$

By (5.2) the function A_n is well approximated by $A : [0,1] \to \mathbb{R}$ given as

$$A(a) = 2ha + dg_\beta(a) + I(a).$$

In particular, we can find positive constants ε_*, δ_* and C_* such that the conditions (A1)–(A3) hold. Can (2019, Claim 1*) and Can (2017, Lemma 2.2) showed that

- (i) if $(\beta, h) \in \mathcal{U} = \{(\beta, h) : \beta > 0, h \neq 0, \text{or } 0 < \beta < \beta_c, h = 0\}$ then A has a unique 2-regular maximizer $a_* \in (0,1)$;
- (ii) if $\beta > \beta_c$ and $h = 0$ then A has two 2-regular maximizers $0 < a_- < a_+ = 1 - a_- < 1$;
- (iii) if $\beta = \beta_c$ and $h = 0$ then A has the unique 4-regular maximizer $a_* = 1/2$.

Here β_c is the critical value of the model $\beta_c = \operatorname{atanh}(1/(d-1))$. We now verify (A4) for the case (ii). Since $h = 0$, the model is symmetric and thus $\mu_n(\omega) = \mu_n(-\omega)$ and

$$\mu_n(X_n = k) = \mu_n(X_n = n-k). \tag{5.3}$$

Let $k_- = [na_-]$ and $k_+ = [na_+]$; we need to show

$$|A_n(k_-/n) - A_n(k_+/n)| = \mathrm{O}(n^{-3/2}). \tag{5.4}$$

Indeed, using (5.3) and (A3)

$$\begin{aligned}
&\big|A_n(k_-/n) - A_n(k_+/n)\big| \\
&\quad= \big|A_n((n-k_-)/n) - A_n(k_+/n)\big| \\
&\quad= \big|A((n-k_-)/n) - A(k_+/n)\big| + \mathrm{O}(|n-k_- - k_+|/n^2) \\
&\quad= \mathrm{O}\big(((n-k_-)/n - a_+)^2\big) + \mathrm{O}\big((k_+/n - a_+)^2\big) + \mathrm{O}\big(|n-k_- - k_+|/n^2\big) \\
&\quad= \mathrm{O}(n^{-2}).
\end{aligned}$$

Here, for the third line, we used Taylor expansion at a_+ and $A'(a_+) = 0$, and for the last one, we used $k_\pm = [na_\pm]$ and $a_- + a_+ = 1$. Therefore, (5.4) holds when $h = 0$ and $\beta > \beta_c$.

In conclusion, all the conditions (A1)–(A4) hold, and thus using Theorems 2.1 and $M_n = 2X_n - n$, we have the following.

Theorem 5.4. *Consider the annealed Ising model on a random regular graph.*

(I) If $(\beta, h) \in \mathcal{U}$ then

$$d_{\mathrm{W}}\Big(\mathscr{L}(W_n), N\Big(0, \tfrac{4}{|A''(a_*)|}\Big)\Big) = \mathrm{O}(n^{-1/2}),$$

where

$$W_n = \frac{M_n - n(2a_* - 1)}{\sqrt{n}}.$$

(II) If $\beta > \beta_c$ and $h = 0$ then

$$M_n/n \xrightarrow{\mathscr{L}} \frac{1}{2}\delta_{2a_- - 1} + \frac{1}{2}\delta_{2a_+ - 1}.$$

Moreover,

$$d_{\mathrm{W}}\Big(\mathscr{L}\big(W_n^{\pm} \,\big|\, |\tfrac{M_n}{n} - (2a_{\pm} - 1)| \leqslant \delta_*\big), N\Big(0, \tfrac{4}{|A''(a_{\pm})|}\Big)\Big) = \mathrm{O}(n^{-1/2}),$$

where

$$W_n^{\pm} = \frac{M_n - n(2a_{\pm} - 1)}{\sqrt{n}}.$$

(III) If $\beta = \beta_c$ and $h = 0$ then

$$d_{\mathrm{W}}\big(W_n, Y\big) = \mathrm{O}(n^{-1/4}), \qquad W_n = \frac{M_n}{n^{3/4}},$$

where $Y \propto \exp(c_* y^4/16)$ with $c_* = A^{(4)}(1/2)/24$.

Parts (I) and (II) are the main results of Can (2019, Theorem 1.3) and Part (III) is the main result of Can (2017, Theorem 1.3) with a convergence rate. The model is not linear in β but linear in h, and hence we can also prove the following.

Theorem 5.5. *Consider the maximum likelihood estimator $\hat{h}_n$ of the annealed Ising model on random regular graphs.*

(I) *If $(\beta, h) \in \mathcal{U}$ then*

$$\sqrt{n}(\hat{h}_n - h) \xrightarrow{\mathscr{L}} N(0, \sigma_h),$$

with σ_h a positive constant.

(II) *If $\beta > \beta_c$ and $h = 0$ then*

$$\sqrt{n}(\hat{h}_n - h) \xrightarrow{\mathscr{L}} U_h,$$

where

$$U_h = p_h^- N^-(0, \sigma_h^-) + p_h^+ N^+(0, \sigma_h^+) + (1 - p_h^- - p_h^+)\delta_0,$$

with $p_h^{\pm}, \sigma_h^{\pm}$ positive constants.

(III) *If $\beta = \beta_c$, $h = 0$ then*

$$n^{3/4}(\hat{h}_n - h) \xrightarrow{\mathscr{L}} Z_h,$$

where Z_h has the distribution as

$$\mathbb{P}[Z_h \leqslant t] = \mathbb{P}[Y_h(0) \leqslant \mathbb{E}[Y_h(t)]]$$

with $Y_h(t) \propto \exp(c_* y^4 + 2ty)$ and $c_* = A^{(4)}(1/2)/24$.

ACKNOWLEDGEMENTS

This project was supported by the Singapore Ministry of Education Academic Research Fund Tier 2 grant MOE2018-T2-2-076, and by Vietnam Academy of Science and Technology grant number CTTH00.02/22-23.

STATEMENTS

The authors of this paper declare the following:

- **Data availability**: All the data related to this study are available within the article.
- **Contribution of authors**: The authors contribute equally to this work.
- **Conflict of interest**: The authors do not have any conflict of interest.

REFERENCES

L. H. Y. Chen and A. Röllin (2010). Stein couplings for normal approximation. arXiv preprint arXiv:1003.6039.

V. H. Can (2019). Annealed limit theorems for Ising model on random regular graphs. *Ann. Appl. Probab.* **29**, 1398–1445.

V. H. Can (2017). Critical behavior of annealed Ising model on random regular graphs. *J. Stats. Phys.* **169**, 480–503.

S. Chatterjee (2007). Estimation in spin glasses: A first step. *Ann. Statist.* **35**, 1931–1946.

S. Chaterjee, Q. M. Shao (2011). Non-normal approximation by Stein's method of exchangeable pairs with application to the Curie-Weiss model. *Ann. App. Probab.* **21**, 464–483.

F. Comets (1992). On consistency of a class of estimators for exponential families of Markov random fields on the lattice. *Ann. Statist.* **20**, 455–468.

F. Comets and B. Gidas (1991). Asymptotics of maximum likelihood estimators for the Curie-Weiss model. *Ann. Statist.* **19**, 557–578.

P. Contucci and C. Giardina (2013). *Perspectives on spin glasses.* Cambridge University Press, Cambridge.

R. S. Ellis (1985). *Entropy, large deviations, and statistical mechanics.* vol. 271 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York.

R. S. Ellis, K. Wang (1992). Limit theorems for maximum likelihood estimators in the Curie- Weiss-Potts model. *Stoch. Process. Appl.* **40**, 251–288.

S. Geman and C. Graffigne (1986). Markov random field image models and their applications to computer vision. *Proceedings of the International Congress of Mathematicians*, 1496–1517.

P. Ghosal and S. Mukherjee (2020). Joint estimation of parameters in Ising model. *Ann. Statist.* **48**, 785–810.

P. J. Green and S. Richardson (2002). Hidden markov models and disease mapping. *J. Amer. Statist. Assoc.* **97**, 1055–1070.

R. van der Hofstad (2021+). *Stochastic Processes on Random Graphs.* In preparation, see http://www.win.tue.nl/rhofstad

J. R. Heringa, H. W. J. Blote and A. Hoogland (1989) Phase transitions in self-dual Ising models with multispin interactions and a field. *Phys. Rev. Lett.* **63**, 1546–1549.

S. Mukherjee, J. Son, B. B. Bhattacharya (2021). Fluctuations of the magnetization in the p-spin Curie–Weiss model. *Comm. Math. Phys.* **378**, 681–728.

S. Mukherjee, J. Son, B. B. Bhattacharya (2021+). Phase transitions of the maximum likelihood estimates in the p-spin Curie-Weiss model. arXiv preprint arXiv:2005.03631.

M. Niss (2005). History of the Lenz-Ising model 1920–1950: from ferromagnetic to cooperative phenomena. *Arch. Hist. Exact Sci.* **59**, 267–318.

M. Niss (2009). History of the Lenz-Ising model 1950–1965: from irrelevance to relevance. *Arch. Hist. Exact Sci.* **63**, 243–287.

D. K. Pickard (1987). Inference for discrete Markov Fields: the simplest nontrivial case. *J. Amer. Statist. Assoc.* **82**, 90–96.

N. Ross (2011). Fundamentals of Stein's method. *Probab. Surv.* **8**, 210–293.

M. Suzuki (1972). Solution and critical behavior of some "Three-Dimensional" Ising Models with a four-spin interaction. *Phys. Rev. Lett.* **28**, 507–510.

Y. Yamashiro, M. Ohkuwa, H. Nishimori and D. A. Lidar (2019). Dynamics of reverse annealing for the fully-connected p-spin model. *Phys. Rev. A.* **100**, 052321.