# REPRESENTATION OF POSITIVE POLYNOMIALS ON A GENERALIZED STRIP AND ITS APPLICATION TO POLYNOMIAL OPTIMIZATION

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Abstract. We study the representation of nonnegative two variable polynomials on a certain class of unbounded closed basic semi-algebraic sets (which are called generalized strips). This class includes the strip $[a, b] \times \mathbb{R}$ which was studied by Marshall in [10]. A denominator-free Nichtnegativstellensatz holds true on a generalized strip when the width of the generalized strip is constant and fails otherwise. As a consequence, we confirm that the standard hierarchy of SDP relaxations defined for the compact case indeed can be adapted to the generalized strip with constant width. For polynomial optimization problems on the generalized strip with non-constant width, we follow Ha-Pham's work: Solving polynomial optimization problems via the truncated tangency variety and sums of squares.

## 1. Introduction

Starting with 17 Hilbert's Problem, many problems have arisen in Real Algebraic Geometry, and many interesting results are known. Given a basic closed semi-algebraic set $K$ in $\mathbb{R}^n$ defined by finitely many polynomial inequalities $\{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$, where each $g_i$ is a real polynomial, Positivstellensätze are results characterizing all polynomials, which are positive on $K$, in terms of sums of squares and the polynomials $g_i$ used to describe $K$. Theorems about the existence of such representations have various applications, notably in problems of optimizing polynomial functions on semi-algebraic sets. For a nice survey and related topics, we refer the reader to [8, 18, 7, 6, 14] with the references therein.

In case $K$ is compact, Schmüdgen [20] has proved that any polynomial, which is positive on $K$, is in the preordering $T = T(g_1, \ldots, g_m)$ generated by the $g_i$'s, i.e., $T$ is the set of finite sums of elements of the form $\sigma_e g_1^{e_1} \cdots g_m^{e_m}$, where $e_i \in \{0, 1\}$ and each $\sigma_e$ is a sum of squares of polynomials. Schmüdgen's Positivstellensätz holds for polynomials, which are positive on $K$ and satisfy certain extra conditions, see [5, 10, 11, 12, 13, 17], etc. Scheiderer has shown that Schmüdgen's Positivstellensätz does not hold if $K$ is not compact and $\dim K \geq 3$, or $\dim K = 2$ and $K$ contains a 2-dimensional cone, see [15].

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A stronger problem which has been attracted much attention: Under which conditions is $T$ saturated? Let us remind that a preordering $T$ is saturated if every polynomial which is nonnegative on $K$ belongs to $T$. If the dimension of $K$ is greater or equal to 3, then Scheiderer has shown that $T$ is not saturated regardless of compactness of $K$ [18, Proposition 3.1.14]. Therefore, we have to look for saturated preorderings in the case where the dimension of $K$ is not greater than 2. Some classes of compact (virtually compact) surfaces (curves, respectively) which have saturated preorderings were given in [16, 17]. A remarkable saturated preordering $T$ for the non-compact case is $T(x(1-x))$ (Marshall's Nichtnegativstellensatz for the strip $[0,1]\times\mathbb{R}$ (see [10])). The main theorem in [10] (or also in [12]) which stated that a real polynomial which is nonnegative on the strip $\mathbb{R}\times[0,1]$ belongs to the preordering $T(y(1-y))$. Recently, Scheiderer and Wenzel [19] have extended [10, Theorem 1.1] on the cylinder $\mathbb{R}\times C$, where $C$ is a nonsingular affine curve over $\mathbb{R}$ with $C(\mathbb{R})$ compact. Note that Schmüdgen's Positivstellensätz fails if $K$ contains a 2-dimensional cone, hence Marshall's Nichtnegativstellensatz is probably an extreme result.

Any semi-algebraic subset of $\mathbb{R}^2$ can be decomposed into a finite union of tentacles and a bounded semi-algebraic set (see [1, Proposition 1.2]). Up to some linear change of coordinates, a tentacle in $\mathbb{R}^2$ is assumed to be of the form:

$$\{(x,y)\in\mathbb{R}^2 \mid \beta_1(x)\le y\le\beta_2(x), x\ge R\},$$

where $R>0$ and $\beta_1,\beta_2$ are convergent Puiseux series at infinity such that the sign of $\beta_1-\beta_2$ is constant on $[R,\infty]$ (see [3, 4]). A tentacle is a semi-algebraic set but need not be *closed basic* and so it is not clear if it has a finitely generated preordering. Therefore, we look for a class of tentacles which are possibly changeable to a closed basic semi-algebraic set. Precisely, we consider a tentacle of the form

$$M=\{(x,y)\in\mathbb{R}^2 \mid \beta_1(x)\le y\le\beta_2(x), x\ge R\},$$

where $\beta_1(x),\beta_2(x)$ have finite terms, that is,

$$\beta_i(x)=\sum_{j=m}^{n} b_{i,j}(\frac{1}{x})^{j/q},\quad m\le n\in\mathbb{Z},\ q\in\mathbb{N},\ b_{i,j}\in\mathbb{R},\ i=1,2.$$

Making the change of variable $z=\sqrt[q]{x}$, we can assume that $q=1$. Then

$$M=\{(x,y)\in\mathbb{R}^2 \mid x^{\max\{n,0\}}\beta_1(x)\le x^{\max\{n,0\}}y\le x^{\max\{n,0\}}\beta_2(x),\ x\ge R\}.$$

Let $g_i(x)=x^{\max\{n,0\}}\beta_i,\ i=1,2$. Then $g_1(x),g_2(x)$ are real polynomials in $x$ and

$$M=\{(x,y)\in\mathbb{R}^2 \mid g_1(x)\le x^{\max\{n,0\}}y\le g_2(x), x\ge R\}.$$

If $M$ is unbounded, then there exists a positive number $N$ such that $g_2(x)-g_1(x)>0$ for every $x>N$. Thus, in this paper, we consider a class of closed basic semi-algebraic sets of the form:

$$S(g_1, g_2, \alpha) := \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \le x^\alpha y \le g_2(x)\},$$

where $g_1, g_2$ are real single variable polynomials. In the case $\alpha = 0$, $g_1 \equiv 0$ and $g_2 \equiv 1$ then $K(0, 1, 0) = \mathbb{R} \times [0, 1]$ is the strip mentioned in [10].

An introduction of a class of (unbounded) closed basic semi-algebraic sets $S(g_1, g_2, \alpha)$ (these sets are called *generalized strips*) is written in Section 2. The main results of this paper are presented in Section 3. The first part of Section 3 presents the representations of polynomials which are nonnegative on these semi-algebraic sets. We define $w(x) := g_2(x) - g_1(x)$ and call it the *width* of $S(g_1, g_2, \alpha)$. In particular, the strip $K(0, 1, 0) = \mathbb{R} \times [0, 1]$ has the width $w(x) = 1$. In this paper, we will point out that the width is a characterization of the saturated property of the preordering $T$ generated by $x^\alpha y - g_1(x), g_2(x) - x^\alpha y$. Precisely, if the width $w(x)$ is finite, then every polynomial $p(x, y)$ which is nonnegative on $S(g_1, g_2, \alpha)$ belongs to the preordering $T$ provided some technical conditions. In the special case, we obtain the Marshall's Nichtnegativstellensatz [10]. On the other hand, if the width $w(x)$ is infinite, then there exists a polynomial $p(x, y)$ which is positive on $S(g_1, g_2, \alpha)$ but does not belong to the preordering $T$.

Thanks to Schmüdgen's and Putinar's Positivstellensätz, the optimal value of a polynomial over a compact semi-algebraic set can be approximated as closely as desired by solving a hierarchy of semidefinite programs (SDP). However, the convergence of the Lasserre's hierarchy approximations may not be ensured on the case where the feasible sets are non-compact. In this work, as mentioned above, we obtain a denominator-free Positivstellensätz on a class of generalized strips with constant width, and as a consequence, the standard hierarchy of SDP relaxations defined for the compact case indeed can be adapted to this class of feasible sets. We cannot obtain a denominator-free Positivstellensätz on the generalized strips with non-constant width. Though, for polynomial optimization problems on the generalized strips with non-constant width, we follow Ha-Pham's work: Solving polynomial optimization problems via the truncated tangency variety and sums of squares [21, 23]. Note that throughout [21, 23], the constraint $S$ is always supposed to be regular, while in this work, this property is removed.

The paper is organized as follows. Some basic notations and the definition of generalized strips in $\mathbb{R}^2$ are presented in Section 2. The main results with proofs are written in Section 3.

## 2. Preliminaries

*Notation.* Throughout this paper, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{N}$ the set of positive integer numbers, $\mathbb{Z}_{\ge 0}$ the set of nonnegative integer numbers and $\mathbb{R}^n$ denotes the Euclidean space of dimension $n$. We let $\mathbb{R}[x]$ denote the ring of real polynomials in $n$ indeterminates and a polynomial we always mean a real polynomial. Without of confusing, in many cases, $x$ stands also for a single variable and so $\mathbb{R}[x]$ means the ring of single variable polynomials.

Given a finite set $\mathcal{G} = \{g_1, g_2, \ldots, g_m\} \subset \mathbb{R}[x, y]$, the basic closed semi-algebraic set in $\mathbb{R}^2$ generated by $\mathcal{G}$, denoted as $K_{\mathcal{G}}$, is $\{(x, y) \in \mathbb{R}^2 \colon g_1(x, y) \geq 0, \ldots, g_m(x, y) \geq 0\}$. The quadratic module $\mathcal{M}(\mathcal{G}) = \mathcal{M}(g_1, \ldots, g_m)$ generated by $\mathcal{G}$ in the ring $\mathbb{R}[x, y]$ is the set

$$\{r_0 + r_1 g_1 + \cdots + r_m g_m \mid r_i \in \sum \mathbb{R}[x, y]^2\},$$

where $\sum \mathbb{R}[x, y]^2$ is the smallest quadratic module in $\mathbb{R}[x, y]$ and is equal to the set of all finite sums of squares of polynomials. The preordering $T(\mathcal{G}) = T(g_1, \ldots, g_m)$ is the quadratic module generated by the set of finitely distinct product of $\{g_1, \ldots, g_m\}$. Hence, $\mathcal{M}(g_1, \ldots, g_m)$ is contained in $T(g_1, \ldots, g_m)$. Some works tried to characterize when $\mathcal{M}(g_1, \ldots, g_m)$ is equal to $T(g_1, \ldots, g_m)$. It is trivial, in the ring $\mathbb{R}[x, y]$, that when $m = 1$ the preordering $T(g_1)$ is the same as the quadratic module $\mathcal{M}(g_1)$.

We say that $\mathcal{M}(\mathcal{G})$ (respectively, $T(\mathcal{G})$) is saturated if for every $f \in \mathbb{R}[x, y]$, $f$ nonnegative on $K_{\mathcal{G}}$ implies $f \in \mathcal{M}(\mathcal{G})$ (respectively, in $T(\mathcal{G})$). Marshall's Theorem says that the quadratic module generated by $y - y^2$ in $\mathbb{R}[x, y]$ is saturated.

**Tentacle sets.** We quote here the definitions and properties of tentacle sets from [1, 2]. By a *Puiseux series at infinity* we will mean a series of the form

$$\beta = \sum_{m}^{\infty} b_j (\frac{1}{x})^{\frac{j}{q}},$$

where $q \in \mathbb{N},\ m \in \mathbb{Z},\ b_j \in \mathbb{R}$ for $j \geq m$.
If $b_j \in \mathbb{C}$ we call $\beta$ a *complex* Puiseux series at infinity. The numbers $b_j$ will be called the coefficients of $\beta$. If $\beta \neq 0$, we can assume that the first coefficient $b_m \neq 0$. If $\beta$ is nonzero, we put $\mathrm{ord}_\infty \beta := m/q$ and call it *the order at infinity* of $\beta$. We also denote $\mathrm{ord}_\infty 0 = +\infty$. The set of Puiseux series at infinity with natural addition and multiplication forms a field.

Suppose $\beta$ is a Puiseux series at infinity. If there exists a closed half-line $I \subset \mathbb{R}$ such that the series $\beta(x)$ is convergent for $x \in I$ we will say that $\beta$ is *a convergent Puiseux series at infinity.* If this is the case, we will consider $\beta :\ I \longrightarrow \mathbb{R}$ both as a Puiseux series and a real function.

Suppose $\Gamma \subset \mathbb{R}^2$ is an unbounded semi-algebraic curve. The convergent Puiseux series at infinity $\beta$ is called a *special Puiseux parametrization of the semi-algebraic curve at infinity* if there exists a closed half-line $I \subset \mathbb{R}$ such that

$$\Gamma = \{(x, \beta(x)) \in \mathbb{R}^2 \mid x \in I\}.$$

For convenience, we will sometime call such a series a *Puiseux parametrization.*

**Definition 2.1.** *An unbounded semi-algebraic set $M \subset \mathbb{R}^2$ is called a tentacle set if for any $r > 0$ the set $M \setminus B(0,r)$ is connected, where $B(0,r)$ is the ball centered at the origin with radius $r$.*

Any semi-algebraic subset in $\mathbb{R}^2$ has a following decomposition (see [1, Proposition 1.2]):

$$S = K \cup M_1 \cup \ldots \cup M_k,$$

where $K$ is a bounded semi-algebraic set and $M_i (i = 1, 2, ..., k)$ are pairwise disjoint tentacle sets which are closed in $S$, i.e. $\overline{M_i} \cap S = M_i$. The above decomposition is unique in the following sense (see [1, Remark 1.3]): Given two tentacle decompositions

$$S = K \cup M_1 \cup \ldots \cup M_k = \tilde{K} \cup \tilde{M}_1 \cup \ldots \cup \tilde{M}_l,$$

Then $k = l$ and there exists a compact set $C$ such that

$$M_i \setminus C = \tilde{M}_i \setminus C$$

for $i = 1, \ldots, k$ possibly after rearranging the indices of the tentacles.

Suppose that $S$ is a closed, unbounded semi-algebraic set which does not contain a quadrant. The fact that a tentacle of the set $S$, after some linear change of coordinates and possibly after leaving out a compact subset, is of the form

$$\{(x, y) \in \mathbb{R}^2 \mid \beta_1(x) \leq y \leq \beta_2(x), x \geq R\}$$

where $R > 0$ and $\beta_1, \beta_2$ are convergent Puiseux series at infinity such that the sign of $\beta_1 - \beta_2$ is constant on $[R, \infty]$ (see [3, 4]). A tentacle is a semi-algebraic set. However, it may not be a *closed basic* semi-algebraic set and so it is not easy to find a preordering $T$ which is finitely generated. To avoid such difficulty, we consider a class of tentacles which are possibly changeable to a closed basic semi-algebraic set. Precisely, we consider a tentacle of the form

$$M = \{(x, y) \in \mathbb{R}^2 \mid \beta_1(x) \leq y \leq \beta_2(x), x \geq R\},$$

where $\beta_1(x), \beta_2(x)$ have finite terms, that is,

$$\beta_i(x) = \sum_{j=m}^{n} b_{i,j} (\frac{1}{x})^{j/q}, \quad m \leq n \in \mathbb{Z},\ q \in \mathbb{N},\ b_{i,j} \in \mathbb{R},\ i = 1, 2.$$

Making the change of variable $z = \sqrt[q]{x}$, we can assume that $q = 1$. Then

$$M = \{(x, y) \in \mathbb{R}^2 \mid x^{\max\{n,0\}} \beta_1(x) \leq x^{\max\{n,0\}} y \leq x^{\max\{n,0\}} \beta_2(x),\ x \geq R\}.$$

Let $\alpha = \max\{n, 0\},\ g_i(x) = x^\alpha \beta_i(x),\ i = 1, 2$. Then $g_1(x), g_2(x)$ are real polynomials in $x$ and

$$M = \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \leq x^\alpha y \leq g_2(x), x \geq R\}. \tag{1}$$

If $M$ is unbounded, then there exists a positive number $N$ such that $g_2(x) - g_1(x) > 0$ for every $x > N$. This implies that the polynomial $g_2(x) - g_1(x)$ is either a positive constant or a polynomial of degree at least 1 with the positive highest coefficient.

**Definition 2.2.** *We will call a closed semi-algebraic set of the form*

$$S(g_1, g_2, \alpha) := \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \leq x^\alpha y \leq g_2(x)\}$$

*a generalized strip, where $g_1, g_2$ are real polynomials in $\mathbb{R}[x]$ and $0 \leq \alpha \in \mathbb{Z}$.*

In the case $\alpha = 0$, $g_1 \equiv 0$ and $g_2 \equiv 1$ then $S(0, 1, 0) = \mathbb{R} \times [0, 1]$ is the strip mentioned in [10].

A quadratic module of $S(g_1, g_2, \alpha)$ is the quadratic module generated by $x^\alpha y - g_1(x), g_2(x) - x^\alpha y$. If a tentacle $M$ is determined by (1), then $M$ is a closed basic semi-algebraic and it is called a *half generalized strip.* In this case $M$ has a quadratic module $\mathcal{M}(x^\alpha y - g_1(x), g_2(x) - x^\alpha y, x - R)$.

## 3. Main results

### 3.1. **Polynomials nonnegative on a generalized strip.**

3.1.1. *The saturation of the preordering of a a generalized strip.* The remarkable result by Marshall that the quadratic module $\mathcal{M}(y(1-y))$ of the strip $S(0, 1, 0) = \mathbb{R} \times [0, 1]$ is saturated, see [10, Theorem 1.1]. We have a generalization on $S(g_1, g_2, 0)$ as the following lemma.

**Lemma 3.1.** *Let $g_1(x)$, $g_2(x)$ be single variable polynomials and the set*

$$S(g_1, g_2) := S(g_1, g_2, 0) = \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \leq y \leq g_2(x)\}.$$

*The following statements hold.*

1. *If $deg(g_2 - g_1) > 0$ and the leading coefficient of $g_2 - g_1$ is positive then there exist a polynomial $f(x, y) \in \mathbb{R}[x, y]$ which is positive on $S(g_1, g_2)$ and does not belong to the preordering generated by $y - g_1(x), g_2(x) - y$.*
2. *If $g_2(x) - g_1(x) = c$, where $c$ is a positive constant then the quadratic module $\mathcal{M}([y - g_1(x)][g_2(x) - y])$ is saturated. That is, if $f(x, y)$ is a two variable polynomial which is nonnegative on $S(g_1, g_2)$ then there exist $r_0(x, y), r_1(x, y) \in \sum \mathbb{R}[x, y]^2$ such that*

$$f(x, y) = r_0(x, y) + r_1(x, y)(y - g_1(x))(g_2(x) - y).$$

*Proof.* (1) We have

$$S(g_1, g_2) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y - g_1(x) \leq g_2(x) - g_1(x)\}.$$

Make the change of variable $z = y - g_1(x)$, we have

$$(x, y) \in S(g_1, g_2) \text{ if and only if } (x, z) \in S(0, w) = \{(x, z) \in \mathbb{R}^2 \mid 0 \leq z \leq w(x)\},$$

where $w(x) = g_2(x) - g_1(x)$.

**Claim:** $S(0, w)$ contains an 2-dimensional cone.

By Claim and [9, Proposition 4.2.3], there exist $\tilde{f}(x, z)$ which is positive on $S(0, w)$ does not belong to $T(z, w(x) - z)$. Set $f(x, y) := \tilde{f}(x, y - g_1(x))$. Then $f(x, y)$ is positive on $S(g_1, g_2)$ but $f(x, y)$ does not belong to $T(y - g_1(x), g_2(x) - y)$.

We need to prove Claim above. Indeed, suppose that $a_w$ is the coefficient of $x^s$, where $s = \deg(w)$. Then $a_w > 0$ and $w(x) \geq ax^s \geq ax$ for all $x \geq \delta$ where $a = \frac{a_w}{2}$ and some sufficiently large $\delta$. Then $S(0, w)$ contains the cone $(\delta, 0) + C$ where $C$ is the convex cone generated by $(1, 0)$ and $(1, a)$. This completes the proof.

(2) If $g_2 - g_1 = c$, we have $S(g_1, g_2) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq c^{-1}(y - g_1(x)) \leq 1\}$. Observe that if $z = c^{-1}(y - g_1(x))$, then we have

$$(x, y) \in S(g_1, g_2) \Leftrightarrow (x, z) \in \mathbb{R} \times [0, 1].$$

For $f(x, y) \in \mathbb{R}[x, y]$, let us define $\tilde{f}(x, z) := f(x, cz + g_1(x)) \in \mathbb{R}[x, z]$. Then $\tilde{f}(x, c^{-1}(y - g_1(x))) = f(x, y)$. If $f(x, y) \geq 0$ on $S(g_1, g_2)$ then for every $(x, z) \in \mathbb{R} \times [0, 1]$, we have $\tilde{f}(x, z) = f(x, cz + g_1(x)) \geq 0$. By [10, Theorem 1.1], there exist $\tilde{r}_0(x, z), \tilde{r}_1(x, z) \in \sum \mathbb{R}[x, z]^2$ such that

$$\tilde{f}(x, z) = \tilde{r}_0(x, z) + \tilde{r}_1(x, z)z(1 - z).$$

Set $r_0(x, y) := \tilde{r}_0(x, c^{-1}(y - g_1(x)))$ and $r_1(x, y) := c^{-2}\tilde{r}_1(x, c^{-1}(y - g_1(x)))$.

Then $r_0, r_1 \in \sum \mathbb{R}[x, y]^2$ and

$$\begin{aligned} f(x, y) &= \tilde{f}(x, c^{-1}(y - g_1(x))) \\ &= \tilde{r}_0(x, c^{-1}(y - g_1(x))) + \tilde{r}_1(x, c^{-1}(y - g_1(x)))c^{-1}[y - g_1(x)][1 - c^{-1}(y - g_1(x))] \\ &= r_0(x, y) + r_1(x, y)[y - g_1(x)][g_2(x) - y]. \end{aligned}$$

□

*Note that* if the leading coefficient of $g_2(x) - g_1(x)$ in Lemma 3.1 is *negative,* then $S(g_1, g_2)$ is compact when the degree of $g_2 - g_1$ is even. In the case that degree of $g_2 - g_1$ is odd, replacing $x$ by $-x$, we can assume that the leading coefficient of $g_2 - g_1$ is positive.

In the polynomial ring $\mathbb{R}[x, y]$, since $y = y^2 + y(1 - y)$ and $1 - y = (1 - y)^2 + y(1 - y)$, the preordering $T(y, 1 - y)$ is the same as the preordering $T(y(1 - y))$ and so is equal to the quadratic module $\mathcal{M}(y(1 - y))$.

In the case that $\alpha > 0$, some further conditions are added and we obtain two following lemmas.

**Lemma 3.2.** *Let $g_1, g_2$ and $S(g_1, g_2, \alpha)$ be as in Definition 2.2. Assume that*

$$g_2(x) - g_1(x) = c > 0 \textit{ and } g_1(0) < 0 < g_2(0).$$

*Then the quadratic module $\mathcal{M}([x^\alpha y - g_1(x)][g_2(x) - x^\alpha y])$ is saturated.*

*Proof.* We write

$$S(g_1, g_2, \alpha) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x^\alpha y - g_1(x) \leq c\}.$$

Observe that if $z = c^{-1}(x^\alpha y - g_1(x))$, then $y = \dfrac{cz + g_1(x)}{x^\alpha}$ for all $x \neq 0$. So, we have

$$\forall x \neq 0 : (x, y) \in S(g_1, g_2, \alpha) \Leftrightarrow (x, z) \in \mathbb{R} \times [0, 1].$$

Take any two variables polynomial $f(x, y)$ which is nonnegative on $S(g_1, g_2, \alpha)$. Let us define $\tilde{f}(x, z) := f(x, \dfrac{cz + g_1(x)}{x^\alpha})$ $\forall x \neq 0$ then $\tilde{f}(x, c^{-1}(x^\alpha y - g_1(x))) = f(x, y)$. Hence, for every $(x, z) \in \mathbb{R}^* \times [0, 1]$, we have $\tilde{f}(x, z) = f(x, \dfrac{cz + g_1(x)}{x^\alpha}) \geq 0$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. There exists $k \in \mathbb{Z}_{\geq 0}$ such that $x^{2k}\tilde{f}(x, z)$ becomes a polynomial in $\mathbb{R}[x, z]$ and $x^{2k}\tilde{f}(x, z) \geq 0$ on $\mathbb{R}^* \times [0, 1]$. Since the density of $\mathbb{R}^* \times [0, 1]$ in $\mathbb{R} \times [0, 1]$ and the continuity of the polynomial $x^{2k}\tilde{f}(x, z)$, we have $x^{2k}\tilde{f}(x, z) \geq 0$ on $\mathbb{R} \times [0, 1]$.

By [10, Theorem 1.1], there exist $\tilde{r}_0(x, z), \tilde{r}_1(x, z) \in \sum \mathbb{R}[x, z]^2$ such that

$$x^{2k}\tilde{f}(x, z) = \tilde{r}_0(x, z) + \tilde{r}_1(x, z)z(1 - z).$$

Set $r_0(x, y) := \tilde{r}_0(x, c^{-1}(x^\alpha y - g_1(x)))$ and $r_1(x, y) := c^{-2}\tilde{r}_1(x, c^{-1}(x^\alpha y - g_1(x)))$. Then $r_0(x, y), r_1(x, y) \in \sum \mathbb{R}[x, y]^2$ and

$$\begin{aligned} x^{2k} f(x, y) &= x^{2k}\tilde{f}(x, c^{-1}(x^\alpha y - g_1(x))) \\ &= r_0(x, y) + r_1(x, y)[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y]. \end{aligned}$$

If $k \neq 0$ then let $x = 0$, we get $r_0(0, y) + r_1(0, y)[-g_1(0)]g_2(0) = 0$ $\forall y \in \mathbb{R}$. By the assumption $g_1(0) < 0 < g_2(0)$, we have $r_0(0, y) = 0, r_1(0, y) = 0$ $\forall y \in \mathbb{R}$. So $r_0(x, y) = x^2\bar{r}_0(x, y);\ r_1(x, y) = x^2\bar{r}_1(x, y)$ and therefore

$$x^{2k-2} f(x, y) = \bar{r}_0(x, y) + \bar{r}_1(x, y)[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y].$$

Repeat this procedure, we obtain the following presentation.

$$f(x, y) = s_0(x, y) + s_1(x, y)[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y],$$

where $s_0, s_1 \in \sum \mathbb{R}[x, y]^2$. $\square$

**Lemma 3.3.** *Let $g_1(x), g_2(x)$ be single variable polynomials and $\alpha$ is a positive integer number. Suppose that the degree of $g_2(x) - g_1(x)$ is at least one, the leading coefficient of $g_2(x) - g_1(x)$ is positive and $g_1(0) < g_2(0)$. Then there exist a polynomial $f(x, y) \in \mathbb{R}[x, y]$ which is positive on $S(g_1, g_2, \alpha)$ and does not belong to the preordering generated by $x^\alpha y - g_1(x), g_2(x) - x^\alpha y$.*

*Proof.* Make the change of variables $z = x^\alpha y$, we have

$$(x, y) \in S(g_1, g_2, \alpha) \Longrightarrow (x, z) \in S(g_1, g_2, 0) = \{(x, z) \in \mathbb{R}^2 \mid g_1(x) \leq z \leq g_2(x)\}$$

and

$$(x, z) \in S(g_1, g_2, 0) \setminus \{0 \times \mathbb{R}\} \Longrightarrow (x, y) \in S(g_1, g_2, \alpha).$$

By Lemma 3.1, there exist $\tilde{f}(x, z)$ which is positive on $S(g_1, g_2, 0)$ does not belong to $T(z - g_1(x), g_2(x) - z)$. Set $f(x, y) := \tilde{f}(x, x^\alpha y)$. We will show that $f(x, y)$ is positive on $S(g_1, g_2, \alpha)$ but $f(x, y)$ does not belong to $T(x^\alpha y - g_1(x), g_2(x) - x^\alpha y)$.

Since $\tilde{f}(x, z) > 0$ on $S(g_1, g_2, 0)$, we have $f(x, y) = \tilde{f}(x, x^\alpha y) > 0$ on $S(g_1, g_2, \alpha)$. We suppose that $f(x, y)$ in $T(x^\alpha y - g_1(x), g_2(x) - x^\alpha y)$, that is

$$\begin{aligned} f(x, y) = r_0(x, y) + & r_1(x, y)[x^\alpha y - g_1(x)] + r_2(x, y)[g_2(x) - x^\alpha y] \\ & + r_3(x, y)[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y], \end{aligned}$$

where $r_i(x, y) \in \sum \mathbb{R}[x, y]^2;\ i = 1, 2, 3$. Hence, for all $(x, z) \in \mathbb{R}^2, x \neq 0$, we have

$$\begin{aligned} \tilde{f}(x, z) = f(x, \frac{z}{x^\alpha}) = r_0(x, \frac{z}{x^\alpha}) + & r_1(x, \frac{z}{x^\alpha})[z - g_1(x)] + r_2(x, \frac{z}{x^\alpha})[g_2(x) - z] \\ & + r_3(x, \frac{z}{x^\alpha})[z - g_1(x)][g_2(x) - z]. \end{aligned} \tag{2}$$

So, there exists $k \in \mathbb{Z}_{\geq 0}$ such that $x^{2k} r_i(x, \frac{z}{x^\alpha}) = \tilde{r}_i(x, z)$, where $\tilde{r}_i(x, z) \in \sum \mathbb{R}[x, z]^2;\ i = 1, 2, 3$ and

$$\begin{aligned} x^{2k} \tilde{f}(x, z) = \tilde{r}_0(x, z) + & \tilde{r}_1(x, z)[z - g_1(x)] + \tilde{r}_2(x, z)[g_2(x) - z] \\ & + \tilde{r}_3(x, z)[z - g_1(x)][g_2(x) - z]. \end{aligned} \tag{3}$$

By the equality (2) is true for all $x \neq 0$, the equality (3) is also true for all $x \neq 0$. However, the equality (3) holds on $\mathbb{R}^2$ since the continuity of $x^{2k}\tilde{f}(x, z)$ and the density of $\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}$ in $\mathbb{R}^2$. Let $x = 0$, by (3), we have

$$0 = \tilde{r}_0(0, z) + \tilde{r}_1(0, z)[z - g_1(0)] + \tilde{r}_2(0, z)[g_2(0) - z] + \tilde{r}_3(0, z)[z - g_1(0)][g_2(0) - z]. \tag{4}$$

Since the equality (4) is true for all $z \in [g_1(0), g_2(0)]$, we have $\tilde{r}_i(0, z) = 0,\ i = 1, 2, 3$. So $\tilde{r}_i(x, z) = x^2 r'_i(x, z)$, where $r'_i(x, z) \in \sum \mathbb{R}[x, y]^2;\ i = 1, 2, 3$. Therefore,

$$\begin{aligned} x^{2k-2} \tilde{f}(x, z) = {r'}_0(x, z) + & {r'}_1(x, z)[z - g_1(x)] + {r'}_2(x, z)[g_2(x) - z] \\ & + {r'}_3(x, z)[z - g_1(x)][g_2(x) - z]. \end{aligned}$$

Repeat this procedure, we get

$$\begin{aligned} \tilde{f}(x, z) = s_0(x, z) + & s_1(x, z)[z - g_1(x)] + s_2(x, z)[g_2(x) - z] \\ & + s_3(x, z)[z - g_1(x)][g_2(x) - z], \end{aligned}$$

where $s_0, s_1, s_2, s_3$ are sums of squares. That is $\tilde{f}(x,z) \in T(z - g_1(x), g_2(x) - z)$, we get a contradiction.

$\square$

Let us denote by $w(x) = g_2(x) - g_1(x)$ and call it the *width* of $S(g_1, g_2, \alpha)$. In particular, the strip $S(0,1,0) = \mathbb{R} \times [0,1]$ has the width $w(x) = 1$. Since $w(x)$ is a polynomial, $\lim\limits_{x\to\infty} w(x)$ is either infinite or constant. The width of $S(x^2, x^3, 0) = \{(x,y) \in \mathbb{R}^2 \colon x^2 \le y \le x^3\}$ is $w(x) = x^3 - x^2$ and $\lim\limits_{x\to+\infty} w(x) = \infty$. This set can not be enclosed in any strip with arbitrary constant width. We see later that the preordering $T(y - x^2, x^3 - y)$ of $S(x^2, x^3, 0)$ is not saturated.

**Theorem 3.1.** *Let $g_1(x), g_2(x), \alpha$ and $S(g_1, g_2, \alpha)$ be as in Definition 2.2. The following statements hold.*

(1) *Suppose that either $\lim\limits_{x\to+\infty} w(x) = +\infty$ or $\lim\limits_{x\to-\infty} w(x) = +\infty$. Then there exist a polynomial $f(x,y) \in \mathbb{R}[x,y]$ which is positive on $S(g_1, g_2, \alpha)$ and $f$ does not belong to the preordering generated by $x^\alpha y - g_1(x), g_2(x) - x^\alpha y$ provided that $w(0) > 0$.*

*In the case $\alpha = 0$, the hypothesis '$w(0) > 0$' can be removed.*

(2) *If $\lim\limits_{x\to\infty} w(x) = c > 0$ and $g_1(0) < 0 < g_2(0)$ then the quadratic module $\mathcal{M}([x^\alpha y - g_1(x)][g_2(x) - x^\alpha y])$ is saturated.*

*In the case $\alpha = 0$, the hypothesis '$g_1(0) < 0 < g_2(0)$' can be removed.*

*Proof.* (1) If $\lim\limits_{x\to+\infty} w(x) = +\infty$ then $deg(w) > 0$ and the leading coefficient of $w$ is positive. So if $w(0) > 0$ then by Lemma 3.3 we get the conclusion. Similarly, in the case $\alpha = 0$, we apply Lemma 3.1 (1) instead of Lemma 3.3.

If $\lim\limits_{x\to-\infty} w(x) = +\infty$, we put $t = -x$ and $\bar{w}(t) = w(-t)$ then $\lim\limits_{t\to+\infty} \bar{w}(t) = +\infty$ and the problem becomes to the above case.

(2) If $\lim\limits_{x\to+\infty} w(x) = c > 0$ then $g_2(x) - g_1(x) \equiv c > 0$. Now we apply Lemma 3.2 and we get the proof. In the case $\alpha = 0$, we use Lemma 3.1 (2) instead of Lemma 3.2.

$\square$

**Example 3.1.** Consider the set $S = S(x^2, x^3, 0) = \{(x,y) \in \mathbb{R}^2 \colon x^2 \le y \le x^3\}$. According to the theorem above, there exists the polynomial $f \ge 0$ on $S$ but $f$ does not belong to the preordering $T(x^3 - y, y - x^2)$.

We know that if a semi-algebraic set $S$ contains a open cone then the Positivstellensätz fails. However, in this case, $S$ does not contain any open cone. Indeed, assume that $S$ contains a convex cone $C$, then $S$ contains a half line $d$. This is impossible.

On the other hand, put $u = y - x^2$, then the set $\tilde{S} = \{(x,u) \in \mathbb{R}^2 \colon 0 \le u \le x^3 - x^2\}$ contains the open cone $\tilde{C} = \{(x,u) \in R^2 \colon 0 \le u \le x - 1; x \ge 1\}$.

**Example 3.2.** Let the set

$$S(x^2, x^2+x+1, 2) = \{(x,y) \in \mathbb{R}^2 | \ x^2 \leq x^2 y \leq x^2+x+1\}.$$

We have $w(x) = x+1$, $\lim\limits_{x \to +\infty} w(x) = +\infty$ and $w(0) = 1 > 0$. Hence, by Theorem 3.1 (1), there exist a polynomial $f(x,y) \in \mathbb{R}[x,y]$ which is positive on $S(x^2, x^2+x+1, 2)$ and $f$ does not belong to the preordering generated by $x^2y - x^2, x^2+x+1-x^2y$.

**Example 3.3.** Consider the set

$$S(x-1, x+1, 2) = \{(x,y) \in \mathbb{R}^2 | \ x-1 \leq x^2 y \leq x+1\}.$$

In this case, $w(x) = 2 > 0$, $g_1(0) = -1 < 0 < g_2(0) = 1$. So, according to Theorem 3.1 (2), the quadratic module $\mathcal{M}(x^2y - x + 1, x + 1 - x^2y)$ is saturated.

### 3.2. **Polynomial Optimization on the generalized strips.**

3.2.1. *Standard hierarchy of semidefinite relaxations for constrained optimization problem.* For reader's convenience, we recall the Lasserre's hierarchy of semidefinite relaxations in polynomial optimization as follows. Fix a basic closed semialgebraic set $K$ in $\mathbb{R}^n$ and $f \in \mathbb{R}[x]$. We wish to compute lower bounds for

$$f_* = \inf\{f(x) \mid x \in K\}.$$

Fix a finite set $\mathcal{G} = \{g_1, g_2, \ldots, g_m\} \subset \mathbb{R}[x]$, recall that the basic closed semi-algebraic set in $\mathbb{R}^n$ generated by $\mathcal{G}$, denoted as $K = K_{\mathcal{G}}$, is $\{x \in \mathbb{R}^n \colon g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$. The quadratic module $\mathcal{M} = \mathcal{M}(\mathcal{G}) := \mathcal{M}(g_1, \ldots, g_m)$ generated by $\mathcal{G}$ in the ring $\mathbb{R}[x]$ is the set

$$\left\{ \sum_{i=0}^{m} s_i g_i \mid s_i \in \sum \mathbb{R}[x]^2, g_0 = 1 \right\},$$

where $\sum \mathbb{R}[x]^2$ is the smallest quadratic module in $\mathbb{R}[x]$ and is equal to the set of all finite sums of squares of polynomials. Denote by $\chi$ the set of all linear maps $L : \mathbb{R}[x] \to \mathbb{R}$ sastisfying $L(1) = 1$ and $L \geq 0$ on $\mathcal{M}$. Set

$$\begin{cases} f_{momt} & := \inf \{L(f) \mid L \in \chi\}, \\ f_{sos} & := \sup \{r \in \mathbb{R} \mid f - r \in \mathcal{M}\}. \end{cases}$$

For convenience, we denote supremum of the empty set is $-\infty$. Fix an integer $d \geq \deg(f)$. Define $\mathcal{M}(\mathcal{G})_{[d]} \subset \mathcal{M}[\mathcal{G}]$ the set of elements of the form $\sum_{i=0}^m s_i g_i$ where $s_i \in \sum \mathbb{R}[x]^2$ and $\deg(s_i) \leq 2d, i = 0, 1, 2, ..., m$. Set

$$f_{sos,d} = \sup \left\{ r \in \mathbb{R} \mid f - r \in \mathcal{M}(\mathcal{G})_{[d]} \right\}.$$

It is clearly that

$$f_{sos,d} \leq f_{sos,d+1} \leq f_{sos} \leq f_{momt} \leq f_*.$$

**Proposition 3.1.** *Suppose that every polynomial which is positive on $K_{\mathcal{G}}$ must belong to $\mathcal{M}(\mathcal{G})$ (Here we do not assumed that $\mathcal{M}(\mathcal{G})$ is Archimedean). Then for $f \in \mathbb{R}[x]$, $f_* = f_{sos} = \lim_{d\to+\infty} f_{sos,d}$.*

Of course, since $f_{sos} \le f_{momt} \le f_*$, this also implies that $f_{momt} = f_*$.

*Proof.* If $f$ is not bounded below on $K_{\mathcal{G}}$, then $\left\{r \in \mathbb{R} \mid f - r \in \mathcal{M}(\mathcal{G})_{[d]}\right\} = \emptyset$. Hence $f_{sos,d} = -\infty$ for all $d$.

Suppose that $f_*$ is finite. For any $r \in \mathbb{R}$ and $r < f_*$, then $f - r > 0$ on $K_{\mathcal{G}}$ so, by the assumption, we have $f - r \in \mathcal{M}(\mathcal{G})$. Thus $f_{sos} \ge r$. That implies $f_{sos} \ge f_*$. □

It is known that $f_{sos,d}$ can be computed by semidefinite programming problems.

3.2.2. *PO on a generalized strip or half strip with constant width.* Given $g_1(x), g_2(x)$ single variable polynomials, $\alpha \in \mathbb{Z}_{\ge 0}$ and $f(x,y)$ two variables polynomial, consider the optimization problem:

$$f_* := \inf \{f(x,y) \mid (x,y) \in S(g_1, g_2, \alpha)\},$$

where $S(g_1, g_2, \alpha) = \{(x,y) \in \mathbb{R}^2 \mid g_1(x) \le x^\alpha y \le g_2(x)\}$ and $\lim_{x\to\infty}(g_2(x) - g_1(x))$ is finite.

By Lemma 3.2 and Proposition 3.1, we can obtain the following corollary.

**Corollary 3.1.** *Let $g_1, g_2$ be single variable polynomials, $\alpha \in \mathbb{Z}_{\ge 0}$ and the set*

$$S(g_1, g_2, \alpha) := \{(x,y) \in \mathbb{R}^2 \mid g_1(x) \le x^\alpha y \le g_2(x)\}.$$

*Assume that $g_2(x) - g_1(x) = c > 0$ and $g_1(0) < 0 < g_2(0)$. If $f \in \mathbb{R}[x,y]$ is bounded below on $S(g_1, g_2, \alpha)$, then the sequence $\{f_{sos,d}\}_{d\in\mathbb{N}}$ converges monotonically increasing to the $f_*$ where*

$$\begin{cases} f_* & := \inf \{f(x,y) \mid (x,y) \in S(g_1, g_2, \alpha)\}, \\ f_{sos,d} & := \sup \left\{ r \in \mathbb{R} \mid f - r \in \mathcal{M}\left((x^\alpha y - g_1(x))(g_2(x) - x^\alpha y)\right)_{[d]} \right\}. \end{cases}$$

*In the case $\alpha = 0$, the hypothesis '$g_1(0) < 0 < g_2(0)$' can be removed.*

3.2.3. *PO on a generalized strip with non-constant width.* In this subsection, we consider the polynomial optimization problem:

$$f_* := \inf \{f(x,y) \mid (x,y) \in S(g_1, g_2, \alpha)\},$$

where $S(g_1, g_2, \alpha) = \{(x,y) \in \mathbb{R}^2 \mid g_1(x) \le x^\alpha y \le g_2(x)\}$, $g_1(x), g_2(x) \in \mathbb{R}[x]$ be single polynomials and $\lim_{x\to\infty}(g_2(x) - g_1(x)) = +\infty$. Then $S(g_1, g_2, \alpha)$ is a nonempty and unbounded set. In this case, By Theorem 3.1, Putinar's Positivstellensätz does not hold anymore. To solve the optimization problem in this case, we will follow Ha-Pham's ideas in [21, 23]. A little improvement here is that we do not require regularity of the constraint. We recall some notations and rewrite the useful results in [21, 23] as follows.

**Definition 3.1.** *For any polynomial $f \in \mathbb{R}[x]$ and subset $K \subset \mathbb{R}^n$, the set $R_\infty(f, K)$ of asymptotic values of $f$ on $K$ consists of all $y \in \mathbb{R}$ for which there exists a sequence $\{x^k\}_{k\in\mathbb{N}}$ of points $x^k \in K$ such that $\lim_{k\to\infty} \|x^k\| = +\infty$ and $\lim_{k\to\infty} f(x^k) = y$.*

**Theorem 3.2.** [22, Theorem 9] *Let $f, g_1, g_2, ..., g_m \in \mathbb{R}[x]$ and the set*

$$K := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, g_2(x) \geq 0, ..., g_m(x) \geq 0\}.$$

*Suppose that*

(i) *$f$ is bounded on $K$;*
(ii) *$R_\infty(f, K)$ is a finite subset of $\mathbb{R}_{>0} := \{y \in \mathbb{R} \mid y > 0\}$; and*
(iii) *$f > 0$ on $K$.*

*Then $f \in T(g_1, g_2, ..., g_m)$.*

We will aplly Theorem 3.2 to obtain a weakly version of denominator-free Positivstellensätz on a generalized strip. To use this theorem, we need to ensure 2 assumptions (i) and (ii). In [24], we studied the boundedness of a polynomial on a generalized strip (so it is bounded on its tangency variety on that strip). For a general polynomial, we replace its tangency variety by its truncated tangency variety (see the difinition by eq. (7)) then the boundedness of the polynomial is still hold. Next, we will show that the asymptotic values set of a given polynomial on its tangency variety is finite.

Let $f(x, y) \in \mathbb{R}[x, y]$ be a non-constant polynomial and $\alpha$ be a non-negative integer number. Then the tangency variety of $f$ on $S(g_1, g_2, \alpha)$ (see the definition in [23]) can be written as

$$\Gamma(f, S(g_1, g_2, \alpha)) = \left\{ (x, y) \in S(g_1, g_2, \alpha) |\ [x^\alpha y - g_1(x)][x^\alpha y - g_2(x)][y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}] = 0 \right\} \quad (5)$$

Hence, we can write $\Gamma(f, S(g_1, g_2, \alpha)) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$\Gamma_1 := \{(x, y) \in S(g_1, g_2, \alpha) |\ x^\alpha y - g_1(x) = 0\}$;

$\Gamma_2 := \{(x, y) \in S(g_1, g_2, \alpha) |\ x^\alpha y - g_2(x) = 0\}$;

$\Gamma_3 := \{(x, y) \in S(g_1, g_2, \alpha) |$ there exists a real number $\lambda$ such that $\nabla f(x, y) = \lambda(x, y)\}$.

**Theorem 3.3.** *The set $\Gamma(f, S(g_1, g_2, \alpha))$ is a nonempty, unbounded algebraic set, and*

$$\inf \{f(x, y) \mid (x, y) \in S(g_1, g_2, \alpha)\} = \inf \{f(x, y) \mid (x, y) \in \Gamma(f, S(g_1, g_2, \alpha))\}. \quad (6)$$

*Proof.* By the definition of $\Gamma(f, S(g_1, g_2, \alpha))$ and by the assumption $\lim\limits_{x\to+\infty} w(x) = +\infty$ or $\lim\limits_{x\to-\infty} w(x) = +\infty$, $\Gamma(f, S(g_1, g_2, \alpha))$ is a nonempty, unbounded and algebraic set.

Now, we prove the equality (6).

Case 1: Assume that $f_* = f(x_*, y_*)$ at some point $(x_*, y_*) \in S(g_1, g_2, \alpha)$. We will show that $(x_*, y_*) \in \Gamma(f, S(g_1, g_2, \alpha))$.

- If $(x_*, y_*)$ belongs to $\partial S(g_1, g_2, \alpha) = \Gamma_1 \cup \Gamma_2$. Then obviously $(x_*, y_*) \in \Gamma(f, S(g_1, g_2, \alpha))$.
- If $(x_*, y_*)$ belongs to $\text{int} S(g_1, g_2, \alpha) = \{(x, y) \in \mathbb{R}^2 \mid g_1(x) < x^\alpha y < g_2(x)\}$. Then $\nabla f(x_*, y_*) = 0$. That means $(x_*, y_*) \in \Gamma_3 \subset \Gamma(f, S(g_1, g_2, \alpha))$.

Case 2: Assume that $f$ does not attain its infimum on $S(g_1, g_2, \alpha)$. For all large enough $r > 0$, we consider the set $B_r := \{(x, y) \in S(g_1, g_2, \alpha) \mid x^2 + y^2 = r^2\}$, then $B_r$ is nonempty and compact. There exists $(x_r, y_r) \in B_r$ such that

$$f(x_r, y_r) = \inf \{f(x, y) \mid (x, y) \in B_r\}.$$

We show that $(x_r, y_r) \in \Gamma(f, S(g_1, g_2, \alpha))$ for all large enough $r$. We divide it into the following cases:

- If $(x_r, y_r)$ belongs to $\partial S(g_1, g_2, \alpha)$ of $S(g_1, g_2, \alpha)$ then $(x_r, y_r)$ obviously belongs to $\Gamma(f, S(g_1, g_2, \alpha))$.
- Otherwise, $(x_r, y_r) \in B'_r := \{(x, y) \in \mathbb{R}^2 \mid g_1(x) < x^\alpha y < g_2(x), x^2 + y^2 = r^2\}$. Then there exists a real number $\lambda$ such that $\nabla f(x_r, y_r) = \lambda(x_r, y_r)$. Therefore, $(x_r, y_r) \in \Gamma(f, S(g_1, g_2, \alpha))$.

It is clear that $f_* = \lim_{r\to\infty} f(x_r, y_r) \geq \inf\{f(x, y) \mid (x, y) \in \Gamma(f, S(g_1, g_2, \alpha))\}$. On the other hand, since $\Gamma(f, S(g_1, g_2, \alpha)) \subset S(g_1, g_2, \alpha)$, we obtain the reverse inequality. □

**Lemma 3.4.** *$R_\infty(f, \Gamma(f, S(g_1, g_2, \alpha)))$ is a finite set.*

*Proof.* We have $\Gamma(f, S(g_1, g_2, \alpha)) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 := \{(x, y) \in S(g_1, g_2, \alpha) |\ x^\alpha y - g_1(x) = 0\};$$

$$\Gamma_2 := \{(x, y) \in S(g_1, g_2, \alpha) |\ x^\alpha y - g_2(x) = 0\};$$

$$\Gamma_3 := \{(x, y) \in S(g_1, g_2, \alpha) |\ \text{there exists a real number } \lambda \text{ such that } \nabla f(x, y) = \lambda(x, y)\}.$$

Therefore,

$$R_\infty(f, \Gamma(f, S(g_1, g_2, \alpha))) = R_\infty(f, \Gamma_1) \cup R_\infty(f, \Gamma_2) \cup R_\infty(f, \Gamma_3).$$

We show that $R_\infty(f, \Gamma_1)$ is a finite set. Let $\{(x^k, y^k)\}_{k\in\mathbb{N}} \subset \Gamma_1$ such that $\lim_{k\to\infty} \|(x^k, y^k)\| = +\infty$.

If there are infinitely many $x^k \neq 0$, then

$$\lim_{k\to\infty} f(x^k, y^k) \in \left\{ \lim_{t\to+\infty} f(t, \frac{g_1(t)}{t^\alpha}), \lim_{t\to-\infty} f(t, \frac{g_1(t)}{t^\alpha}), \lim_{t\to 0} f(t, \frac{g_1(t)}{t^\alpha}) \right\}.$$

Otherwise, if $x^k = 0$ for all but finitely many $k$, then

$$\lim_{k\to\infty} f(x^k, y^k) \in \left\{ \lim_{t\to+\infty} f(0, t), \lim_{t\to-\infty} f(0, t) \right\}.$$

Hence, the set $R_\infty(f,\Gamma_1)$ is finite. Similarly, we can show that $R_\infty(f,\Gamma_2)$ is a finite set.

Since $\Gamma_3 \subset \Gamma(f) := \{(x,y) \in \mathbb{R}^2 |$ there exists a real number $\lambda$ such that $\nabla f(x,y) = \lambda(x,y)\}$, $R_\infty(f,\Gamma_3) \subset R_\infty(f,\Gamma(f))$. By [23, Corollary 2], $R_\infty(f,\Gamma(f))$ is a finite set. So $R_\infty(f,\Gamma_3)$ is also a finite set. □

In what follows, we shall fix a real number $M = f(x,y)$ for some $(x,y) \in S(g_1,g_2,\alpha)$. Then by truncated tangency variety of $f$ on $S(g_1,g_2,\alpha)$ we mean the set

$$\begin{aligned}\Gamma_M(f,S(g_1,g_2,\alpha)) :=&\{(x,y) \in \Gamma(f,S(g_1,g_2,\alpha)) \mid M - f(x,y) \geq 0\} \\ =&\{(x,y) \in \mathbb{R}^2 | x^\alpha y - g_1(x) \geq 0,\ g_2(x) - x^\alpha y \geq 0, \\ &[x^\alpha y - g_1(x)][x^\alpha y - g_2(x)][y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}] = 0,\ M - f(x,y) \geq 0\}.\end{aligned} \tag{7}$$

**Corollary 3.2.** *We have*

$$\inf\{f(x,y) \mid (x,y) \in S(g_1,g_2,\alpha)\} = \inf\{f(x,y) \mid (x,y) \in \Gamma_M(f,S(g_1,g_2,\alpha))\}. \tag{8}$$

*Proof.* By Theorem 3.3, we have

$$\begin{aligned}\inf\{f(x,y) \mid (x,y) \in S(g_1,g_2,\alpha)\} &= \inf\{f(x,y) \mid (x,y) \in \Gamma(f,S(g_1,g_2,\alpha))\} \\ &= \inf\{f(x,y) \mid (x,y) \in \Gamma_M(f,S(g_1,g_2,\alpha))\}.\end{aligned}$$

□

**Lemma 3.5.** *Let $f$ be a real polynomial in two variables. If*

$$\inf\{f(x,y) \mid (x,y) \in \Gamma_M(f,S(g_1,g_2,\alpha))\} > 0$$

*then*

$$\begin{aligned}f(x,y) =& s_0(x,y) + s_1(x,y)\,[x^\alpha y - g_1(x)] + s_2(x,y)\,[g_2(x) - x^\alpha y] \\ &+ s_3(x,y)\,[M - f(x,y)] + t(x,y)[x^\alpha y - g_1(x)][x^\alpha y - g_2(x)][y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}],\end{aligned} \tag{9}$$

*where the $s_i(x,y)$ are sum of squares in $\mathbb{R}[x,y]$ and $t(x,y) \in \mathbb{R}[x,y]$.*

*Proof.* It is clear from the assumption that $f$ is bounded and strictly positive on $\Gamma_M(f,S(g_1,g_2,\alpha))$. Moreover, the following inclusion holds:

$$R_\infty(f,\Gamma_M(f,S(g_1,g_2,\alpha))) \subset R_\infty(f,\Gamma(f,S(g_1,g_2,\alpha))).$$

Thus, Lemma 3.4 implies that $R_\infty(f,\Gamma_M(f,S(g_1,g_2,\alpha)))$ is a finite set of $\mathbb{R}_{>0}$. Then the lemma follows now from Theorem 3.2. □

Now, we are ready to state the main results of this article.

**Theorem 3.4.** *Let $S(g_1,g_2,\alpha)$ be as above and $f$ be a real polynomial in two variables. Then the following conditions are equivalent:*

(i) $f \geq 0$ *on* $S(g_1, g_2, \alpha)$*;*

(ii) $f \geq 0$ *on* $\Gamma_M(f, S(g_1, g_2, \alpha))$*;*

(iii) *For every* $\epsilon > 0$, *there are sums of squares* $s_i$ *(*$0 \leq i \leq 3$*) and a polynomial* $t$ *in* $\mathbb{R}[x, y]$ *such that*

$$f(x,y) + \epsilon = s_0(x,y) + s_1(x,y)\,[x^\alpha y - g_1(x)] + s_2(x,y)\,[g_2(x) - x^\alpha y] \\ + s_3(x,y)\,[M - f(x,y)] + t(x,y)[x^\alpha y - g_1(x)][x^\alpha y - g_2(x)][y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}].$$

*Proof.* The implication $(i) \iff (ii)$ is straightforward by Corollary 3.2. The implication $(iii) \Longrightarrow (ii)$ is immediate. For the implication $(ii) \Longrightarrow (iii)$, we only have to apply Lemma 3.5 to $f + \epsilon$ instead of $f$. □

*Note that* in Theorem 3.4, we do not need the regularity of $S(g_1, g_2, \alpha)$ (compare with [21, Theorem 4.1]). For example,

$$S(x^2, x^3, 0) = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leq y \leq x^3\}$$

is not regular, so we cannot apply [21, Theorem 4.1]. However, we can use the above theorem to obtain a representation of polynomial positive on $S(x^2, x^3, 0)$.

**Definition 3.2.** *Let* $f \in \mathbb{R}[x, y]$ *and* $k \in \mathbb{N}$. *Define* $f_*^k \in \mathbb{R} \cup \{\pm\infty\}$ *as the supremum over all* $a \in \mathbb{R}$ *such that* $f - a$ *can be written as a sum*

$$f(x,y) - a = s_0(x,y) + s_1(x,y)\,[x^\alpha y - g_1(x)] + s_2(x,y)\,[g_2(x) - x^\alpha y] \\ + s_3(x,y)\,[M - f(x,y)] + t(x,y)[x^\alpha y - g_1(x)][x^\alpha y - g_2(x)][y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}],$$

*where* $s_0, s_1, s_2, s_3, t$ *are polynomials of degree at most* $2k$ *and* $s_0, s_1, s_2, s_3$ *are sums of squares in* $\mathbb{R}[x, y]$.

As is well known, the problem of computing the supremum $f_*^k$ can be reduced to an SDP. Moreover, by Theorem 3.4, we have

$$f_*^k \leq f_*^{k+1} \leq f_*, \ \forall k \in \mathbb{N}.$$

We have the following general result concerning the convergence of lower bounds.

**Theorem 3.5.** *Let* $S(g_1, g_2, \alpha)$ *be as above and* $f$ *a real polynomial in two variables. Then the sequence* $\{f_*^k\}_{k \in \mathbb{N}}$ *converges monotonically increasing to the infimum* $f_*$.

*Proof.* The same proof of [21, Theorem 5.2]. □

**Example 3.4.** Consider the set $S := S(x^2, x^3 + 2, 2) = \{(x, y) \in \mathbb{R}^2 \colon x^2 \leq x^2 y \leq x^3 + 2\}$. We have $w(x) = x^3 - x^2 + 2$, $\lim_{x \to +\infty} w(x) = +\infty$ and $w(0) = 2 > 0$. By Theorem 3.1, $T(x^2 y - x^2, x^3 + 2 - x^2 y)$ is not saturated. On the other hand, $S$ is not regular in the sense of [21, Definition 3.1].

(a) Let $f(x,y) = x+1 \in \mathbb{R}[x,y]$. It is clear that the polynomial $f$ is nonnegative on $S(x^2, x^3+2, 2)$. However, we show that $f$ does not belong to $T(x^2y-x^2, x^3+2-x^2y)$. Indeed, assume that $x+1 \in T(x^2y-x^2, x^3+2-x^2y)$, then

$$\begin{aligned} x+1 =& \sigma_0(x,y) + \sigma_1(x,y)(x^2y-x^2) + \sigma_2(x,y)(x^3-x^2y+2) \\ &+ \sigma_3(x,y)(x^2y-x^2)(x^3-x^2y+2), \end{aligned}$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in \sum \mathbb{R}[x,y]^2$. Evaluating at $y=1$, this yields

$$x+1 = \sigma_0(x,1) + \sigma_2(x,1)(x^3-x^2+2).$$

Setting $x=-1$, we have $0 = \sigma_0(-1,1) + \sigma_2(-1,1)$, hence $\sigma_0(-1,1) = \sigma_2(-1,1) = 0$ since $\sigma_0, \sigma_2 \in \sum \mathbb{R}[x,y]^2$. It follows that $x+1$ divides $\sigma_0(x,1)$ and $\sigma_2(x,1)$. Since they are sums of squares, we have $(x+1)^2$ divides $\sigma_0(x,1)$ and $\sigma_2(x,1)$. This implies that there exist $\beta_0, \beta_2 \in \sum \mathbb{R}[x]^2$ such that

$$x+1 = \beta_0(x)(x+1)^2 + \beta_2(x)(x+1)^2(x^3-x^2+2).$$

Dividing both sides by $x+1$ yields

$$1 = (x+1)(\beta_0(x) + \beta_2(x)(x^3-x^2+2)).$$

This is the contradiction.

Now, by using MATLAB2014a and SOSTOOLS.303 [25] with the following procedure, we can approximate the infimum of $f(x,y) = x+1$ on $S(x^2, x^3+2, 2)$.

```
clear; echo on;
syms x y gam;
vartable = [x, y];
degree=10
prog = sosprogram(vartable);
prog = sosdecvar(prog,[gam]);
f =x+1;
prog = sosineq(prog,(f-gam));
prog = sossetobj(prog,-gam);
solver_opt.solver = 'sedumi';
prog = sossolve(prog,solver_opt);
% Finally, get solution
SOLgamma = sosgetsol(prog,gam)
[gam,vars,opt] = findbound(f,[1-f,y*x^2-x^2,x^3+2-x^2*y,
(x^2*y-x^2)*(x^3+2-x^2*y)*y,
-(x^2*y-x^2)*(x^3+2-x^2*y)*y],degree);
echo off
```

Then, gam $= 6.065753531728744e - 08 = f_{10}$. It is straightforward to see that

$$f_* = \min\{x+1|\ (x,y) \in S(x^2, x^3+2, 2)\} = f(-1,1) = 0.$$

(b) Consider the infimum problem of $g(x,y) = x^2 + (xy-1)^2$ on $S := S(x^2, x^3+2, 2)$. We see that $g(x,y) \geq 0$ on $S$ but there is not $(x,y) \in S$ such that $g(x,y) = 0$. However, there is the sequence $\{(\frac{1}{k}, k)\}_{k\in\mathbb{N}} \subset S$ and $\lim_{k\to+\infty} g(\frac{1}{k}, k) = 0$. Hence, $\inf\{g(x,y)|(x,y) \in S\} = 0$ and there is no minimizer.

Take $M = g(0,0) = 1$, then

$$\begin{aligned}\Gamma_M(g,S) =& \{(x,y) \in \mathbb{R}^2 | M - g(x,y) \geq 0, x^2y - x^2 \geq 0; x^3 + 2 - x^2y \geq 0;\\ & (x^2y - x^2)(x^3+2-x^2y)[(x^2-y^2)(xy-1) - xy] = 0\}\end{aligned}$$

By using MATLAB2014a and SOSTOOLS.303 [25], we can approximate the infimum of $g(x,y)$ on $S(x^2, x^3+2, 2)$ by $g_{10} = 0.0384; g_{14}$ and no optimizer.

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