MULTIPLE EULERIAN POLYNOMIALS

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ABSTRACT. In this page, we give some related properties of a class of polynomials which are said the multiple Eulerian polynomials. After that, we use these polynomials to regularize the polyzetas at non-positive integer multi-indices.

1. INTRODUCTION

The Eulerian polynomials[1], denoted by $A_n(z),\ n \in \mathbb{N}$, play an interesting role in the theory of quadrature formulas and the enumerative combinatorics, as well as in other areas since they first appeared in the works of Euler about the alternating sums $\sum_{n=1}^{N}(-1)^n n^k$ for any $N \in \mathbb{N}^*$ and $k \in \mathbb{Z}$ [16, 17] (also see in [14, 15, 19]). In these works, Euler gave the following general result:

$$\sum_{n=1}^{N} n^k z^n = \sum_{i=1}^{k}(-1)^{k+i}\binom{k}{i}\frac{z^{N+1}N^i}{(z-1)^{k-i+1}}A_{k-i}(z) + (-1)^k\frac{z(z^k-1)}{(z-1)^{k+1}}A_k(z)$$

where the Eulerian polynomials $\{A_k(z)\}_{k\in\mathbb{N}}$ are recursively defined by

$$A_0(z) = 1 \text{ and } A_k(z) = \sum_{j=0}^{k-1}\binom{k}{j}A_j(z)(z-1)^{k-1-j}. \tag{1.1}$$

Moreover, as $N \to +\infty$, for any $k \in \mathbb{N}$, we have [15]:

$$\mathrm{Li}_{-k}(z) = \frac{zA_k(z)}{(1-z)^{k+1}} \tag{1.2}$$

where $\mathrm{Li}_s(z)$ denotes the polylogarithm at the index $s \in \mathbb{C}$, *i.e.*,

$$\mathrm{Li}_s(z) = \sum_{n=1}^{+\infty}\frac{z^n}{n^s},\ \text{for any } |z| < 1. \tag{1.3}$$

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[1] In fact, this name was introduced by Scherk in 1825 [25].

In addition, for any $k \in \mathbb{N}$, Euler also stated that the $k^{th}-$Eulerian polynomial $A_k(z)$ can be rewritten as a generating polynomial of Eulerian numbers, denoted by $\{A_{k,i} \mid i = 0, \ldots, k-1\}$ [17], as follows:

$$A_k(z) = \sum_{i=0}^{k-1} A_{k,i}\ z^i. \tag{1.4}$$

On the other hand, there is a combinatorial definition of Eulerian polynomial[2] [14, 28]. The $k^{th}-$Eulerian polynomial can be understood as the generating function of the descent statistic over the symmetric group G_k:

$$A_k(z) = \sum_{\sigma \in G_k} z^{\mathrm{des}(\sigma)+1} \tag{1.5}$$

where $\mathrm{des}(\sigma) = \sharp\{1 \leq i \leq k-1 \mid \ \sigma(i) > \sigma(i+1)\}$. By this way, we can show that all the roots of Eulerian polynomials are real, distinct, and negative. This was already noted by Frobenius[3] [12], and is not an isolated phenomenon as surprisingly many polynomials appearing in combinatorics are real-rooted [4, 5, 7, 32]. Furthermore, another intersting result is that the polynomials $P_k(z)$ are conjectured irreducible over $\mathbb{Q}$ where $P_k(z)$ denote $A_k(z)$ if k is odd and $\dfrac{A_k(z)}{z+1}$ if k is even. To date it has not been possible either to verify this conjecture completely or to give a counterexample. However, the works of Heidrich in 1982 proved that for $k > 3$, even if $P_k(z)$ is not irreducible over $\mathbb{Q}$, it must nevertheless possess an irreducible factor of degreed $d \geq p-1$ where p is the largest prime not exceeding k [19].

Up to the present, the Eulerian polynomials have been extended by many different directions.

We must mention the results of Stanley in his thesis [33] in 1972. In those works, he introduced an extension of the Eulerian polynomials to labeled posets P which were called the $P-$Eulerian polynomials and denoted by $A_P(z)$. After that, these polynomials were studied in [4, 5, 7]. The Neggers-Stanley conjecture asserted that for each labeled poset P, $A_P(x)$ is also real-rooted [4]. In fact, this conjecture was disproved by Brändén in [6], and for natural labelings it was disproved by Stembrigde in [34]. However, the

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[2]The definition of Eulerian numbers in this way was introduced by Riordan in the 1950's [37].

[3]In fact, Frobenius also stated that -1 is a root of $A_k(z)$ if k is even, and that when $k > 2$ the roots of $A_{k+1}(z)$ are separated by those of $A_k(z)$ [12, 19].

conjecture was proved for several classes of posets by Brenti [7] and Wagner [35].

In 1974, Carlitz also gave an extension of Eulerian polynomials which associcate to an arithmetic progression [9] $\{a, a+d, a+2d, a+3d, \ldots\}$ for any $a, d \in \mathbb{R}$. After that, the properties of these polynomials have been completed by some other mathematicians as Foata [15] or Tingyao Xiong and al. [37].

Another interesting result was claimed by Gérard and al. in 2017 when they studied about the polygarithms at negative multi-indices [10, 11]. In that work, they showed that, for any $s_1, \ldots, s_r \in \mathbb{N}$, there exists a family of polylomials, denoted by $A_{s_1,\ldots,s_r}(z)$, such that

$$\mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \frac{z^r}{(1-z)^{s_1+\ldots+s_r+r}} A_{s_1,\ldots,s_r}(z),\ |z| < 1 \tag{1.6}$$

where $\mathrm{Li}_{t_1,\ldots,t_r}(z)$ is denoted the polylogarithm at $(t_1, \ldots, t_r) \in \mathbb{C}^r$ and defined by[4]

$$\mathrm{Li}_{t_1,\ldots,t_r}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{t_1} \ldots n_r^{t_r}}, |z| < 1. \tag{1.7}$$

The polynomials $A_{s_1,\ldots,s_r}(z)$ for any $s_1, \ldots, s_r \in \mathbb{N}$ as in (1.6) are called the multiple Eulerian polynomials.

In this paper, we will clarify the properties of the class of polynomials as in (1.6). Specifically, like the traditional Eulerian polynomials, we are concerned with the description of a recursive formula of multiple Eulerian polynomials. After that, we are interested in the properties of roots of multiple Eulerian polynomials. Then it is the same for the traditional Euler polynomials, we also obtained that the multiple Eulerian polynomials are real-rooted. These results are described as in two following theorems.

Theorem 1. For any $r \in \mathbb{N}^*$ and $s_1, \ldots, s_r \in \mathbb{N}$, we have

(i) If $s_1 = 0$ then

$$A_{s_1,\ldots,s_r}(z) = A_{s_2,\ldots,s_r}(z).$$

(ii) If $s_1 \geq 1$ then

$$A_{s_1,\ldots,s_r}(z) = \sum_{k=1}^{s_1} S(s_1, k)(1-z)^{s_1-k} A_{k,s_2,\ldots,s_r}(z) \tag{1.8}$$

[4] In fact, the polylogarithms are defined on the unit disc $|z| < 1$ and they are extended as a meromorphism functions on $\mathbb{C}$ by continuation [27, 29].

where $S(s_1,k)$ is the Stirling numbers of second kind.

Theorem 2. For any $s_1,\dots,s_r\in\mathbb{N}$, the polynomial $A_{s_1,\dots,s_r}(z)$ is real-rooted. In addition, all roots of $A_{s_1,\dots,s_r}(z)$ are negative for any $s_1,\dots,s_r\in\mathbb{N}$. Moreover, these polynomials are unimodal and log-concave.

Finally, in the rest of this paper, using the above results, we give some applications of multiple Eulerian polynomials in the theory of strong uniqueness polynomials and also connected them to regularize the polyzetas at non-positive integer multiple indices.

2. The proofs and some consequences

We start this section by some basic concepts which will used in our proofs. Firstly, we consider the differential operators which are denoted by

$$\theta_0 := z\frac{d}{dz} \text{ and } \theta_1 := (1-z)\frac{d}{dz}.$$

Note that, for any function $f\in\mathcal{C}^\infty$, we get

$$\begin{aligned}
\theta_0(f) &= z\frac{df}{dz},\\
\theta_0^2(f) &= \theta_0\left(\theta_0(f)\right) = z\frac{d}{dz}\left(z\frac{df}{dz}\right) = z\left(\frac{df}{dz}+z\frac{d^2f}{dz^2}\right) = z\frac{df}{dz}+z^2\frac{d^2f}{dz^2},\\
\theta_0^3(f) &= \theta_0\left(\theta_0^2(f)\right) = z\frac{df}{dz}+3z^2\frac{d^2f}{dz^2}+z^3\frac{d^3f}{dz^3},\\
\theta_0^4(f) &= \theta_0\left(\theta_0^3(f)\right) = z\frac{df}{dz}+7z^2\frac{d^2f}{dz^2}+6z^3\frac{d^3f}{dz^3}+z^4\frac{d^4f}{dz^4}.
\end{aligned}$$

By the recurrence way, it is seen that

$$\theta_0^s(f) = \sum_{k=1}^{s} S(s,k)z^k\frac{d^kf}{dz^k} \tag{2.1}$$

where $S(s,k)$ is the $(s,k)-$Stirling number of second kind[5] for any $s\in\mathbb{N}^*$.

We are now in a position to prove the first theorem.

Proof of theorem 1:

(i) Note that

$$\mathrm{Li}_{0,-s_2,\dots,-s_r}(z) = \frac{z}{1-z}\,\mathrm{Li}_{-s_2,\dots,-s_r}(z)$$

[5]The interested readers may find the concept of Stirling numbers of second kind in [10].

for any $r \in \mathbb{N}^*$ and $s_2, \ldots, s_r \in \mathbb{N}$. Since the equation (1.6), we get

$$\frac{z^r}{(1-z)^{s_2+\ldots+s_r+r}} A_{0,s_2,\ldots,s_r}(z) = \frac{z}{1-z} \frac{z^{r-1}}{(1-z)^{s_2+\ldots+s_r+r-1}} A_{s_2,\ldots,s_r}(z).$$

Then by some simple transformations, we obtain the first statement.

(ii) For this statement, on the class of analytic functions on $\mathbb{C}$, we will denote the sections of the operators θ_0 and θ_1 by ι_1 and ι_2, respectively. This means that

$$\iota_0(f)(z) = \int_{z_0}^{z} f(s) \frac{ds}{s} \text{ and } \iota_1(f)(z) = \int_{z_0}^{z} f(s) \frac{ds}{1-s}$$

where z_0 is a such point in the continuous domains of $f(z)$ [10]. Recall that

$$\mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \left(\theta_0^{s_1+1}\iota_1\right) \mathrm{Li}_{-s_2,\ldots,-s_r}(z) = \theta_0^{s_1}\left(\mathrm{Li}_{0,-s_2,\ldots,-s_r}(z)\right)$$

and then since the equation (2.1), we get

$$\begin{aligned} \mathrm{Li}_{-s_1,\ldots,-s_r}(z) &= \sum_{k=1}^{s_1} S(s_1,k) z^k \frac{d^k}{dz^k}\left[\mathrm{Li}_{0,-s_2,\ldots,-s_r}(z)\right] \\ &= \sum_{k=1}^{s_1} S(s_1,k)\, \mathrm{Li}_{-k,-s_2,\ldots,-s_r}(z). \end{aligned}$$

for any $s_1, \ldots, s_r \in \mathbb{N}$. Finnaly, from the definition of polynomials $A_{s_1,\ldots,s_r}(z)$ in (1.6), we obtain that

$$\frac{z^r}{(1-z)^{s_1+\ldots+s_r+r}} A_{s_1,\ldots,s_r}(z) = \sum_{k=1}^{s_1} S(s_1,k) \frac{z^r}{(1-z)^{s_2+\ldots+s_r+k}} A_{k,s_2,\ldots,s_r}(z).$$

Then after some computations, the second statement is proved, *i.e.*,

$$A_{s_1,\ldots,s_r}(z) = \sum_{k=1}^{s_1} S(s_1,k)(1-z)^{s_1-k} A_{k,s_2,\ldots,s_r}(z).$$

The proof is completed now.

In the simplest case, for r=1, we obtain the usual formula of the traditional Eulerian polynomials as follows:

Corollary 1 ([14]). For any $n \geq 0$, we get

$$A_n(z) = \sum_{k=1}^{n} k! S(n,k)(z-1)^{k-1}.$$

Recall that, for any $r \in \mathbb{N}^*$ and $s_1, \ldots, s_r \in \mathbb{N}$,

$$\mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \sum_{n_1>\ldots>n_r>0} n_1^{s_1} \ldots n_r^{s_r} z^{n_1}.$$

Hence

$$\frac{d}{dz}\,\mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \sum_{n_1>\ldots>n_r>0} n_1^{s_1+1} \ldots n_r^{s_r} z^{n_1-1} = \frac{1}{z}\,\mathrm{Li}_{-s_1-1,-s_2,\ldots,-s_r}(z).$$

From the equation (1.6), after some computations, we obtain the following recurrence relation:

$$[(s_1+\ldots+s_r)\,z+r]\,A_{s_1,s_2,\ldots,s_r}(z) + z\,(1-z)\,A^{'}_{s_1,s_2,\ldots,s_r}(z) = A_{s_1+1,s_2,\ldots,s_r}(z) \tag{2.2}$$

for any $s_1, \ldots, s_r \in \mathbb{N}$. Furthermore, we also get [10]

$$\mathrm{Li}_{\underbrace{0,\ldots,0}_{r \text{ times } 0}}(z) = \left(\frac{z}{1-z}\right)^r, \tag{2.3}$$

and then $A_{\underbrace{0,\ldots,0}_{r \text{ times } 0}}(z) = 1$ for any $r \in \mathbb{N}^*$. Thus, from the equation (2.2), it is easily seen that the degree of $A_{s_1,s_2,\ldots,s_r}(z)$ is $s_1+\ldots+s_r-1$ for any $s_1, \ldots, s_r \in \mathbb{N}$.

In fact, the equation (2.2) also provides a useful alternative in the computations to the original definition of multiple Eulerian polynomials. Indeed, for any $r \in \mathbb{N}^*$ and $s_1, \ldots, s_r \in \mathbb{N}$, we assume that

$$A_{s_1,\ldots,s_r}(z) = \sum_{k=0}^{s_1+\ldots+s_r-1} a^k_{s_1,\ldots,a_r} z^k. \tag{2.4}$$

Then since the first statement of Theorem 1 and the equation (2.2), after some simple computations, we obtain the recurrence relationship of the coefficients of multiple Eulerian polynomials as follows:

Corollary 2. For any $r \in \mathbb{N}^*$ and $s_1, \ldots, s_r \in \mathbb{N}$,

$$\begin{array}{rcl}
a^0_{s_1+1,s_2,\ldots,s_r} & = & r a^0_{s_1,s_2,\ldots,s_r}, \\
a^1_{s_1+1,s_2,\ldots,s_r} & = & (r+1)s^1_{s_1,s_2,\ldots,s_r} + (s_1+\ldots+s_r)a^0_{s_1+1,s_2,\ldots,s_r}, \\
a^k_{s_1+1,s_2,\ldots,s_r} & = & (r+k)a^k_{s_1,s_2,\ldots,s_r} - (k-1)a^{k-1}_{s_1,s_2,\ldots,s_r} \\
 & & +(s_1+\ldots+s_r)a^{k-1}_{s_1+1,s_2,\ldots,s_r}, \forall k \geq 2.
\end{array}$$

This corollary helps us computering the multiple Eulerian polynomials.

Example 1.

$$\begin{aligned}
A_1(z) &= 1,\\
A_2(z) &= 1+z,\\
A_3(z) &= 1+4z+z^2,\\
A_4(z) &= 1+11z+11z^2+z^3,\\
A_5(z) &= 1+26z+66z^2+26z^3+z^4,\\
A_6(z) &= 1+57z+302z^2+302z^3+57z^4+z^5,\\
A_{2,3}(z) &= z^4+29z^5+93z^2+53z+4,\\
A_{3,4}(z) &= z^6+127z^5+1458z^4+3654z^3+2429z^2+387z+8,\\
A_{4,6}(z) &= z^9+1028z^8+51900z^7+548492z^6+1816214z^5+2177484z^4,\\
&\quad +961052z^3+141140z^2+5073z+16,\\
A_{1,2,2}(z) &= z^4+34z^3+133z^2+100z+12,\\
A_{2,2,2}(z) &= z^5+75z^4+603z^3+1065z^2+460z+36.
\end{aligned}$$

Moreover, using Corollary 2, we also get another important result of coefficients of multiple Eulerian polynomials as in the following corollary.

Corollary 3. For any $s_1,\ldots,s_r \in \mathbb{N}$, the coefficients of $A_{s_1,\ldots,s_r}(z)$ are positive.

For the proof of second theorem, we begin by the following definition:

Definition 1. Let

$$P(z) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

be a real polynomial. It is called unimodal if there is some j such that

$$a_0 \le a_1 \le \ldots \le a_{j-1} \le a_j \ge a_{j+1} \ge \ldots \ge a_n. \tag{2.5}$$

Suppose now all coefficients of P are positive. We say that P is log-concave if

$$a_{j-1}a_{j+1} \le a_j^2 \tag{2.6}$$

for every $j = 1,\ldots,n$.

Finally, if P has only real roots, it is called real-rooted.

In fact, every real-rooted polynomial with positive coeficients is log-concave and then is so unimodal [4]. Moreover, since the works of Brenti in 2000 [8] (see also [31]), he gave a following important lemma.

Lemma 1 ([8]). Suppose that $A_n(z,q)$ are the family of polynomials which are defined by the recursion relation as follows:

$$A_{n+1}(z,q) = (nx+q)A_{n-1}(z,q) + z(1-z)\frac{d}{dz}A_{n-1}(z,q) \tag{2.7}$$

for any $n \in \mathbb{N}$ with the initial condition $A_0(z,q) = z$. Then the polynomials $A_n(z,q)$ have only real nonpositive simple roots for any $q \in \mathbb{Q}$ and $q > 0$. In particular, these polynomials are log-concave and unimodal.

We will not send Brenti's proof for Lemma 1 in here. Interested readers can find it from [8]. We will use that techniques to prove a more generally results as follows:

Lemma 2. Suppose that $P_n(z,q)$ are the family of polynomials which are defined by the recursion relation as follows:

$$P_{n+1}(z,q) = (a_n x+q)P_{n-1}(z,q) + z(1-z)\frac{d}{dz}P_{n-1}(z,q) \tag{2.8}$$

where $a_n > 0$ for any $n \in \mathbb{N}$ and $P_0(z,q)$ satisfies the following conditions:

(i) 0 is a root of $P_0(z,q)$, *i.e.*, $P_0(0,q) = 0$.

(ii) $P_0(z,q)$ has only real nonpositive simple roots.

If the leading coefficient of $P_n(z,q)$ is positive for every $n \in \mathbb{N}$ then the polynomials $P_n(z,q)$ have only real nonpositive simple roots for any $q \in \mathbb{Q}$ and $q > 0$.

Proof of Lemma 2: We will continue this proof by the induction on $n \in \mathbb{N}$ and reuse Brenti's technique in the proof of Lemma 1 in [8]. From our from our induction hypothesis, $P_0(z,q)$ has only real nonpositive simple roots. The lemma is clearly true for $n = 0$. Suppose that

$$\xi_{d-1} < \xi_{d-2} < \ldots < \xi_2 < \xi_1 = 0$$

be the roots of $P_{n-1}(z,q)$ where d is the degree of $P_n(z,q)$. From Rolle's theorem, the polynomial $P'_{n-1}(z,q)$ has only simple real roots, denoted by $\nu_{d-2} < \ldots < \nu_1$. Moreover, we also get

$$\xi_{d-1} < \nu_{d-2} < \xi_{d-2} < \ldots \xi_{d-2} < \ldots < \xi_3 < \nu_2 < \xi_2 < \nu_1 < \xi_1 = 0.$$

Recall that the leading coefficient of $P'_{n-1}(z)$ are positive. Then it is easily seen that

$$(-1)^j P_{n-1}(\nu_j) > 0, \ \forall j = 1, 2, \ldots, d-2, \tag{2.9}$$

and

$$(-1)^{j+1}P_{n-1}^{'}(\xi_j) > 0,\ \forall j = 1, 2, \ldots, d-1. \tag{2.10}$$

Given $\alpha = -\dfrac{q}{a_n} < 0$. If $\alpha < \nu_1$ then there is an index i such that $2 \leq i \leq d-1$ and

$$\nu_i < \alpha \leq \nu_{i-1}$$

where we denote that $\nu_{d-1} = -\infty$. Since the equations (2.18), (2.9) and (2.10), we obtain that

$$(-1)^i P_n(\alpha, q) \geq 0, \tag{2.11}$$

$$(-1)^j P_n(\nu_j, q) \begin{cases} > 0 & \text{if} \quad j = 1, \ldots, i-2 \\ \geq 0 & \text{if} \quad j = i-1 \\ < 0 & \text{if} \quad i \leq j \leq d-1 \end{cases}, \tag{2.12}$$

$$(-1)^j P_n(\xi_j, q) > 0,\ \forall j = 1, \ldots, d-1. \tag{2.13}$$

In here, we denote that $P_n(-\infty, q) = \lim\limits_{z \to -\infty} P_n(z, q)$. Then we get

$$P_n(\nu_j, q) P_n(\xi_{j+1}, q) < 0,\ \forall j = 1, \ldots, i-2 \tag{2.14}$$

and

$$P_n(\xi_j, q) P_n(\nu_j, q) < 0,\ \forall j = i+1, \ldots, d-1. \tag{2.15}$$

The equations (2.14) and (2.15) prove that there are the numbers $A_j \in (\xi_{j+1}, \nu_j),\ \forall j = 1, \ldots, i-2$ and $B_j \in (\nu_j, \xi_j),\ \forall j = i+1, \ldots, d-1$ which are the roots of $P_n(z, q)$.

On the other hand, from the equation (2.11), (2.12) and (2.13), it is easily seen that the polynomial $P_n(z, q)$ has a root $A_{i-1} \in (\nu_i, M)$ and another root $B_i \in (N, \nu_i)$ with $M = \min(\xi_i, \alpha)$ and $N = \max(\xi_i, \alpha)$. In fact, the roots $A_1, \ldots, A_{i-1}, B_i, \ldots, B_{d-1}$ are distinct since $P_n(\xi_j, q) \neq 0$ by (2.8) for any $j = 2, \ldots, d-1$. In addition, since $P_n(0, q) = 0$ by (2.8), it implies that this polynomial has d real nonpositive simple roots.

For the case $\alpha > \nu_1$, we must recall some following base concepts.

Definition 2. Given two nonzero real-rooted polynomials $f, g \in \mathbb{R}[x]$. Suppose that the roots of f are $\alpha_1 < \ldots < \alpha_n$, and the roots of g are $beta_l < \ldots < \beta_m$. We say that f alternates left of g, denoted by $f \ll g$, if and only if $m = n$ and

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \ldots \leq \alpha_n \leq \beta_n. \tag{2.16}$$

In 1992, Wagner proved the important result as in the following lemma [36].

Lemma 3. [36] Let $f, g \in \mathbb{R}[x]$ be standard, real-rooted, and with $f \ll g$. Then for all $a, b > 0$, one has $f \ll af + bg \ll g$.

Complete the proof of Lemma 2: If $\alpha > \nu_1$, then $[a_n z + q]\, P_n(z, q)$ alternates left of $z(1-z)P'_{n-1}(z, q)$. And then, as a directly consequence of Lemma 3, $P_n(z, q)$ has d real nonpositive simple roots. The proof is completed now.

At the moment, we will start to prove Theorem 2.

Proof of Theorem 2: For any $r > 0$, setting now

$$P_{s_1,\ldots,s_r}(z) = zA_{s_1,\ldots,s_r}(z) \tag{2.17}$$

for any $s_1, \ldots, s_r \in \mathbb{N}$. Since $P'_{s_1,\ldots,s_r}(z) = A_{s_1,\ldots,s_r}(z) + xA'_{s_1,\ldots,s_r}(z)$ and the equation (2.2), we obtain that

$$P_{s_1+1,\ldots,s_r}(z) = [(s_1 + \ldots + s_r + 1)z + r - 1]\, P_{s_1,\ldots,s_r}(z) + z(1-z)P'_{s_1,\ldots,s_r}(z), \tag{2.18}$$

for any $s_1, \ldots, s_r \in \mathbb{N}$.

For this proof, we will proceed by the induction on $r \in \mathbb{N}^*$.

For $r = 1$, the theorem is a direct consequence of Lemma 1.

Fixe $r > 1$. Suppose that the result is true for any $k < r$, *i.e.*, the polynomial $P_{s_1,\ldots,s_k}(z)$ has only real nonpositive simple roots for every $s_1, \ldots, s_k \in \mathbb{N}$.

Fixe the sequence of indices $s_2, \ldots, s_r \in \mathbb{N}$. For any $s_1 \in \mathbb{N}$, it is easily seen that the polynomial $P_{s_1,\ldots,s_r}(z)$ is a polynomial of degree $s_1 + \ldots + s_r$ (by (2.18)) such that $P_{0,s_2\ldots,s_r}(0) = 0$. Note that

$$P_{0,s_2\ldots,s_r}(z) = xA_{0,s_2\ldots,s_r}(z) = xA_{s_2\ldots,s_r}(z) = P_{s_2\ldots,s_r}(z).$$

Then by our induction hypothesis, $P_{0,s_2\ldots,s_r}(z)$ has only real nonpositive simple roots. Furthermore, the leading coefficient of $P_{s_1,\ldots,s_r}(z)$ is positive. Thus the polynomial $P_{s_1,\ldots,s_r}(z)$ has only real nonpositive simple roots for all $s_1, s_2, \ldots, s_r \in \mathbb{N}$ by Lemma 2. This implies that $A_{s_1,\ldots,s_r}(z)$ has only real negative simple roots and then it is unimodal and log-concave because its coefficients are positive for all $s_1, s_2, \ldots, s_r \in \mathbb{N}$.

The proof is completed now.

Example 2. The polynomial $A_{1,0,2}(z) = z^2 + 6z + 3$, has 2 real roots.

The polynomial $A_{1,1,2}(z) = z^3 + 15z^2 + 26z + 6$ has 3 real roots.

By the computer, the polynomial

$$A_{3,4}(z) = z^6 + 127z^5 + 1458z^4 + 3654z^3 + 2429z^2 + 387z + 8,$$

has 6 real roots $\alpha_1 \in (-115; -114)$, $\alpha_2 \in (-10; -9)$, $\alpha_3 \in (-3; -2)$, $\alpha_4 \in (-1; -\frac{1}{2})$, $\alpha_5 \in (-\frac{1}{2}; -\frac{1}{10})$ and $\alpha_6 \in (-\frac{1}{10}; 0)$.

3. Some applications

3.1. **A relationship between the multiple Eulerian polynomials and the harmonic sums.** Remark that, for any $s_1, \ldots, s_r \in \mathbb{N}$, the function $\dfrac{\mathrm{Li}_{-s_1,\ldots,-s_r}(z)}{1-z}$ is an analytic function of unity disc $|z| < 1$. Moreover, its Taylor expansion at $z = 0$ is that

$$\frac{\mathrm{Li}_{-s_1,\ldots,-s_r}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{-s_1,\ldots,-s_r}(N) z^N, \ |z| < 1 \tag{3.1}$$

where the coefficient[6] $\mathrm{H}_{-s_1,\ldots,-s_r}(N)$ is defined by

$$\mathrm{H}_{-s_1,\ldots,-s_r}(N) = \sum_{N \geq n_1 > n_2 \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}, \ \forall N \in \mathbb{N}.$$

Proposition 1. Given $s_1, \ldots, s_r \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $0 \leq n \leq s_1 + \ldots + s_r - 1$. We get

$$a^n_{s_1,\ldots,s_r} = \sum_{i=0}^{n} \binom{s_1 + \ldots s_r + r}{i} (-1)^i \mathrm{H}_{-s_1,\ldots,-s_r}(n - i + r).$$

Proof. Since the equation (3.1), we get

$$\frac{z^r}{(1-z)^{s_1+\ldots+s_r+r+1}} A_{s_1,\ldots,s_r}(z) = \sum_{N \geq 0} \mathrm{H}_{-s_1,\ldots,-s_r}(N) z^N.$$

Note that

$$\mathrm{H}_{-s_1,\ldots,-s_r}(N) := 0$$

[6]For any $s_1, \ldots, s_r \in \mathbb{Z}$, the numbers

$$\mathrm{H}_{s_1,\ldots,s_r}(N) := \sum_{N \geq n_1 > n_2 \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, N \in \mathbb{N}^*$$

define an arithmetic function which is also called the harmonic sum at $(s_1, \ldots, s_r)$ [3, 10, 23].

for any $N \in \mathbb{N}$ such that $0 \leq N < r$. Thus

$$\begin{aligned}
A_{s_1,\ldots,s_r}(z) &= (1-z)^{s_1+\ldots+s_r+r+1} \sum_{N \geq r} \mathrm{H}_{-s_1,\ldots,-s_r}(N) z^{N-r} \\
&= \sum_{n=0}^{s_1+\ldots+s_r+r+1} \binom{s_1+\ldots+s_r+r+1}{n} (-1)^n z^n \sum_{n \geq 0} \mathrm{H}_{-s_1,\ldots,-s_r}(n+r) z^n \\
&= \sum_{n=0}^{\infty} c_n z^n
\end{aligned}$$

where for any $n \in \mathbb{N}$,

$$c_n = \sum_{i=0}^{\min(n, s_1+\ldots+s_r+r+1)} \binom{s_1+\ldots+s_r+r+1}{i} (-1)^i \mathrm{H}_{-s_1,\ldots,-s_r}(n-i+r).$$

On the other hand, $A_{s_1,\ldots,s_r}(z)$ is a polynomial of degree $s_1+\ldots+s_r-1$. Hence

$$a^n_{s_1,\ldots,s_r} = \sum_{i=0}^{n} \binom{s_1+\ldots+s_r+r+1}{i} (-1)^i \mathrm{H}_{-s_1,\ldots,-s_r}(n-i+r)$$

for any $n = 0, \ldots, s_1+\ldots+s_r-1$. $\square$

Example 3. Given

$$A_{2,3}(z) = a^4_{2,3} z^4 + a^3_{2,3} z^3 + a^2_{2,3} z^2 + a^1_{2,3} z + a^0_{2,3}.$$

Using Proposition 1, we get

$$\begin{aligned}
a^4_{2,3} &= \sum_{i=0}^{4} \binom{8}{i} (-1)^i \mathrm{H}_{-2,-3}(6-i) \\
&= 70\mathrm{H}_{-2,-3}(2) - 56\mathrm{H}_{-2,-3}(3) + 28\mathrm{H}_{-2,-3}(4) - 8\mathrm{H}_{-2,-3}(5) + \mathrm{H}_{-2,-3}(6) \\
a^3_{2,3} &= \sum_{i=0}^{3} \binom{8}{i} (-1)^i \mathrm{H}_{-2,-3}(5-i) \\
&= -56\mathrm{H}_{-2,-3}(2) + 28\mathrm{H}_{-2,-3}(3) - 8\mathrm{H}_{-2,-3}(4) + \mathrm{H}_{-2,-3}(5) \\
a^2_{2,3} &= \sum_{i=0}^{2} \binom{8}{i} (-1)^i \mathrm{H}_{-2,-3}(4-i) = 28\mathrm{H}_{-2,-3}(2) - 8\mathrm{H}_{-2,-3}(3) + \mathrm{H}_{-2,-3}(4)
\end{aligned}$$

$$\begin{aligned}
a^1_{2,3} &= \sum_{i=0}^{1} \binom{8}{i} (-1)^i \mathrm{H}_{-2,-3}(3-i) = -8\mathrm{H}_{-2,-3}(2) + \mathrm{H}_{-2,-3}(3) \\
a^0_{2,3} &= \sum_{i=0}^{0} \binom{8}{i} (-1)^i \mathrm{H}_{-2,-3}(2-i) = \mathrm{H}_{-2,-3}(2).
\end{aligned}$$

Remark that

$$\mathrm{H}_{-2,-3}(N) = \frac{N(N-1)(N+1)(30N^4+35N^3-33N^2-35N+2)}{840},\ N \in \mathbb{N}.$$

And then $\mathrm{H}_{-2,-3}(2) = 4$, $\mathrm{H}_{-2,-3}(3) = 85$, $\mathrm{H}_{-2,-3}(4) = 661$, $\mathrm{H}_{-2,-3}(5) = 3161$ and $\mathrm{H}_{-2,-3}(6) = 11261$. It implies that

$$a^0_{2,3} = 4;\ a^1_{2,3} = 53;\ a^2_{2,3} = 93;\ a^3_{2,3} = 29;\ a^4_{2,3} = 1.$$

Hence

$$A_{2,3}(z) = z^4 + 29z^3 + 93z^2 + 53z + 4.$$

In fact, from the proof of Proposition 1, we also obtain that

Corollary 4. Let $s_1, \ldots, s_r$ be the non-negative integers. For any $n \geq s_1 + \ldots + s_r + r + 1$, we get

$$\sum_{i=0}^{s_1+\ldots+s_r+r} \binom{s_1+\ldots+s_r+r+1}{i} (-1)^i \mathrm{H}_{-s_1,\ldots,-s_r}(s_1+\ldots+s_r-i+2r) = 0$$

and

$$\sum_{i=0}^{s_1+\ldots+s_r+r+1} \binom{s_1+\ldots+s_r+r+1}{i} (-1)^i \mathrm{H}_{-s_1,\ldots,-s_r}(n-i+r) = 0.$$

3.2. **The strong uniqueness polynomials for $L-$functions.** Suppose that $\mathcal{F}$ is a subset of meromorphic functions. Recall that a polynomial $P(z)$ is called a strong uniqueness polynomial for $\mathcal{F}$ [2, 24] if for any two non-constant meromorphic functions $f, g \in \mathcal{F}$, then

$$(P(f) = cP(g);\ c \neq 0) \Longrightarrow (f = g)\,. \tag{3.2}$$

In particular, $P(z)$ is called a uniqueness polynomial if for any two non-constant meromorphic functions $f, g \in \mathcal{F}$, then

$$(P(f) = P(g)) \Longrightarrow (f = g)\,. \tag{3.3}$$

Theorem 3 ([24]). Let $P(z)$ be a polynomial satisfying $P(1)P'(1) \neq 0$. Then $P(z)$ is a strong uniqueness polynomial for $L-$functions.

Note that the coefficients of $A_{s_1,\ldots,s_r}(z)$ are positive for every $s_1, \ldots, s_r \in \mathbb{N}$. Thus $A_{s_1,\ldots,s_r}(1)A'_{s_1,\ldots,s_r}(1) \neq 0$. It led us to the following corollary.

Corollary 5. For any $s_1, \ldots, s_r \in \mathbb{N}$, the polynomial $A_{s_1,\ldots,s_r}(z)$ is a strong uniqueness polynomial for $L-$functions.

Given f to be a meromorphic function in $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_f(a)$ the set of $a-$points of f counted with its multiplicities. Moreover, for any nonempty subset S of $\mathbb{C} \cup \{\infty\}$, set that

$$E_f(S) = \bigcup_{a \in S} E_f(a). \tag{3.4}$$

And then, the subset S is called a unique range set, counting multiplicities for $\mathcal{F}$ if for any $f, g \in \mathcal{F}$ then

$$(E_f(S) = E_g(S)) \Longrightarrow (f = g). \tag{3.5}$$

Thank so much the works of authors in [24] who sent to the important results about the set of roots of a strong uniqueness polynomial.

Theorem 4. ([24]) Let $P(z)$ be a uniqueness polynomial for $L-$functions satisfying that $P(z)$ has no multiple zeros and $P(1) \neq 0$. Then the zero set of $P(z)$ is a unique range set for $L-$functions, counting multiplicities.

As an immediate consequence of Corollary 5, Theorem 2 and Theorem 4, we obtain that

Proposition 2. For any $s_1, \ldots, s_r \in \mathbb{N}$, the set of roots of the polynomial $A_{s_1,\ldots,s_r}(z)$ is unique range set, counting multiplicities for $L-$functions.

3.3. **Polyzetas at non-positive integer multiple indices.** In the general case, for any $s_1, \ldots, s_r \in \mathbb{C}$, the polylogarithm at $(s_1, \ldots, s_r)$ is defined by[7]

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}, |z| < 1 \tag{3.6}$$

and the polyzeta at $(s_1, \ldots, s_r)$ is defined by [1, 10, 11]:

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}. \tag{3.7}$$

The convergent domain $\mathcal{H}_r$ [13, 38] of the series in (3.7) is well-defined as [8]

$$\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \ \mathrm{Re}(s_1) + \ldots + \mathrm{Re}(s_m) > m; \forall m = 1, \ldots, r\}.$$

For any $(s_1, \ldots, s_r) \in \mathcal{H}_r$, by a theorem of Abel, we get

$$\lim_{z \to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \zeta(s_1, \ldots, s_r). \tag{3.8}$$

---

[7]In fact, the polylogarithms are defined on the unit disc $|z| < 1$ and they are extended as a meromorphism functions on $\mathbb{C}$ by continuation.

[8]In here, $\mathrm{Re}(z)$ is the real part of complex number z.

However, the equation (3.8) is no more valid in the divergent cases which require the renormalization of the corresponding divergent polyzetas. This problem has been studied by Hoang Ngoc Minh [20, 21, 22], L. Gou [18], K. Matsumoto [13] or D. Manchon [26],...

In this section, we would like to regularize the value of polyzetas at non-positive integer multi-indices by applying the properties of multiple Eulerian polynomials. The history is started by the works of Euler when he described a method of computing values of the zeta function at negative integers[9] [17]. The technique used the function

$$\zeta_2(k) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^k} = -\operatorname{Li}_k(-1),\ \forall k \in \mathbb{Z}.$$

This looks not too different from $\zeta(k)$, but has the advantage as an alternating series of converging for all positive k. For $k > 1$,

$$\zeta_2(k) = (1 - 2^{1-k})\zeta(k).$$

Note that

$$\operatorname{Li}_{-k}(z) = \frac{zA_k(z)}{(1-z)^{k+1}},\ |z| < 1;\ k \in \mathbb{N}.$$

Hence, taking $z = -1$,

$$\operatorname{Li}_{-k}(-1) = -\frac{A_k(-1)}{2^{k+1}},$$

and then, we obtain that

$$\zeta(-k) = \frac{1}{1-2^{k+1}}\zeta_2(-k) = \frac{A_k(-1)}{2^{k+1}(2^{k+1}-1)}, \forall k \in \mathbb{N}.$$

Like a progression in the method of Euler, we extend the fucntion ζ_2 for the multivariate case. Setting now

$$\zeta_2(s_1, \ldots, s_r) := \sum_{n_1 > n_2 > \ldots > n_r > 0} \frac{(-1)^{n_1 - 1}}{n_1^{s_1} \ldots n_r^{s_r}}$$

for any $r \in \mathbb{N}^*$ and $s_1, \ldots, s_r \in \mathbb{Z}$. Then it is easily seen that

$$\zeta_2(s_1, \ldots, s_r) = -\sum_{n_1 > n_2 > \ldots > n_r > 0} \frac{(-1)^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} = -\operatorname{Li}_{s_1, \ldots, s_r}(-1)$$

---

[9] In fact, Euler introduced the Eulerian polynomials in an attempt to evaluate the Dirichlet eta function at $-1, -2, \ldots$ and this led him to conjecture the functional equation of the eta function (which immediately implies the functional equation of the zeta function). Recently, this technique was also used by C. S. Ryoo in his works about the Euler Zeta Function [30].

for any $s_1, \ldots, s_r \in \mathbb{Z}$. In particular, for $s_1, \ldots, s_r \in \mathbb{N}$, note that

$$\mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \frac{z^r}{(1-z)^{s_1+\ldots+s_r+r}} A_{s_1,\ldots,s_r}(z),$$

taking $z = -1$, we get

$$\zeta_2(-s_1, \ldots, -s_r) = -\mathrm{Li}_{-s_1,\ldots,-s_r}(-1) = \frac{(-1)^{r+1}}{2^{s_1+\ldots+s_r+r}} A_{s_1,\ldots,s_r}(-1). \tag{3.9}$$

On the other hand, for any $s_1, \ldots, s_r \in \mathbb{Z}$, we also get

$$\zeta_2(s_1, \ldots, s_r) = \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1-1}}{n_1^{s_1}} \mathrm{H}_{s_2,\ldots,s_r}(n_1 - 1).$$

Hence

$$\begin{aligned}\zeta(s_1, \ldots, s_r) - \zeta_2(s_1, \ldots, s_r) &= \sum_{n_1=1}^{\infty} \frac{2}{(2n_1)^{s_1}} \mathrm{H}_{s_2,\ldots,s_r}(n_1 - 1) \\ &= 2^{1-s_1} \zeta(s_1, \ldots, s_r),\end{aligned}$$

or

$$\zeta(s_1, \ldots, s_r) = \frac{1}{1 - 2^{1-s_1}} \zeta_2(s_1, \ldots, s_r) \tag{3.10}$$

for any $s_1, \ldots, s_r \in \mathbb{Z}$.

From the formulas (3.9) and (3.10), we obtain a formula in the type of Euler which includes the polyzeta at the non-positive integer multi-indices.

Proposition 3. For any $s_1, \ldots, s_r \in \mathbb{N}$ and $r \in \mathbb{N}^*$, we get

$$\zeta(-s_1, \ldots, -s_r) = \frac{(-1)^r}{2^{s_1+\ldots+s_r+r}(2^{s_1+1} - 1)} A_{s_1,\ldots,s_r}(-1).$$

Example 4.

$$\begin{aligned}
\zeta(-1) &= \frac{(-1)^1}{2^{1+1}(2^{1+1}-1)}A_1(-1) = -\frac{1}{12},\\
\zeta(-2) &= \frac{(-1)^1}{2^{2+1}(2^{2+1}-1)}A_2(-1) = 0,\\
\zeta(-3) &= \frac{(-1)^1}{2^{3+1}(2^{3+1}-1)}A_3(-1) = \frac{1}{120},\\
\zeta(-4) &= \frac{(-1)^1}{2^{4+1}(2^{4+1}-1)}A_4(-1) = 0,\\
\zeta(-5) &= \frac{(-1)^1}{2^{5+1}(2^{5+1}-1)}A_5(-1) = -\frac{1}{252},\\
\zeta(-1,-2) &= \frac{(-1)^2}{2^{1+2+2}(2^{1+1}-1)}A_{1,2}(-1) = -\frac{1}{48},\\
\zeta(-2,-3) &= \frac{(-1)^2}{2^{2+3+2}(2^{2+1}-1)}A_{2,3}(-1) = \frac{1}{56},\\
\zeta(-3,-4) &= \frac{(-1)^2}{2^{3+4+2}(2^{3+1}-1)}A_{3,4}(-1) = -\frac{17}{480},\\
\zeta(-1;-3;-2) &= \frac{(-1)^3}{2^{1+3+2+3}(2^{1+1}-1)}A_{1,3,2}(-1) = -\frac{1}{16},\\
\zeta(-2,0,-2) &= \frac{(-1)^3}{2^{2+0+2+3}(2^{2+1}-1)}A_{2,0,2}(-1) = \frac{1}{112},\\
\zeta(-2;-2;-2) &= \frac{(-1)^3}{2^{2+2+2+3}(2^{2+1}-1)}A_{2,2,2}(-1) = -\frac{1}{32}.
\end{aligned}$$

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