New Formulas for Subdifferentials of Perturbed Distance Functions

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Abstract. We give exact formulas for the subdifferentials of perturbed distance functions in a normed space. Our method, seemingly novel and different from existing ones, is to turn the involved problem equivalently to a parametric optimization problem and then apply variational analysis technique to the optimal value function. In the convex setting, we obtain new representations for the subdifferential of perturbed distance functions, which do not depend on the relative position of the reference point with respect to the input set, and which are described directly via the input data. Our results complement those of Wang et al. [J. Global Optim. 46 (2010), 489–501] and of Li and Bounkhel [Nonlinear Anal. 108 (2014), 173–188] which were established by different methods.

Keywords: perturbed optimization problem, perturbed distance function, distance function, optimal value function, subdifferential

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1 Introduction

Let X be a real Banach space endowed with a norm $\|\cdot\|$, $S \subset X$ be a nonempty closed subset, and $J : S \to \mathbb{R}$ be a lower semicontinuous function. For each $x \in X$, consider the following *perturbed optimization problem*, which is denoted by $\min_J(x, S)$,

$$\min\{\|x - y\| + J(y) \ : \ y \in S\}. \tag{1.1}$$

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The perturbed optimization problem of this type was first presented and investigated by Baranger [1]. Since then, it has been studied extensively and applied to *optimal control* problems governed by partial differential equations; see, for example, [2, 3, 4, 5]. Generic results on the solution existence and/or well-posedness of perturbed optimization problems have been established in [6, 7, 8, 9, 10, 11, 12, 13, 14]. Apart from these studies, there is another main research stream, which focuses on differential stability of the optimal value function of $\min_J(x, S)$. The function is called *perturbed distance function*, denoted by $d_S^J : X \to \overline{\mathbb{R}}$, and given by

$$d_S^J(x) := \inf\{\|x - y\| + J(y) \ : \ y \in S\}, \quad x \in X. \tag{1.2}$$

In the case when $J(\cdot) \equiv 0$, the perturbed distance function $d_S^J(\cdot)$ is reduced to the well-known *distance function* $d_S : X \to \mathbb{R}$ defined by $d_S(x) := \inf\{\|x - y\| \ : \ y \in S\}$ for each $x \in X$. The latter is a backbone in the theory of optimization and variational analysis (see, e.g., the papers [15, 16, 17, 18, 19] for original results and the books [20, 21, 22, 23] for connections among different theories). It also plays an important role in the analysis of PDEs of Monge–Kantorovich type arising from problems in optimal transportation theory and shape optimization [24, 25]. In the other case when $S = X$, the *exact penalization* [26, Theorem 2.5], which plays a key role in algorithms for *convex composite optimization* [27] as a natural extension of the so-called *big-M* method of linear programming to nonlinear constrained programming can be seen as a perturbed distance function. In a similar manner, the *bounded approximants for monotone operators* using sequences of infimal convolutions of a function with $n\|\cdot\|$ (instead of $n\|\cdot\|^2$ as in Moreau–Yosida approximation method, and thus can work in non-reflexive Banach spaces) proposed by Fitzpatrick and Phelps [28] are other examples of functions in the form of (1.2) with $S = X$.

Various subdifferentials (including the subdifferential in the sense of convex analysis, the proximal subdifferential, the Fréchet subdifferential, the Fréchet type ε-subdifferential, the limiting/Mordukhovich subdifferential, as well as the Clarke subdifferential) of distance functions have been investigated in [29, 30, 31, 32] and the references therein. Likewise, results on lower or upper estimations, exact representations, and regularities for subdifferentials of perturbed distance functions have been obtained in [33, 34, 35, 36, 37]. Many of these results have been extended to the corresponding ones for subdifferentials of distance functions in Banach spaces. Meanwhile, several results have just been developed for perturbed distance functions in *Riemannian manifolds* in [38, 39], where the perturbed optimization problems were introduced and considered in a more general setting with Riemannian manifolds, instead of the setting with real Banach spaces.

We pay attention to *exact representations* for subdifferentials of the perturbed distance function $d_S^J(\cdot)$ given by (1.2) in the Banach space setting. The first exact formulas for subdifferentials of $d_S^J(\cdot)$ are due to Wang et al. in [33]. Herein, the reference points are assumed to be in a subset of S called the *target set* consisting of points that are solutions of the corresponding perturbed minimization problem (see, formula (3.8) below). In the convex case, when the input data S and $J(\cdot)$ are both convex, the function $d_S^J(\cdot)$ is convex. Thus, its subdifferential in the sense of convex analysis at a given point in the target set was considered and represented as *the intersection of the corresponding subdifferential of the function* $(J + \delta_S)(\cdot)$ *at that point and the closed unit ball in the topological dual space* (see formula (3.6) in [33] or (3.9) in this paper). Here, $(J + \delta_S)(\cdot)$ is the function that coincides with $J(\cdot)$ on S and takes the value $+\infty$ otherwise. Let us mention this type of representation as R_1*-type*. In the nonconvex case, representations for proximal and Fréchet subdifferentials of $d_S^J(\cdot)$ were given in R_1-type, under assumptions on the *well-posedness* (in the sense of Tykhonov) of the perturbed optimization problem and on the *center locally Lipschitz constant* of the input function at the reference point. Similar results were given by Nam in [37] for Fréchet and Hölder subdifferentials by an approach from infimal–convolution theory. The R_1-type representation for limiting subdifferential of $d_S^J(\cdot)$ was also obtained by Li et al. in [34], under an additional assumption that X *is finite-dimensional*.

Note that even if $J(\cdot)$ can be defined only on S, $d_S^J(\cdot)$ is well-defined on the whole space, not just on S; therefore, not just on the target set. So, formulas for computing subdifferentials of $d_S^J(\cdot)$ at points outside the target set are obviously desired. Unfortunately, R_1-type representations for subdifferentials of $d_S^J(\cdot)$ at points outside the target set are no longer true in general. For instance, if $J(\cdot)$ is nonexpansive on S, then the target set equals to S by definition. At points outside S, subdifferentials of $J+\delta_S$ are just empty sets; hence, so are the intersections in R_1-type representations. Meanwhile, subdifferentials of $d_S^J(\cdot)$ at those points are not always empty as can be shown by simple examples. This leads to the need of finding alternative representations, instead of the R_1-type one, for subdifferentials of $d_S^J(\cdot)$ at points outside the target set.

It turns out that for subdifferentials of $d_S^J(\cdot)$ at points outside the target set, though quite rich results on lower or upper estimations were obtained in [34, 35, 36], few exact representations can be found in the paper [35] by Li and Bounkhel. Herein, proximal and Fréchet subdifferentials of $d_S^J(\cdot)$ at points outside the target set were represented in terms of *enlargement sets* by distinguishing two prior assumptions on the input function. The first one is that $J(\cdot)$ *has a continuous extension on* X. In this situation, extra requirements are put on the well-posedness of the perturbed optimization problem, the center Lipschitz constant, and the second center Lipschitz constant of $J(\cdot)$ on enlargement sets at the reference point. The second one is that $J(\cdot)$ *is continuous on* S. In this situation, an additional assumption related to an asymptotic behavior of $J(\cdot)$ on S is needed. The obtained representations in [35] for proximal and Fréchet subdifferentials of $d_S^J(\cdot)$ at points outside the target set are sharp, though a bit complicated because the enlargement sets are not easy to describe directly from input data.

From the above observations, a natural question arises: *Is there any representation for subdifferentials of* $d_S^J(\cdot)$ *at a point that does not depend on the relative position between the reference point and the input set and that is described directly via the input data?* This paper aims at giving some answers to the question. Toward this aim, *we propose a new approach to the studying of subdifferentials of* $d_S^J(\cdot)$, the idea of which is as follows. The perturbed optimization problem $\min_J(x, S)$ is viewed as a parametric optimization problem with x playing the role of a parameter. Therefore, one can estimate or compute subdifferentials of its optimal value function $d_S^J(\cdot)$ via the differential information of the objective function and the constraint set. From (1.1), we notice that the objective function of $\min_J(x, S)$ is given as a sum of the norm and the input function. So, sum rules for computing subdifferentials in variational analysis should be exploited; hence, the subdifferential of the norm could be fully taken into account.

Note also that $\min_J(x, S)$ is well-defined in any normed space X (not necessarily Banach space). Thus, in this paper, we will just assume that $(X, \|\cdot\|)$ *is a normed space*. Besides, to avoid an overwhelming presentation of different subdifferentials, *we devote this paper for implementing the above-mentioned approach with a basic assumption on input data, that is, S and $J(\cdot)$ are both convex*. The first obtained result (Theorem 3.1 in Section 3) allows us to compute the subdifferential (in the sense of convex analysis) of $d_S^J(\cdot)$ at any given point, as long as its corresponding perturbed minimization problem has a solution. Apart from using directly information of input data as in the R_1-type, the new representation is established by utilizing the information of an arbitrary solution and of subdifferential of the norm. Interestingly, the R_1-type representation for the subdifferential of $d_S^J(\cdot)$ at points in the target set and formulas for the subdifferential of the distance function $d_S(\cdot)$ can be recovered as corollaries of this result. Besides, in a combination with a special property of solution sets, this new representation reveals a relation among subdifferentials of $d_S^J(\cdot)$ at points on the segment connecting the reference point and points in its corresponding solution set.

The rest of this paper is organized as follows. In Section 2, we recall some notations and state our assumptions. The results, together with illustrative examples, are included in Section 3. The approach for studying subdifferentials of perturbed distance functions is presented as a preliminary for the proof of the main result in Section 4. Some concluding remarks and open topics are discussed in Section 5.

2 Notations and assumptions

Throughout this paper, let $(X, \|\cdot\|)$ be a normed space with its topological dual space denoted by X^*. The notation $\langle x^*, x\rangle$ is used to indicate the value of a bounded linear functional $x^* \in X^*$ at a given point $x \in X$. The symbols $\mathbb{B}$ and $\mathbb{B}^*$ stand for the closed unit balls in X and X^*, respectively.

Let $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. One calls the sets $\operatorname{dom} f := \{x \in X \ : \ f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x, \lambda) \in X \times \mathbb{R} \ : \ \lambda \geq f(x)\}$ the *effective domain* and *epigraph* of f, respectively. The function f is said to be *lower semicontinuous* (resp., *convex*) if $\operatorname{epi} f$ is a closed (resp., convex) set in $X \times \mathbb{R}$. When f is a convex function, the *subdifferential* (in the sense of convex analysis) of f at a point $\bar{x}$ is the empty set if $\bar{x} \in X \setminus \operatorname{dom} f$ and is the set

$$\partial f(\bar{x}) := \{x^* \in X^* \ : \ \langle x^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x}), \quad \forall x \in X\}$$

if $\bar{x} \in \operatorname{dom} f$.

Let S be a nonempty subset of X. The set S is associated with an extended real-valued function on X by the so-called *indicator function* $\delta_S : X \to \overline{\mathbb{R}}$ of S, where $\delta_S(x) := 0$ if

x in S and $\delta_S(x) := +\infty$ otherwise. It is clear that $\delta_S(\cdot)$ is lower semicontinuous (resp., convex) iff S is closed (resp., convex). When S is a convex set, the *normal cone* to S at $\bar{x} \in S$ is defined by $N(\bar{x}, S) := \partial\delta_S(\bar{x})$. It is easy to verify that

$$N(\bar{x}, S) = \{x^* \in X^* \ : \ \langle x^*, x - \bar{x}\rangle \leq 0, \quad \forall x \in S\}.$$

Now, if a function $h : S \to \mathbb{R}$ is given only on S, it can be linked with the function $h + \delta_S : X \to \overline{\mathbb{R}}$ defined on the whole space X by

$$(h + \delta_S)(x) := \begin{cases} h(x), & \text{if } x \in S \\ +\infty, & \text{if } x \notin S. \end{cases}$$

In this situation, one says that h is *lower semicontinuous* (resp., *convex*) if $h + \delta_S$ is lower semicontinuous (resp., convex).

For a systematical treatment of convex analysis, the interested reader is referred to the books by Penot [22] or by Ioffe and Tihomirov [40], for example.

In this paper, *we consider the problem* $\min_J(x, S)$ *defined in* (1.1) *and the perturbed distance function* $d_S^J(\cdot)$ *given by* (1.2) *with the assumptions:* $(X, \|\cdot\|)$ *is a normed space,* $S \subset X$ *is a nonempty convex set, and* $J : S \to \mathbb{R}$ *is a convex function.* (The completeness of $(X, \|\cdot\|)$, the closedness of S and the lower semicontinuity of $J(\cdot)$, which were usually supposed in the literature, are not necessary herein.) By the convexity of S and $J(\cdot)$, it is not hard to see that the perturbed distance function $d_S^J : X \to \overline{\mathbb{R}}$ given by (1.2) is convex (see, e.g., the comments after Lemma 4.1). We aim at providing exact formulas for the subdifferential of the perturbed distance function $d_S^J(\cdot)$.

3 Results

The following theorem allows us to compute the subdifferential of $d_S^J(\cdot)$ at any point x in X, as long as its corresponding problem $\min_J(x, S)$ has a solution, i.e., the *solution set*

$$P_S^J(x) := \big\{y \in S \ : \ d_S^J(x) = \|x - y\| + J(y)\big\}$$

is nonempty.

Theorem 3.1. *Suppose that S is a nonempty convex subset of a normed space $(X, \|\cdot\|)$ and $J : S \to \mathbb{R}$ is a convex function. Let $\bar{x} \in X$ be such that $P_S^J(\bar{x}) \neq \emptyset$. Then, for any $\bar{y} \in P_S^J(\bar{x})$ one has*

$$\partial d_S^J(\bar{x}) = \partial(J + \delta_S)(\bar{y}) \cap \partial\|\cdot\|(\bar{x} - \bar{y}), \tag{3.3}$$

where

$$\partial\|\cdot\|(\bar{x} - \bar{y}) = \begin{cases} \mathbb{B}^*, & \textit{if } \bar{y} = \bar{x} \\ \{x^* \in X^* \ : \ \|x^*\| = 1, \langle x^*, \bar{x} - \bar{y}\rangle = \|\bar{x} - \bar{y}\|\}, & \textit{if } \bar{y} \neq \bar{x}. \end{cases} \tag{3.4}$$

The proof of Theorem 3.1 will be presented in the next section of this paper.

As we can see from formula (3.3), at a point $\bar{x}$ such that $P_S^J(\bar{x}) \neq \emptyset$, the subdifferential of $d_S^J(\cdot)$ is computed not only by exploiting the information of S and $J(\cdot)$, which are initial data of the problem, but also by taking into account the subdifferential of the norm appearing in the objective function of the problem. This makes the representation (3.3) become totally novel in the existing literature on subdifferentials of the perturbed distance function $d_S^J(\cdot)$.

Regarding the sets in the right-hand side of formula (3.4), the reader is referred to (says) [41, Section 4.6] for subdifferential of the norm in a normed space. Discussions on or detailed descriptions of subdifferentials of the norm in infinite-dimensional spaces can be found in [40, Subsection 4.4.3] for the case where the norm is differentiable at non-zero points, such as $L_p([t_0, t_1])$ with $1 < p < +\infty$ or Hilbert spaces and in [40, Subsection 4.5.1] for the spaces $C(T)$ and $L_p^n([t_0, t_1])$ with $p \in \{1, +\infty\}$ where the norm is non-differentiable at non-zero points. To keep it simple to verify, let us now present an illustrative example for Theorem 3.1 in the following setting.

Example 3.1. Consider the problem $\min_J(x, S)$ with $X := \mathbb{R}$, $S := [0, +\infty)$, $J(y) := y$ for all $y \in S$. For every $x \in X$, it is not hard to verify that

$$d_S^J(x) = |x| \quad \text{and} \quad P_S^J(x) = \begin{cases} [0, x], & \text{if } x > 0 \\ \{0\}, & \text{if } x \le 0. \end{cases} \tag{3.5}$$

The graphs of $d_S^J(\cdot)$ and $P_S^J(\cdot)$ are illustrated in Figure 1.

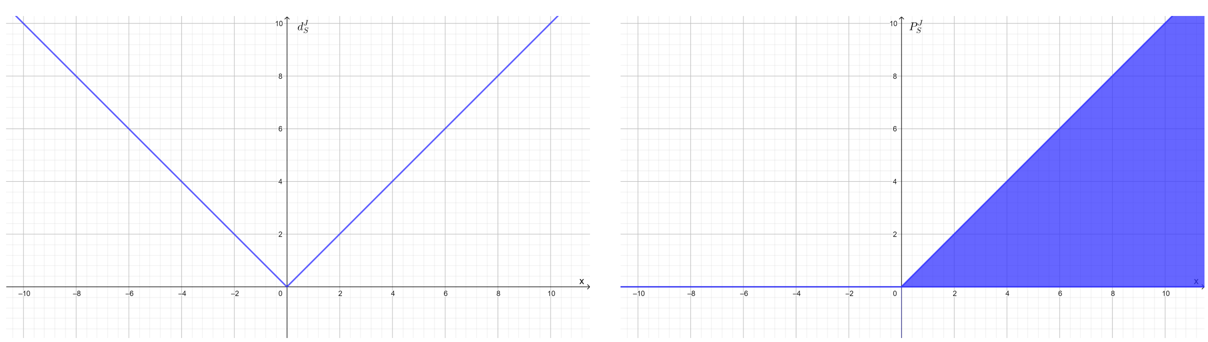

Figure 1: The perturbed distance function $d_S^J : \mathbb{R} \to \mathbb{R}$ and the perturbed projection $P_S^J : \mathbb{R} \rightrightarrows \mathbb{R}$ in (3.5).

By direct computing, we have

$$\partial d_S^J(x) = \begin{cases} \{1\}, & \text{if } x > 0 \\ [-1, 1], & \text{if } x = 0 \\ \{-1\}, & \text{if } x < 0, \end{cases} \tag{3.6}$$

$$\partial(J+\delta_S)(y) = \begin{cases} \{1\}, & \text{if } y>0 \\ (-\infty,1], & \text{if } y=0, \end{cases} \quad \text{and} \quad \partial\|\cdot\|(x-y) = \begin{cases} \{1\}, & \text{if } x>y \\ [-1,1], & \text{if } x=y \\ \{-1\}, & \text{if } x<y. \end{cases} \tag{3.7}$$

Fix an $\bar{x} \in X$ and take any $\bar{y} \in P_S^J(\bar{x})$. We are going to show that (3.3) holds.

Consider first the situation where $\bar{x} > 0$. Then, we have $P_S^J(\bar{x}) = [0, \bar{x}]$ by (3.5) and $\partial d_S^J(\bar{x}) = \{1\}$ by (3.6). The inclusion $\bar{y} \in [0,\bar{x}]$ is separated into two cases: $\bar{y} \in [0,\bar{x})$ and $\bar{y} = \bar{x}$. If $\bar{y} \in [0,\bar{x})$, then from (3.7) we get that $\partial(J+\delta_S)(\bar{y})$ equals to either $(-\infty,1]$ or $\{1\}$ while $\partial\|\cdot\|(\bar{x}-\bar{y}) = \{1\}$. Thus, $\partial(J+\delta_S)(\bar{y}) \cap \partial\|\cdot\|(\bar{x}-\bar{y}) = \{1\}$ and therefore (3.3) is valid. Similarly, if $\bar{y} = \bar{x}$, then it follows from (3.7) that $\partial(J+\delta_S)(\bar{y}) = \{1\}$ and $\partial\|\cdot\|(\bar{x}-\bar{y}) = [-1,1]$. So, $\partial(J+\delta_S)(\bar{y}) \cap \partial\|\cdot\|(\bar{x}-\bar{y}) = \{1\}$; hence (3.3) is fulfilled.

Next, in the situation where $\bar{x} = 0$, we have $P_S^J(\bar{x}) = \{0\}$ by (3.5). By using (3.6) and (3.7), we see that the equality (3.3) is valid with $\bar{y} = 0$, as both sides of the equality equal to $[-1,1]$.

The final situation is $\bar{x} < 0$, where we have $P_S^J(\bar{x}) = \{0\}$ by (3.5). Thanks to (3.6) and (3.7), one can similarly verify (3.3) with $\bar{y} = 0$. In this situation, both $d_S^J(\bar{x})$ and $\partial(J+\delta_S)(\bar{y}) \cap \partial\|\cdot\|(\bar{x}-\bar{y})$ equal to $\{-1\}$.

Note that the sets on the right-hand side of the equality in (3.3) do not depend on a particular choice $\bar{y}$ in $P_S^J(\bar{x})$. Thus, if $\bar{x}$ is a point in the *target set* (see, [11, 13])

$$S_0 := \{x \in S \ : \ x \in P_S^J(x)\}, \tag{3.8}$$

one can choose $\bar{y} := \bar{x}$ in (3.3) to obtain the representation for $\partial d_S^J(\bar{x})$ in the paper by Wang, Li, and Xu [33, Theorem 3.1], which we mentioned in the introduction as the R_1-*type representation*, as a corollary. Keep in mind that the completeness of $(X, \|\cdot\|)$, the closedness of S and the lower semicontinuity of $J(\cdot)$ required in [33] are not necessary herein.

Corollary 3.1. *If $\bar{x} \in S_0$, then*

$$\partial d_S^J(\bar{x}) = \partial(J+\delta_S)(\bar{x}) \cap I\!B^*. \tag{3.9}$$

Proof. If $\bar{x} \in S_0$, then $\bar{x} \in P_S^J(\bar{x})$. Applying Theorem 3.1 for $\bar{y} := \bar{x}$, we obtain (3.9). □

The next corollary gives formulas for computing subdifferential of the *distance function* $d_S(x) := \inf\{\|x-y\| : y \in S\}$, $x \in X$, as a special case of the perturbed distance function $d_S^J(\cdot)$ with $J(\cdot) \equiv 0$. These formulas can be found in [29, Theorem 1] under the additional assumption that S is a closed subset of a normed space X for the second formula and of a reflexive Banach space X for the first one, and in [42, Example 2.130] or in [33, Corollary 3.1] with the extra requirement that S is a closed subset of a Banach space X.

Corollary 3.2. *Let $\bar{x} \in X$ with $P_S(\bar{x}) := \{y \in S \ : \ d_S(\bar{x}) = \|\bar{x}-y\|\} \neq \emptyset$. Then for any $\bar{y} \in P_S(\bar{x})$, one has*

$$\partial d_S(\bar{x}) = N(\bar{y}, S) \cap \partial\|\cdot\|(\bar{x}-\bar{y}).$$

In particular, if $\bar{x} \in S$, then

$$\partial d_S(\bar{x}) = N(\bar{x}, S) \cap I\!B^*.$$

Proof. Note that when $J(\cdot) \equiv 0$, $(J+\delta_S)(\cdot) \equiv \delta_S(\cdot)$ and $S_0 = S$ by definitions. Therefore, the first claim of the corollary is straightforward from Theorem 3.1, while the second one follows directly from Corollary 3.1. □

Corollary 3.3. *Suppose that $S = X$. Let $\bar{x} \in X$ be such that $P_S^J(\bar{x}) \neq \emptyset$. Then for any $\bar{y} \in P_S^J(\bar{x})$, it holds that*

$$\partial d_S^J(\bar{x}) = \partial J(\bar{y}) \cap \partial \|\cdot\|(\bar{x} - \bar{y}). \tag{3.10}$$

Proof. It is clear that when $S = X$, one has $(J+\delta_S)(\cdot) \equiv J(\cdot)$. Thus, the desired formula comes instantly from Theorem 3.1 and Corollary 3.1. □

Let us present an illustrative example for Corollary 3.3.

Example 3.2. Consider the following nonlinear optimization problem

$$\min\left\{\sqrt{|x_1-y_1|^2+|x_2-y_2|^2}+|y_1-y_2| \ : \ (y_1,y_2)\in \mathbb{R}^2\right\}, \quad (x_1,x_2)\in\mathbb{R}^2.$$

Clearly, the above problem is in the form of $\min_J(x,S)$ in the Euclidean space $X := \mathbb{R}^2$ with $S := X$ and $J(y) := |y_1 - y_2|$ for all $y = (y_1, y_2) \in S$. It is not hard to see that $\bar{x} := (1, 0)$ is a point outside of the target set S_0. Besides, $\bar{y} := (\frac{1}{2}, \frac{1}{2}) \in P_S^J(\bar{x})$. Thus, we can compute the subdifferential $d_S^J(\bar{x})$ by (3.10) and get

$$d_S^J(\bar{x}) = ([-1,1]\times\{(1,-1)\}) \cap \left\{\frac{\sqrt{2}}{2}(1,-1)\right\} = \left\{(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\right\}.$$

The conclusions in the next proposition on a special property of the solution sets and of the perturbed distance function were obtained in [34, Proposition 4.1] for the problem $\min_J(x,S)$ under the assumptions that X is a *Banach* space, $S \subset X$ is a nonempty *closed* set, and $J : S \to \mathbb{R}$ is a *lower semicontinuous* function. However, we found from the proof therein that those assumptions are not necessary to get the same conclusions. For the sake of completeness, we include some details below.

Proposition 3.1. *Consider the problem $\min_J(x,S)$ defined by a nonempty (not necessary closed nor convex) subset S of a normed space X and a (not necessary lower semicontinuous nor convex) function $J : S \to \mathbb{R}$. Let $\bar{x} \in X$ be such that $P_S^J(\bar{x}) \neq \emptyset$ and $\bar{y} \in P_S^J(\bar{x})$. Then it holds for any $\lambda \in [0,1]$ that*

$$\bar{y} \in P_S^J(\lambda\bar{y} + (1-\lambda)\bar{x}) \tag{3.11}$$

and

$$d_S^J(\lambda\bar{y} + (1-\lambda)\bar{x}) = (1-\lambda)d_S^J(\bar{x}) + \lambda J(\bar{y}). \tag{3.12}$$

Proof. Let $\bar{x} \in X$ be such that $P_S^J(\bar{x}) \neq \emptyset$ and $\bar{y} \in P_S^J(\bar{x})$ and fix any $\lambda \in [0,1]$. Then, for any $y \in S$, using basic properties of the norm and the inclusion $\bar{y} \in P_S^J(\bar{x})$, we have

$$\begin{aligned}\|\lambda\bar{y} + (1-\lambda)\bar{x} - \bar{y}\| + J(\bar{y}) &= \|(1-\lambda)(\bar{x} - \bar{y})\| + J(\bar{y}) \\ &= d_S^J(\bar{x}) - \lambda\|\bar{y} - \bar{x}\| \\ &\leq \|\bar{x} - y\| + J(y) - \lambda\|\bar{y} - \bar{x}\| \\ &\leq \|\lambda\bar{y} + (1-\lambda)\bar{x} - y\| + J(y).\end{aligned}$$

Thus, we get the inclusion (3.11). As a consequence,

$$\begin{aligned}d_S^J(\lambda\bar{y} + (1-\lambda)\bar{x}) &= \|\lambda\bar{y} + (1-\lambda)\bar{x} - \bar{y}\| + J(\bar{y}) \\ &= \|(1-\lambda)(\bar{x} - \bar{y})\| + J(\bar{y}) \\ &= (1-\lambda)\|\bar{x} - \bar{y}\| + (1-\lambda)J(\bar{y}) + \lambda J(\bar{y}) \\ &= (1-\lambda)d_S^J(\bar{x}) + \lambda J(\bar{y}),\end{aligned}$$

which shows the equality (3.12) and completes the proof. □

Thanks to the above proposition, we are able to obtain a relationship among subdifferentials of $d_S^J(\cdot)$ at points on a segment connecting a reference point and points in its corresponding solution set in the next theorem.

Theorem 3.2. *Suppose that S is a nonempty convex subset of a normed space $(X, \|\cdot\|)$ and $J : S \to \mathbb{R}$ is a convex function. Let $\bar{x} \in X$ be such that $P_S^J(\bar{x}) \neq \emptyset$. Then, for any $\bar{y} \in P_S^J(\bar{x})$ and $\lambda \in [0,1)$, one has*

$$\partial d_S^J(\lambda\bar{y} + (1-\lambda)\bar{x}) = \partial d_S^J(\bar{x}). \tag{3.13}$$

Proof. Let $\bar{x} \in X$ be such that $P_S^J(\bar{x}) \neq \emptyset$. Fix any $\bar{y} \in P_S^J(\bar{x})$ and $\lambda \in [0,1)$. If $\bar{y} = \bar{x}$ or $\lambda = 0$, then (3.13) holds trivially. If $\bar{y} \neq \bar{x}$ and $\lambda \in (0,1)$, then by (3.4), we have

$$\partial\|\cdot\|((1-\lambda)(\bar{x} - \bar{y})) = \partial\|\cdot\|(\bar{x} - \bar{y}). \tag{3.14}$$

Besides, as $\bar{y} \in P_S^J(\bar{x})$ and $\lambda \in (0,1)$, it follows from (3.11) that $\bar{y} \in P_S^J(\lambda\bar{y} + (1-\lambda)\bar{x})$. Thus, applying Theorem 3.1 for $\bar{x} := \lambda\bar{y} + (1-\lambda)\bar{x}$, we have

$$\begin{aligned}\partial d_S^J(\lambda\bar{y} + (1-\lambda)\bar{x}) &= \partial(J + \delta_S)(\bar{y}) \cap \partial\|\cdot\|(\lambda\bar{y} + (1-\lambda)\bar{x} - \bar{y}) \\ &= \partial(J + \delta_S)(\bar{y}) \cap \partial\|\cdot\|((1-\lambda)(\bar{x} - \bar{y})).\end{aligned}$$

This and (3.14) imply that $\partial d_S^J(\lambda\bar{y} + (1-\lambda)\bar{x}) = \partial(J + \delta_S)(\bar{y}) \cap \partial\|\cdot\|(\bar{x} - \bar{y})$, which in a combination with (3.3) yields (3.13). The proof is complete. □

Let us verify Theorem 3.2 via the next example.

Example 3.3. Consider the problem $\min_J(x, S)$ with the same setting as in Example 3.1. Let $\bar{x} \in X$. From (3.5), we have $P_S^J(\bar{x}) = \{0\}$ if $\bar{x} \leq 0$ and $P_S^J(\bar{x}) = [0, \bar{x}]$ if $\bar{x} > 0$. As the equality (3.13) automatically holds when $P_S^J(\bar{x})$ is a singleton, i.e., when $\bar{x} \leq 0$, to illustrate Theorem 3.2, we only need to verify the equality (3.13) in the situation where $\bar{x} > 0$. Since $\bar{x} > 0$, it follows from (3.6) that $\partial d_S^J(\bar{x}) = \{1\}$. Taking any $\bar{y} \in P_S^J(\bar{x}) = [0, \bar{x}]$ and $\lambda \in [0, 1)$, one has $\lambda\bar{y} + (1 - \lambda)\bar{x} > 0$. Thus, (3.6) yields $\partial d_S^J(\lambda\bar{y} + (1 - \lambda)\bar{x}) = \{1\}$; hence (3.13) is verified.

We close this section by providing a proposition regarding properties of the solution set of the perturbed minimization problem, including the nonemptiness, which plays a key assumption for all the above results.

Proposition 3.2. *We have, for each $x \in X$, $P_S^J(x)$ is a convex set. Suppose, in addition, that X is a reflexive Banach space, S is closed, $J(\cdot)$ is lower semicontinuous, and either S is bounded or $J(\cdot)$ satisfies the coercivity condition*

$$\lim_{\substack{\|y\| \to +\infty \\ (y \in S)}} J(y) = +\infty. \tag{3.15}$$

Then $P_S^J(x)$ is nonempty.

Proof. Let $x \in X$. The fact that the solution set $P_S^J(x)$ of the problem $\min_J(x, S)$ is convex follows from the convexity of S and $J(\cdot)$. To see the nonemptiness, suppose that X is a reflexive Banach space. If $J(\cdot)$ is a lower semicontinuous and convex function on the nonempty, closed, and convex set S, then so is the function $y \mapsto f(y) := \|x - y\| + J(y)$, $y \in S$. Thus, when either S is bounded or $J(\cdot)$ satisfies (3.15), it follows from [43, Corollary 3.23] that $f(\cdot)$ attains minimum on S, i.e., the problem $\min_J(x, S)$ has a solution. □

4 Proof of the main result

To prove the exact formula for the subdifferential of the perturbed distance function $d_S^J(\cdot)$ in Theorem 3.1, we will first transform $\min_J(x, S)$ to a parametric minimization problem.

4.1 $\min_J(x, S)$ as a parametric minimization problem

Let φ_1, φ_2, and φ be functions from $X \times X$ to $\overline{\mathbb{R}}$ with

$$\varphi_1(x, y) := \|x - y\|, \quad (x, y) \in X \times X, \tag{4.16}$$

$$\varphi_2(x, y) := \begin{cases} J(y), & \text{if } (x, y) \in X \times S \\ +\infty, & \text{if } (x, y) \notin X \times S, \end{cases} \tag{4.17}$$

and

$$\varphi(x, y) := \varphi_1(x, y) + \varphi_2(x, y), \quad (x, y) \in X \times X. \tag{4.18}$$

Clearly, $\operatorname{dom}\varphi_1 = X \times X$, $\operatorname{dom}\varphi_2 = \operatorname{dom}\varphi = X \times S$, and

$$\varphi(x,y) = \begin{cases} \|x-y\| + J(y), & \text{if } (x,y) \in X \times S \\ +\infty, & \text{if } (x,y) \notin X \times S. \end{cases}$$

Besides, by the convexity of S and $J(\cdot)$, we see that φ_1, φ_2, and φ are convex functions. For each $x \in X$, consider the following unconstrained minimization problem

$$\min\{\varphi(x,y) \ : \ y \in X\}. \tag{P_x}$$

We see that (P_x) is a *parametric minimization problem* with x playing the role of the parameter. The *optimal value function* $\mu : X \to \overline{\mathbb{R}}$ and the *solution map* $M : X \rightrightarrows X$ of (P_x) are, respectively, given by

$$\mu(x) := \inf\{\varphi(x,y) \ : \ y \in X\}, \quad x \in X$$

and

$$M(x) := \{y \in X \ : \ \varphi(x,y) = \mu(x)\}, \quad x \in X.$$

The easy proof of the next lemma on the equivalence between the constrained optimization problem $\min_J(x,S)$ and the unconstrained one (P_x) is omitted. (See, e.g., [22, Excercise 2, p. 47].)

Lemma 4.1. *The problems* $\min_J(x,S)$ *and* (P_x) *are equivalent in the sense that, for each* $x \in X$*, one has* $d_S^J(x) = \mu(x)$ *and* $P_S^J(x) = M(x)$*.*

As φ is a convex function, [22, Lemma 1.51] yields that μ is a convex function. Hence, so is $d_S^J(\cdot)$ by Lemma 4.1. Moreover, due to considering $d_S^J(\cdot)$ in the role of the optimal value function of (P_x), we can now provide a rough estimate for subdifferential of $d_S^J(\cdot)$ via subdifferential of the objective function φ of (P_x). Similar approaches for parametric optimization problems under inclusion constraints can be found in [44, 45, 46].

Lemma 4.2. *Let* $\bar{x} \in X$ *be such that* $P_S^J(\bar{x}) \neq \emptyset$*. Then, for any* $\bar{y} \in P_S^J(\bar{x})$ *one has*

$$\partial d_S^J(\bar{x}) = \{x^* \in X^* \ : \ (x^*,0) \in \partial\varphi(\bar{x},\bar{y})\}. \tag{4.19}$$

Proof. Let $\bar{x} \in X$ be such that $P_S^J(\bar{x}) \neq \emptyset$ and fix any $\bar{y} \in P_S^J(\bar{x})$. By Lemma 4.1, one has $\bar{y} \in M(\bar{x})$. So, it follows from [22, Proposition 3.37] that

$$\partial\mu(\bar{x}) = \{x^* \in X^* \ : \ (x^*,0) \in \partial\varphi(\bar{x},\bar{y})\}.$$

This and the fact in Lemma 4.1 that $d_S^J(x) = \mu(x)$ for all $x \in X$ imply the desired formula (4.19). □

By (4.19), to represent the subdifferential of $d_S^J(\cdot)$ in terms of initial data of the problem $\min_J(x,S)$, we need to explore the relation $(x^*,0) \in \partial\varphi(\bar{x},\bar{y})$. As $\varphi(\cdot)$ is defined by the sum in (4.18), we will apply the sum rule from [22, Theorem 3.39] to compute subdifferential of $\varphi(\cdot)$ via subdifferentials of functions $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$, which are given in the next two lemmas.

Lemma 4.3. *Setting $A(x,y) = x - y$ for all $(x,y) \in X \times X$, one has*

$$\partial\varphi_1(\bar{x},\bar{y}) = A^*(\partial\|\cdot\|(\bar{x}-\bar{y})), \quad (\bar{x},\bar{y}) \in X \times X, \tag{4.20}$$

where $A^* : X^* \to X^* \times X^*$ stands for the adjoint operator of the continuous linear operator $A : X \times X \to X$.

Proof. Since $\varphi_1(x,y) = \|A(x,y)\|$ for all $(x,y) \in X \times X$, one has $\varphi_1 = \|\cdot\| \circ A$. Thus, applying the chain rule from [22, Theorem 3.40] for the linear operator $A : X \times X \to X$ and the convex function $\|\cdot\| : X \to \mathbb{R}$, we get formula (4.20). □

Lemma 4.4. *For any $(\bar{x},\bar{y}) \in X \times S$, one has*

$$\partial\varphi_2(\bar{x},\bar{y}) = \{0\} \times \partial(J+\delta_S)(\bar{y}). \tag{4.21}$$

Proof. It follows from (4.17) that $\varphi_2(x,y) = (J+\delta_S)(y)$ for all $(x,y) \in X \times X$. So, formula (4.21) is straightforward from the definition of subdifferential. □

Lemma 4.5. *Let $(\bar{x},\bar{y}) \in X \times S$. Then, for any $x^* \in X^*$, one has $(x^*,0) \in \partial\varphi(\bar{x},\bar{y})$ if and only if there exists $y^* \in \partial(J+\delta_S)(\bar{y})$ such that $(x^*,-y^*) \in A^*(\partial\|\cdot\|(\bar{x}-\bar{y}))$.*

Proof. Let $(\bar{x},\bar{y}) \in X \times S$. Then, $(\bar{x},\bar{y}) \in \operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2$. From formulas (4.16)–(4.18), we see that φ is the sum of two convex functions φ_1 and φ_2. Moreover, φ_1 and φ_2 are finite at $(\bar{x},\bar{y})$, and φ_1 is continuous on the whole space $X \times X$ containing $\operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2$. So, applying the sum rule from [22, Theorem 3.39] for the functions φ_1 and φ_2, we get

$$\partial\varphi(\bar{x},\bar{y}) = \partial\varphi_1(\bar{x},\bar{y}) + \partial\varphi_2(\bar{x},\bar{y}).$$

Combining this with (4.20) and (4.21) yields

$$\partial\varphi(\bar{x},\bar{y}) = A^*(\partial\|\cdot\|(\bar{x}-\bar{y})) + \{0\} \times \partial(J+\delta_S)(\bar{y}).$$

Thus, for any $x^* \in X^*$, $(x^*,0) \in \partial\varphi(\bar{x},\bar{y})$ if and only if there exist $(x_1^*,y_1^*) \in A^*(\partial\|\cdot\|(\bar{x}-\bar{y}))$ and $y^* \in \partial(J+\delta_S)(\bar{y})$ such that $(x^*,0) = (x_1^*,y_1^*)+(0,y^*)$. The latter equality yields $x_1^* = x^*$ and $y_1^* = -y^*$; hence the claim of the lemma is proved. □

We are now in a position to give a proof to Theorem 3.1.

4.2 Proof of Theorem 3.1

Proof of Theorem 3.1. Let $\bar{x} \in X$ be such that $P_S^J(\bar{x}) \neq \emptyset$ and fix a point $\bar{y} \in P_S^J(\bar{x})$. Since (3.4) directly follows from formulas for subdifferential of the norm function (see, e.g., [41, Section 4.6]), we only need to show that the equality in (3.3) holds. To do so, we will prove the validity of both inclusions $\subseteq$ and $\supseteq$.

[$\subseteq$] Take any $x^* \in \partial d_S^J(\bar{x})$. By (4.19), one has $(x^*,0) \in \partial\varphi(\bar{x},\bar{y})$. Hence, it follows from Lemma 4.5 that there exists $y^* \in \partial(J+\delta_S)(\bar{y})$ such that $(x^*,-y^*) \in A^*(\partial\|\cdot\|(\bar{x}-\bar{y}))$. So, there is some $z^* \in \partial\|\cdot\|(\bar{x}-\bar{y})$ satisfying $(x^*,-y^*) = A^*(z^*)$. Thus, using the definition of the adjoint operator A^*, we have

$$\langle (x^*,-y^*),(x,y)\rangle = \langle A^*(z^*),(x,y)\rangle = \langle z^*, A(x,y)\rangle$$

for all $(x,y) \in X \times X$. By the definition of A, the latter means that

$$\langle x^*, x\rangle - \langle y^*, y\rangle = \langle z^*, x-y\rangle, \quad \forall (x,y) \in X \times X.$$

This yields that $x^* = y^* = z^*$. Combining this and the properties that $y^* \in \partial(J+\delta_S)(\bar{y})$ and $z^* \in \partial\|\cdot\|(\bar{x}-\bar{y})$, we obtain $x^* \in \partial(J+\delta_S)(\bar{y}) \cap \partial\|\cdot\|(\bar{x}-\bar{y})$.

[$\supseteq$] Fix some $x^* \in \partial(J+\delta_S)(\bar{y}) \cap \partial\|\cdot\|(\bar{x}-\bar{y})$. Since $x^* \in \partial\|\cdot\|(\bar{x}-\bar{y})$, one has $A^*(x^*) \in A^*(\partial\|\cdot\|(\bar{x}-\bar{y}))$. Besides, using the definitions of A^* and A, one gets

$$\langle A^*(x^*),(x,y)\rangle = \langle x^*, A(x,y)\rangle = \langle x^*, x-y\rangle = \langle (x^*,-x^*),(x,y)\rangle$$

for all $(x,y) \in X \times X$. This means that $(x^*,-x^*) = A^*(x^*)$. Combining this with the fact that $A^*(x^*) \in A^*(\partial\|\cdot\|(\bar{x}-\bar{y}))$ yields $(x^*,-x^*) \in A^*(\partial\|\cdot\|(\bar{x}-\bar{y}))$. Hence, remembering that $x^* \in \partial(J+\delta_S)(\bar{y})$ and using Lemma 4.5, we obtain $(x^*,0) \in \partial\varphi(\bar{x},\bar{y})$. So, by (4.19), we have $x^* \in \partial d_S^J(\bar{x})$. The proof is complete. □

5 Conclusion and open topics

In this paper, we have proposed a new approach for studying subdifferentials of perturbed distance functions in normed spaces. The involved perturbed optimization problem is viewed as a parametric optimization problem of which the optimal value function represents the perturbed distance function under investigation. Hence, subdifferentials of the perturbed distance function can be estimated/computed directly via differential information of all input data, which are the constraint set, the perturbed function, and the norm. We have shown that this approach performed well for the convex case where both the constraint set and the perturbed function are convex. Namely, the main result (Theorem 3.1) allows us to compute the subdifferential (in the sense of convex analysis) of the perturbed distance function at any given point without depending on its relative position w.r.t. the so-called target set, as long as its corresponding perturbed minimization problem has a solution. As straightforward consequences, existing results on the subdifferential of the perturbed distance function at points in the target set and on the subdifferential of the well-known distance function can be recovered under milder assumptions. Last, but not least, in a combination with a special property of solution sets, the new representation in Theorem 3.1 reveals a special relation among subdifferentials of the perturbed distance function at points on the segment connecting the reference point and points in its corresponding solution set in Theorem 3.2.

The nice performance of the proposed approach to the convex case allows us to hope that similar results for estimating or computing the Fréchet/Mordukhovich/Clarke subdifferentials of the perturbed distance function in the nonconvex case can be attainable, and are therefore deserved for further investigation. Another interesting topic is the setting of the underlying space. In this paper we worked out our approach in the normed space setting; however, we do not know whether our approach is viable for the Riemannian manifold setting, as done in [38, 39]. This turns out to be an interesting open topic for future study.

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