Convex and Nonconvex Sweeping Processes with Velocity Constraints: well-posedness and insights

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Abstract

In this paper, we study some classes of sweeping processes with velocity constraints in the moving set. In addition to the solution existence and the solution uniqueness for the case of a moving convex constraint set, some results on the solution existence and the solution multiplicity where the constraint set is a finite union of disjoint convex sets are also obtained. Our main tool is a theorem on the solution sensitivity of parametric variational inequalities. Beside the traditional requirement that the constraint set moves continuously in the Hausdorff distance sense, we intensively use a new assumption on the local Lipschitz-likeness of the constraint set-valued mapping. The obtained results are compared with the existing ones and analyzed by several examples.

Keywords: Sweeping process, velocity constraint, local Lipschitz-likeness, Bochner integration, parametric variational inequality, uniform prox-regularity, proximal normal cone.

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1 Introduction

The notion of a sweeping process originates from the pioneering work of Jean-Jacques Moreau in the 1970s for the modeling of quasi-static evolution of elastoplastic systems in unilateral mechanics. He has written more than 25 papers devoted to the treatment of theoretical and numerical aspects of the sweeping process and its applications in unilateral mechanics [40–45]. Let $\mathcal{H}$ be a real Hilbert space and $C : [0, T] \rightrightarrows \mathcal{H},\ t \mapsto C(t) \subset \mathcal{H}$, be a set-valued mapping. Moreau's sweeping processes consist in finding a trajectory $t \in [0, T] \mapsto u(t) \in C(t)$ satisfying the following generalized Cauchy problem:

$$\text{(SWP)} \qquad \begin{cases} \dot{u}(t) \in -\mathcal{N}_{C(t)}(u(t)) & \text{a.e. } t \in [0, T] \\ u(0) = u_0 \in C(0), \end{cases}$$

where $\mathcal{N}_{C(t)}(u(t))$ denotes the normal cone (in the sense of convex analysis) associated to the moving nonempty convex and closed set $C(t)$ at the point $u(t)$. Translating the above inclusion to

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a mechanical language, we have the following interpretation in the case of quasi-static evolution systems (by neglecting any inertial effects):

- If the position $u(t)$ at the time t of a material point lies in the interior of the moving set $C(t)$, then the particle remains at rest, since the normal cone is reduced to the singleton $\{0\}$ in this case.

- If the material point is in contact with the boundary at a certain time t, then it is pushed in a normal inward direction by the boundary to stay inside the moving set and satisfies the viability constraint $u(t) \in C(t)$. This mechanical visualization leads Moreau to call this problem the sweeping process: the particle is swept by the moving set.

Several extensions of Moreau's sweeping process in diverse ways have been studied in the literature (see, e.g., [1, 31, 32, 34, 37, 56] and references therein). Recently, Krejčí, Monteiro, and Recupero [30] have obtained existence and uniqueness results for explicit and implicit nonconvex sweeping processes. The motion of the constraint set is separated into translation part and shape-change part. Rewriting the problems in terms of Kurzweil integral, the authors investigated the case where no compactification or other kinds of regularization are required.

Studied firstly by Siddiqi and Manchanda [52] and Bounkhel [12] in some simple forms, *sweeping processes with velocity constraint in moving sets* encompass a class of evolution variational inequalities, which have numerous applications in mechanics and physics (see [5, p. 8] and [23, Section 6.4]). Adopting a more general setting than the ones in [12, 52], Adly, Haddad and Thibault [5, Theorem 5.1] obtained a result on the solution existence of sweeping processes in separable Hilbert spaces with velocity in a moving bounded convex set. Afterwards, Adly and Le [6, Theorem 1] proved that a similar result can be established for the case where the moving set is unbounded and convex. In addition, by constructing an example (see [6, Example 1]), the authors showed that the sweeping process in question may not have solutions if one of the assumptions of the existence theorem is violated. Vilches and Nguyen [57, Section 5] have improved the result of [6] by weakening the continuity condition of the moving constraint set. The solution existence in [57] has been obtained by applying an existence result on evolution inclusions governed by time-dependent maximal monotone operators with a full domain.

The interested reader is referred to [6, pp. 840–842] for an application of the solution existence results to nonregular electrical circuits.

Adly and Haddad [3] have proved the equivalence between sweeping processes with velocity constraints and quasistatic evolution variational inequalities. In fact, convex implicit sweeping processes can be seen as the dual of a quasi-static evolution variational inequality involving positively homogeneous convex functionals. The result in [3] was extended by Migórski, Sofonea and Zend in [36] to nonlinear implicit sweeping processes by using a discrete approximation and a fixed-point argument for history-dependent operators. Focusing on the case of convex constraint sets (the convex case), Jourani and Vilches [27] have established the existence and uniqueness of the solution to the sweeping process in a very general framework by equivalently transforming the problem in question to an ordinary differential equation on a Hilbert space. The obtained results have been applied to quasistatic evolution variational inequalities and nonsmooth electrical circuits [27, Sections 7 and 8]. Among other things, the authors have shown [27, p. 5169] that one solution existence result in [12] can be proved by noting that the velocity vector at each time instance is uniquely defined as the projection of the origin of the Hilbert space on the moving constraint set. As a consequence, the corresponding results on the solution existence and uniqueness in [52], which are applicable to the case of moving convex constraint sets, also can be derived in this way.

Recently, Adly and Haddad [4] have obtained existence and uniqueness results for sweeping processes with velocity constraints in the convex case where the constraint set depends on both time and state.

Let $A_0, A_1 : \mathcal{H} \to \mathcal{H}$ be bounded symmetric linear operators and $f : [0,T] \to \mathcal{H}$ be a continuous mapping. Recall that a linear operator $A : \mathcal{H} \to \mathcal{H}$ is said to be symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}$. Following [5, 6], we consider the sweeping process

$$\begin{cases} A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{P}$$

where $\mathcal{N}^P_{C(t)}(\dot{u}(t))$ is the proximal normal cone (see, e.g., [13, p. 21] and Section 2 below) to $C(t)$ at $\dot{u}(t)$. An *absolutely continuous* function $u : [0,T] \to \mathcal{H}$ is said to be a *solution* of (P) if it satisfies the differential inclusion and the initial value condition in the formulation of the problem. Note that if $u : [0,T] \to \mathcal{H}$ is an absolutely continuous function, then u is Fréchet differentiable almost everywhere on $[0,T]$ with respect to the Lebesgue measure of the segment (see Subsection 2.1 below). Since every Lipschitz function $u : [0,T] \to \mathcal{H}$ is absolutely continuous, it is desirable to have sufficient conditions for (P) to have a Lipschitz solution.

For concrete examples of sweeping processes with velocity in a moving set we refer to [5, Examples 1 and 2] and [6, Example 1].

The solution existence theorem in [5, Theorem 5.1] for (P) was obtained under the following assumptions:

(a) *$C(t)$ is closed convex bounded for every $t \in [0,T]$;*

(b) *A_1 is positive semidefinite, i.e., $\langle A_1x, x\rangle \geq 0$ for all $x \in \mathcal{H}$.*

For the sweeping process (P), the authors of [6] showed that the next two assumptions guarantee the solution existence:

($\widetilde{\text{a}}$) *$C(t)$ is closed convex for every $t \in [0,T]$;*

($\widetilde{\text{b}}$) *A_1 is positive semidefinite and there exist positive constants α, β such that*

$$\langle A_1x, x\rangle \geq \alpha\|x\|^2 - \beta \ \text{ for all } \ x \in C(0).$$

It is worth to emphasize that the settings and results of [5, 6, 27, 57] require the separability of the Hilbert space $\mathcal{H}$.

As far as we know, nonconvex sweeping processes with velocity constraints have only been addressed by Bounkhel [12], who assumed that $A_0 \equiv 0$ (identically null), $A_1 = \text{Id}$ is the identity operator, and the sets $C(t)$ are uniformly prox-regular and contained in a convex compact set for all $t \in [0,T]$.

Our aim is to study the sweeping process (P) where $C(t)$ is not necessarily convex for every $t \in [0,T]$. Firstly, by using a result of Yen [59] on the solution sensitivity of parametric variational inequalities, we investigate (P) in the case where the set-valued mapping $t \mapsto C(t)$, $t \in [0,T]$, has nonempty closed convex values and is locally Lipschitz-like. Thanks to this approach, the vital requirements of the separability of $\mathcal{H}$ and of the linearity of the operator A_1 in most of the preceding works can be omitted. Note also that a locally Lipschitz-like set-valued mapping with nonempty closed convex values can be not continuous in the Hausdorff distance sense. Secondly, we obtain several solution existence results for the case where $C(t)$ is a finite union of disjoint convex sets.

Assuming that the operator A_0 in (P) is coercive and the constraint sets are convex, the authors in [5] have given a condition for the solution uniqueness. Herein, we will prove that (P) can have at most one solution if the operator A_1 is coercive. However, the coerciveness of both A_0 and A_1 does not imply the solution uniqueness of (P) even in the case of a fixed nonconvex constraint set, which is compact, uniformly prox-regular, and connected (see Remark 5.4 below). We think that the solution uniqueness of (P) deserves further investigations. Besides, due to the wide range of applications of (P), other properties of the solutions of that problem are also of great interest.

The present paper is organized as follows. Section 2 gives some preliminaries that we will use in the rest of the paper. Sweeping processes with convex constraint sets are discussed in Section 3. Three theorems on the solution existence of (P) in the nonconvex case are given in Section 4. Several illustrative examples are presented in Section 5. Among other things, Section 6 establishes some significant generalizations of two theorems from Section 3. These generalizations are mainly due to one of the two anonymous referees, who has solved two questions raised in the submitted version of our paper. We formulate, in Section 7, five open questions related to the results in the previous sections. Concluding remarks are given in Section 8. For the convenience of the reader, an appendix providing detailed proofs of the equivalences of three norms in the space $W^{1,\infty}((0,T),\mathcal{H})$ is included.

2 Preliminaries

By $\mathbb{N}$ we denote the set of positive integers. The notation $[a,b]$ (resp., (a,b)) stands for a closed interval (resp., an open interval) in the real line $\mathbb{R}$. Throughout this paper, let $\mathcal{H}$ be a real Hilbert space equipped with the norm $\|\cdot\|$ and the scalar product $\langle\cdot,\cdot\rangle$. The open ball (resp., closed ball) in $\mathcal{H}$ with center x and radius r is denoted by $\mathbb{B}_{\mathcal{H}}(x,r)$ (resp., $\bar{\mathbb{B}}_{\mathcal{H}}(x,r)$). If the space is itself clear by the context, we will omit the subscripts in these notations. The closure, the interior, the boundary, and the convex hull of a set $\Omega\subset\mathcal{H}$ are denoted respectively by $\mathrm{cl}(\Omega)$, $\mathrm{int}(\Omega)$, $\partial\Omega$, and $\mathrm{co}(\Omega)$. The distance from x to Ω is $d(x,\Omega):=\inf\limits_{y\in\Omega}\|x-y\|$. The *projection* of a point $x\in\mathcal{H}$ on Ω is defined by $\mathrm{proj}_\Omega(\mathrm{x})=\big\{\mathrm{y}\in\Omega\mid \mathrm{d}(\mathrm{x},\Omega)=\|\mathrm{x}-\mathrm{y}\|\big\}$. The *Hausdorff distance* between nonempty subsets Ω_1, Ω_2 of $\mathcal{H}$ is given by $d_H(\Omega_1,\Omega_2)=\max\left\{\sup\limits_{x\in\Omega_1} d(x,\Omega_2),\ \sup\limits_{y\in\Omega_2} d(y,\Omega_1)\right\}$. The Banach space of continuous functions from $[a,b]$ to $\mathcal{H}$ is denoted by $C^0([a,b],\mathcal{H})$ and its norm of uniform convergence is given by $\|x\|_{C^0}=\max\limits_{t\in[a,b]}\|x(t)\|$.

2.1 Notations Related to (P)

Definition 2.1. A function $x:[a,b]\to\mathcal{H}$ is said to be *absolutely continuous* on $[a,b]$ if, for any $\varepsilon>0$, there is $\delta>0$ such that $\sum\limits_{k=1}^{\ell}\|x(b_k)-x(a_k)\|<\varepsilon$ for every finite system of pairwise disjoint subintervals $(a_k,b_k)\subset[a,b]$, $k=1,\dots,\ell$, with the total length $\sum\limits_{k=1}^{\ell}(b_k-a_k)$ less than δ.

It is a well-known fact (see [21, Corollary 13 of Chapter 3, Theorem 2 on p. 107, and Section 6 of Chapter VII] or [10, Corollary 5.12 and Theorem 5.21]) that any absolutely continuous function $u : [0,T] \to \mathcal{H}$ is Fréchet differentiable almost everywhere on $[0,T]$ with respect to the Lebesgue measure of the segment.

Definition 2.2. (See, e.g., [13, p. 21]) The *proximal normal cone* $\mathcal{N}^P_\Omega(x)$ to $\Omega \subset \mathcal{H}$ at $x \in \Omega$ is defined by setting

$$\mathcal{N}^P_\Omega(x) = \{\xi \in \mathcal{H} \mid \exists \alpha > 0 \ \text{ such that } \ x \in \mathrm{proj}_\Omega(\mathrm{x} + \alpha\xi)\}.$$

Remark 2.3. (See, e.g., [18, Proposition 1.3]) Let $x \in \Omega \subset \mathcal{H}$ and $\xi \in \mathcal{N}^P_\Omega(x) \setminus \{0\}$. If α is a positive number such that $x \in \mathrm{proj}_\Omega(\mathrm{x} + \alpha\xi)$, then $x \in \mathrm{proj}_\Omega(\mathrm{x} + \mathrm{t}\xi)$ for every $t \in (0, \alpha)$.

Remark 2.4. (See, e.g., [18, Proposition 1.5]) Proximal normal cone is a local structure. Namely, for any $x \in \Omega \subset \mathcal{H}$ and $\rho > 0$, the proximal normal cone to $\Omega \subset \mathcal{H}$ at x coincides with the proximal normal cone to $\Omega \cap \bar{\mathbb{B}}(x,\rho)$ at x, i.e.,

$$\mathcal{N}^P_\Omega(x) = \mathcal{N}^P_{\Omega \cap \bar{\mathbb{B}}(x,\rho)}(x). \tag{2.1}$$

Definition 2.5. For some $r \in (0, +\infty]$, a nonempty closed set $\Omega \subset \mathcal{H}$ is called *r-prox-regular* (or uniformly prox-regular with radius r) if for all $x \in \Omega$, for all $t \in (0, r)$ and for all $\xi \in \mathcal{N}^P_\Omega(x)$ such that $\|\xi\| = 1$, one has $x \in \mathrm{proj}_\Omega(\mathrm{x} + \mathrm{t}\xi)$.

It is a simple matter to verify that every nonempty closed convex set is uniformly prox-regular with radius $r = +\infty$. According to [19, Proposition 7], if a nonempty closed set Ω is uniformly prox-regular, then $\mathcal{N}^P_\Omega(x) = \mathcal{N}^{Cl}_\Omega(x)$ with $\mathcal{N}^{Cl}_\Omega(x)$ being the Clarke normal cone to Ω at x. In particular, if Ω is a nonempty closed convex set, then $\mathcal{N}^P_\Omega(x)$ coincides with the normal cone $\mathcal{N}_\Omega(x)$ to Ω at x in the sense of convex analysis, i.e., $\mathcal{N}^P_\Omega(x) = \mathcal{N}_\Omega(x) := \{x^* \in \mathcal{H} \mid \langle x^*, y - x\rangle \le 0 \text{ for all } y \in \Omega\}$.

It is worth noting that the r-prox-regularity of Ω is equivalent to the hypomonotonicity of the truncated proximal normal cone, i.e., for all $x_1,\ x_2 \in \Omega$ and for all $\xi_i \in \mathcal{N}^P_\Omega(x_i) \cap \bar{\mathbb{B}}$, $i = 1, 2$, we have

$$\langle \xi_2 - \xi_1, x_2 - x_1\rangle \ge -\frac{1}{r}\|x_2 - x_1\|^2,$$

where $\bar{\mathbb{B}}$ denotes the closed unit ball in $\mathcal{H}$.

Let us also mention that if Ω is r-prox-regular, then the projection operator proj_Ω is well-defined (single-valued) and locally Lipschitz continuous on the r-open enlargement $\mathcal{U}_r(\Omega) := \{x \in \mathcal{H} \ : \ d(x,\Omega) < r\}$ of Ω.

The interested reader is referred to [7, 14, 19] for other properties, as well as various characterizations, of uniformly prox-regular sets.

Example 2.6. Let $\mathcal{H} = \mathbb{R}^2$, the set $\Omega = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \le x_1^2\right\}$ is unbounded, closed, nonconvex, and $\frac{1}{2}$-prox-regular. To prove the r-prox-regularity of Ω with $r = \frac{1}{2}$, observe by the closedness of Ω that the projection of any $u \in \mathbb{R}^2 \setminus \Omega$ on Ω exists and belongs to the boundary $\partial\Omega$. Let us set $f(x) = \|u - x\|^2$ and $g(x) = -x_1^2 + x_2$, and consider the following two-dimensional constrained optimization problem

$$\min\{f(x) \mid g(x) \le 0\}. \tag{2.2}$$

Since $\nabla g(x) = (-2x_1, 1)$ is nonzero for every $x \in \mathbb{R}^2$, there is some $v \in \mathbb{R}^2$ such that $\langle \nabla g(x), v\rangle < 0$. Applying the Lagrange multiplier rule (see [47, Theorem 1, p. 260] and [17]) to (2.2), one can prove that the problem has a unique solution x_u for each $u \in \mathbb{R}^2 \setminus \Omega$, i.e., $\mathrm{proj}_\Omega(\mathrm{u}) = \{\mathrm{x_u}\}$. Moreover, a careful analysis of the necessary optimality conditions given by the Lagrange multiplier rule shows that, for each $\bar{x} \in \partial\Omega \setminus \{(0,0)\}$, the equality $\bar{x} = \mathrm{proj}_\Omega(\bar{\mathrm{u}})$ holds for $\bar{u} \in \mathbb{R} \setminus \Omega$ if and only if $\bar{u} = \bar{x} + t\nabla g(\bar{x})$ with $t \in (0, \frac{1}{2})$. Therefore, we have $\mathcal{N}^P_\Omega(\bar{x}) = \mathbb{R}_+\nabla g(\bar{x})$ for every $\bar{x} \in \partial\Omega \setminus \{(0,0)\}$. For $\bar{x} \in (0,0)$, the equality $\bar{x} = \mathrm{proj}_\Omega(\bar{\mathrm{u}})$ holds for $\bar{u} \in \mathbb{R} \setminus \Omega$ if and only if $\bar{u} = \bar{x} + t\nabla g(\bar{x}) = (0,t)$ with $t \in (0,+\infty)$. Hence, $\mathcal{N}^P_\Omega((0,0)) = \{0\} \times \mathbb{R}_+$. To find a modulus $r > 0$ for the uniform prox-regularity of Ω, we can argue as follows. Fix a point $\bar{x} \in \partial\Omega \setminus \{(0,0)\}$ and let $\bar{u} = \bar{x} + \tau\nabla g(\bar{x})$ for some $\tau \in (0, \frac{1}{2})$. Since

$$\bar{u} - \bar{x} = \tau\|\nabla g(\bar{x})\| \frac{\nabla g(\bar{x})}{\|\nabla g(\bar{x})\|} = \tau\sqrt{4\bar{x}_1^2 + 1}\,\frac{\nabla g(\bar{x})}{\|\nabla g(\bar{x})\|},$$

for $\xi := \dfrac{\nabla g(\bar{x})}{\|\nabla g(\bar{x})\|}$ one has $\bar{x} \in \mathrm{proj}_\Omega(\bar{\mathrm{x}} + \mathrm{t}\xi)$ if and only if $t := \tau\sqrt{4\bar{x}_1^2+1}$ belongs to the interval $(0, \frac{1}{2}\sqrt{4\bar{x}_1^2+1})$. Clearly, the infimum of $\frac{1}{2}\sqrt{4\bar{x}_1^2+1}$ over the set $\bar{x}_1 \in \mathbb{R} \setminus \{0\}$ is $\frac{1}{2}$. In addition, at $\bar{x} \in (0,0)$, one has $\bar{x} = \mathrm{proj}_\Omega(\bar{\mathrm{x}} + \mathrm{t}(0,1))$ for all $t \in (0,+\infty)$. So, in agreement with Definition 2.5, we can conclude that $r := \frac{1}{2}$ is the best radius or modulus for the uniform prox-regularity of Ω.

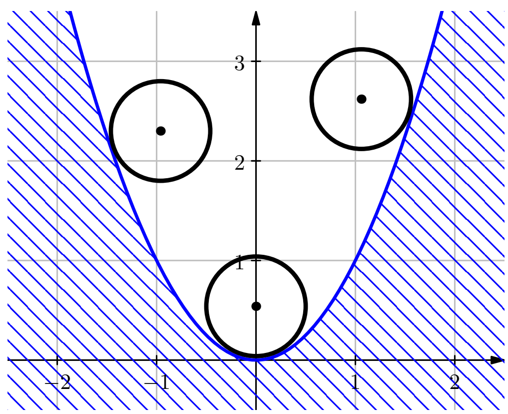

Figure 1: Illustration of Example 2.6

From the result established in Example 2.6 we get the following useful examples of uniformly prox-regular sets in spaces of higher dimensions.

Example 2.7. The set $\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_2 \le x_1^2\}$, where $n \ge 3$, is unbounded, closed, nonconvex, and $\frac{1}{2}$-prox-regular.

Example 2.8. The set $\{x = (x_1, x_2, x_3, \ldots) \in \ell^2 \mid x_2 \le x_1^2\}$ is unbounded, closed, nonconvex, and $\frac{1}{2}$-prox-regular.

Remark 2.9. Let I be a finite index set. The union Ω of disjoint nonempty closed convex subsets $\Omega_i \subset \mathcal{H}, i \in I$, is nonconvex if I has more than one element. If all the numbers $\alpha_{ij} := \inf\{\|x - y\| \mid$

$x \in \Omega_i,\ y \in \Omega_j\}$, with $i, j \in I$ and $i \neq j$, are positive, then Ω is uniformly prox-regular. More precisely, Ω is r-prox-regular, where $r > 0$ is any number satisfying the condition $r \leq \frac{1}{2}\alpha_{ij}$ for all $i, j \in I$ with $i \neq j$. In addition, Ω is not uniformly prox-regular if $\alpha_{ij} = 0$ for a pair $(i, j) \in I \times I$ with $i \neq j$. These assertions can be easily proved by using Definition 2.5 and the fact that the proximal normal cone coincides with the normal cone in the sense of convex analysis if the set under consideration is convex.

Remark 2.10. Closed and convex sets constitute an important class of sets in convex analysis and optimization. To go beyond convexity, the class of uniform prox-regular sets was introduced and shares with convex sets many nice properties (we refer to [7] for more details). The prox-regularity is known in the literature under different names (positively reached sets, weakly convex sets or proximally smooth sets). It plays an important role in the context of Moreau's sweeping processes. In fact, L. Thibault in [54] extended known Moreau's existence and uniqueness results for (SWP) to prox-regular sets (see also Edmond and Thibault [24]). The perturbed version of the dynamical system (SWP) with prox-regular sets $C(t)$ has been recently used by Maury and Venel [35] and Maury and Faure [33] for the modeling of crowd motion and the evacuation of individuals in case of an emergency situation (in both discrete and continuous dynamics).

2.2 The Bochner Integration

We now recall the definition of Bochner integral.

Definition 2.11. (See [21, pp. 44–45]) Let (Ω, Σ, μ) be a finite measurable space and X be a Banach space. A μ-measurable function $f : \Omega \to X$ is called *Bochner integrable* if there exists a sequence of simple functions $\{f_k\}$ such that

$$\lim_{k\to\infty} \int_\Omega \|f_k(\omega) - f(\omega)\|_X d\mu = 0.$$

In this case, $\int_E f(\omega)d\mu$ is defined for each $E \in \Sigma$ by $\int_E f(\omega)d\mu = \lim_{k\to\infty} \int_E f_k(\omega)d\mu$, where $\int_E f_k(\omega)d\mu$ is defined in an obvious way.

As noted in [21, p. 45], the limit in Definition 2.11 exists and is independent of the defining sequence $\{f_k\}$.

According to [21, Theorem 2, p. 45], a μ-measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f(\omega)\|_X d\mu < \infty$. If $1 \leq p < \infty$, the Bochner space $L^p(\Omega, X)$ consists of all μ-measurable functions $f : \Omega \to X$ satisfying

$$\|f\|_p = \left(\int_\Omega \|f(\omega)\|_X^p d\mu\right)^{1/p} < \infty$$

(see, e.g., [21, pp. 49–50]). For more details on Bochner integration, we refer to [61, p. 132], [21, Chapter II], and [15, p. 116].

Some useful facts on Bochner integration of absolutely continuous functions will be given in Section 3 (see Remark 3.9).

2.3 The Space $W^{1,\infty}((0,T),\mathcal{H})$

Following Cazenave and Haraux [16, Definition 1.4.33], by $W^{1,\infty}((0,T),\mathcal{H})$ we denote the space of (equivalent classes of) functions $f \in L^\infty((0,T),\mathcal{H})$ that $f' \in L^\infty((0,T),\mathcal{H})$, in the sense of $\mathcal{D}'((0,T),\mathcal{H})$. For $f \in W^{1,\infty}((0,T),\mathcal{H})$, we set $\|f\|_{W^{1,\infty}} = \|f\|_{L^\infty} + \|f'\|_{L^\infty}$. It is well known (see [16, Proposition 1.4.34]) that $\big(W^{1,\infty}((0,T),\mathcal{H}), \|.\|_{W^{1,\infty}}\big)$ is a Banach space. The just mentioned norm is equivalent to the norm $\|f\| = \|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \|\dot{f}(t)\|$, which in turn is equivalent to following one:

$$\|f\|_M = \|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \Big(e^{-Mt}\|\dot{f}(t)\|\Big) \tag{2.3}$$

with M being a positive constant. Detailed proofs of these useful facts can be found in Appendix of this paper.

2.4 Parametric Variational Inequalities

Let (M, d_M) and (Λ, d_Λ) be two metric spaces. Let $F : \mathcal{H} \times M \to \mathcal{H}$ be a vector-valued function, and $K : \Lambda \rightrightarrows \mathcal{H}$ be a set-valued map with nonempty closed convex values. For each pair of parameters $(\mu, \lambda) \in M \times \Lambda$, we consider the problem of finding a vector $x \in K(\lambda)$ such that

$$\langle F(x,\mu), y - x\rangle \geq 0 \quad \forall y \in K(\lambda), \tag{2.4}$$

which is a *parametric variational inequality* with a perturbed constraint set. We note that (2.4) can be rewritten as

$$0 \in F(x,\mu) + \mathcal{N}_{K(\lambda)}(x).$$

The *pseudo-Lipschitz property* of set-valued mappings introduced by Aubin [9, p. 98] is a crucial concept in set-valued and variational analysis. This property is also known under other names: the *Aubin continuity property* [22], the *sub-Lipschitzian property* [49], and the *Lipschitz-like property* [38]. Complete characterizations of the property can be found in [38, 39, 49, 50] and the references therein. For the study of the Aubin property to the solution map of a composite parametric variational systems using the coderivative approach and its applications in nonsmooth mechanics and nonregular electrical circuits, we refer to [2, 8].

Definition 2.12. (See [38, Definition 1.40] and [39, Definition 3.1]) K is said to be *Lipschitz-like* around $(\tilde{\lambda}, \tilde{x})$, where $\tilde{x} \in K(\tilde{\lambda})$, if there exist a neighborhood V of $\tilde{\lambda}$, a neighborhood W of $\tilde{x}$ and a constant $\kappa > 0$ such that

$$K(\lambda) \cap W \subset K(\lambda') + \kappa d_\Lambda(\lambda, \lambda')\bar{\mathbb{B}}(0,1), \quad \forall \lambda, \lambda' \in V.$$

Remark 2.13. If there exist a neighborhood V of $\tilde{\lambda}$ and a constant $\kappa > 0$ such that

$$K(\lambda) \subset K(\lambda') + \kappa d_\Lambda(\lambda, \lambda')\bar{\mathbb{B}}(0,1), \quad \forall \lambda, \lambda' \in V, \tag{2.5}$$

then one says that K is *locally Lipschitz* around $\tilde{\lambda}$. If the inclusion in (2.5) holds for some $\kappa > 0$ and for all $\lambda, \lambda' \in \Lambda$, then K is said to be a *Lipschitz* set-valued mapping. It is well known that if K is locally Lipschitz around $\tilde{\lambda}$, then K is Lipschitz-like around $(\tilde{\lambda}, \tilde{x})$ for every $\tilde{x} \in K(\tilde{\lambda})$. In particular, a Lipschitz set-valued mapping is Lipschitz-like around every point of its graph.

Consider the parametric variational inequality (2.4). Let $\bar{x}$ be a solution to it at given parameters $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$. We make two assumptions on the behavior of the function $F(x, \mu)$ around the point $(\bar{x}, \bar{\mu})$. Namely, we assume that there exist a closed convex neighborhood X of $\bar{x}$, a neighborhood U of $\bar{\mu}$, and two positive constants α, l such that

$$\|F(x', \mu') - F(x, \mu)\| \leq l(\|x' - x\| + d_M(\mu', \mu)), \quad \forall \mu, \mu' \in U, \; x, x' \in X, \tag{2.6}$$

and

$$\langle F(x', \mu) - F(x, \mu), x' - x \rangle \geq \alpha \|x' - x\|^2, \quad \forall \mu \in U, \; x, x' \in X. \tag{2.7}$$

The following result was originally stated in [59] in finite dimensional spaces. However, it is still valid for a general Hilbert space $\mathcal{H}$ and two metric spaces (M, d_M) and (Λ, d_Λ) of perturbation parameters (see [59, Remark 2.3] for more details).

Theorem 2.14. ([59, Theorem 2.1]) *Assume that $\bar{x}$ is a solution to* (2.4) *with respect to the given parameters $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$, conditions* (2.6) *and* (2.7) *hold, and the set-valued map $K : \Lambda \rightrightarrows \mathcal{H}$ has nonempty closed convex values and is Lipschitz-like around $(\bar{\lambda}, \bar{x})$. Then, there exist positive constants $\kappa_{\bar{u}}$ and $\kappa_{\bar{\lambda}}$, and neighborhoods $\tilde{U}$ of $\bar{\mu}$ and $\tilde{V}$ of $\bar{\lambda}$ such that*

(i) *For every $(\mu, \lambda) \in \tilde{U} \times \tilde{V}$, there exists a unique solution to* (2.4) *in X, denoted by $x(\mu, \lambda)$;*

(ii) *For all $(\mu', \lambda'), (\mu, \lambda) \in \tilde{U} \times \tilde{V}$, one has*

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq \kappa_{\bar{\mu}} d_M(\mu', \mu) + \kappa_{\bar{\lambda}} d_\Lambda(\lambda', \lambda)^{1/2}. \tag{2.8}$$

3 The Case of Convex Constraint Sets

For studying the problem (P), the next two assumptions were used in [5, 6].

Assumption (H1). *The constraint sets $C(t)$, $t \in [0, T]$, are nonempty, closed, and convex.*

Assumption (H2). *The set-valued mapping C is continuous in the Hausdorff distance sense, i.e., there exists a continuous function $g : [0, T] \to \mathbb{R}$ such that*

$$d_H(C(s), C(t)) \leq |g(s) - g(t)| \;\; \textit{for all } s, t \in [0, T]. \tag{3.1}$$

The results of Adly, Haddad, and Thibault [5] also require the following assumption.

Assumption (H3a). *The constraint set $C(0)$ is bounded.*

Later, to deal with possibly unbounded constraint sets, Adly and Le [6], have used the next semicoercivity assumption.

Assumption (H3b). *For the bounded symmetric linear operator $A_1 : \mathcal{H} \to \mathcal{H}$, there exist positive constants c_1, c_2 such that*

$$\langle A_1 x, x \rangle \geq c_1 \|x\|^2 - c_2, \quad \forall x \in C(0). \tag{3.2}$$

Remark 3.1. If (H3a) is satisfied, then there exist $c_1 > 0$ and $c_2 > 0$ such that (3.2) is fulfilled, i.e., (H3b) is also satisfied. Indeed, if $C(0)$ is bounded, then we can find $\rho > 0$ such that $C(0) \subset \rho\bar{\mathbb{B}}(0, 1)$. Since A_1 is bounded, we have $|\langle A_1 x, x \rangle| \leq \|A_1\| \|x\|^2$. Hence, the inequality $\langle A_1 x, x \rangle \geq$

$-\|A_1\|\rho^2$ holds for any $x \in \mathcal{H}$. If $A_1 = 0$, then by choosing $c_1 = \dfrac{1}{2}$ and $c_2 = \rho^2$ we get (3.2). If $A_1 \neq 0$, then we choose $c_1 = \|A_1\|$ and $c_2 = 2\|A_1\|\rho^2$. For any $x \in C(0)$, we have

$$c_1\|x\|^2 - c_2 \leq \|A_1\|(\|x\|^2 - \rho^2) - \|A_1\|\rho^2 \leq -\|A_1\|\rho^2 \leq \langle A_1x, x\rangle,$$

which justifies (3.2).

Remark 3.2. If (H2) and (H3b) are satisfied, then exist positive constants $\hat{c}_1, \hat{c}_2$ such that $\langle A_1x, x\rangle \geq \hat{c}_1\|x\|^2 - \hat{c}_2$ for all $t \in [0, T]$ and $x \in C(t)$. We omit the detailed proof, whose main idea is to construct a barrier $\gamma > 0$ and the positive constants $\hat{c}_1, \hat{c}_2$ such that for every $y \in C(t)$, $t \in [0, T]$, one can deduce the desired estimate $\langle A_1y, y\rangle \geq \hat{c}_1\|y\|^2 - \hat{c}_2$ from (3.2) if $\|y\| > \gamma$. If $\|y\| \leq \gamma$, then the estimate is valid thanks to the suitable choice of $\hat{c}_1$ and $\hat{c}_2$.

Remark 3.3. It can be shown that if the assumptions (H1) and (H2) are satisfied then the *recession cone* [48, pp. 61–63] of $C(t)$, which is denoted by $0^+C(t)$, is invariant with respect to t, i.e., $0^+C(t) = 0^+C(0)$ for every $t \in [0, T]$. Indeed, if $C(0)$ is bounded, then the assumption (H2) implies that $C(t)$ is bounded for every $t \in [0, T]$. It follows that $0^+C(t) = 0^+C(0) = \{0\}$ for every $t \in [0, T]$. If $C(0)$ is unbounded, then $C(t)$ is also unbounded for all $t \in [0, T]$. Fix any $t \in [0, T]$, let $x \in C(t)$ and $d \in 0^+C(t)\backslash\{0\}$ be chosen arbitrarily. Then, $x + \lambda d \in C(t)$ for all $\lambda \geq 0$. Let $\{\lambda_k\}_{k=0}^{\infty}$ be an increasing sequence of positive real numbers satisfying $\lim\limits_{k\to\infty} \lambda_k = \infty$. For each $k \geq 0$, by (H2) we have $x + \lambda_k d \in C(0) + |g(t) - g(0)|\bar{\mathbb{B}}$. So, there exist $y_k \in C(0)$ and $v_k \in |g(t) - g(0)|\bar{\mathbb{B}}$ such that $x + \lambda_k d = y_k + v_k$. Since $\{v_k\}$ is bounded, $d \neq 0$, and $\lim\limits_{k\to\infty} \lambda_k = \infty$, there exists an integer $\bar{k}$ such that $y_k \neq y_0$ for all $k \geq \bar{k}$. Then, by the boundedness of $\{v_k\}$ we have

$$\lim_{k\to\infty} \frac{y_k - y_0}{\|y_k - y_0\|} = \lim_{k\to\infty} \frac{v_0 - v_k + (\lambda_k - \lambda_0)d}{\|v_0 - v_k + (\lambda_k - \lambda_0)d\|} = \frac{d}{\|d\|}.$$

Since $C(0)$ is nonempty, closed, and convex, applying [25, Lemma 2.10] (the proof of that lemma works not only for closed convex sets in $\mathbb{R}^n$, but also for closed convex sets in any normed space), one obtains $\dfrac{d}{\|d\|} \in 0^+C(0)$, which implies that $d \in 0^+C(0)$. Hence, $0^+C(t) \subset 0^+C(0)$ for all $t \in [0, T]$. Arguing similarly, we can show that $0^+C(0) \subset 0^+C(t)$ for all $t \in [0, T]$. We have thus obtained the desired result.

The solution existence and solution uniqueness results of [5] for sweeping processes with velocity constraints of the form (P) can be stated as follows.

Theorem 3.4. (The moving constraint set is bounded and continuous in the Hausdorff distance sense; see [5, Theorems 5.1 and 5.2]) *Suppose that $\mathcal{H}$ is separable and A_0, A_1 are bounded positive semidefinite linear operators. If the assumptions* (H1), (H2), (H3a) *are satisfied, then* (P) *has at least one Lipschitz solution. If A_0 is coercive, i.e., there exists a constant $\alpha_0 > 0$ such that $\langle A_0x, x\rangle \geq \alpha_0\|x\|^2$ for all $x \in \mathcal{H}$, and* (H1) *is satisfied, then* (P) *has at most one solution.*

The above results of Adly, Haddad, and Thibault have been extended by Adly and Le [6] to the case of possibly unbounded closed convex sets $C(t)$, $t \in [0, T]$. In fact, there is no statement on solution uniqueness of (P) in [6] in the unbounded and semicoercive case. However, it is not difficult to see that the proof of Theorem 5.2 in [5] is also valid for the case of unbounded closed convex constraint sets.

Theorem 3.5. (The moving constraint set is continuous in the Hausdorff distance sense; cf. [6, Theorem 1]) *Suppose that $\mathcal{H}$ is separable and A_0, A_1 are positive semidefinite. If the assumptions* (H1)*,* (H2)*,* (H3b) *are satisfied, then* (P) *has at least one Lipschitz solution. If A_0 is coercive and* (H1) *is satisfied, then* (P) *has at most one solution.*

The separability of $\mathcal{H}$ and the continuity in the Hausdorff distance sense of the set-valued mapping C are vital assumptions in Theorems 3.4 and 3.5, which were proved by Moreau's time discretization techniques and the catching-up algorithm. Besides, as it has been noted in Remark 3.3, if (H1) and (H2) are satisfied then the recession cone $0^+C(t)$ of $C(t)$ is invariant with respect to t. By using the concept of parametric variational inequality and Theorem 2.1 from [59], which have been recalled in Section 2, we now establish a new result on the solution existence and solution uniqueness of (P). Here, *$\mathcal{H}$ can be a non-separable Hilbert space, the constraint set $C(t)$ can be unbounded, and the recession cone of $C(t)$ can vary when t changes in $[0,T]$ and the operator A_1 is allowed to be nonlinear.* More precisely, let us consider the following more general problem:

$$\begin{cases} A_1(\dot{u}(t)) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}(t)) & \text{a.e. } t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{$\tilde{\text{P}}$}$$

where $A_1 : \mathcal{H} \to \mathcal{H}$ is an operator (possibly nonlinear) satisfying the following strong monotonicity and Lipschitz continuity assumptions

Assumption (H3c). *There exist positive constants $\alpha_1 > 0, k_1 > 0$ such that*

$$\langle A_1(x') - A_1(x), x' - x\rangle \geq \alpha_1 \|x' - x\|^2, \tag{3.3}$$

$$\|A_1(x') - A_1(x)\| \leq k_1 \|x' - x\|, \ \ \forall x',\ x \in \mathcal{H}. \tag{3.4}$$

Applied to the set-valued mapping $C : [0,T] \rightrightarrows \mathcal{H}$ in the formulations of (P) and ($\tilde{\text{P}}$) around a point $(\bar{t}, \bar{x})$ in the graph of C, the notion of Lipschitz-like set-valued map recalled in Definition 2.12 means that the set $C(t) \cap W$ is close to $C(t')$ with t, t' from a neighborhood V of $\bar{t}$ and W being a neighborhood of $\bar{x}$. This is a natural requirement, because the velocity sets $C(t)$, $t \in [0,T]$, must have some continuity property.

Remark 3.6. Until now, the crucial assumption on sweeping process with velocity constraint of the form (P) is the continuity of the set-valued mapping $C : [0,T] \rightrightarrows \mathcal{H}$, which has been formulated as **(H2)**. The typical situation is that $C(t)$ is the solution set of an inequality and equality system depending on the parameter $t \in [0,T]$, say,

$$C(t) = \{x \in \mathcal{H} \mid g_i(x,t) \leq 0,\ i = 1, \ldots, m,\ h_j(x,t) = 0,\ j = 1, \ldots, s\},$$

where $g_i : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ and $h_j : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ are certain (smooth or nonsmooth) continuous functions such that $g_i(.,t),\ i = 1, \ldots, m$, are convex and $h_j(.,t) = 0,\ j = 1, \ldots, s$, are affine for each $t \in [0,T]$. As far as we know, there are no criteria to verify whether **(H2)** is satisfied, or not. Meanwhile, there are effective criteria for checking Lipschitz-like property of the set-valued mapping $C : [0,T] \rightrightarrows \mathcal{H}$ around every point in its graph. Such criteria can be found, e.g., in book of Aubin and Frankowska [11, Theorem 3.4.3], the book of Mordukhovich [38], the book of Rockafellar and Wets [50, Theorem 9.40], the papers by Dien and Yen [20], Yen [58, 60].

Theorem 3.7. (The moving constraint set is locally Lipschitz-like) *Let $\mathcal{H}$ be a Hilbert space, $A_1 : \mathcal{H} \to \mathcal{H}$ satisfying the assumptions* (3.3)–(3.4)*, and $f : [0,T] \to \mathcal{H}$ a continuous mapping.*

Assume that $C : [0,T] \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed convex values, which is Lipschitz-like around every point in its graph. Then $(\tilde{\mathrm{P}})$ has a unique solution u, which is a Lipschitz function. Moreover, the unique solution is a continuously differentiable function (provided that one identifies $\dot{u}(0)$ with the right derivative of u at 0 and $\dot{u}(T)$ with the left derivative of u at T).

Proof. Let us set $M = \mathcal{H}$, $\Lambda = [0,T]$, $F(x,\mu) = A_1(x) + \mu$ for $(x,\mu) \in \mathcal{H} \times M$, $K(\lambda) = C(\lambda)$ for $\lambda \in \Lambda$. Using (3.3)–(3.4) and choosing $X = \mathcal{H}$, $U = M$, and $l = \max\{k_1, 1\}$, we see that the conditions (2.6) and (2.7) are satisfied. For each pair $(\mu,\lambda) \in M \times \Lambda$, by the well-known solution existence theorem for strongly monotone variational inequality (see, e.g., Theorem 4.1 in [26], which has the origin in [28, Theorem 2.1, p. 24]) we know that (2.4) has a unique solution. The latter is denoted by $x(\mu,\lambda)$. For every $\lambda \in \Lambda$, we define a vector $\mu(\lambda) = -f(\lambda)$. Fix a value $\bar{\lambda} = \bar{t} \in [0,T]$ and let $\bar{\mu} = \mu(\bar{\lambda}) = -f(\bar{t})$, $\bar{x} = x(\bar{\mu},\bar{\lambda})$. Since the set-valued mapping $K(\cdot) = C(\cdot)$ is Lipschitz-like around $(\bar{\lambda},\bar{x})$, Theorem 2.14 asserts that there exist positive constants $\kappa_{\bar{u}}$ and $\kappa_{\bar{\lambda}}$, and neighborhoods $\tilde{U}$ of $\bar{\mu}$ and $\tilde{V}$ of $\bar{\lambda}$ such that the inequality (2.8) holds for all $(\mu',\lambda'), (\mu,\lambda) \in \tilde{U} \times \tilde{V}$. As $\tilde{U}$ is a neighborhood of $\bar{\mu} = \mu(\bar{\lambda}) = -f(\bar{t})$, $\mu(\lambda) = -f(\lambda)$, and $f(\cdot)$ is continuous at $\bar{t}$, we can find a neighborhood V_0 of $\bar{t}$ in $[0,T]$ such that $V_0 \subset \tilde{V}$ and $\mu(\lambda) \in \tilde{U}$ for all $\lambda = t$ with $t \in V_0$. Then, by (2.8) one has

$$\begin{aligned} \|x(\mu(t),t) - x(\mu(\bar{t}),\bar{t})\| &\le \kappa_{\bar{\mu}} \|\mu(t) - \mu(\bar{t})\| + \kappa_{\bar{\lambda}} |t - \bar{t}|^{1/2} \\ &= \kappa_{\bar{\mu}} \|f(t) - f(\bar{t})\| + \kappa_{\bar{\lambda}} |t - \bar{t}|^{1/2} \end{aligned}$$

for every $t \in V_0$. It follows that $\lim\limits_{t \to \bar{t}} \|x(\mu(t),t) - x(\mu(\bar{t}),\bar{t})\| = 0$. Therefore, the formula $z(t) = x(\mu(t),t)$ defines a continuous function $z : [0,T] \to \mathcal{H}$.

Summing up all the above, we can assert that, for every $t \in [0,T]$, the variational inequality (2.4) with the chosen function F, the set-valued mapping K, where $(\mu,\lambda) := (-f(t),t)$, has the unique solution $z(t)$, and the function $z(\cdot)$ is continuous on $[0,T]$. In particular, for every $t \in [0,T]$, one has

$$\langle A_1(z(t)) - f(t), y - z(t) \rangle \ge 0 \quad \forall y \in C(t).$$

Or equivalently,

$$A_1(z(t)) - f(t) \in -\mathcal{N}_{C(t)}(z(t)). \tag{3.5}$$

Conversely, since the inclusion $A_1(z) - f(t) \in -\mathcal{N}_{C(t)}(z)$ is equivalent to the condition

$$\langle A_1(z) - f(t), y - z \rangle \ge 0 \quad \forall y \in C(t),$$

one has $A_1(\dot{u}(t)) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}(t))$ if and only if $\dot{u}(t) = z(t)$. Since $z(\cdot)$ is continuous on $[0,T]$, the norm $\|z(t)\|$ is bounded for every $t \in [0,T]$. So, the Lebesgue integral $\displaystyle\int_0^T \|z(\tau)\| d\tau$ exists. By [21, Theorem 2, p. 45], z is Bochner integrable over the interval $[0,T]$ with respect to the Lebesgue measure. Setting

$$u(t) = u_0 + \int_0^t z(\tau) d\tau \quad (\forall t \in [0,T]), \tag{3.6}$$

we have $\dot{u}(t) = z(t)$ for all $t \in [0,T]$. Indeed, applying Theorem 9, p. 49, from [21] and the arguments in its proof (recalling that the Lebesgue integral of a continuous real-valued function coincides with the Riemann integral [29, Theorem 1, p. 368]), for all $t \in (0,T)$, the limit

$\lim_{h\to 0}\left[\frac{1}{h}\int_t^{t+h} z(\tau)d\tau\right]$ exists and it is equal to $z(t)$. So, from the relation $\lim_{h\to 0}\frac{u(t+h)-u(t)}{h} = \lim_{h\to 0}\left[\frac{1}{h}\int_t^{t+h} z(\tau)d\tau\right]$ it follows that, for all $t \in (0, T)$, the derivative $\dot{u}(t)$ exists and one has $\dot{u}(t) = z(t)$. Moreover, for any $t, s \in [0, T]$ with $s \le t$,

$$\begin{aligned}\|u(t)-u(s)\| = \left\|\int_0^t z(\tau)d\tau - \int_0^s z(\tau)d\tau\right\| &\le \int_s^t \|z(\tau)\|d\tau \\ &\le \max\{\|z(\tau)\| \mid \tau \in [0,T]\}(t-s).\end{aligned}$$

Thus, this function u is Lipschitz continuous with the rank $L = \max_{\tau\in[0,T]} \|z(\tau)\|$. The fulfillment of (3.5) for all $t \in [0, T]$ and the equality $u(0) = u_0$ assure that u is a Lipschitz solution of $(\tilde{\mathrm{P}})$. It remains to prove that $u(\cdot)$ is the unique solution of $(\tilde{\mathrm{P}})$. Arguing by contradiction, suppose that $(\tilde{\mathrm{P}})$ has another solution $v(\cdot)$ for which there is $\bar{t} \in [0, T]$ such that $v(\bar{t}) \neq u(\bar{t})$. Set $w(t) = v(t) - u(t)$ for all $t \in [0, T]$. Clearly, w is absolutely continuous on $[0, T]$ and $w(0) = 0$. Since $\dot{v}(t) = z(t)$ for almost every $t \in [0, T]$, we have $\dot{w}(t) = \dot{v}(t) - \dot{u}(t) = 0$ for almost every $t \in [0, T]$. As $w(\bar{t}) \neq 0$, there exists $x^* \in \mathcal{H}$ such that $\langle x^*, w(\bar{t})\rangle > 0$. Consider the function $\varphi(t) := \langle x^*, w(t)\rangle$. Note that φ is absolutely continuous on $[0, T]$, $\varphi(0) = 0$, and $\dot{\varphi}(t) = \langle x^*, \dot{w}(t)\rangle = 0$ for almost every $t \in [0, T]$. Applying [29, Theorem 6, p. 40] for the scalar function φ, one has $\varphi(t) = \varphi(0) + \int_0^t \dot{\varphi}(\tau)d\tau = 0$ for each $t \in [0, T]$. In particular, $\varphi(\bar{t}) = 0$. Hence, one gets $\langle x^*, w(\bar{t})\rangle = 0$, which is a contradiction. We have thus established the solution uniqueness of $(\tilde{\mathrm{P}})$. So, formula (3.6) defines the unique solution of $(\tilde{\mathrm{P}})$, which is a Lipschitz function on $[0, T]$. Moreover, the unique solution is a continuously differentiable function. The proof is thereby completed. □

When the operator $A_1 : \mathcal{H} \to \mathcal{H}$ is assumed to be linear we have the following direct consequence from Theorem 3.7.

Corollary 3.8. (The moving constraint set is locally Lipschitz-like) *Let $\mathcal{H}$ be a Hilbert space, $A_0 = 0$, $A_1 : \mathcal{H} \to \mathcal{H}$ a symmetric coercive bounded linear operator, and $f : [0, T] \to \mathcal{H}$ a continuous mapping. Assume that $C : [0, T] \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed convex values, which is Lipschitz-like around every point in its graph. Then* (P) *has a unique solution u, which is a Lipschitz function. Moreover, the unique solution is a continuously differentiable function.*

Remark 3.9. By the arguments in the final part of the above proof, we obtain the following useful facts on the Bochner integration:

(a) If $z : [0, T] \to X$, where X is a Banach space, is a continuous function, then the formula $u(t) = u_0 + \int_0^t z(\tau)d\tau$ defines a continuously differentiable function $u : [0, T] \to X$ and we have $\dot{u}(t) = z(t)$ for all $t \in [0, T]$.

(b) Let $u, v : [0, T] \to X$, where X is a reflexive Banach space, be absolutely continuous functions. If $u(0) = v(0)$ and $\dot{u}(t) = \dot{v}(t)$ for a.e. $t \in [0, T]$, then $u(t) = v(t)$ for all $t \in [0, T]$.

(c) (See the proof of Theorem 2 on p. 107 in [21]) Let $u : [0, T] \to X$, where X is a reflexive Banach space, be an absolutely continuous function. Then,

$$u(t) = u_0 + \int_0^t \dot{u}(\tau)d\tau \quad (\forall t \in [0, T]).$$

(d) If $z : [0,T] \to X$, where X is a Banach space, is a Bochner integrable function with respect to the Lebesgue measure, then the formula $u(t) = u_0 + \displaystyle\int_0^t z(\tau)d\tau$ defines a function $u : [0,T] \to X$, which is Fréchet differentiable a.e. on $[0,T]$ and we have $\dot{u}(t) = z(t)$ for a.e. $t \in [0,T]$.

To prove (c), it suffices to put $v(t) = u_0 + \displaystyle\int_0^t \dot{u}(\tau)d\tau$ for $t \in [0,T]$, and apply the assertion (b). The fact that the function $\dot{u}(\cdot)$ is Bochner integrable on $[0,T]$ is shown with detailed explanations in the proof of [21, Theorem 2, p. 107]. The assertion (d) follows from [21, Theorem 9, p. 49] which asserts that, under the assumptions made, $\displaystyle\lim_{h\to 0}\left[\frac{1}{h}\int_t^{t+h} z(\tau)d\tau\right] = z(t)$.

For any Hilbert space $\mathcal{H}$ of dimension greater or equal 2, there exist set-valued mappings $C : \mathbb{R} \rightrightarrows \mathcal{H}$ with nonempty closed convex values, Lipschitz-like around every point in their graphs, which are *not* continuous in the Hausdorff distance sense on any interval $[a,b] \subset \mathbb{R}$, where $a < b$. The forthcoming example justifies our observation.

Example 3.10. Let $\mathcal{H} = \mathbb{R}^2$, $\Lambda = \mathbb{R}$, and

$$K(\lambda) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \lambda x_1\right\} \quad (\forall \lambda \in \mathbb{R}).$$

For any $\bar{\lambda} \in \Lambda$ and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in K(\bar{\lambda})$, K is Lipschitz-like around $(\bar{\lambda}, \bar{x})$. This assertion can be verified by using a formula for computing the *limiting normal cone*, the notion of *coderivative*, and *the Mordukhovich criterion* (see, e.g., [38, Theorems 1.17 and Theorem 4.10]) as follows. (The related notations and definitions can be easily found in [38].) First, note that the graph of the set-valued mapping $K : \mathbb{R} \rightrightarrows \mathbb{R}^2$ coincides with the solution set of an equation given by a continuously differentiable function, namely $\operatorname{gph} K = \left\{z = (\lambda, x_1, x_2) \in \mathbb{R}^3 \mid f(z) = 0\right\}$, where $f(z) := x_2 - \lambda x_1$ for all $z = (\lambda, x_1, x_2) \in \mathbb{R}^3$. For $\bar{z} := (\bar{\lambda}, \bar{x}_1, \bar{x}_2)$, since $\nabla f(\bar{z}) = (-\bar{x}_1, -\bar{\lambda}, 1)$, the derivative $\nabla f(\bar{z}) : \mathbb{R}^3 \to \mathbb{R}$ is surjective. Therefore, applying Theorems 1.17 from [38] to f, $\bar{z}$, and $\Theta := \{0\} \subset \mathbb{R}$, we have

$$N(\bar{z}; \operatorname{gph} K) = N(\bar{z}; f^{-1}(\Theta)) = \nabla f(\bar{z})^*(N(f(\bar{z}); \Theta)).$$

As $N(f(\bar{z}); \Theta) = N(0; \Theta) = \mathbb{R}$, we get

$$N(\bar{z}; \operatorname{gph} K) = \left\{(-\mu\bar{x}_1, -\mu\bar{\lambda}, \mu) \mid \mu \in \mathbb{R}\right\}.$$

Next, by the definition of mixed coderivative, which coincides with the normal coderivative because all the spaces in question are finite-dimensional, we have

$$\begin{aligned} D_M^*K(\bar{z})(0) &= D_N^*K(\bar{z})(0) \\ &= \{\xi \in \mathbb{R} \mid (\xi, 0) \in N(\bar{z}; \operatorname{gph} K)\} \\ &= \left\{\xi \in \mathbb{R} \mid \exists \mu \in \mathbb{R} \ \text{ s.t. } -\mu\bar{x}_1 = \xi,\ -\mu\bar{\lambda} = 0,\ \mu = 0\right\} \\ &= \{0\}. \end{aligned}$$

Finally, it remains to apply the equivalence (a) $\Leftrightarrow$ (c) in [38, Theorem 4.10] to conclude that K is Lipschitz-like around $(\bar{\lambda}, \bar{x})$.

It is well known that any Hilbert space $\mathcal{H}$ of dimension greater or equal 2 admits the representation $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$ and $\mathcal{H}_1$ are orthogonal subspaces, and $\dim(\mathcal{H}_0) = 2$. Fixing a coordinate system in $\mathcal{H}_0$, we can identify $\mathcal{H}_0$ with $\mathbb{R}^2$. Define a set-valued mapping $C : \mathbb{R} \rightrightarrows \mathcal{H}$ by setting $C(t) = K(t) \oplus \mathcal{H}_1$ for all $t \in \mathbb{R}$. Then, from the above analysis it follows that C has nonempty closed convex values, and C is Lipschitz-like around every point in its graphs. For any interval $[a, b] \subset \mathbb{R}$, where $a < b$, C is not continuous in the Hausdorff distance sense on $[a, b]$. Indeed, one has $0^+C(t) = C(t)$ for every $t \in [a, b]$ and $C(t) \neq C(t')$ for any $t, t' \in [a, b]$ with $t' \neq t$. Hence the condition $0^+C(t) = 0^+C(a)$ for every $t \in [a, b]$, which is necessary for the continuity of C in the Hausdorff distance sense on $[a, b]$, is violated (see Remark 3.3).

Remark 3.11. Example 3.10 shows that the Lipschitz-like property is easily verifiable by using the Mordukhovich criterion, which fully characterizes the Lipschitz-like property of set-valued mappings with closed graphs. One can also use *the Robinson regularity condition* for an inequality and equality system described by smooth functions or its extensions which work for generalized inequality systems given by nonsmooth functions (see, e.g., the papers [20, 58, 60]).

The next example is designed to show how Theorem 3.7 and Corollary 3.8 can be used for solving concrete problems.

Example 3.12. Consider the sweeping process (P) with $\mathcal{H} = \mathbb{R}^2$, $T = 1$, $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f(t) = \begin{pmatrix} 1 + \sqrt{t} \\ t\sqrt{t} \end{pmatrix}$, and $u_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let $C(t) = K(t)$ with K being the set-valued mapping defined in Example 3.10. For each $t \in [0, 1]$, since $C(t)$ is the straight line $tx_1 - x_2 = 0$, one has $\mathcal{N}^P_{C(t)}(\dot{u}(t)) = \mathbb{R}\begin{pmatrix} t \\ -1 \end{pmatrix}$. Then, (P) is equivalent to the following initial value problem for an ordinary differential equation:

$$\begin{cases} \dot{u}(t) = \mathbb{P}_{C(t)}(f(t)), \\ u(0) = (0, 0). \end{cases} \tag{3.7}$$

As shown in Example 3.10, C is Lipschitz-like around every point in its graph. So, all the assumptions of Corollary 3.8 are satisfied and, by that theorem, problem (3.7) has a unique solution $u(\cdot) : [0, 1] \to \mathbb{R}^2$, which is a continuously differentiable function. To find an explicit formula for $u(t)$, we observe from the proof of Theorem 3.7 that $\dot{u}(t) = z(t)$ for all $t \in [0, 1]$, where $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$ is the unique solution of the parametric variational inequality

$$\langle A_1 z(t) - f(t), y - z(t) \rangle \geq 0 \quad \forall y \in C(t).$$

The latter is equivalent to $z(t) + \mathcal{N}_{C(t)}(z(t)) \ni f(t)$. This means $z(t) = \mathbb{P}_{C(t)}(f(t))$. A simple computation gives

$$z(t) = \begin{pmatrix} 1 + \sqrt{t} - \dfrac{t^2}{1 + t^2} \\ t\sqrt{t} + \dfrac{t}{1 + t^2} \end{pmatrix} = \begin{pmatrix} \sqrt{t} + \dfrac{1}{1 + t^2} \\ t\sqrt{t} + \dfrac{t}{1 + t^2} \end{pmatrix}$$

for all $t \in [0, 1]$. Using Remark 3.9(c), we have $u(t) = u_0 + \displaystyle\int_0^t z(\tau)d\tau$ for each $t \in [0, 1]$. Therefore,

$$u(t) = \begin{pmatrix} \dfrac{2}{3}t\sqrt{t} + \arctan t \\ \dfrac{2}{5}t^2\sqrt{t} + \dfrac{1}{2}\ln(1 + t^2) \end{pmatrix} \qquad (t \in [0, 1]),$$

a continuously differentiable function on $[0, 1]$, is the unique solution of (3.7).

The solution uniqueness result established in Theorem 3.7 is new, because the operator $A_0 = 0$ is positive semidefinite, but not coercive and the operator A_1 is allowed to be nonlinear. Thus, in some sense, our result complements those given in Theorem 3.4 and 3.5. A natural question arises: *Whether the coerciveness of A_1 also guarantees the solution uniqueness of* (P) *in the case where $A_0 \neq 0$?* The following theorem, whose proof is based on some ideas of [5], solves this question in the affirmative.

Theorem 3.13. *If $C(t)$ is nonempty and convex for every $t \in [0, T]$, A_1 is coercive, and A_0 is positive semidefinite, then* (P) *can have at most one solution.*

Proof. Suppose that $u(\cdot)$ and $v(\cdot)$ are two solutions of (P), where $C(t)$ is nonempty and convex for every $t \in [0, T]$, A_1 is coercive, and A_0 is positive semidefinite. Then $u, v : [0, T] \to \mathcal{H}$ are absolutely continuous functions, $u(0) = v(0) = u_0$,

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z\rangle \leq 0 \quad \forall z \in C(t) \tag{3.8}$$

and

$$\langle A_1\dot{v}(t) + A_0v(t) - f(t), \dot{v}(t) - z\rangle \leq 0 \quad \forall z \in C(t) \tag{3.9}$$

for a.e. $t \in [0, T]$. Since $\dot{u}(t)$ and $\dot{v}(t)$ belong to $C(t)$ for almost every $t \in [0, T]$, substituting $z = \dot{v}(t)$ to the inequality in (3.8) and $z = \dot{u}(t)$ to the inequality in (3.9) yields

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - \dot{v}(t)\rangle \leq 0 \;\text{ and }\; \langle A_1\dot{v}(t) + A_0v(t) - f(t), \dot{v}(t) - \dot{u}(t)\rangle \leq 0$$

for almost every $t \in [0, T]$. Adding the last inequalities gives

$$\langle A_1(\dot{u}(t) - \dot{v}(t)) + A_0(u(t) - v(t)), \dot{u}(t) - \dot{v}(t)\rangle \leq 0 \tag{3.10}$$

for almost every $t \in [0, T]$. Since A_1 is coercive, there exists $\alpha_1 > 0$ such that $\langle A_1x, x\rangle \geq \alpha_1\|x\|^2$ for all $x \in \mathcal{H}$. Thus, (3.10) implies that

$$\alpha_1\|\dot{u}(t) - \dot{v}(t)\|^2 + \langle A_0(u(t) - v(t)), \dot{u}(t) - \dot{v}(t)\rangle \leq 0 \qquad \text{a.e. } t \in [0, T]. \tag{3.11}$$

Taking the Lebesgue integral of both sides of (3.11) and applying [51, Remarks 11.23(c)], we obtain

$$\int_0^T \alpha_1\|\dot{u}(\tau) - \dot{v}(\tau)\|^2 d\tau + \int_0^T \langle A_0(u(\tau) - v(\tau)), \dot{u}(\tau) - \dot{v}(\tau)\rangle d\tau \leq 0. \tag{3.12}$$

Since $\frac{d}{d\tau}\langle A_0(u(\tau) - v(\tau)), u(\tau) - v(\tau)\rangle = 2\langle A_0(u(\tau) - v(\tau)), \dot{u}(\tau) - \dot{v}(\tau)\rangle$ at every point τ where both derivatives $\dot{u}(\tau), \dot{v}(\tau)$ exist, using [29, Theorem 6, p. 340] and noting that $u(0) = v(0)$, one has

$$\langle A_0(u(T) - v(T)), u(T) - v(T)\rangle = 2\int_0^T \langle A_0(u(\tau) - v(\tau)), \dot{u}(\tau) - \dot{v}(\tau)\rangle d\tau.$$

Thus, (3.12) is equivalent to

$$\int_0^T \alpha_1\|\dot{u}(\tau) - \dot{v}(\tau)\|^2 d\tau + \frac{1}{2}\langle A_0(u(T) - v(T)), u(T) - v(T)\rangle \leq 0.$$

Since A_0 is positive semidefinite, the latter implies

$$\int_0^T \|\dot{u}(\tau) - \dot{v}(\tau)\|^2 d\tau \leq 0. \tag{3.13}$$

As $\int_0^T \|\dot{u}(\tau)-\dot{v}(\tau)\|^2 d\tau \geq 0$, by (3.13) we have $\int_0^T \|\dot{u}(\tau)-\dot{v}(\tau)\|^2 d\tau = 0$. Hence, by [29, Corollary of Theorem 5, pp. 299–300], $\dot{u}(t) = \dot{v}(t)$ almost everywhere on $[0, T]$. So, thanks to Remark 3.9(b), we obtain $u(t) = v(t)$ for all $t \in [0, T]$. Thus, (P) can have at most one solution. □

4 The Case of Nonconvex Constraint Sets

The question of the solution existence of the velocity constraint sweeping process (P) beyond the convexity assumption of the constraint set $C(t)$ is an open question in the literature. Using the results in Section 3, we will prove some facts about solution existence for sweeping processes with nonconvex constraint sets. The obtained results differ from those of Bounkhel [12]. Note that the union of convex sets are not convex in general. Let $I = \{1, \ldots, m\}$ be a finite index set with $m \geq 2$. Let $C_i : [0, T] \rightrightarrows \mathcal{H}$, $i \in I$, be set-valued mappings with nonempty closed convex values such that, for any $t \in [0, T]$ and $i, j \in I$ with $i \neq j$, $C_i(t)$ does not intersect $C_j(t)$. Then, the set $C(t) := \bigcup_{i \in I} C_i(t)$ is closed and nonconvex for every $t \in [0, T]$. The uniform prox-regularity of such kind of sets has been discussed in Remark 2.9. In this section, we will study (P) with $C : [0, T] \rightrightarrows \mathcal{H}$ being the just defined set-valued mapping. To do so, for each $i \in I$, we consider the problem

$$\begin{cases} A_1\dot{u}(t) + A_0 u(t) - f(t) \in -\mathcal{N}_{C_i(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) = u_0. \end{cases} \tag{P_{C_i}}$$

The following theorems establish the solution existence for three classes of nonconvex sweeping processes with velocity constraints. *The key point here is that the problems in question admit multiple solutions.*

Theorem 4.1. (The moving constraint set is bounded and continuous in the Hausdorff distance sense) *Suppose $\mathcal{H}$ be separable and A_0, A_1 are positive semidefinite. If every set-valued mapping C_i, $i \in I$, satisfies the assumptions* (H1), (H2), *and* (H3a), *then* (P) *has an uncountable number of Lipschitz solutions, among them there are m solutions $u^{(i)}$, $i \in I$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0, T]$.*

Proof. Let $i \in I$ be chosen arbitrarily. Since C_i satisfies the conditions (H1), (H2), and (H3a), under the assumptions made, (P_{C_i}) has a Lipschitz solution $u^{(i)}(\cdot)$ by Theorem 3.4. If $\dot{u}_i(t) \in C_i(t)$, then the condition $C_i(t) \cap C_j(t) = \emptyset$ for $j \in I \setminus \{i\}$ and the closedness of $C_j(t)$, $j \in I \setminus \{i\}$, assure that there is a number $\rho_i(t) > 0$ satisfying $C_j(t) \cap \bar{\mathbb{B}}\left(\dot{u}^{(i)}(t), \rho_i(t)\right) = \emptyset$ for all $j \in I \setminus \{i\}$. So, one gets

$$C(t) \cap \bar{\mathbb{B}}\left(\dot{u}^{(i)}(t), \rho_i(t)\right) = C_i(t) \cap \bar{\mathbb{B}}\left(\dot{u}^{(i)}(t), \rho_i(t)\right).$$

Therefore, thanks to Remark 2.4 and the fact that the viability condition $\dot{u}^{(i)}(t) \in C_i(t)$ holds for almost every $t \in [0, T]$, we have

$$\begin{aligned} \mathcal{N}^P_{C(t)}(\dot{u}^{(i)}(t)) = \mathcal{N}^P_{C(t)\cap\bar{\mathbb{B}}\left(\dot{u}^{(i)}(t),\rho_i(t)\right)}(\dot{u}^{(i)}(t)) &= \mathcal{N}^P_{C_i(t)\cap\bar{\mathbb{B}}\left(\dot{u}^{(i)}(t),\rho_i(t)\right)}(\dot{u}^{(i)}(t)) \\ &= \mathcal{N}^P_{C_i(t)}(\dot{u}^{(i)}(t)) \end{aligned}$$

for almost every $t \in [0, T]$. Since $u^{(i)}(\cdot)$ is a Lipschitz continuous solution of (P_{C_i}), this yields

$$\begin{cases} A_1 \dot{u}^{(i)}(t) + A_0 u^{(i)}(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}^{(i)}(t)) \quad \text{a.e. } t \in [0, T], \\ u^{(i)}(0) = u_0. \end{cases}$$

Hence, $u^{(i)}(\cdot)$ is a Lipschitz continuous solution of (P).

Next, fix a pair $(i, j) \in I \times I$ with $i \neq j$, and let $u^{(i)}$ be a Lipschitz solution of (P_{C_i}), $u^{(j)}$ be a Lipschitz solution of (P_{C_j}). Then both functions $u^{(i)}$ and $u^{(j)}$ are Lipschitz solutions of (P). These functions are distinct. Indeed, if $u^{(i)}(t) = u^{(j)}(t)$ for all $t \in [0, T]$ then, since the inclusions $\dot{u}^{(i)}(t) \in C_i(t)$ and $\dot{u}^{(j)}(t) \in C_j(t)$ hold for a.e. $t \in [0, T]$, we find $\bar{t} \in (0, T)$ such that the derivatives $\dot{u}^{(i)}(\bar{t})$ and $\dot{u}^{(j)}(\bar{t})$ exist, $\dot{u}^{(i)}(\bar{t}) \in C_i(\bar{t})$ and $\dot{u}^{(j)}(\bar{t}) \in C_j(\bar{t})$. This is impossible because $\dot{u}^{(i)}(\bar{t}) = \dot{u}^{(j)}(\bar{t})$ and $C_i(\bar{t}) \cap C_j(\bar{t}) = \emptyset$. We have proved the existence of pairwise distinct Lipschitz solutions $u^{(1)}, \ldots, u^{(m)}$ of (P), for which one has $\dot{u}^{(i)}(t) \in C_i(t)$ for every $i \in I$ and for almost every $t \in [0, T]$.

Let $\tau \in (0, T)$ be arbitrarily chosen. By Theorem 3.4, the problem

$$\begin{cases} A_1 \dot{u}(t) + A_0 u(t) - f(t) \in -\mathcal{N}^P_{C_1(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0, \tau], \\ u(0) = u_0, \end{cases} \tag{4.1}$$

has a Lipschitz solution, which we denote by $u_{1,\tau}(\cdot)$. Similarly, the problem

$$\begin{cases} A_1 \dot{u}(t) + A_0 u(t) - f(t) \in -\mathcal{N}^P_{C_2(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [\tau, T], \\ u(\tau) = u_{1,\tau}(\tau), \end{cases} \tag{4.2}$$

has a Lipschitz solution, which is denoted by $u_{2,\tau}(\cdot)$. Setting

$$u_\tau(t) = \begin{cases} u_{1,\tau}(t) & \text{if } t \in [0, \tau], \\ u_{2,\tau}(t) & \text{if } t \in (\tau, T], \end{cases}$$

we see that u_τ is Lipschitz continuous function satisfying $u_\tau(0) = u_0$. As noted at the beginning of this proof, if $z \in C_1(t)$ (resp., $z \in C_2(t)$), then $\mathcal{N}^P_{C_1(t)}(z) = \mathcal{N}^P_{C(t)}(z)$ (resp., $\mathcal{N}^P_{C_2(t)}(z) = \mathcal{N}^P_{C(t)}(z)$). Therefore, from (4.1) and (4.2) it follows that $A_1 \dot{u_\tau}(t) + A_0 u_\tau(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u_\tau}(t))$ for almost every $t \in [0, T]$. Hence, u_τ is a Lipschitz solution of (P). Now, take any $\tau_1, \tau_2 \in (0, T)$ with $\tau_1 < \tau_2$. Since $u_{\tau_1}(\tau_1) = u_{\tau_2}(\tau_1)$, arguing similarly as in the above proof of the pairwise distinctness of the solutions $u^{(1)}, \ldots, u^{(m)}$ of (P), we can show that the restrictions of u_{τ_1} and u_{τ_2} on $[\tau_1, \tau_2]$ are two different functions. So, the family $\{u_\tau \mid \tau \in (0, T)\}$ consists of pairwise distinct Lipschitz functions. Hence, by the uncountability of $(0, T)$ we can assert that (P) has an uncountable number of Lipschitz continuous solutions. □

Theorem 4.2. (The moving constraint set is continuous in the Hausdorff distance sense) *Suppose $\mathcal{H}$ is separable and A_0, A_1 are positive semidefinite. If every set-valued mapping C_i, $i \in I$, satisfies the assumptions* (H1), (H2), *and* (H3b), *then* (P) *has an uncountable number of Lipschitz solutions, among them there are m solutions $u^{(i)}$, $i \in I$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0, T]$.*

Proof. Using the same arguments as the ones in the proof of Theorem 4.1 and applying Theorem 3.5 instead of Theorem 3.4, we then obtain the desired results. □

Theorem 4.3. (The moving constraint set is locally Lipschitz-like) *Suppose that $\mathcal{H}$ is a Hilbert space, $A_1 : \mathcal{H} \to \mathcal{H}$ satisfies the assumptions* (3.3)–(3.4)*, and $f : [0,T] \to \mathcal{H}$ is a continuous mapping. Assume that, for $i \in I$, the set-valued mapping C_i has nonempty closed convex values and is Lipschitz-like around every point in its graph. Then* $(\tilde{\mathrm{P}})$ *has an uncountable number of Lipschitz solutions, among them there are m continuously differentiable solutions $u^{(i)}$, $i \in I$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0,T]$.*

Proof. It suffices to follow the proof scheme of Theorem 4.1 and use Theorem 3.7 instead of Theorem 3.4. □

Remark 4.4. If $A_1 : \mathcal{H} \to \mathcal{H}$ is a symmetric coercive bounded linear operator, then the assertions of Theorem 4.3 are valid.

5 Illustrative Examples

In general, problem (P) does not have a unique solution even in the case where $C(t)$ is convex; see [5, Example 1]. For the convex case, Adly, Haddad, and Thibault [5, Theorem 5.2] (see Theorem 3.4 in Section 3) have proved that if A_0 is coercive, then (P) can have at most one solution. By constructing an example, we will show that this condition is not enough to obtain the solution uniqueness in the case where $C(t)$ is r-prox-regular and connected for each $t \in [0,T]$. We now give an example with a moving constraint set which is compact, with smooth boundary, connected, nonconvex, and uniformly prox-regular, where the problem has multiple solutions.

Example 5.1. Consider problem (P) with $T = 1$, $\mathcal{H} = \mathbb{R}^2$, $A_0 = A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f(t) \equiv 0$, $u_0 = (0,0)$, and

$$C(t) = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \mid (1+t)^2 \le x_1^2 + x_2^2 \le 9 \right\}.$$

Clearly, A_0 and A_1 are coercive, $C(t)$ is an annulus, which is r-prox-regular with $r = 1$ and connected for each $t \in [0,T]$. As the condition (3.1) is fulfilled with $g(t) := t$ and $C(0)$ is bounded, the assumptions (H2) and (H3a) are satisfied. Since $C(t)$, $t \in [0,T]$, are nonempty and closed, the assumption (H1) is partially satisfied. Nevertheless, here Theorem 3.4 cannot be applied, because the set-valued mapping C has nonconvex values. So, the solution existence of (P) is under question. Let $u_1(t) = \left(\frac{1}{2}(1+t)^2 - \frac{1}{2}, 0 \right)$ for $t \in [0,T]$. We see that $\dot{u}_1(t) = (1+t, 0) \in C(t)$ and $\mathcal{N}^P_{C(t)}(\dot{u}_1(t)) = \mathbb{R}_- \times \{0\}$ for $t \in [0,T]$. Since

$$A_1 \dot{u}_1(t) + A_0 u(t) - f(t) = \begin{pmatrix} 1+t \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(1+t)^2 - \frac{1}{2} \\ 0 \end{pmatrix} \in -\mathcal{N}^P_{C(t)}(\dot{u}_1(t))$$

for all $t \in [0,T]$ and $u_1(0) = (0,0)$, u_1 is a continuously differentiable solution of (P). Now, let

$$u_2(t) = \frac{1}{2\sqrt{2}} \left((1+t)^2 - 1, (1+t)^2 - 1 \right) \quad (\forall t \in [0,T]).$$

We have $u_2(0) = (0,0)$, $\dot{u}_2(t) = \frac{1}{\sqrt{2}}(1+t, 1+t) \in C(t)$ and

$$\mathcal{N}^P_{C(t)}(\dot{u}_2(t)) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2 \le 0\}.$$

Then,

$$
\begin{aligned}
A_1\dot{u}_1(t) + A_0 u(t) + f(t) &= \begin{pmatrix} \frac{1}{\sqrt{2}}(1+t) \\ \frac{1}{\sqrt{2}}(1+t) \end{pmatrix} + \begin{pmatrix} \frac{1}{2\sqrt{2}}(1+t)^2 - \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}}(1+t)^2 - \frac{1}{2\sqrt{2}} \end{pmatrix} \\
&\in -\mathcal{N}^P_{C(t)}(\dot{u}_2(t)).
\end{aligned}
$$

Therefore, $u_2(\cdot)$ is also a continuously differentiable solution of (P). So, (P) has multiple solutions.

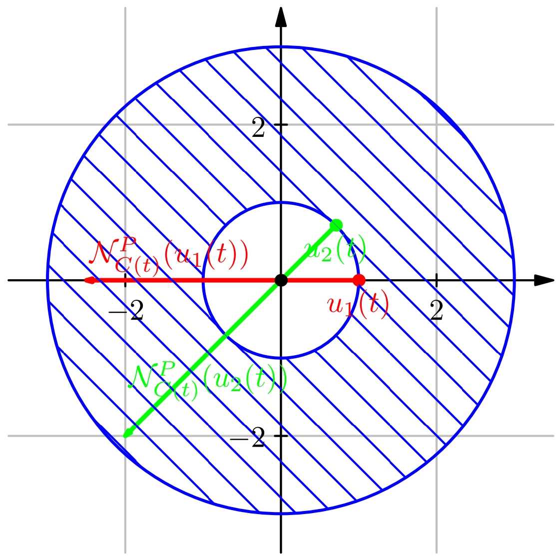

Figure 2: Illustration of Example 5.1

The next two examples will shed light on the assertions about solution uniqueness in Theorem 3.5 and Theorem 3.7. It turns out that the convexity assumption on the sets $C(t)$, $t \in [0,T]$, cannot be replaced by uniform prox-regularity and connectedness.

Example 5.2. Let T, $\mathcal{H}$, A_0, A_1, and f be as in the preceding example. Let

$$C(t) = \left\{x = (x_1,x_2) \in \mathbb{R}^2 \mid (1+t)^2 \le x_1^2 + x_2^2\right\} \text{ for all } t \in [0,T].$$

Then, $C(t)$ is unbounded, r-prox-regular with $r = 1$ and connected for each $t \in [0,T]$. The assumptions (H2) and (H3b) are fulfilled. Since the assumption (H1) is just partially satisfied, Theorem 3.5 cannot be used. Set $u(t) = (\frac{1}{2}t^2 + t)a$ for $t \in [0,T]$, where a is any point in $\partial C(0)$. By a direct verification, we can show that u is a continuously differentiable solution of (P). So, (P) has multiple solutions.

Example 5.3. Let T, $\mathcal{H}$, A_1, f, and $C(\cdot)$ be the same as in Example 5.1. The fulfillment of (3.1) with $g(t) := t$ shows that C is a Lipschitz set-valued mapping. Hence, as noticed in Remark 2.13, C is Lipschitz-like around every point in its graph. Choosing $A_0 = 0$, we see that, except for the required convexity of each $C(t)$, all other assumptions of Corollary 3.8 are satisfied. It is easy to verify that the formula $u(t) = (\frac{1}{2}t^2 + t)a$, where $a \in \mathbb{R}^2$ and $\|a\| = 1$, defines a continuously differentiable solution of (P). So, (P) has multiple solutions.

Remark 5.4. In Examples 5.1 and 5.3, if the formula of $C(t)$ is changed to

$$C(t) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid 1 \le x_1^2 + x_2^2 \le 9\right\},$$

then one has a problem with a fixed constraint set. The formula $u(t) = ta$, where $a \in \mathbb{R}^2$ and $\|a\| = 1$, defines a continuously differentiable solution of the problem (P). So, (P) can have multiple solutions even in the case of a fixed nonconvex constraint set, which is compact, uniformly prox-regular, and connected. This observation is also valid for Example 5.2, if the constraint set is kept fixed, i.e., one takes

$$C(t) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid 1 \le x_1^2 + x_2^2\right\}$$

for all $t \in [0, T]$.

If a person uses a motorbike to go on a road starting from A on a time interval $[0, T]$ then, roughly speaking, at every time instant he/she can choose one level of velocity from the set $\{0, 1, 2, 3\}$ of the motorcycle gear levels. Different choices of the velocity level $\dot{u}(t)$ for various disjoint segments of $[0, T]$ generate different path length functions $u(t)$. Here one has $u(0) = 0$. The following example will put this very common daily nonconvex sweeping process with velocity constraints in an abstract setting.

Example 5.5. Consider problem (P) with A_1, A_0, f, u_0 being given arbitrarily, and $C(t) = \{v^1, \dots v^m\}$ for all $t \in [0, T]$, where $m \ge 2$ and v^i, $i \in I := \{1, \dots, m\}$, are pairwise distinct points in $\mathcal{H}$. By Remark 2.9, we know that C is uniformly prox-regular. Let $\tau_0 = 0 < \tau_1 < \cdots < \tau_k = T$ be a partition of the interval $[0, T]$. Let $\dot{u}(t)$ be a step function that takes just one value from $\{v^1, \dots v^m\}$ on each interval (τ_j, τ_{j+1}), $j = 0, \dots, k-1$. The formula $u(t) = u_0 + \int_0^t \dot{u}(s)ds$ gives a Lipschitz function defined on $[0, T]$. It is obvious that, for any $z \in \{v^1, \dots v^m\}$ and $t \in [0, T]$, one has $\mathcal{N}^P_{C(t)}(z) = \mathcal{H}$. Hence, the two conditions in the formulation of (P) are satisfied. Thus, $u(t)$ is a solution of (P). We have shown that (P) has uncountable number of Lipschitz solutions.

The next example can serve as an illustration for Theorem 4.3.

Example 5.6. Consider problem (P) where $\mathcal{H} = \mathbb{R}^2$, $A_0 = 0$, $A_1 \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix, $f : [0, T] \to \mathbb{R}^2$ is a continuous function,

$$C_1(t) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \ge e^{-x_1 + t}\right\},$$

$C_2(t) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \le 0\right\}$, and $C(t) = C_1(t) \cup C_2(t)$ for $t \in [0, T]$. According to Remark 2.9, $C(t)$ is not uniformly prox-regular for any $t \in [0, T]$. Meanwhile, each mapping C_i, $i \in \{1, 2\}$, is Lipschitz-like around every point in its graph. To verify this property for C_1, one can apply a suitable implicit function theorem for set-valued mappings (for instance, [49, Theorem 3.2] and [60, Theorem 3.3]). Since all the assumptions of Theorem 4.3 are satisfied, we can assert that (P) has an uncountable number of Lipschitz solutions, among them there are two continuously differentiable solutions $u^{(i)}$, $i \in \{1, 2\}$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0, T]$.

To verify the local Lipschitz-likeness of an implicit set-valued mapping defined by a generalized inequality system in infinite-dimensional Hilbert spaces or Banach spaces, one can use, e.g., some results in [20, 58].

Interestingly, Theorem 4.2 can be applied to the sweeping process considered in Example 5.6.

Example 5.7. Let $\mathcal{H}$, A_0, A_1, $f(\cdot)$, and $C_1(\cdot)$, $C_2(\cdot)$, and $C(\cdot)$ be the same as in Example 5.6. To show that every set-valued mapping C_i, $i \in \{1,2\}$, satisfies the assumptions (H1), (H2), and (H3b), it suffices to verify the continuity of C_1 in the Hausdorff distance sense. To do so, take any $t, s \in [0,T]$ with $s < t$. Then, one has $C_1(t) \subset C_1(s)$. Given any $y = (y_1, y_2) \in C_1(s)$, we define $x = (x_1, x_2)$, where $x_1 = y_1 + t - s$ and $x_2 = y_2$. Since

$$e^{-x_1+t} = e^{-(y_1+t-s)+t} = e^{-y_1+s} \le y_2 = x_2,$$

we get $x \in C_1(t)$. As $\|x - y\| = t - s$, it follows that $d_H(C_1(s), C_1(t)) \le |t-s|$ for all $t, s \in [0,T]$. Therefore, by Theorem 4.2, (P) has an uncountable number of Lipschitz solutions, among them there are 2 solutions $u^{(i)}$, $i \in \{1,2\}$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0,T]$. Note that, to apply Theorem 4.2 for this sweeping process, as A_0 one can choose an arbitrary symmetric positive semidefinite 2×2 matrix (i.e., it is not necessary to put $A_0 = 0$).

6 Generalizations of Theorems 3.7 and 3.13

Let $\mathcal{H}$ be a Hilbert space, $A_0 : \mathcal{H} \to \mathcal{H}$ a symmetric positive semidefinite bounded linear operator, $A_1 : \mathcal{H} \to \mathcal{H}$ a symmetric coercive bounded linear operator, and $f : [0,T] \to \mathcal{H}$ a continuous mapping. Assume that $C : [0,T] \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed convex values, which is Lipschitz-like around every point in its graph. Then, according to Theorem 3.13, the sweeping process (P) can have at most one solution. If $A_0 = 0$, by Corollary 3.8 we know that (P) has a unique solution, which is a continuously differentiable function. The first open question is about the case where A_0 is a nonzero operator.

(Q1) *In the case $A_0 \neq 0$, are the above assumptions sufficient for the solution existence of* (P)*?*

If **(Q1)** can be solved in the affirmative, it is of interest to have an iteration scheme to find the unique solution of (P). Based on Theorem 3.7, we can propose such a scheme. At the initial step $k = 0$, one solves the problem $(\tilde{\mathrm{P}})$ and denotes the unique solution by $u^{(0)}$. Clearly, $u^{(0)}$ is a *rough* approximate solution of (P), because the operator $A_0 \neq 0$ had no role in creating the function. If u is the exact solution of (P), which is to be found, and $u^{(k)}$ is an approximate solution of (P) at a step $k \in \{0, 1, 2, \dots\}$, then

$$A_1\dot{u}(t) + A_0u(t) - f(t) \approx A_1\dot{u}(t) + A_0u^{(k)}(t) - f(t) \quad \text{a.e. } t \in [0,T].$$

Hence, setting $\tilde{f}_{k+1}(t) = -A_0u^{(k)}(t) + f(t)$ for all $t \in [0,T]$, we have

$$A_1\dot{u}(t) + A_0u(t) - f(t) \approx A_1\dot{u}(t) - \tilde{f}_k(t) \quad \text{a.e. } t \in [0,T].$$

So, the approximate problem of (P) at step $k+1$ is

$$\begin{cases} A_1\dot{u}(t) - \tilde{f}_{k+1}(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{$\mathrm{P}_{1,k+1}$}$$

Since $\tilde{f}_{k+1} : [0,T] \to \mathcal{H}$ is a continuous function, problem $(\mathrm{P}_{1,k+1})$ is of the form $(\tilde{\mathrm{P}})$. Therefore, by Corollary 3.8, it has a unique solution, which is denoted by $u^{(k+1)}$. The just described iteration scheme yields a sequence of continuously differentiable functions $\{u^{(k)}\}_{k\in\mathbb{N}}$. The second open question is as follows.

(Q2) *Whether the sequence* $\{u^{(k)}\}_{k\in\mathbb{N}}$ *converges to a solution of* (P)*?*

The above two questions were raised in our submitted paper. One of the two anonymous referees has proposed a detailed affirmative answer for **(Q1)** which even works for the situation where A_1 needs not to be a linear operator. Moreover, as noted by the referee, this solution of **(Q1)** also solves **(Q2)** in the affirmative. The results of the referee are given herein with some slight modifications to make the presentation easy for reading.

Significant generalizations of Theorems 3.7 and 3.13 are given in the following theorem.

Theorem 6.1. *Let $\mathcal{H}$ be a Hilbert space, $A_0 : \mathcal{H} \to \mathcal{H}$ a bounded linear operator, $A_1 : \mathcal{H} \to \mathcal{H}$ a mapping satisfying the assumptions* (3.3)–(3.4)*, and $f : [0,T] \to \mathcal{H}$ a continuous mapping. Assume that $C : [0,T] \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed convex values, which is Lipschitz-like around every point in its graph. Then the sweeping process*

$$\begin{cases} A_1(\dot{u}(t)) + A_0u(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) & a.e.\ t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{P1}$$

has a unique solution u, which is continuously differentiable.

Proof. Let V be the set

$$V = \left\{ v \in W^{1,\infty}((0,T),\mathcal{H}) \mid v(0) = u_0,\ \dot{v}(t) \in C(t) \text{ a.e. } t \in [0,T] \right\}. \tag{6.1}$$

For an arbitrary $v \in V$, we consider the following variational inequality

$$\langle A_1(\dot{u}(t)) + A_0v(t) - f(t), \dot{u}(t) - z\rangle \le 0 \quad \forall z \in C(t), \tag{6.2}$$

where $t \in [0,T]$ plays the role of a parameter. Since A_0 and v are continuous, the arguments of the proof of Theorem 3.7 show that the parametric variational inequality (6.2) admits a unique solution $u \in V$ and u is continuously differentiable. We thus can define the solution mapping $S : V \to V$, which maps a function $v \in V$ to the unique solution of (6.2) corresponding to v. It can easily be seen that the set V is convex and closed in $W^{1,\infty}((0,T),\mathcal{H})$. Clearly, u is a solution of (P1) if and only if it is a fixed point of S. In other words, the problem of solving (P1) reduces to that of finding a fixed point of S. So, to prove the solution existence of (P1), it suffices to prove that S has a fixed point in V. To this end, let us consider $v_1, v_2 \in V$ and the corresponding solutions $u_i = S(v_i)$, $i = 1,2$. By (6.2) we have

$$\langle A_1(\dot{u}_1(t)) + A_0v_1(t) - f(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \le 0$$

and

$$\langle A_1(\dot{u}_2(t)) + A_0v_2(t) - f(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle \le 0$$

for every $t \in [0,T]$. Adding the last inequalities yields

$$\langle A_1(\dot{u}_1(t)) - A_1(\dot{u}_2(t)) + A_0(v_1(t) - v_2(t)), \dot{u}_1(t) - \dot{u}_2(t)\rangle \le 0$$

for every $t \in [0,T]$. Therefore,

$$\langle A_1(\dot{u}_1(t)) - A_1(\dot{u}_2(t)), \dot{u}_1(t) - \dot{u}_2(t)\rangle \le \langle A_0(v_1(t) - v_2(t)), \dot{u}_2(t) - \dot{u}_1(t)\rangle \quad \forall t \in [0,T].$$

So, by (3.3) we have

$$\begin{aligned} \alpha_1\|\dot{u}_1(t) - \dot{u}_2(t)\|^2 &\le \langle A_1(\dot{u}_1(t)) - A_1(\dot{u}_2(t)), \dot{u}_1(t) - \dot{u}_2(t)\rangle \\ &\le \langle A_0(v_1(t) - v_2(t)), \dot{u}_2(t) - \dot{u}_1(t)\rangle \\ &\le \|A_0\|\|v_1(t) - v_2(t)\|\|\dot{u}_1(t) - \dot{u}_2(t)\| \end{aligned}$$

for every $t \in [0,T]$. It follows that

$$\|\dot{u}_1(t) - \dot{u}_2(t)\| \le M\|v_1(t) - v_2(t)\| \le M\int_0^t \|\dot{v}_1(s) - \dot{v}_2(s)\|ds \quad \forall t \in [0,T], \tag{6.3}$$

where $M := \dfrac{\|A_0\|}{\alpha_1}$. Let us introduce in the space $W^{1,\infty}((0,T),\mathcal{H})$ an equivalent norm $\|f\|_M$ by the formula (2.3) (see Subsection 2.3). Since $v_1(0) - v_2(0) = 0$, by virtue of (6.3) we have

$$\begin{aligned} e^{-Mt}\|\dot{u}_1(t) - \dot{u}_2(t)\| &\le Me^{-Mt}\int_0^t e^{Ms}\left[e^{-Ms}\|\dot{v}_1(s) - \dot{v}_2(s)\|\right]ds \\ &\le Me^{-Mt}\|v_1 - v_2\|_M\int_0^t e^{Ms}ds = (1 - e^{-Mt})\|v_1 - v_2\|_M \end{aligned}$$

for all $t \in (0,T)$. Hence, recalling that $u_1(0) = u_2(0) = u_0$, we get

$$\|u_1 - u_2\|_M \le (1 - e^{-MT})\|v_1 - v_2\|_M.$$

This means that the mapping S is a contraction on V with respect to the norm $\|\cdot\|_M$. By the Banach fixed point theorem (see, e.g., [29, Theorem 1, p. 66]), there is a unique $u \in V$ such that

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z\rangle \le 0 \quad \forall z \in C(t)$$

for almost every $t \in [0,T]$. So u is the unique solution of (P1) and u is continuously differentiable by Theorem 3.7. □

The next theorem gives a comprehensive solution of the question **(Q2)**.

Theorem 6.2. *Suppose that the assumptions of Theorem 6.1 are satisfied. Then, if $u^{(0)}$ is the unique solution of the problem* ($\tilde{\mathrm{P}}$) *and, for every $k \in \mathbb{N}$, $u^{(k+1)}$ denotes the unique solution of the $(k+1)-$th approximate problem*

$$\begin{cases} A_1\dot{u}(t) + A_0u^{(k)}(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) & a.e.\ t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{6.4}$$

of (P1)*, then the iteration sequence $\{u^{(k)}\}_{k\in\mathbb{N}}$ converges in $W^{1,\infty}((0,T),\mathcal{H})$ to the unique solution of* (P1)*.*

Proof. Define the closed convex set V by (6.1) and let the mapping $S : V \to V$ be defined as in the proof of Theorem 6.1. Then, in the distance induced by the equivalent norm (2.3) with $M := \dfrac{\|A_0\|}{\alpha_1}$ of $W^{1,\infty}((0,T),\mathcal{H})$, $S : V \to V$ is a contraction. Clearly, $u^{(0)} \in V$. Moreover, by (6.4) we can infer that $u^{(k+1)} = S(u^k)$. Hence, the desired conclusion follows from the convergence of the iterative sequence usually associated with the proof of the Banach fixed point theorem (see, e.g., [29, p. 67]). □

7 Open Questions

Several open questions related to the results given in Sections 3–6 will be formulated in this section.

7.1 A Regularization Method

It is appealing to study the problem $(\tilde{\mathrm{P}})$ in the setting of Corollary 3.8 with A_1 being only a symmetric positive semidefinite bounded linear operator. Let us denote the problem by (P_0) and its solution set by S_0.

(Q3) *Can we obtain a solution existence result for the problem* (P_0)*?*

If $S_0 \neq \emptyset$, then it would be reasonable to try to get a solution by the Tikhonov regularization method, which has been successfully applied for monotone variational inequalities (see, e.g., [53, Theorem 2.3]). For each $\varepsilon > 0$, the operator $A_1 + \varepsilon \mathrm{Id}$, where Id denotes the identity function, is coercive. Therefore, by Theorem 3.7, the regularized problem

$$\begin{cases} (A_1 + \varepsilon \mathrm{Id})\dot{u}(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u_0, \end{cases}$$

of (P_0) has a unique solution, which is denoted by u_ε. The following questions deserve further considerations:

(Q4) *If $S_0 \neq \emptyset$, then the solution u_ε of the regularized problem converges in $C^0([0,T],\mathcal{H})$ to a solution of the original problem as $\varepsilon \to 0^+$?*

(Q5) *If $S_0 \neq \emptyset$, then the limit of u_ε as $\varepsilon \to 0^+$, if exists, is a solution of* (P_0) *whose derivative has the smallest $L^2([0,T],\mathcal{H})$ norm?*

Another regularization method has been proposed by Moreau when he introduced the sweeping process [41]. The idea is that one can regularize the normal cone $\mathcal{N}_\Omega(\cdot)$ in term of the gradient of the square distance function $\nabla d^2(\cdot,\Omega)$. This method has been generalized and applied to nonconvex sweeping processes (see [46, 55] and references therein). It is natural to adopt this method to sweeping processes with velocity constraints. More precisely, we consider the following regularized problem of (P):

$$\begin{cases} A_1\dot{u}_\lambda(t) + A_0 u_\lambda(t) - f(t) = -\dfrac{1}{2\lambda}\nabla d^2(\dot{u}_\lambda(t), C(t)) \quad \text{a.e. } t \in [0,T], \\ u_\lambda(0) = u_0. \end{cases} \tag{R_P}$$

The next question arises:

(Q6) *Whether the differential equation (R_P) has a solution and there exists a sequence of solutions (depending on the parameter λ) of (R_P) which converges to a solution of* (P) *as $\lambda \to +\infty$?*

7.2 Problems Having a Fixed Connected Uniformly Prox-Regular Constraint Set

Several examples of sweeping processes with uniformly prox-regular constraint sets have been given in Section 5. In Example 5.5, despite of the fact that the constraint set is fixed and finite, (P) has multiple solutions for any choice of A_1, A_0, and f. In addition, from Remark 5.4 where the constraint set of the problem under consideration is fixed and both operators A_0, A_1 are coercive, we see that the solution uniqueness cannot be guaranteed. Thus, the following question seems to be interesting.

(Q7) *Under which conditions, can we obtain the solution existence and uniqueness for* (P) *when the constraint set is fixed, uniformly prox-regular, and connected?*

8 Conclusions

We have established the solution existence for some classes of sweeping processes in Hilbert spaces with velocity constraints where the constraint sets can be either convex or nonconvex. For the convex case, a new result on the solution uniqueness has been obtained. For the nonconvex case, we have proved that there are many classes of problems having an uncountable number of solutions.

Using a theorem on the solution sensitivity of parametric variational inequalities, we have proposed a new approach to the solution existence and solution uniqueness of sweeping processes with velocity constraints. Among other things, being locally Lipschitz-like, the constraint set mapping needs not to be continuous in the Hausdorff distance sense. An example has been given to show the advantage of the new results. Other illustrative examples, where the focus was made on uniform prox-regularity of the constraint sets, have been presented.

Five open problems deserving further investigations have been formulated.

Appendix: Equivalent Norms

Claim 1: *In the space $W^{1,\infty}((0,T),\mathcal{H})$, the norm $\|f\|_{W^{1,\infty}} = \|f\|_{L^\infty} + \|f'\|_{L^\infty}$ is equivalent to the following one: $\|f\| = \|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \|\dot f(t)\|$.*

Proof. Noting that $W^{1,\infty}((0,T),\mathcal{H}) \subset W^{1,1}((0,T),\mathcal{H})$, one has f is an absolutely continuous function, which means

$$f(t) = f(0) + \int_0^t \dot f(s)ds, \quad \forall t \in [0,T].$$

From the proof of [16, Corollary 1.4.31] we can deduce that $\dot f(t) = f'(t)$ almost everywhere on $(0,T)$. Then,

$$\begin{aligned}
\|f\|_{W^{1,\infty}} = \|f\|_{L^\infty} + \|f'\|_{L^\infty} &= \|f\|_{L^\infty} + \|\dot f\|_{L^\infty} \\
&= \operatorname*{ess\,sup}_{t\in(0,T)} \|f(t)\| + \|\dot f\|_{L^\infty} \\
&= \sup_{t\in(0,T)} \|f(t)\| + \|\dot f\|_{L^\infty} \qquad (\|f(\cdot)\| \text{ is continuous on } (0,T)) \\
&\geq \|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \|\dot f(t)\|.
\end{aligned} \tag{A.1}$$

We also have

$$\|f\|_{W^{1,\infty}} = \|f\|_{L^\infty} + \|f'\|_{L^\infty} = \|f\|_{L^\infty} + \|\dot f\|_{L^\infty}$$

$$
\begin{aligned}
&= \operatorname*{ess\,sup}_{t\in(0,T)} \|f(0) + \int_0^t \dot f(s)ds\| + \|\dot f\|_{L^\infty} \\
&\le \|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \|\int_0^t \dot f(s)ds\| + \|\dot f\|_{L^\infty} \\
&\le \|f(0)\| + \int_0^T \|\dot f(s)\|ds + \|\dot f\|_{L^\infty} \qquad \text{(using Holder's inequality)} \\
&\le \|f(0)\| + 2\operatorname*{ess\,sup}_{t\in(0,T)} \|\dot f(t)\| \\
&\le 2(\|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \|\dot f(t)\|). \qquad \text{(A.2)}
\end{aligned}
$$

From (A.1) and (A.2), we obtain the desired result. □

Claim 2: *If M be a positive number, then the norm $\|\cdot\|_{W^{1,\infty}}$ on the space $W^{1,\infty}((0,T),\mathcal{H})$ is equivalent to the norm $\|\cdot\|_M$ defined by*

$$
\|f\|_M = \|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \left(e^{-Mt}\|\dot f(t)\|\right), \quad \forall f \in W^{1,\infty}((0,T),\mathcal{H}).
$$

Proof. As we have mentioned in the previous proof, $\dot f = f'$ almost everywhere on $(0,T)$. From (A.1) it follows that

$$
\begin{aligned}
\|f\|_{W^{1,\infty}} &\ge \|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \|\dot f(t)\| \\
&\ge \|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} (e^{-Mt}\|\dot f(t)\|) \qquad (\text{since } e^{-Mt} \le e^0 = 1, \text{ for all } t \in [0,T]).
\end{aligned} \tag{A.3}
$$

By (A.2), we get

$$
\begin{aligned}
\|f\|_{W^{1,\infty}} &\le 2(\|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \|\dot f(t)\|) \\
&= 2\left[\|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \left(e^{Mt}e^{-Mt}\|\dot f(t)\|\right)\right] \\
&\le 2\left[\|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \left(e^{MT}e^{-Mt}\|\dot f(t)\|\right)\right] \\
&= 2\left[\|f(0)\| + e^{MT}\operatorname*{ess\,sup}_{t\in(0,T)} \left(e^{-Mt}\|\dot f(t)\|\right)\right] \\
&\le 2e^{MT}\left[\|f(0)\| + \operatorname*{ess\,sup}_{t\in(0,T)} \left(e^{-Mt}\|\dot f(t)\|\right)\right].
\end{aligned} \tag{A.4}
$$

Combining (A.3) and (A.4) yields the equivalence of the two norms under consideration. □

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References

[1] S. Adly, A Variational Approach to Nonsmooth Dynamics. Applications in Unilateral Mechanics and Electronics, Springer Briefs in Mathematics, Springer International Publishing, Cham, 2017.

[2] S. Adly, A coderivative approach to the robust stability of composite parametric variational systems: applications in nonsmooth mechanics, J. Optim. Theory Appl. 180 (2019), 62–90.

[3] S. Adly, T. Haddad, An implicit sweeping process approach to quasistatic evolution variational inequalities, SIAM J. Math. Anal. 50 (2018), 761–778.

[4] S. Adly, T. Haddad, On evolution quasi-variational inequalities and implicit state-dependent sweeping processes, Discrete Contin. Dyn. Syst. 6 (2020), 1791–1801.

[5] S. Adly, T. Haddad, L. Thibault, Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities, Math. Program. 148 (2014), Ser. B, 5–47.

[6] S. Adly, B.K. Le, On semicoercive sweeping process with velocity constraint. Optim. Lett. 12 (2018), 831–843.

[7] S. Adly, F. Nacry, L. Thibault, Preservation of prox-regularity of sets with applications to constrained optimization, SIAM J. Optim. 26 (2016), 448–473.

[8] S. Adly, J.V. Outrata, Qualitative stability of a class of non-monotone variational inclusions. Application in electronics, J. Convex Anal. 20 (2013), 43–66.

[9] J.-P. Aubin, Lipschitz behavior of solutions to convex minimization problems, Math. Oper. Res. 9 (1984), 87–111.

[10] Y. Benyamini, J. Lindenstrauss, Geometric Nonlinear Functional Analysis, American Mathematical Society, 1998.

[11] J.-P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser Boston, 1990.

[12] M. Bounkhel, Existence and uniqueness of some variants of nonconvex sweeping processes, J. Nonlinear Convex Anal. 8 (2007), 311–323.

[13] M. Bounkhel, Regularity Concepts in Nonsmooth Analysis, Springer, New York, 2012.

[14] M. Bounkhel, L. Thibault, Nonconvex sweeping process and prox-regularity in Hilbert space, J. Nonlinear Convex Anal. 6 (2005), 359–374.

[15] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2010.

[16] T. Cazenave, A. Haraux, An Introduction to Semilinear Evolution Equations, Translated from the 1990 French original by Yvan Martel and revised by the authors, Oxford University Press, New York, 1998.

[17] F.H., Clarke, A new approach to Lagrange multipliers, Math. Oper. Res. 1 (1976), 165–174.

[18] F.H. Clarke, Y.S. Ledyaev, R.J. Stern, P.R. Wolenski, Nonsmooth Analysis and Control Theory, Springer-Verlag, New York, 1998.

[19] G. Colombo, L. Thibault, Prox-regular sets and applications, In: Handbook of Nonconvex Analysis and Applications, pp. 99–182, Int. Press, Somerville, MA, 2010.

[20] P.H. Dien, N.D. Yen, On implicit function theorems for set-valued maps and their application to mathematical programming under inclusion constraints, Appl. Math. Optim. 24 (1991), 35–54.

[21] J. Diestel, J.J. Uhl, Jr., Vector Measures, American Mathematical Society, Providence, R.I., 1977.

[22] A.L. Dontchev, R.T. Rockafellar, Characterizations of strong regularity for variational inequalities over polyhedral convex sets, SIAM J. Optim. 6 (1996), 1087–1105.

[23] G. Duvaut, J.-L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, Berlin, 1976.

[24] J.F. Edmond, L. Thibault, BV solutions of nonconvex sweeping process differential inclusions with perturbation, J. Diff. Eqs. 226 (2006) 135–179.

[25] N.T.T. Huong, J.-C. Yao, N.D. Yen, Geoffrion's proper efficiency in linear fractional vector optimization with unbounded constraint sets. J. Global Optim. 78 (2020), 545–562.

[26] N.Q. Huy, N.D. Yen, Minimax variational inequalities, Acta Math. Vietnam. 36 (2011), 265–281.

[27] A. Jourani, E. Vilches, A differential equation approach to implicit sweeping processes, J. Differential Equations 266 (2019), 5168–5184.

[28] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, Inc., New York-London, 1980.

[29] A.N. Kolmogorov, S.V. Fomin, Introductory Real Analysis, Courier Corporation, 1975.

[30] P. Krejčí, G.A. Monteiro, V. Recupero, Explicit and implicit non-convex sweeping processes in the space of absolutely continuous functions. Appl. Math. Optim. 84 (2021), suppl. 2, 1477–1504.

[31] M. Kunze, M.D.P. Monteiro Marques, On the discretization of degenerate sweeping processes, Port. Math. 55 (1998), 219–232.

[32] M. Kunze, M.D.P. Monteiro Marques, An introduction to Moreau's sweeping process, in "Impacts in Mechanical Systems. Analysis and Modelling", pp. 1–60, Springer, Berlin, 2000.

[33] B. Maury, S. Faure, Crowds in Equations: An Introduction to the Microscopic Modeling of Crowds, Advanced Textbooks In Mathematics, World Scientific Europe Ltd., 2018.

[34] B. Maury, J. Venel, A mathematical framework for a crowd motion model, C. R. Math. Acad. Sci. Paris, 346 (2008), 1245–1250.

[35] B. Maury, J. Venel, A discrete contact model for crowd motion ESAIM Math. Model. Numer. Anal. 45 (2011), 145–168.

[36] S. Migórski, M. Sofonea, S. Zeng, Well-posedness of history-dependent sweeping processes, SIAM J. Math. Anal. 51 (2019), 1082–1107.

[37] M.D.P. Monteiro Marques, Differential Inclusions in Nonsmooth Mechanical Problems, Shocks and Dry Friction, Progr. Nonlinear Differential Equations Appl. 9, Birkhäuser, Boston, 1993.

[38] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. Vol. I: Basic Theory. Vol. II: Applications, Springer, Berlin, 2006.

[39] B.S. Mordukhovich, Variational Analysis and Applications, Springer Monographs in Mathematics, Springer, Cham, 2018.

[40] J.J. Moreau, Sur l'évolution d'un système élastoplastique, C. R. Acad. Sci. Paris Sér. A-B, 273 (1971), A118–A121.

[41] J.J. Moreau, Rafle par un convexe variable I, Sém. Anal. Convexe Montpellier (1971), Exposé 15.

[42] J.J. Moreau, Rafle par un convexe variable II, Sém. Anal. Convexe Montpellier (1972), Exposé 3.

[43] J.J. Moreau, On unilateral constraints, friction and plasticity, in "New Variational Techniques in Mathematical Physics" (G. Capriz and G. Stampacchia, Eds.), 173–322, C.I.M.E. II Ciclo 1973, Edizioni Cremonese, Roma, 1974.

[44] J.J. Moreau, Intersection of moving convex sets in a normed space, Math. Scand. 36 (1975), 159–173.

[45] J.J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, J. Diff. Eqs. 26, (1977), 347–374.

[46] F. Nacry, L. Thibault, Regularization of sweeping process: old and new. Pure Appl. Funct. Anal. 4 (2019), 59–117.

[47] B.T. Polyak, Introduction to optimization, Optimization Software, Inc., Publications Division, New York, 1987.

[48] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.

[49] R.T. Rockafellar, Lipschitzian properties of multifunctions, Nonlinear Anal. TMA, 9 (1985), 867–885.

[50] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, Springer, New York, 1998.

[51] W. Rudin, Principles of Mathematical Analysis, Third edition, McGraw-Hill Book Co., New York, 1976.

[52] A.H. Siddiqi, P. Manchanda, Variants of Moreau's sweeping process, Adv. Nonlinear Var. Inequal. 5 (2002), 1–16.

[53] N.N. Tam, J.-C. Yao, N.D. Yen, Solution methods for pseudomonotone variational inequalities, J. Optim. Theory Appl. 138 (2008), 253–273.

[54] L. Thibault, Sweeping process with regular and nonregular sets. J. Diff. Eqs. 193 (2003), 1–26.

[55] L. Thibault, Regularization of nonconvex sweeping process in Hilbert space. Set-Valued Anal. 16 (2008), 319–333.

[56] J. Venel, A numerical scheme for a class of sweeping processes, Numer. Math. 118 (2011), 367–400.

[57] E. Vilches, B.T. Nguyen, Evolution inclusions governed by time-dependent maximal monotone operators with a full domain, Set-Valued Var. Anal., 28 (2020), 569–581.

[58] N.D. Yen, Implicit function theorems for set-valued maps, Acta Math. Vietnam. 12 (1987), No. 2, 17–28.

[59] N.D. Yen, Hölder continuity of solutions to a parametric variational inequality, Appl. Math. Optim. 31 (1995), 245–255.

[60] N.D. Yen, Stability of the solution set of perturbed nonsmooth inequality systems and application, J. Optim. Theory Appl. 93 (1997), 199–225.

[61] K. Yosida, Functional Analysis, Springer-Verlag, Berlin-New York, 1980.