NON-SELF SIMILAR BLOWUP SOLUTIONS TO THE HIGHER DIMENSIONAL YANG MILLS HEAT FLOWS

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Abstract. In this paper, we consider the Yang-Mills heat flow on $\mathbb{R}^d \times SO(d)$ with $d \geq 11$. Under a certain symmetry preserved by the flow, the Yang-Mills equation can be reduced to the following nonlinear equation:

$$\partial_t u = \partial_r^2 u + \frac{d+1}{r}\partial_r u - 3(d-2)u^2 - (d-2)r^2u^3, \text{ and } (r,t) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

We are interested in describing the singularity formation of this parabolic equation. More precisely, we aim to construct non self-similar blowup solutions in higher dimensions $d \geq 11$, and prove that the asymptotic of the solution is of the form

$$u(r,t) \sim \frac{1}{\lambda_\ell(t)} \mathcal{Q}\left(\frac{r}{\sqrt{\lambda_\ell(t)}}\right), \text{ as } t \to T,$$

where $\mathcal{Q}$ is the steady state corresponding to the boundary conditions $\mathcal{Q}(0) = -1, \mathcal{Q}'(0) = 0$ and the blowup speed λ_ℓ verifies

$$\lambda_\ell(t) = (C(u_0) + o_{t\to T}(1))\,(T-t)^{\frac{2\ell}{\alpha}} \text{ as } t \to T,\ \ell \in \mathbb{N}_+^*,\ \alpha > 1.$$

In particular, the case $\ell = 1$ corresponds to the stable type II blowup regime, whereas for the cases $\ell \geq 2$ corresponds to a finite co-dimensional stable regime.

Our approach here is not based on energy estimates but on a careful construction of time dependent eigenvectors and eigenvalues combined with maximum principle and semigroup pointwise estimates.

1. Introduction

Recently, geometric heat flows received a lot of attention from both the mathematics and physics communities. Among these geometric flows, the Yang-Mills heat flow is of a great interest. Let us give a brief survey of the physics behind it (more details can be found in [21] and [18]). The Yang-Mills theory is in some sense a *non-commutative* version of Maxwell's electromagnetism where in the latter, the gauge group is the *abelian* group $U(1)$. In order to describe the weak nuclear force, governing the nuclear decay of some particles, Yang and Mills proposed to substitute for the Maxwell's gauge group $U(1)$ the *non-abelian* gauge group $SU(2)$. Let us describe the mathematical setting of the theory. Consider a Riemannian manifold M of dimension d, with a structure group G (i.e., a semi-simple Lie group) and denote by π the canonical projection. Let $\mathcal{G}$ be the Lie

2010 *Mathematics Subject Classification.* Primary: 35K50, 35B40; Secondary: 35K55, 35K57.

Key words and phrases. Blowup solutions, Finite time blowup, Type II blowup, Geometric heat flows, Singularity.

Acknowledgement: The work by Giao Ky Duong was supported by the International Center for Research and Postgraduate Training in Mathematics, Institute of Mathematics, Vietnam Academy of Science and Technology, code ICRTM04_2021.05. The work by Tej-Eddine Ghoul is based upon work supported by Tamkeen under the NYU Abu Dhabi Research Institute grant CG002.

algebra of G and E a principal fibre bundle over M. Let D_A be a covariant derivative from E to $Ad(E) \otimes T^*M$. On each coordinate chart U_α, D_A can be represented by the $\mathcal{G}$-valued 1-form of $\kappa + A_\alpha$ where κ is some fixed reference connection (e.g. usual exterior derivative), and $A_{(\alpha)}$ a $\mathcal{G}$-valued 1-form

$$A_{(\alpha)} = \sum_{\mu=1}^{d} A_{\alpha,\mu} dx^\mu.$$

Since the transition functions are smooth, we can set $A_{(\alpha)} = A$. Physically, the vector A represents the electromagnetic potential.

Let the curvature F_A be the tensor $D_A D_A$. By using a local chart U_α, one can represent F_A by the $\mathcal{G}$-valued 2-form

$$F_A = \sum_{\mu,\nu} F_{\mu,\nu} dx^\mu \wedge dx^\nu,$$

where

$$F_{\mu,\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The second rank covariant tensor $F_{\mu,\nu}$ is the well-known electromagnetic tensor. The Yang-Mills connections are defined as the *critical points* of Yang-Mills functional $\mathcal{F}_A$ given by

$$\mathcal{F}_A := \int_M |F_A|^2 \, dvol_M.$$

The Euler Lagrange equations corresponding to these critical points are

$$\sum_{\nu=1}^{d} D^\nu F_{\mu,\nu} = 0, \forall \mu = 1, ..., d,$$

where $D_\nu = \partial_\nu + [A_\nu, \cdot]$.

The Yang-Mills *heat flow* is defined as the gradient flow associated to the above problem where A is the Yang-Mills connection. By using a local chart, the time-dependent connection locally satisfies

$$\left\{ \begin{array}{l} \partial_t A_\mu(x,t) + \partial^\nu F_{\mu\nu}(x,t) + [A^\nu, F_{\mu,\nu}](x,t) = 0, \ t > 0, \\ A_\mu(x,0) = A_{\mu,0}(x). \end{array} \right. \tag{1.1}$$

Note that equation (1.1) is invariant under the following scaling

$$A_\lambda(x,t) = \lambda A\left(\lambda x, \lambda^2 t\right), \text{ for } \lambda > 0. \tag{1.2}$$

However, the Yang-Mills *functional* is invariant under scaling symmetry for $d = 4$, this is why we refer to this dimension as the energy critical one. For $d \geq 5$, we say that the equation is supercritical. Results on the long time existence and uniqueness were obtained in [28] for $d = 2, 3$, [24, 31] for $d = 4$ for weak solutions (see also [29] and [30] for the existence of smooth solutions). In particular, in the case $d = 4$, the authors in [30] conjectured finite time singularities do not occur on a compact manifold which recently confirmed by [33]. For the energy *supercritical* problem, i.e. $d \geq 5$, there is few results on the global existence and this due to the the gauge invariance of the Yang-Mills heat flow.

Let us restrict ourselves to a special case where $M = \mathbb{R}^d$ and $E = \mathbb{R}^d \otimes SO(d)$ is the trivial bundle. In this case, the Yang-Mills connection $A_\mu(\mu \in \{1, .., d\})$ is globally given by its $\mathcal{SO}(d)$-valued coefficient functions $A_\mu(\mu = 1, ..., d)$. In particular, the Lie algebra $\mathcal{SO}(d)$ is simply the space of *skew-symmetric* $d \times d$ matrices endowed with the commutator bracket. Let us denote the coefficient functions by $A_\mu = A_\mu^{i,j}$ and make (as in [14]) the following $\mathcal{SO}(d)$-equivariant ansatz

$$A_\mu^{i,j}(x,t) = u(|x|,t)\sigma_\mu^{i,j}(x), \text{ where } \sigma_\mu^{i,j}(x) = \delta_\mu^i x^j - \delta_\mu^j x^i, i, j \in \{1, ..., d\}.$$

We emphasize here that the covariant derivative of σ is zero, so that the ansatz amounts to consider the problem in the Lorentz gauge. By following these settings (see [21]), it reduces to

$$\partial_t u = \partial_r^2 u + \frac{d+1}{r}\partial_r u - 3(d-2)u^2 - (d-2)r^2u^3, \text{ and } (r,t) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{1.3}$$

The solution to this equation is invariant under the scaling

$$u_\lambda(x,t) = \frac{1}{\lambda}u\left(\frac{x}{\sqrt{\lambda}}, \frac{t}{\lambda}\right) \tag{1.4}$$

for $\lambda > 0$. Let us remark that (1.3) is locally well posed in some weighted L^∞ spaces as

$$L^\infty_{1+r^\alpha}(\mathbb{R}_+) = \left\{f \text{ measurable on } \mathbb{R}_+ \text{ such that } \|(1+r^\alpha)f(r)\|_{L^\infty(\mathbb{R}_+)} < +\infty, \alpha \geq \frac{2}{3}\right\},$$

by following a fixed point argument and an extension to a $\mathbb{R}^{d+2}$-problem. Consequently, with an arbitrary initial data in $L^\infty_{1+r^\alpha}$, the corresponding solution is either global or develops singularity in finite time T, in the sense that

$$\limsup_{t\to T} \|u(\cdot,t)\|_{L^\infty_{1+r^\alpha}(\mathbb{R}_+)} = +\infty.$$

In this paper, we are interested in the blowup phenomenon and a variety of papers were devoted to the study of singularity formation. First, in [18], the author constructed self-similar blowup solutions with $5 \leq d \leq 9$. Besides that, the authors in [35] also gave explicit examples (so-called Weinkove solutions)

$$u_W(x,t) = \frac{1}{T-t}W\left(\frac{r}{\sqrt{T-t}}\right),$$

with

$$W(r) = -\frac{1}{a_1(d)r^2 + a_2(d)}.$$

Here $a_1(d) = \frac{\sqrt{d-2}}{2\sqrt{2}}, a_2(d) = \frac{1}{2}\left(6d - 16 - (d+2)\sqrt{2d-4}\right)$. Recently, the authors in [13] have constructed non trivial solutions in the range $5 \leq d \leq 9$ which approach u_W in $L^\infty(\mathbb{R}^+)$ and these solutions corresponding to similar blowup setting. The stability of Weinkove solutions was also proved by [13] and [22]. For higher dimension $d \geq 10$, the authors in [6] excluded the existence of self similar blowup solutions and then non selfsimilar solutions are expected.

We have been successful in constructing non-self similar blowup solutions (so-called Type II blowup solutions). Our results are stated in the following.

Theorem 1.1 (Existence of stable blowup solution)**.** *Let $d \geq 11$ be an integer. Then, there exist initial data $u_0 \in C_0^\infty(\mathbb{R}_+, \mathbb{R})$ such that the corresponding solution to (1.3) blows up in finite time $T(u_0)$. Moreover, the following decomposition holds true*

$$u(r,t) = \lambda^{-1}(t)Q\left(\frac{r}{\sqrt{\lambda(t)}}\right) + \tilde{u}(r,t), t \in [0,T), \tag{1.5}$$

where Q is the ground state of (1.3) satisfying $Q(0) = -1$ and $Q'(0) = 0$; and the error $\tilde{u}(r,t)$ satisfies

$$\lambda(t)\|\tilde{u}(.,t)\|_{L^\infty(\mathbb{R}^+)} \to 0 \text{ as } t \to T, \tag{1.6}$$

and the blowup speed $\lambda(t)$ exactly behaves as follows

$$\lambda(t) = C(u_0)(1 + o(1))(T-t)^{\frac{2}{\alpha}}. \tag{1.7}$$

as $t \to T$ and α defined in (2.27). In particular, the constructed blowup behavior is stable.

By a suitable expansion the construction technique in Theorem 1.1, we can construct *unstable* blowup solutions with different blowup speeds. More precisely, the result reads.

Theorem 1.2 (Existence of unstable blowup solutions)**.** *Let us consider integer numbers $\ell \geq 2$ and $d \geq 11$. Then, there exist initial data $u_{0,\ell} \in C_0^\infty(\mathbb{R}_+, \mathbb{R})$ such that the corresponding solution u_ℓ to (1.3) blows up in finite time $T(u_{0,\ell})$. Moreover, the following decomposition holds true*

$$u_\ell(r,t) = \lambda_\ell^{-1}(t) Q\left(\frac{r}{\sqrt{\lambda_\ell(t)}}\right) + \tilde{u}_\ell(r,t), \tag{1.8}$$

where Q is the ground state satisfying $Q(0) = -1$ and $Q'(0) = 0$; and the error $\tilde{u}_\ell(r,t)$ satisfies

$$\lambda_\ell(t)\|\tilde{u}(.,t)\|_{L^\infty(\mathbb{R}^+)} \to 0 \text{ as in } t \to T, \tag{1.9}$$

and the blowup speed $\lambda_\ell(t)$ exactly behaves as follows

$$\lambda_\ell(t) = C(u_0)(1 + o(1))(T-t)^{\frac{2\ell}{\alpha}} \text{ as } t \to T. \tag{1.10}$$

Remark 1.3 (Related blowup results for PDE's problem)**.** Note that the Yang-Mills heat flow (1.3) has a lot of similarities with the harmonic map heat flow (under corotational symmetry):

$$\partial_t u = \partial_r^2 u + \frac{d+1}{r}\partial_r u - (d-1)\frac{\sin(2u)}{2r^2}, \text{ and } (r,t) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{1.11}$$

The harmonic map heat flow forms also singularity in finite time, and the self-similar nature of the singularity appears only when $3 \leq d \leq 6$, and for $d \geq 7$ self-similar blowup solutions don't exist [6]. For $3 \leq d \leq 6$, the existence of the self-similar solutions is known [17] and the stability has been proved only in the case $d = 3$ as in [1]. When $d = 7$ the blowup is not self-similar and the speed λ has a log correction [19], it turns out that the non-self-similar regime is stable when $d = 7$. If $d \geq 8$, in [20] the authors proved similar results. The results in [20], also in [4], have been proved with a different method. In [20], the result is based on an energy based method, whereas in [4] is based on the maximum principle which does not allow to abtain the stability. In the present paper, we present a new method that has been introduced previously in [7, 10] but combined with ideas from [4]. We also mention that the author in [36] has obtained the same results for (1.3) in comparising with our paper, by following a robust map based on energy estimates as in [20] which one of co-author is also a co-author this new paper. Even though our paper presents the same results and appears later for four months (noted on *Arxiv*), our one is more orginality that we have built up a new technique to adapt to more general models that we will explain those novelties in the remark below.

Remark 1.4 (Novelty of the paper)**.** We point out that the approach pursued here is more intuitive than the one in [20] for the heat flow map as it is based on a spectral approach rather than an energy method. Note that here, the selection by the flow of the blow up speed is linked to the eigenvalue λ_ℓ of the time dependent linearized operator $\mathscr{L}_b$, after perturbing initially Q in the direction of the eigenvectors ϕ_ℓ. Such an idea was not clear in [20]. The length of the paper is due to the heavy and technical construction of the eigenvectors and eigenvalues of $\mathscr{L}_b$. In comparison with [20], the use of maximum principle reduce considerably the difficulty of the control of the infinite dimensional part ε_-. We believe that this method can be adapted to a large class of parabolic problems.

Remark 1.5 (Structure of the paper)**.** To be more convenient for the readers, we aim to give the structure of the paper here: We introduce and explain the importance of the different set of variables: self-similar and blowup variables in the second section. In the third and fourth sections we explain the strategy of the proof, and the time dependent spectral analysis strategy. The fifth section aims to provide a proof of the main theorem without technical details where we show that the infinite dimensional problem can be reduced to a finite dimensional one. In other words, we show that the solution can be split into two parts a finite dimensional part and an infinite dimensional one. In the sixth section we study the dynamic of the finite dimensional part under the assumption that the infinite dimensional part of the solution is decaying in a suitable weighted

L^2 norm. The seventh section shows that the assumption made in the section 6 on the infinite dimensional part holds after assuming an L^∞ bound. In the 8th section we prove this L^∞ bound assumed in the previous section by using maximum principle and pointwise estimates which is based on the semigroup associating to the linearised operator. The 9th section is devoted to prove the existence of the ground state Q which solves an non-autonomous second order ODE. To do so, we prove the existence of an heteroclinic trajectory by finding an appropriate trapping set. In the 10th section we sketch the proof of the existence of the unstable blowup solutions and the last section is devoted to the diagonalisation of the time dependent linearised operator $\mathscr{L}_b$.

2. Mathematical setting

Let u be a solution to the following equation on $[0, T)$ for some $T > 0$

$$\partial_t u = \partial_r^2 u + \frac{d+1}{r}\partial_r u - 3(d-2)u^2 - (d-2)r^2 u^3, \text{ and } (r,t) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{2.1}$$

Let λ be an unknown blow-up speed satisfying $\lambda(t) \to 0$ as $t \to T$ and write

$$u(t,r) = \frac{1}{\lambda(t)} v(\xi, s) \tag{2.2}$$

where the blow-up variables s and ξ are such that

$$\frac{ds}{dt} = \frac{1}{\lambda}, \quad \xi = \frac{r}{\sqrt{\lambda}}.$$

Simple computation yields

$$\partial_s v = \partial_\xi^2 v + \frac{d+1}{\xi}\partial_\xi v + \frac{1}{2}\frac{\lambda_s}{\lambda}\Lambda_\xi v - 3(d-2)v^2 - (d-2)\xi^2 v^3. \tag{2.3}$$

We anticipate that $\frac{\lambda_s}{\lambda} \to 0$ as $s \to \infty$, since the blow-up mechanism is non-self similar, thus, v is expected to converge to the ground state Q, which is a solution to

$$Q''(\xi) + \frac{d+1}{\xi}Q'_\xi - 3(d-2)Q^2 - (d-2)\xi^2 Q^3 = 0 \tag{2.4}$$

with the boundary conditions $Q(0) = -1$ and $Q'(0) = 0$.
In order to establish the convergence of v to the stationary solution Q, we linearize around the latter and study the operator

$$H_\xi + \frac{1}{2}\frac{\lambda_s}{\lambda}\Lambda_\xi, \tag{2.5}$$

where

$$\Lambda_\xi = 2 + \xi\partial_\xi, \tag{2.6}$$

and

$$H_\xi = \partial_\xi^2 + \frac{d+1}{\xi}\partial_\xi - 3(d-2)(2Q(\xi) + \xi^2 Q^2(\xi)). \tag{2.7}$$

More precisely, we would like to determine the eigenvectors and eigenvalues of the linearized operator which depend on time. To do so, one has to switch to the so-called self-similar variables, i.e., we write the solution u as

$$u(r,t) = \frac{1}{T-t} w\left(\frac{r}{\sqrt{T-t}}, \tau\right), \quad \tau = -\log(T-t). \tag{2.8}$$

One then finds that w satisfies

$$\partial_\tau w = \partial_y^2 w + \frac{d+1}{y}\partial_y w - \frac{1}{2}\Lambda_y w - 3(d-2)w^2 - (d-2)y^2 w^3. \tag{2.9}$$

Now, introduce a function b of time such that

$$b = \frac{\lambda}{T-t}. \tag{2.10}$$

If the blow-up is self-similar, b would be a (non-zero) constant. In our case, the blow-up is foreseen to be non-self-similar and b has then to tend to zero as $t \to T$.

Stepping on the fact that our problem is invariant under time translation, we allow the blow-up time to vary. That is, we replace $T-t$ by some function μ and we prove that it behaves like $T-t$ for $t \to T$. Hence we relax $b = \frac{\lambda}{\mu}$ instead of $\frac{\lambda}{T-t}$. The parameter b is measuring the non-self similarity of the solution.

- *Notation:* Based on the above, we write

$$u(r,t) = \frac{1}{\mu(t)} w(y,\tau), \quad y = \frac{r}{\sqrt{\mu(t)}} \quad \text{and } \frac{d\tau}{dt} = \frac{1}{\mu(t)}. \tag{2.11}$$

The function w now satisfies

$$\partial_\tau w = \partial_y^2 w + \frac{d+1}{y}\partial_y w - \beta(\tau)\Lambda_y w - 3(d-2)w^2 - (d-2)y^2 w^3, \tag{2.12}$$

where

$$\beta(\tau) = -\frac{1}{2}\frac{\mu_\tau}{\mu(\tau)}, \tag{2.13}$$

and

$$\Lambda_y f = y\partial_y f + 2f. \tag{2.14}$$

Note that in the self-similar scale μ, one needs to linearise around Q_b instead of Q, where

$$Q_{b(\tau)}(y) = \frac{1}{b(\tau)} Q\left(\frac{y}{\sqrt{b(\tau)}}\right). \tag{2.15}$$

In addition, w is global but blows up in infinite time. Indeed, introduce the error

$$\varepsilon(y,\tau) = w(y,\tau) - Q_{b(\tau)}(y). \tag{2.16}$$

By a simple calculations, it leads to

$$\partial_\tau \varepsilon = \mathscr{L}_b(\varepsilon) + B(\varepsilon) + \Phi(y), \tag{2.17}$$

where

$$\mathscr{L}_b = \partial_y^2 + \frac{d+1}{y}\partial_y - \beta(\tau)\Lambda_y - 3(d-2)\left(2Q_b + Q_b^2|y|^2\right), \tag{2.18}$$

and

$$B(\varepsilon) \quad = \quad -3(d-2)(1+|y|^2 Q_b)\varepsilon^2 - (d-2)|y|^2\varepsilon^3, \tag{2.19}$$

and

$$\Phi(\cdot,\tau) \quad = \quad \frac{1}{2}\Lambda_y Q_{b(\tau)}\left[\frac{b'(\tau)}{b(\tau)} - 2\beta(\tau)\right]. \tag{2.20}$$

From the expression of the operator H_ξ, we have the relation

$$\mathscr{L}_b w(y,\tau) = \frac{1}{b}\left(H_\xi - b\beta\Lambda_\xi\right) v(\xi,\tau). \tag{2.21}$$

From Lemma 9.1, we infer that

$$3(d-2)\left[2Q_b(y) + y^2 Q_b^2\right] \to -\frac{3(d-2)}{y^2} \text{ as } b \to 0 \text{ with } y \neq 0.$$

We next introduce the limit operator

$$\mathscr{L}_\infty^\beta = \partial_y^2 + \frac{d+1}{y}\partial_y - \beta\Lambda_y + \frac{3(d-2)}{y^2}, \tag{2.22}$$

and we set $\mathscr{L}_\infty^{\frac{1}{2}} = \mathscr{L}_\infty$

$$\mathscr{L}_\infty = \partial_y^2 + \frac{d+1}{y}\partial_y - \frac{1}{2}\Lambda_y + \frac{3(d-2)}{y^2}. \tag{2.23}$$

Let $\rho = y^{d+1}e^{-\beta\frac{y^2}{2}}$. Then a simple computation yields

$$\mathscr{L}_\infty^\beta \phi = \frac{1}{\rho}\frac{d}{dy}(\rho\phi') + \frac{3(d-2)}{y^2}\phi - 2\beta\phi. \tag{2.24}$$

In the present paper, we use weighted Sobolev spaces $L^2_{\rho_\beta}$ and $H^1_{\rho_\beta}$ where the weight ρ_β is defined by

$$\rho_\beta(y) = \frac{(2\beta)^{\frac{d+2}{2}}}{(4\pi)^{\frac{d+2}{2}}} y^{d+1} e^{-(2\beta)\frac{y^2}{4}}. \tag{2.25}$$

We also denote $\rho_{\frac{1}{2}} = \rho$.

The space $L^2_{\rho_\beta}$ is equipped with the norm

$$\|f\|^2_{L^2_{\rho_\beta}(\mathbb{R}^+)} = \int_{\mathbb{R}_+} f^2(y)\rho_\beta(y)dy,$$

and $H^1_{\rho_\beta}(\mathbb{R}^+)$ has the norm

$$\|f\|^2_{H^1_{\rho_\beta}(\mathbb{R}^+)} = \|f\|^2_{L^2_{\rho_\beta}(\mathbb{R}^+)} + \|\partial_y f\|^2_{L^2_{\rho_\beta}(\mathbb{R}^+)}.$$

We also define some special constants in our paper and we assume the dimension $d \geq 11$. Let

$$\gamma = \frac{1}{2}(d - \sqrt{d^2 - 12d + 24}), \tag{2.26}$$

$$\alpha = \gamma - 2, \tag{2.27}$$

and

$$a_{i,j} = \frac{(-1)^{i-j}}{(i-j)!}4^{i-j}\frac{i!}{j!}\frac{(\frac{d}{2}-\gamma)_i!}{(\frac{d}{2}-\gamma)_j!} = c_{i,j}C_j, \text{ for all } 0 \leq j \leq i,$$

with $c_{i,j}$ and C_j defined as follows

$$c_{i,j} = \frac{(-1)^{i-j}4^i i!\left(\frac{d}{2}-\gamma\right)_i!}{(i-j)!}, \tag{2.28}$$

$$C_j = \frac{1}{4^j j!\left(\frac{d}{2}-\gamma\right)_j!}, \tag{2.29}$$

where

$$\left(\frac{d}{2}-\gamma\right)_i! = \left(\frac{d}{2}-\gamma+1\right)\left(\frac{d}{2}-\gamma+2\right)...\left(\frac{d}{2}-\gamma+i\right) \text{ and } \left(\frac{d}{2}-\gamma\right)_0! = 1.$$

We also use the notation

$$\langle y\rangle = \sqrt{1+|y|^2}.$$

3. Strategy of the proof

We aim to summarize in this paragraph our strategy for the proof of our results. As mentioned above, our goal is to prove that $v \to Q$ as $s \to \infty$ which is equivalent to the control

$$\|\varepsilon(\cdot,\tau)\|_{L^\infty} \ll \|Q_{b(\tau)}\|_{L^\infty} = b^{-1}(\tau) \text{ as } \tau \to \infty, \tag{3.1}$$

where Q_b defined as in (2.15), b determined as in (2.10), and $w = Q_b + \varepsilon$ with w defined in (2.8). Our problem mainly focuses on the perturbative problem (2.17). In addition, the perturbative spectral properties of the linear operator $\mathscr{L}_b$ is studied in Proposition 4.2 which allows us to expand the error ε along its eigenmodes $\phi_{i,b,\beta}, i \in \{0,1,...,\ell\}$. More precisely, we arrive at the following decomposition

$$\varepsilon(\tau) = \sum_{j=1}^{\ell-1} \varepsilon_j(\tau)\phi_{j,b(\tau),\beta(\tau)} + \varepsilon_\ell(\tau)\left[\frac{\phi_{\ell,b(\tau),\beta(\tau)}}{c_{\ell,0}} - \phi_{0,b(\tau),\beta(\tau)}\right] + \varepsilon_-(\tau), \tag{3.2}$$

where $c_{\ell,0}$ defined in (2.28), ε_- is the orthogonal part of ε to $\phi_{i,b,\beta}$ for all $i \le \ell$ i.e.

$$\langle \varepsilon_-, \phi_{i,b,\beta}\rangle_{L^2_{\rho_\beta}} = 0, \forall j = 0, ..., \ell. \tag{3.3}$$

Note that the decomposition in (3.3) is crucial to our approach, as first introduced in [9] (see also [7]). On the one hand, this decomposition provides a good approximation to our solution, including the main perturbative term i.e.

$$\varepsilon_\ell(\tau)\left[\frac{\phi_{\ell,b(\tau),\beta(\tau)}}{c_{\ell,0}} - \phi_{0,b(\tau),\beta(\tau)}\right]$$

which offers a better approximation compared to the profile when the solution is far from the singular domain. On the other hand, it plays an important role in driving the law of the blowup speed, $b(\tau)$. In order to ensure the decomposition (3.2) be unique, we couple the problem (2.17) with

$$c_{\ell,0}\|\phi_{\ell,\phi,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \varepsilon, \phi_{\ell,b,\beta}\rangle_{L^2_{\rho_\beta}} = -\|\phi_{0,\phi,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \varepsilon, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} i.e. \varepsilon_\ell = -c_{\ell,0}\varepsilon_0, \tag{3.4}$$

and the following compatibility condition (for only the case $\ell = 1$)

$$\varepsilon_\ell(\tau) = -\frac{2}{\alpha} m_0 b^{\frac{\alpha}{2}}(\tau). \tag{3.5}$$

Finally, the main issue is to control (ε, b, β) by a suitable asymptotic behaviors. Specifically, we employ the concept of shrinking set, $V_\ell[A, \eta, \tilde{\eta}]$ as defined in Definition 5.1 to handle the problem. It's worth noting that the set bears resemblance to recent studies on Type I blowup constructions, such as those found in [3], [25], [27], [11], [16], [15], [12]. More precisely, we control $\varepsilon_j, j = 0, .., \ell$, ε_-, the blowup speed b, the parameter $\beta(\cdot)$. Due to the nonlinearity $y^2\varepsilon^3$ in equation (2.17), we need to control $\|\varepsilon_-\|_{L^2_\beta}$ to derive *a priori estimates* on ε_j and ε_-. Besides that, it is not enough to imply (3.1) from ε_j and ε_-, since the eigenmodes $\phi_{i,b,\beta}, i \in \mathbb{N}$ are not bounded as $y \to +\infty$. To address this challenge, we also regulate the outer part ε_e introduce in (5.9). Furthermore, we propose a simpler way for constructing Type II blowup solutions for parabolic problems, as an alternative to a direct brute force energy method.

Additionally, we also point out main ideas of the proofs of Theorems 1.1 and 1.2.

- *For* $\ell = 1$. This case involves Theorem 1.1. It is sufficient to control $(\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau)$, for all $\tau \ge \tau_0$ for some τ_0 a sufficient large value. The maim idea is to construct a suitable initial choice $(\varepsilon, b, \beta)(\tau_0)$ (see more in subsection 5.2), then we reply in *a priori estimates* provided in Lemmas 6.1, 7.1, 7.2 and 7.3 to improve the bounds in the $V_1[A, \eta, \tilde{\eta}]$. Thus, by continuity of the solution in time, we easily conclude that the maximum time trapped in the shrinking set is $+\infty$. Finally, using the renormalisation in time given in (5.50), we conclude the proof of Theorem 1.1. We also mention some interesting points in our proof. First, the control of ε_- which we sufficiently do on the interval $[0, b^{-\tilde{\eta}}(\tau)]$. On the one hand, on $[0, b^{\frac{\eta}{4}}]$ we use the maximum principle, initialled in [4], to control it in avoiding a heavy control from energy method. On the other hand, on $[b^{\frac{\eta}{4}}, b^{-\tilde{\eta}}]$ which is far the origin enough. Then, the result follows pointwise estimates based on the Poisson semigroup, see more in section C. Second, the control of the outer part ε_e, follows pointwise estimates based on the semigroup $\mathcal{K}_\beta(\tau, \tau')$.

- *For* $\ell \geq 2$. This case is related to Theorem 1.2. Similar to the first one. We control the solution to be trapped in the shrinking set $V_\ell[A, \eta, \tilde{\eta}](\tau)$ by *a priori estimates.* However, this case includes unstable modes that ε_j for all $j \in \{0, ..., \ell\}$. Thus, we reduce our problem to a finite dimensional one which is solvent by a classical topological argument.

4. Spectral analysis

The aim of this section is to study the linear operator $\mathscr{L}_b^\beta$. In order to do so, we begin with the limit operator $\mathscr{L}_\infty^\beta$.

Proposition 4.1 (Diagonalisation of $\mathscr{L}_\infty^\beta$, [23], [4], [9])**.** *Let $d \geq 11$, $\beta \in \left(\frac{1}{4}, \frac{3}{4}\right)$ and $\mathscr{L}_\infty^\beta$ defined as in (2.22). Then, $\mathscr{L}_\infty^\beta$ admits a unique Friedrichs extension, still denoted by $\mathscr{L}_\infty^\beta$ with domain $\mathcal{D}(\mathscr{L}_\infty^\beta) \subset H^1_{\rho_\beta}$ and $H^2_{\rho_\beta} \subset \mathcal{D}(\mathscr{L}_\infty^\beta)$, is self-adjoint with compact resolvent. Moreover, the following hold:*

(i) Spectrum property: $\mathscr{L}_\infty^\beta$ consists of countable many eigenvalues. More precisely, the eigenvalues and eigenfunctions are given by

$$\lambda_{i,\infty,\beta} = 2\beta\left(\frac{\alpha}{2} - i\right), i \in \mathbb{N}, \tag{4.1}$$

$$\begin{aligned} \phi_{i,\infty,\beta}(y) &= \mathcal{N}_i\left(\sqrt{2\beta}y\right)^{-\gamma} L_i^{\left(\frac{d}{2}-\gamma\right)}\left(\frac{\beta y^2}{2}\right) = \sum_{j=0}^{i} a_{i,j}(2\beta)^j y^{2j-\gamma} \\ &= \begin{cases} a_{i,0}y^{-\gamma}\left(1 + O(y^2)\right) & \text{as } y \to 0, \\ a_{i,i}(2\beta)^i y^{2i-\gamma}\left(1 + O(y^{-2})\right) & \text{as } y \to +\infty, \end{cases} \end{aligned} \tag{4.2}$$

where $L_i^{(\nu)}(z)$ denotes the generalized Laguerre polynomial, $\mathcal{N}_i$ is a normalization constant and $\gamma, \alpha, a_{i,j}$ are defined in (2.26), (2.27) and (2.28), respectively.

(ii) Spectral gap estimate: for all $u \in H^1_{\rho_\beta}$ satisfying $\langle \phi_{i,\beta,\infty}, u\rangle_{L^2_{\rho_\beta}} = 0, \forall i \in \{1, ..., \ell\}$, then

$$\langle \mathscr{L}_\infty^\beta u, u\rangle_{L^2_{\rho_\beta}} \leq \lambda_{\ell+1,\infty,\beta}\|u\|^2_{L^2_{\rho_\beta}}.$$

As has been noted above, $\mathscr{L}_\infty^\beta$ is formally the limit $(b \to 0)$ of $\mathscr{L}_b$ defined in (2.18), and a priori it is a good approximation of the latter for large values of τ. However, such an approximation is good only for y large enough since $\phi_{i,\infty}$ is singular when y approaches 0. Hence, to understand well the operator $\mathscr{L}_b$ around zero (i.e., y small), one has to use the blow-up variables (ξ, s) and our operator then reads

$$\mathscr{L}_b = \frac{1}{b}\left(H_\xi - b\beta\Lambda_\xi\right). \tag{4.3}$$

The strategy is is to construct the eigenvalues and eigenvectors of $\mathscr{L}_b$ in two different regions, namely, for $y > y_0$ (outer region) using the self-similar scale and and for $\xi \leq \xi_0$ (inner region) using the variable ξ. Once such a construction is achieved, we glue at y_0 and in a C^1-manner the obtained eigenvalues and eigenvectors. The result is summarized in the following proposition.

Proposition 4.2 (Diagonalisation of $\mathscr{L}_b$)**.** *Let $d \geq 11, b > 0, \beta \in \left(\frac{1}{4}, \frac{3}{4}\right), \ell \in \mathbb{N}^*$ and $\mathscr{L}_b$ be defined as in (2.18). Then, $\mathscr{L}_b$ admits a unique Friedrichs extension, still denoted by $\mathscr{L}_b$ with domain $\mathcal{D}(\mathscr{L}_b) \subset H^1_{\rho_\beta}$ and $H^1_{\rho_\beta} \subset \mathcal{D}(\mathscr{L}_b)$, is self-adjoint with compact resolvent. Moreover, for all $\ell \in \mathbb{N}^*$ there exists $b^*(\ell) \ll 1$ such that for all $b \in (0, b^*)$ and $j \leq \ell$, the following hold:*

(I) Spectrum: the eigenvalues and eigenfunctions are given by

$$\lambda_{i,\beta,b} = 2\beta\left(\frac{\alpha}{2} - i\right) + \tilde{\lambda}_{i,\beta,b}, \forall i \in \mathbb{N}, \tag{4.4}$$

$$\phi_{i,\beta,b}(y) = \sum_{j=0}^{i} c_{i,j}(2\beta)^j(\sqrt{b})^{2j-\gamma}T_j\left(\frac{y}{\sqrt{b}}\right) + \tilde{\phi}_{i,\beta,b}, \tag{4.5}$$

where

$$\|\tilde{\phi}_{i,\beta,b}\|_{H^1_{\rho_\beta}} \lesssim b^{1-\frac{\epsilon}{2}}, \quad \rho_\beta \text{ defined in (2.25)}.$$

In particular, we have

$$|\tilde{\lambda}_{i,\beta,b}| \lesssim b^{1-\frac{\epsilon}{2}} \text{ and } \left|\partial_b\tilde{\lambda}_{i,\beta,b}\right| \lesssim b^{-\frac{\epsilon}{2}}, \text{ and } \left|\partial_\beta\tilde{\lambda}_{i,\beta,b}\right| \lesssim 1. \tag{4.6}$$

(II) Difference estimate:

$$\|\phi_{i,\beta,b} - \phi_{i,\beta,\infty}\|_{H^1_\rho} \lesssim b^{1-\frac{\epsilon}{2}}, \tag{4.7}$$

where $\phi_{i,b,\infty}$ defined as in (4.2)

(III) Pointwise estimate: for $k \in \{0,1\}$ we have

$$\left|\partial_y^k\phi_{i,\beta,b}(y)\right| \lesssim \frac{\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma+k}}, \tag{4.8}$$

$$\left|\partial_y^k b\partial_b\phi_{i,\beta,b}(y)\right| \lesssim \frac{\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma+k}}, \tag{4.9}$$

and

$$\left|\partial_y^k\tilde{\phi}_{i,\beta,b}(y)\right| + \left|\partial_y^k b\partial_b\tilde{\phi}_{i,\beta,b}(y)\right| + \left|\partial_y^k\partial_\beta\tilde{\phi}_{i,\beta,b}(y)\right| \lesssim \frac{b^{1-\frac{\epsilon}{2}}\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma+k}}. \tag{4.10}$$

(iv) Spectral gap estimate: assume that $u \in H^1_{\rho_\beta}(\mathbb{R}^+)$ satisfies

$$\langle u, \phi_{i,\beta,b}\rangle_{L^2_{\rho_\beta}} = 0, \forall i \in \{0,1,...,\ell\},$$

then, there exists $c(\ell) > 0$ such that

$$\langle \mathscr{L}_b u, u\rangle_{L^2_{\rho_\beta}} \leq -\left(\lambda_{\ell,b,\beta} + c(\ell)\right)\|u\|^2_{L^2_{\rho_\beta}}. \tag{4.11}$$

Proof. The spectral analysis is quite the same as in [8] and [9]. We kindly refer the reader to check the details. In addition, we also give the matching ODE approach and the pointwise estimates in Section 10. □

5. Proof in the stable case without technical details

In this section, we aim to give the proof of Theorem 1.1 without technical details.

5.1. Shrinking set

We define below the shrinking set that controls the asymptotic behavior of (ε, b, β) leading to the global existence of the solution, and deriving Theorem 1.1. Let b and β be positive functions satisfying the hypothesises in Proposition 4.2, then we decompose ε as in (3.2) by taking $\ell = 1$

$$\varepsilon(\tau) = \varepsilon_1(\tau)\left(\frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta}\right) + \varepsilon_-(\tau) = \varepsilon_+(\tau) + \varepsilon_-(\tau). \tag{5.1}$$

Definition 5.1 (Shrinking set)**.** Let $A, \eta, \tilde{\eta}$ and τ_0 be positive constants. For each $\bar{\tau} > \tau_0$ we introduce $V_1[A, \eta, \tilde{\eta}](\bar{\tau})$ as the set of all triple time-dependent functions (ε, b, β) on $[\tau_0, \bar{\tau}]$ such that $(\varepsilon, b, \beta)(\tau) \in L^\infty(\mathbb{R}_+) \times \mathbb{R}^2$ for all $\tau' \in [\tau_0, \bar{\tau}]$ and the following estimates are satisfied:

(i) The dominating mode ε_1 satisfies

$$\varepsilon_1(\tau) = -\frac{2}{\alpha} m_0 b^{\frac{\alpha}{2}}(\tau), \forall \tau \in [\tau_0, \bar{\tau}], \tag{5.2}$$

and functions b and β satisfy

$$\frac{1}{2} \le b(\tau) \exp\left(\left(\frac{2}{\alpha} - 1\right)\left(\int_{\tau_0}^{\tau} 2\beta(\tilde{\tau}) d\tilde{\tau} + \tau_0\right)\right) \le 2, \forall \tau \in [\tau_0, \bar{\tau}], \tag{5.3}$$

and

$$\left|\beta(\tau) - \frac{1}{2}\right| \le A I^\eta(\tau_0), \forall \bar{\tau} \in [\tau_0, \bar{\tau}], \tag{5.4}$$

where m_0 is given by (6.15), and $I(\tau)$ is defined by

$$I(\tau) = e^{\left(1 - \frac{2}{\alpha}\right)\tau}. \tag{5.5}$$

(iii) The part ε_- of ε defined as in (5.1) satisfies

$$\|\varepsilon_-(.,\tau)\|_{L^2_{\rho_{\beta(\tau)}}} \le A^2 b^{\frac{\alpha}{2}+\eta}(\tau), \forall \tau \in [\tau_0, \bar{\tau}], \tag{5.6}$$

and

$$\left\| y^\gamma \frac{\varepsilon_-(.,\tau)}{\langle y \rangle^4} \right\|_{L^\infty[0, b^{-\tilde{\eta}}(\tau)]} \le A^3 b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau), \forall \tau \in [\tau_0, \bar{\tau}]. \tag{5.7}$$

(iv) The part ε_e satisfy

$$\||y|\varepsilon_e(.,\tau)\|_{L^\infty} \quad \le \quad A^4 b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tau), \forall \tau \in [\tau_0, \bar{\tau}], \tag{5.8}$$

where

$$\varepsilon_e(y,\tau) = (1 - \chi_0(2yb^{\tilde{\eta}}(\tau)))\varepsilon(y,\tau), \text{ and } \operatorname{supp}(\varepsilon_e) \subset \left\{|y| \ge \frac{1}{2} b^{-\tilde{\eta}}\right\}, \tag{5.9}$$

and χ_0 defined by

$$\chi_0 \in C^\infty, \chi_0(x) = 1, \forall x \in [0,1], \text{ and } \chi_0(x) = 0, \forall x \ge 2. \tag{5.10}$$

Consequently, once (ε, b, β) belongs to $V_1[A, \eta, \tilde{\eta}]$, one can easily deduce the following pointwise estimates.

Lemma 5.2 (Pointwise estimates)**.** *For all $A \ge 1$ and $0 < \tilde{\eta} < \eta \ll 1$, then there exists $\tau_1(A, \tilde{\eta}, \eta) \ge 1$ such that for all $\tau_0 \ge \tau_1$ the following holds: Assume $(\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau)$ for all $\tau \in [\tau_0, \bar{\tau}]$ with $\bar{\tau} > \tau_0$ arbitrarily given, then we have*

$$I^{1+\frac{\tilde{\eta}}{10}}(\tau) \le b(\tau) \le I^{1-\frac{\tilde{\eta}}{10}}(\tau), \forall \tau \in [\tau_0, \bar{\tau}]. \tag{5.11}$$

Accordingly (5.1), ε_+ and ε_- satisfy the following pointwise estimates

$$|\varepsilon_+(y,\tau)| \le \frac{C b^{\frac{\alpha}{2}}}{y^\gamma}\left\{y^2 + b^{10\eta}(\tau)\langle y \rangle^4\right\}, \forall y > 0, \tau \in [\tau_0, \bar{\tau}], \tag{5.12}$$

and

$$|\varepsilon_-(y,\tau)| \le CA^4 b^{\frac{\alpha}{2}+\tilde\eta}(\tau)\frac{\langle y\rangle^4}{y^\gamma}, \forall y>0, \tau\in[\tau_0,\bar\tau]. \tag{5.13}$$

Proof. The results immediately follow from the bounds given in Definition 5.1 of the shrinking set $V_1[A,\eta,\tilde\eta](\tau)$. □

5.2. Preparing initial data

In this part, we aim to construct a suitable falimy of initial data $(\varepsilon,b,\beta)(\tau_0)$ such that the solution to the problem (2.17-3.4-3.5) globally exists and satisfies

$$(\varepsilon,b,\beta)\in V_1[A,\eta,\tilde\eta](\bar\tau), \forall\bar\tau>\tau_0.$$

Let us define $\beta_0=\beta(\tau_0)=\frac12$, $b_0=b(\tau_0)=I^{\frac{\alpha}{2}}(\tau_0)$ where $I(\tau)$ introduced as in (5.5), $\delta\ll1$ satisfying $0<\tilde\eta\ll\eta\ll\delta\ll1$. In addition, we recall χ_0 defined as in (5.10) and we introduce then

$$\psi(\tau_0)=\chi_0\left(yb_0^\delta\right)\left(1-\chi_0\left(\frac{y}{b_0^\delta}\right)\right)\left(-\frac{2}{\alpha}m_0b_0^{\frac{\alpha}{2}}\right)\left\{[1+\hat\psi(\tau_0)]\frac{\phi_{1,b_0,\beta_0}}{c_{1,0}}-\left[1+\tilde\psi(\tau_0)\right]\phi_{0,b_0,\beta_0}\right\}, \tag{5.14}$$

where the corrections $\tilde\psi(\tau_0)$ and $\hat\psi(\tau_0)$ are uniquely determined such that (3.4-3.5) are satisfied at $\tau=\tau_0$. More precisely, via a direct computation, they satisfy

$$\left|\tilde\psi(\tau_0)\right|+\left|\hat\psi(\tau_0)\right|\lesssim b^\delta(\tau_0).$$

Thus, our initial data is of the form

$$(\varepsilon,b,\beta)(\tau_0)=(\psi(\tau_0),b_0,\beta_0). \tag{5.15}$$

In addition, the initial data for problem (2.12) will be of the form

$$w(y,\tau_0)=Q_{b(\tau_0)}(y)+\varepsilon(\tau_0). \tag{5.16}$$

In the sequel, we prove by using modulation that we can propagate (3.4) and (3.5).

Lemma 5.3 (Modulation technique)**.** *There exists $\delta_2\ll1$ such that for all $\delta\le\delta_2$ there exists $A_2\ge1$ such that for all $A\ge A_2$ there exists $\eta_2(A,\delta)$ such that for all $\eta\le\eta_2$ there exists $\tilde\eta_2(A,\delta,\eta)$ such that for all $\tilde\eta\le\tilde\eta_2$ there exists $\tau_2(A,\delta,\eta,\tilde\eta)\ge1$ such that the following property holds: Assume that initial datum is of the form in (5.14), then there exists $\tau^*_{loc}>\tau_0$ and smooth functions $(b,\beta)\in\left(C\left[\tau_0,\tau^*_{loc}\right],\mathbb{R}^2\right)\cap C^1\left((\tau_0,\tau^*_{loc}],\mathbb{R}^2\right)$ such that the solution w (corresponding to initial data in (5.16)) to equation (2.12), locally exists on $[\tau_0,\tau^*_{loc}]$ and uniquely admits the following decomposition*

$$w(\tau)=Q_{b(\tau)}+\varepsilon(\tau), \tag{5.17}$$

where (ε,b,β) satisfying (2.17-3.4-3.5) and $Q_{b(\tau)}$ defined as in (2.15). In addition, it holds that $(\varepsilon,b,\beta)\in V_1[A,\eta,\tilde\eta](\bar\tau)$, for all $\bar\tau\in[\tau_0,\tau^*_{loc}]$. In particular, the existence of (ε,b,β) can be propagated to the interval $[\tau_0,\tau^*_{loc}+\tilde\sigma]$ for some $\tilde\sigma$ small thank to the bounds in V_1.

Proof. Let us consider initial data $w(\tau_0)$ defined as in (5.16). Thanks to the local well-posedness in $L^\infty_{1+r^\alpha},\alpha\ge\frac23$ of the problem (1.3), there exists $\tilde\tau>\tau_0$ such that the solution w to equation (2.12) uniquely exists on $[\tau_0,\tilde\tau]$. We mention that the existence of modulations b and β and decomposition (5.17) is a direct consequence of the implicit function theorem. Let us introduce the following maps

$$\left\{\begin{array}{rcl} F_1(\tau,b,\beta) & := & \langle w(\tau)-Q_b, \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}c_{1,0}\phi_{1,b,\beta}+\|\phi_{0,b,\beta}\|^{-2}_{\rho_\beta}\phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}},\\ F_2(\tau,b,\beta) & := & \langle w(\tau)-Q_b,\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}}+\frac{2}{\alpha}m_0\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}b^{\frac{\alpha}{2}}(\tau).\end{array}\right. \tag{5.18}$$

Since $\varepsilon(\tau_0)=\psi(\tau_0)$ defined as in (5.14), it immediately follows

$$F(\tau_0,b_0,\beta_0)=(F_1,F_2)(\tau_0,b_0,\beta_0)=0.$$

Now we admit the following expansion (the proof will be given below)

$$\begin{aligned}\mathrm{Det}(\mathbf{J})(\tau_0,b_0,\beta_0) &= m_0\|\phi_{1,b_0,\beta_0}\|^2_{L^2_{\rho_{\beta_0}}} b_0^{\frac{\alpha}{2}-1}\varepsilon_1(\tau_0)\left(\frac{\|\phi_{1,b_0,\beta_0}\|^2_{L^2_{\rho_{\beta_0}}}\|\phi_{0,b_0,\beta_0}\|^{-2}_{L^2_{\rho_{\beta_0}}}}{4\beta_0}\right) \\ &+ o(b_0^{\alpha-1}),\end{aligned} \tag{5.19}$$

which implies

$$\mathrm{Det}(\mathbf{J})(\tau_0,b_0,\beta_0)\neq 0,$$

provided that b_0 small enough i.e. $\tau_0\geq\tau_{2,1}(\delta)$. Finally, we apply the implicit function theorem to conclude the unique existence of the functions $(b,\beta)\in C([\tau_0,\tilde\tau_1],\mathbb{R}^2)\cap C^1((\tau_0,\tilde\tau_1],\mathbb{R}^2)$ for some $\tilde\tau_1>\tau_0$ such that

$$F(\tau,b(\tau),\beta(\tau))=0,\forall\tau\in[\tau_0,\tilde\tau_1].$$

Now, we define $\tau^*_{loc}=\min(\tilde\tau_1,\tilde\tau)$ and we define $\varepsilon(\tau)=w(\tau)-Q_{b(\tau)},\tau\in[\tau_0,\tau_1]$. Thus, (ε,b,β) reads (2.17-3.4-3.5) for all $\tau\in[\tau_0,\tau^*_{loc}]$.

Besides that, from definition (5.14) and the continuity of the solution, there exists $A_2(\delta)$ such that for all $A\geq A_2$ there exists $\eta_2(A,\delta)$ such that for all $\eta\leq\eta_2$ there exists $\tilde\eta_2(A,\delta,\eta)$ such that for all $\tilde\eta\leq\tilde\eta_2$ there exists $\tau_2(A,\delta,\eta,\tilde\eta)\geq 1$ such that for all $\tau_0\geq\tau_2$ and $\tau^*_{loc}>\tau_0$ such that $(\varepsilon,b,\beta)\in V_1[A,\eta,\tilde\eta](\bar\tau),\forall\bar\tau\in(\tau_0,\tau^*_{loc}]$. To finish the proof we now complete the proof of (5.19) provided that $\delta\leq\delta_2,\eta\leq\eta_2(\delta),\tilde\eta\leq\tilde\eta_2(\delta,\eta)$ and $\tau_0\geq\tau_2(\delta,\eta,\tilde\eta)$.

Let us recall Jacobian matrix $\mathbf{J}$ defined by

$$\mathbf{J}(\tau,b,\beta)=\begin{pmatrix}\frac{\partial F_1}{\partial b} & \frac{\partial F_1}{\partial\beta}\\ \frac{\partial F_2}{\partial b} & \frac{\partial F_2}{\partial\beta}\end{pmatrix}(\tau,b,\beta).$$

We now explicitly write the partial derivatives:

$$\begin{aligned}\frac{\partial F_1}{\partial b} &= \int\frac{1}{2b}\Lambda_yQ_b\left(\|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}c_{\ell,0}\phi_{1,b,\beta}+\|\phi_{0,b,\beta}\|^{-2}_{\rho_\beta}\phi_{0,b,\beta}\right)\rho_\beta dy\\ &+ \int(w(\tau)-Q_b)\partial_b\left(\|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}c_{\ell,0}\phi_{1,b,\beta}+\|\phi_{0,b,\beta}\|^{-2}_{\rho_\beta}\phi_{0,b,\beta}\right)\rho_\beta dy,\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}\frac{dF_1}{\partial\beta} &= \partial_\beta(\|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}})c_{\ell,0}\int(w-Q_b)\phi_{1,b,\beta}\rho_\beta dy+\|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}c_{1,0}\int(w-Q_b)\partial_\beta\phi_{1,b,\beta}\rho_\beta dy\\ &+ \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}c_{1,0}\int(w-Q_b)\phi_{1,b,\beta}\partial_\beta\rho_\beta dy+\partial_\beta(\|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}})\int(w-Q_b)\phi_{0,b,\beta}\rho_\beta dy\\ &+ \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\int(w-Q_b)\partial_\beta\phi_{0,b,\beta}\rho_\beta dy+\|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\int(w-Q_b)\phi_{0,b,\beta}\partial_\beta\rho_\beta dy,\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}\frac{\partial F_2}{\partial b} &= \int\frac{1}{2b}\Lambda Q_b\phi_{1,b,\beta}\rho_\beta dy+\int(w(\tau)-Q_\beta)\partial_b\phi_{1,b,\beta}\rho_\beta dy+\frac{2}{\alpha}m_0\partial_b\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}b^{\frac{\alpha}{2}}\\ &+ m_0\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}b^{\frac{\alpha}{2}-1},\end{aligned} \tag{5.22}$$

and

$$\frac{\partial F_2}{\partial\beta}=\int(w(\tau)-Q_b)\partial_\beta\phi_{1,b,\beta}\rho_\beta dy+\int(w(\tau)-Q_b)\phi_{1,b,\beta}\partial_\beta\rho_\beta dy+\frac{2}{\alpha}m_0\partial_\beta\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}b^{\frac{\alpha}{2}}. \tag{5.23}$$

We now claim the following (which will be proved later)

$$\frac{\partial F_1}{\partial b}(\tau_0, b(\tau_0), \beta(\tau_0)) = m_0 b^{\frac{\alpha}{2}-1} + o(b^{\frac{\alpha}{2}-1}), \tag{5.24}$$

$$\frac{\partial F_1}{\partial \beta}(\tau_0, b(\tau_0), \beta(\tau_0)) = -\tfrac{1}{\beta}\varepsilon_1(\tau_0) - \|\phi_{1,b,\beta}\|^2 \|\phi_{0,b,\beta}\|^{-2} \tfrac{\varepsilon_1(\tau_0)}{4\beta} + o(b^{\frac{\alpha}{2}}) \ , \tag{5.25}$$

$$\frac{\partial F_2}{\partial b}(\tau_0, b(\tau_0), \beta(\tau_0)) = m_0 \|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} b^{\frac{\alpha}{2}-1} + o(b^{\frac{\alpha}{2}-1}), \tag{5.26}$$

$$\frac{\partial F_2}{\partial \beta}(\tau_0, b(\tau_0), \beta(\tau_0)) = -\frac{1}{\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} \varepsilon_1 + o(b^{\frac{\alpha}{2}}). \tag{5.27}$$

Indeed, using estimates ((5.24)-(5.27)) with $(b(\tau_0), \beta(\tau_0)) = (b_0, \beta_0)$, we derive

$$\mathrm{Det}(\mathbf{J})(\tau_0, b_0, \beta_0) = m_0 \|\phi_{1,b_0,\beta_0}\|^2_{L^2_{\rho_{\beta_0}}} b^{\frac{\alpha}{2}-1} \varepsilon_1(\tau_0) \left(\frac{\|\phi_{1,b_0,\beta_0}\|^2_{L^2_{\rho_{\beta_0}}} \|\phi_{0,b_0,\beta_0}\|^{-2}_{L^2_{\rho_{\beta_0}}}}{4\beta_0} \right) + o(b^{\alpha-1}).$$

Thus, we get

$$\mathrm{Det}(\mathbf{J})(\tau_0, b(\tau_0), \beta(\tau_0)) \neq 0,$$

provided that b_0 is small enough. Finally, we apply the implicit function theorem to get existence of $(b, \beta) \in C([\tau_0, \tau_1], \mathbb{R}^2) \cap C^1((\tau_0, \tau_1], \mathbb{R}^2)$ for some $\tau_1 > \tau_0$ and $\varepsilon = w - Q_b$ satisfying the decomposition (7.6) and the compatibility (3.5) for $\tau \in [\tau_0, \tau_1]$. In addition to that, since $\varepsilon = w - Q_b$, ε evidently solves (2.17)

Let us now give the details of the computation.

- For (5.24):
we have

$$\int \frac{1}{2b} \Lambda_y Q_b \left(\|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} c_{1,0} \phi_{1,b,\beta} + \|\phi_{0,b,\beta}\|^{-2}_{\rho_\beta} \phi_{0,b,\beta} \right) \rho_\beta dy = m_0 b^{\frac{\alpha}{2}-1} + o(b^{\frac{\alpha}{2}-1}).$$

Next, we estimate

$$\partial_b \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} = -\|\phi_{1,b,\beta}\|^{-4}_{L^2_{\rho_\beta}} \left(2 \int \phi_{\ell,b,\beta} \partial_b \phi_{\ell,b,\beta} \rho_\beta dy \right).$$

From (6.25), we get

$$b \partial_b \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} = O(b^{1-\frac{\epsilon}{2}}). \tag{5.28}$$

Since we choose an initial data $\varepsilon(\tau_0) = \psi(\tau_0)$ defined as in (5.14) and satisfying $w(\tau_0) - Q_{b_0} = \varepsilon(\tau_0)$, we obtain that

$$|\varepsilon(\tau_0)| = |w(\tau_0) - Q_b| \leq C b^{\frac{\alpha}{2}} \frac{(1+y^4)}{y^\gamma}.$$

Hence, from (6.25), we infer that

$$\left| \int (w - Q_b) \partial_b \phi_{\ell,b,\beta} \rho_\beta dy \right| = o(b^{\frac{\alpha}{2}-1}).$$

It follows that,

$$\int (w(\tau_0) - Q_b) \partial_b \left(\|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} c_{1,0} \phi_{1,b,\beta} \right) \rho_\beta dy = o(b^{1-\frac{\epsilon}{2}}).$$

Similarly,

$$\int (w(\tau_0) - Q_b) \partial_b \left(\|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} \phi_{0,b,\beta} \right) \rho_\beta dy = o(b^{1-\frac{\epsilon}{2}}).$$

Finally, by adding all integrals in (5.20), (5.24) follows.

- For (5.25): from (5.21), we will establish the following estimates

$$\partial_\beta \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} = \left[\frac{\left(\frac{d}{2}-\gamma+1\right)}{\beta} - \frac{d+2}{2\beta}\right] \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}}), \tag{5.29}$$

$$\partial_\beta \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} = \left[\frac{\left(\frac{d}{2}-\gamma+1\right)}{\beta} - \frac{d+2}{2\beta}\right] \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}}). \tag{5.30}$$

We remark that these estimates are similar, so we only give the proof of (5.29). Indeed, we write

$$\partial_\beta \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} = -\|\phi_{1,b,\beta}\|^{-4}_{L^2_{\rho_\beta}} \left(2\int \phi_{1,b,\beta}\partial_\beta\phi_{1,b,\beta}\rho_\beta dy + \int \phi_{1,b,\beta}\phi_{1,b,\beta}\partial_\beta\rho_\beta dy\right).$$

From the construction of $\phi_{\ell,b,\beta}$ in Proposition 4.2, we have

$$\begin{aligned}
\partial_\beta \phi_{1,b,\beta} &= \frac{1}{\beta} c_{1,1}(2\beta)(\sqrt{b})^{2-\gamma} T_1(\xi) + \partial_\beta \tilde{\phi}_{\ell,b,\beta} \\
&= \frac{1}{\beta}\phi_{1,b,\beta} + \sum_{j=0}^{\ell-1} \tilde{c}_j(\beta)\phi_{j,b,\beta} + \tilde{\Phi}_{1,b,\beta}, \text{ with } \|\tilde{\Phi}_1\|_{L^2_{\rho_\beta}} \le Cb^{1-\frac{\epsilon}{2}}.
\end{aligned} \tag{5.31}$$

Then,

$$\int_0^\infty 2\phi_{1,b,\beta}\partial_\beta\phi_{1,b,\beta}\rho_\beta dy = 2\left(\frac{1}{\beta}\|\phi_{1,b,\beta}\|^2_{\rho_\beta} + O(b^{1-\frac{\epsilon}{2}})\right). \tag{5.32}$$

For the second integral, we use the identity

$$\partial_\beta \rho_\beta = \frac{d+2}{2\beta}\rho_\beta - \frac{y^2}{2}\rho_\beta, \tag{5.33}$$

to derive

$$\int_0^\infty \phi^2_{1,b,\beta}\partial_\beta\rho_\beta dy = \frac{d+2}{2\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} - \int_0^\infty \phi_{1,b,\beta}\phi_{1,b,\beta}\frac{y^2}{2}\rho_\beta dy.$$

Besides that, by Proposition 4.2 we have

$$\|\phi_{1,b,\beta} - \phi_{1,\infty,\beta}\|_{L^2_{\rho_\beta}} \le Cb^{1-\frac{\epsilon}{2}},$$

which yields

$$\int_0^\infty \phi_{1,b,\beta}\phi_{1,b,\beta}\frac{y^2}{2}\rho_\beta dy = \int_0^\infty \phi_{1,\infty,\beta}\phi_{1,\infty,\beta}\frac{y^2}{2}\rho_\beta dy + O(b^{1-\frac{\epsilon}{2}}). \tag{5.34}$$

In addition, we use $\phi_{\ell,\infty,\beta}$ as in (4.2) to get

$$\begin{aligned}
\frac{y^2}{2}\phi_{1,\infty,\beta} &= \frac{y^2}{2}\left\{a_{1,1}(2\beta)y^{2-\gamma} + a_{1,0}(2\beta)^{\ell-1}y^{-\gamma}\right\} \\
&= \frac{1}{4\beta}(2\beta)^2 y^{4-\gamma} - \frac{1}{\beta}(\frac{d}{2}-\gamma+1)(2\beta)y^{2-\gamma} \\
&= \frac{1}{4\beta}\phi_{2,\infty,\beta} - \frac{1}{4\beta}a_{2,1}\phi_{1,\infty,\beta} - \frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\phi_{1,\infty,\beta} \\
&= \frac{1}{4\beta}\phi_{2,\infty,\beta} + \left(\frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right) - \frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right)\phi_{1,\infty,\beta}.
\end{aligned}$$

Then

$$\int_0^\infty \phi_{1,\infty,\beta}\phi_{1,\infty,\beta}\frac{y^2}{2}\rho_\beta dy = \left(\frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right)-\frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right)\|\phi_{1,\infty,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}})$$

$$= \left(\frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right)-\frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right)\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}}). \quad (5.35)$$

This concludes the proof of (5.29).

Next, we will prove the following

$$\int_0^\infty \phi_{0,b,\beta}\partial_\beta\phi_{1,b,\beta}\rho_\beta dy = \frac{1}{4\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}}). \quad (5.36)$$

Indeed, using the orthogonality between $\phi_{0,b,\beta}$ and $\phi_{1,b,\beta}$ we get

$$\begin{aligned} 0 &= \partial_\beta \int \phi_{0,b,\beta}\phi_{1,b,\beta}\rho_\beta dy = \int \partial_\beta\phi_{0,b,\beta}\phi_{1,b,\beta}\rho_\beta dy + \int \phi_{0,b,\beta}\partial_\beta\phi_{1,b,\beta}\rho_\beta dy \\ &+ \int \phi_{0,b,\beta}\phi_{1,b,\beta}\partial_\beta\rho_\beta dy \\ &= \int \phi_{0,b,\beta}\partial_\beta\phi_{1,b,\beta}\rho_\beta dy + \int \phi_{0,b,\beta}\phi_{1,b,\beta}\partial_\beta\rho_\beta dy + O(b^{1-\frac{\epsilon}{2}}). \end{aligned}$$

From (5.33), we obtain

$$\int \phi_{0,b,\beta}\partial_\beta\phi_{1,b,\beta}\rho_\beta dy = \int \frac{y^2}{2}\phi_{0,b,\beta}\phi_{1,b,\beta}\rho_\beta dy + O(b^{1-\frac{\epsilon}{2}}). \quad (5.37)$$

In addition to that, we have the following identity

$$\frac{y^2}{2}\phi_{0,\infty,\beta} = \frac{1}{4\beta}\phi_{1,\infty,\beta} + \frac{\frac{d}{2}-\gamma+1}{\beta}\phi_{0,\infty,\beta}, \quad (5.38)$$

and (5.36) follows.

- For (5.26): from (5.22), we have

$$\int \frac{\Lambda Q_b}{b}\phi_{1,b,\beta}\rho_\beta dy = o(b^{\frac{\alpha}{2}-1}), \quad (5.39)$$

from (4.5) and the orthogonality between $\phi_{0,b,\beta}$ and $\phi_{1,b,\beta}$. Moreover, (6.25) ensures that

$$\int (w-Q_b)\partial_b\phi_{1,b,\beta}\rho_\beta = o(b^{\frac{\alpha}{2}-1}),$$

and (5.28) implies

$$\frac{2}{\alpha}m_0\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} b^{\frac{\alpha}{2}} = o(b^{\frac{\alpha}{2}-1}).$$

Finally, we get

$$\frac{\partial F_2}{\partial b}(\tau_0, b(\tau_0), \beta(\tau_0)) = m_0\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} b^{\frac{\alpha}{2}-1} + o(b^{\frac{\alpha}{2}-1}), \quad (5.40)$$

which concludes (5.26).

- For (5.27) we use

$$|w(\tau_0) - Q_b| \le Cb^{\frac{\alpha}{2}}\frac{(1+y^4)}{y^\gamma},$$

and

$$\partial_\beta\phi_{1,b,\beta} = \frac{1}{\beta}\phi_{1,b,\beta} + \tilde{\Phi}_{1,b,\beta}, \text{ with } \|\tilde{\Phi}_1\|_{L^2_{\rho_\beta}} \le Cb^{1-\frac{\epsilon}{2}}.$$

Using

$$w(\tau_0)-Q_b=\varepsilon(\tau_0)=\sum_{j=0}^{1}\varepsilon_j(\tau_0)\phi_{j,b,\beta}+\varepsilon_-(\tau_0),$$

we infer

$$\int(w(\tau_0)-Q_b)\partial_\beta\phi_{1,b,\beta}\rho_\beta dy=\varepsilon_\ell(\tau_0)\|\phi_{\ell,b,\beta}\|^2+o(b^{\frac{\alpha}{2}}).$$

It follows that

$$\int(w(\tau_0)-Q_b)\phi_{1,b,\beta}\partial_\beta\rho_\beta dy=\frac{d+2}{2\beta}\int(w(\tau_0)-Q_b)\phi_{1,b,\beta}\rho_\beta dy-\frac{1}{2}\int(w(\tau_0)-Q_b)\phi_{1,b,\beta}y^2\rho_\beta dy.$$

Since $\varepsilon_j=O(b^{\frac{\alpha}{2}+\eta}(\tau_0)), |\varepsilon_-|_{L^2_\rho}\le Cb^{\frac{\alpha}{2}+\tilde\eta}$, we get for the first integral is

$$\int(w(\tau_0)-Q_b)\phi_{1,b,\beta}\rho_\beta dy=\varepsilon_1(\tau_0)\|\phi_{1,b,\beta}\|^2. \tag{5.41}$$

For the second integral, we use the expansion of $w(\tau_0)-Q_b$ to obtain

$$\int(w(\tau_0)-Q_b)\phi_{1,b,\beta}\frac{y^2}{2}\rho_\beta dy=\varepsilon_0(\tau_0)\int\phi_{0,b,\beta}\phi_{1,b,\beta}\frac{y^2}{2}\rho_\beta dy+\varepsilon_1(\tau_0)\int\phi_{1,b,\beta}^2\frac{y^2}{2}\rho_\beta dy+o(b^{\frac{\alpha}{2}}). \tag{5.42}$$

In addition, we have that

$$\int_0^\infty\phi_{1,b,\beta}\phi_{1,b,\beta}\frac{y^2}{2}\rho_\beta dy=\int_0^\infty\phi_{1,\infty,\beta}\phi_{1,\infty,\beta}\frac{y^2}{2}\rho_\beta dy+O(b^{1-\frac{\epsilon}{2}})$$

and

$$\int_0^\infty\phi_{1,\infty,\beta}\phi_{1,\infty,\beta}\frac{y^2}{2}\rho_\beta dy \;=\; \left(\frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right)-\frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right)\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}+O(b^{1-\frac{\epsilon}{2}}).$$

Hence

$$\int(w(\tau_0)-Q_b)\phi_{1,b,\beta}\frac{y^2}{2}\rho_\beta dy=\left(\frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right)-\frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right)\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}+O(b^{1-\frac{\epsilon}{2}}). \tag{5.43}$$

It remains to estimate the last term in $\frac{dF_2}{d\beta}$, namely, $\frac{2}{\alpha}m_0\partial_\beta\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}b^{\frac{\alpha}{2}}$. Indeed, we have

$$\partial_\beta\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}=2\int\partial_\beta\phi_{1,b,\beta}\phi_{1,b,\beta}\rho_\beta+\int\phi_{1,b,\beta}^2\partial_\beta\rho_\beta.$$

Arguing as above, we get

$$\partial_\beta\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}=\left(2\frac{1}{\beta}+\frac{d+2}{2\beta}-\left(\frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right)-\frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right)\right)\|\phi_{1,b,\beta}\|^2_{\rho_\beta}+O(b^{1-\frac{\epsilon}{2}}) \tag{5.44}$$

Putting the different contributions of $\frac{dF_2}{d\beta}$ together, we arrive at

$$\frac{dF_2}{d\beta}=-\frac{1}{\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}\varepsilon_1+o(b^{\frac{\alpha}{2}}).$$

as claimed.

□

5.3. The proof of Theorem 1.1

In this part, we focus on the proof of Theorem 1.1 which immediately the following result:

Proposition 5.4. *There exist $A, \eta, \tilde{\tau}$ such that we can find $\delta \gg \eta, \tilde{\eta}$ and $\tau_0(A, \eta, \tilde{\eta}, \delta)$ small enough such that with initial data $\varepsilon(\tau_0)$ defined as in (5.14) and $(b, \beta)(\tau_0) = \left(e^{\left(1-\frac{2}{\alpha}\right)\tau_0}, \frac{1}{2}\right)$, the solution (ε, b, β) exists for all $\tau \geq \tau_0$ and satisfies*

$$(\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau), \ \forall \tau \geq \tau_0.$$

Proof. Let us define τ^* by

$$\tau^* = \sup\{\tau_1 \geq \tau_0 \text{ such that } (\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau), \forall \tau \in [\tau_0, \tau_1]\}. \tag{5.45}$$

By contradiction we suppose that $\tau^* < +\infty$. Lemmas 7.1, 7.2 and 7.3, the bounds in $V_1[A, \eta, \tilde{\eta}](\tau^*)$ involving $\|\varepsilon\|_{L^2_{\rho_{\beta(\tau)}}}$, ε_- and ε_e are improved by a factor $\frac{1}{2}$. In addition to that the improvement for b and β comes from Lemma 6.1. Indeed, we have

$$|\beta(\tau^*) - \beta(\tau_0)| \lesssim A \int_{\tau_0}^{\tau^*} b^{4\eta}(\tau')d\tau' \leq \frac{A}{2} b^{\eta}(\tau_0), \tag{5.46}$$

provided that τ_0 is large enough. For the bound on $b(\tau)$, we introduce

$$\Psi(\tau) = b(\tau) \exp\left(\left(\frac{2}{\alpha} - 1\right)\left(\int_{\tau_0}^{\tau} 2\beta(\tau')d\tau' + \tau_0\right)\right) \text{ and } \Psi(\tau_0) = 1, \tag{5.47}$$

and from (6.3), we get

$$|\Psi(\tau^*) - 1| \lesssim \int_{\tau_0}^{\tau^*} \left|\Psi(\tau')b^{4\eta}(\tau')\right| d\tau' \leq \frac{1}{10}, \tag{5.48}$$

provided that τ_0 large enough. Thus, the bound of b in the shrinking set is improved by the factor $\frac{1}{2}$. Besides that, by continuity of the solution, there exists $\nu > 0$ small such that $(\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau), \forall \tau \in [\tau^*, \tau^* + \nu]$ which contradicts to τ^*'s definition. ☐

Now, we aim to give a proof of Theorem 1.1: Let us consider suitable constants such that Proposition 5.4 holds that $(\varepsilon, b, \beta) \in V[A, \eta, \tilde{\eta}](\tau)$ for all $\tau > \tau_0$. Next, we derive the laws of b and β as follows:

(i): The law of $b(\tau)$: Let us introduce

$$\Psi(\tau) = b(\tau) \exp\left(\left(\frac{2}{\alpha} - 1\right)\left[\int_{\tau_0}^{\tau} 2\beta(\tau') + \tau_0\right]\right), \tau \in [\tau_0, +\infty), \text{ with } \Psi(\tau_0) = 1.$$

From Lemma 6.1, we have

$$\left|\Psi'(\tau)\right| \lesssim |\Psi(\tau)| b^{4\eta}(\tau), \forall \tau \geq \tau_0,$$

since $(\varepsilon, b, \beta) \in V_1[A, \eta, \tilde{\eta}](\tau)$ for all $\tau > \tau_0$, we get

$$|\Psi(\tau)| \leq C, \text{ and } b^{4\eta}(\tau) \lesssim I^{\eta}(\tau) \text{ where } I(\tau) \text{ defined in (5.5)},$$

which yields

$$\Psi(\tau) = \Psi(\tau_0) + \int_{\tau_0}^{\infty} \Psi'(\zeta)d\zeta - \int_{\tau}^{\infty} \Psi'(\zeta)d\zeta = \Psi_\infty + O(I^{\eta}(\tau)) \text{ as } \tau \to +\infty,$$

with $\Psi_\infty = \Psi(\tau_0) + \int_{\tau_0}^{\infty} \Psi'(\zeta)d\zeta = 1 + \int_{\tau_0}^{\infty} \Psi'(\zeta)d\zeta$. Thus, we get

$$b(\tau) = \Psi_\infty \exp\left(\left(1 - \frac{2}{\alpha}\right)\left(\int_{\tau_0}^{\tau} 2\beta(\zeta)d\zeta + \tau_0\right)\right)\left[1 + O\left(I^{\eta}(\tau)\right)\right] \text{ as } \tau \to +\infty. \tag{5.49}$$

(ii) Renormalized flow $\beta(\tau)$ and derivation the law of $\mu(t)$ defined in (2.11): Using Lemma 6.1 again, we have

$$|\beta'(\tau)| \lesssim b^{4\eta}(\tau),$$

then we deduce

$$\beta(\tau) = \beta(\tau_0) + \int_{\tau_0}^{\infty} \beta'(\tau')d\tau' - \int_{\tau}^{\infty} \beta'(\tau')d\tau' = \beta_\infty + \int_{\tau}^{\infty} \beta'(\tau')d\tau' = \beta_\infty + O(I^\eta(\tau)), \text{ as } \tau \to +\infty,$$

with

$$\beta_\infty = \frac{1}{2} + \int_0^\infty \beta'(\zeta)d\zeta.$$

We introduce the renormalized time $\tilde{\tau}$ by

$$\tilde{\tau} = 2\beta(\tau)\tau, \tag{5.50}$$

which is an invertible function of τ. Indeed,

$$\tau = (2\beta_\infty)^{-1}\tilde{\tau}(1 + O(I^\eta(\tilde{\tau}))), \text{ as } \tilde{\tau} \to +\infty. \tag{5.51}$$

We shall remark that we will make an abuse of notation $\mu(\tau) = \mu(\tilde{\tau}) = \mu(t)$. The relation

$$\frac{d\mu}{d\tilde{\tau}} = \frac{d\mu}{d\tau}\frac{d\tau}{d\tilde{\tau}},$$

implies, from the fact that $\mu_\tau = -2\beta\mu$ and (5.50)

$$\frac{d\mu}{d\tilde{\tau}} = -\mu(\tilde{\tau})\left[1 + O(I^\eta(\tilde{\tau}))\right] \text{ as } \tilde{\tau} \to +\infty.$$

Thus, we get

$$\mu(\tilde{\tau}) = e^{-\tilde{\tau}}(1 + O(I^\eta(\tilde{\tau}))), \text{ as } \tilde{\tau} \to +\infty.$$

In addition, we derive from (2.11) that

$$\frac{d\tilde{\tau}}{dt} = \frac{d\tilde{\tau}}{d\tau}\frac{d\tau}{dt} = 2\beta_\infty e^{\tilde{\tau}(t)}(1 + O(I^\eta(\tilde{\tau}(t)))),$$

which implies

$$\tilde{\tau}(t) = -\ln(2\beta_\infty(T-t))(1 + O((T-t)^{\tilde{\eta}})), \text{ as } t \to T,$$

for some $T = T(\tau_0) > 0$. From (5.51), we have

$$\tau(t) = (2\beta_\infty)^{-1}\tilde{\tau}(t)(1 + I^{\tilde{\eta}}(\tilde{\tau}(t))) = -(2\beta_\infty)^{-1}\ln(2\beta_\infty(T-t))(1 + O((T-t)^{\tilde{\eta}})), \text{ as } t \to T.$$

Substituting $\mu(t)$'s formula, we get

$$\mu(t) = \mu(\tilde{\tau}(t)) = 2\beta_\infty(T-t)\left(1 + O((T-t)^{\tilde{\eta}})\right) \text{ as } t \to T.$$

Recall that

$$\int_{\tau_0}^{\tau} 2\beta(\tau')d\tau' = 2\beta_\infty\tau(1 + O(\tau^{-1})) \text{ as } \tau \to +\infty,$$

from which we deduce, with the use of (5.49), that

$$\begin{aligned} b(t) &= \Psi_\infty \exp\left(\left(1 - \frac{2}{\alpha}\right)\left(\int_{\tau_0}^{\tau} 2\beta(\zeta)d\zeta + \tau_0\right)\right)(1 + O(I^\eta)(\tau(t))) \\ &= \Psi_\infty\,(2\beta_\infty)^{\frac{2}{\alpha}-1}\,(T-t)^{\frac{2}{\alpha}-1}\left(1 + O(|\ln(T-t)|^{-1})\right) \text{ ast} \to T. \end{aligned}$$

Introducing $\lambda(t) = \mu(t)b(t)$ which satisfies

$$\lambda(t) = C(u_0)(T-t)^{\frac{2}{\alpha}}(1 + O(|\ln(T-t)|^{-1})) \text{ as } t \to T. \tag{5.52}$$

Finally, the conclusion of the Theorem 1.1 immediately follows (2.11), (2.16), the fact $(\varepsilon, b, \beta) \in V_1[A, \eta, \tilde{\eta}](\tau)$ for all $\tau > \tau_0$, and (5.52). □

6. Finite dimensional system

In this part, we study the dynamics of finite modes $\varepsilon_j(\tau)$ and the modulation parameters b and β.

Lemma 6.1. *Consider $A \geq 1, \eta > 0, \tilde{\eta} > 0$, there exists $\tau_2(A, \eta, \tilde{\eta})$ such that for all $\tau_0 \geq \tau_2$, the following holds: Assume that $(\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau), \forall \tau \in [\tau_0, \tau_1]$, for some $\tau_1 > \tau_0$, then, we have*

(i) The dominating mode ε_1 satisfies

$$\left\{ \begin{array}{l} \partial_\tau \varepsilon_1 - \left[2\beta \left(\frac{\alpha}{2} - 1 \right) \right] \varepsilon_1 = O\left(b^{\frac{\alpha}{2} + 4\eta}(\tau) [|\beta'| + \left| \frac{b_\tau}{b} \right| + 1] \right), \\ \partial_\tau \varepsilon_1 - \left[2\beta \left(\frac{\alpha}{2} \right) \right] \varepsilon_1 + m_0 \left(\frac{b_\tau}{b} - 2\beta \right) b^{\frac{\alpha}{2}} = O\left(b^{\frac{\alpha}{2} + 4\eta}(\tau) [|\beta'| + \left| \frac{b_\tau}{b} \right| + 1] \right), \end{array} \right. \tag{6.1}$$

for all $\tau \in [\tau_0, \tau_1]$.

(iii) For the b and β, we obtain

$$\left| \beta'(\tau) \right| \leq C A b^{4\eta}(\tau), \tag{6.2}$$

and

$$\left| \frac{b'(\tau)}{b(\tau)} - 2\beta \left(1 - \frac{2}{\alpha} \right) \right| \leq C A b^{4\eta}(\tau), \tag{6.3}$$

for all $\tau \in (\tau_0, \tau_1)$.

Proof. Let us consider $(\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau), \forall \tau[\tau_0, \tau_1]$ and $\varepsilon(\tau)$ decomposed as in (5.1), and we also recall that

$$\varepsilon_j = \|\phi_{j,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \varepsilon, \phi_{j,b,\beta} \rangle_{L^2_{\rho_\beta}}. \tag{6.4}$$

Then, we obtain from (3.4) that

$$\left\{ \begin{array}{lll} \varepsilon_1(\tau) & = & c_{1,0} \|\phi_{\ell,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \varepsilon, \phi_{1,b,\beta} \rangle_{L^2_{\rho_\beta}}, \\ \varepsilon_1(\tau) & = & -\|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \varepsilon, \phi_{0,b,\beta} \rangle_{L^2_{\rho_\beta}}. \end{array} \right. \tag{6.5}$$

By taking τ-derivative of the second equation of the above system, we get

$$\begin{aligned} -\partial_\tau \varepsilon_1 &= \partial_\tau \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \varepsilon, \phi_{0,b,\beta} \rangle_{L^2_{\rho_\beta}} + \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \partial_\tau \varepsilon, \phi_{0,b,\beta} \rangle_{L^2_{\rho_\beta}} + \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \varepsilon, \partial_\tau \phi_{0,b,\beta} \rangle_{L^2_{\rho_\beta}} \\ &+ \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} \left\langle \varepsilon, \frac{\partial_\tau \rho_\beta}{\rho_\beta} \right\rangle_{L^2_{\rho_\beta}}, \end{aligned} \tag{6.6}$$

where ρ_β defined as in (2.25). A direct computation gives

$$\begin{aligned} \partial_\tau \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}} &= \partial_\tau \frac{1}{\int_{\mathbb{R}_+} \phi^2_{0,b,\beta} \rho_\beta dy} = -\frac{2 \int_{\mathbb{R}_+} \phi_{0,b,\beta} \partial_\tau \phi_{0,b,\beta} \rho_\beta dy + \int_{\mathbb{R}_+} \phi_{0,b,\beta} \phi_{0,b,\beta} \partial_\tau \rho_\beta dy}{\left(\int_{\mathbb{R}_+} \phi^2_{0,b,\beta} \rho_\beta dy \right)^2} \\ &= -\|\phi_{0,b,\beta}\|^{-4}_{L^2_{\rho_\beta}} \left(2 \frac{b'}{b} \langle \phi_{0,b,\beta}, b \partial_b \phi_{0,b,\beta} \rangle_{L^2_{\rho_\beta}} + 2\beta' \langle \phi_{0,b,\beta}, \partial_\beta \phi_{0,b,\beta} \rangle_{L^2_{\rho_\beta}} \right. \\ &+ \left. \beta' \left\langle \phi_{0,b,\beta}, \phi_{0,b,\beta} \left(\frac{d+2}{2\beta} - \frac{y^2}{2} \right) \right\rangle_{L^2_{\rho_\beta}} \right), \end{aligned}$$

and

$$\begin{aligned}\langle \varepsilon, \partial_\tau \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} &= \frac{b'}{b}\langle \varepsilon, b\partial_b \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + \beta'\langle \varepsilon, \partial_\beta \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}},\\ \left\langle \varepsilon, \phi_{0,b,\beta}\frac{\partial_\tau \rho_\beta}{\rho_\beta}\right\rangle_{L^2_{\rho_\beta}} &= \beta'\left\langle \varepsilon, \phi_{0,b,\beta}\left(\frac{d+2}{2\beta}-\frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}.\end{aligned}$$

We plug these equalities into (6.6) to derive

$$-\partial_\tau \varepsilon_1 = \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\langle \partial_\tau \varepsilon, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + \tilde{K}_0, \tag{6.7}$$

where

$$\begin{aligned}\tilde{K}_0 &= -\|\phi_{0,b,\beta}\|^{-4}_{L^2_{\rho_\beta}}\langle \varepsilon, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}}\left\{2\frac{b'}{b}\langle \phi_{0,b,\beta}, b\partial_b \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + 2\beta'\langle \phi_{0,b,\beta}, \partial_\beta \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}}\right.\\ &+ \left.\beta'\left\langle \phi_{0,b,\beta}, \phi_{0,b,\beta}\left(\frac{d+2}{2\beta}-\frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}\right\} + \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\left\{\frac{b'}{b}\langle \varepsilon, b\partial_b\phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}}\right.\\ &+ \left.\beta'\langle \varepsilon, \partial_\beta \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + \beta'\left\langle \varepsilon, \phi_{0,b,\beta}\left(\frac{d+2}{2\beta}-\frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}\right\}.\end{aligned} \tag{6.8}$$

Similarly, we derive from the first one in (6.31) that

$$c_{1,0}^{-1}\partial_\tau \varepsilon_1 = \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\langle \partial_\tau \varepsilon, \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} + \tilde{K}_1,$$

where

$$\begin{aligned}\tilde{K}_1 &= -\|\phi_{1,b,\beta}\|^{-4}_{L^2_{\rho_\beta}}\langle \varepsilon, \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}}\left\{2\frac{b'}{b}\langle \phi_{1,b,\beta}, b\partial_b \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} + 2\beta'\langle \phi_{1,b,\beta}, \partial_\beta \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}}\right.\\ &+ \left.\beta'\left\langle \phi_{1,b,\beta}, \phi_{1,b,\beta}\left(\frac{d+2}{2\beta}-\frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}\right\} + \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\left\{\frac{b'}{b}\langle \varepsilon, b\partial_b\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}}\right.\\ &+ \left.\beta'\langle \varepsilon, \partial_\beta \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} + \beta'\left\langle \varepsilon, \phi_{1,b,\beta}\left(\frac{d+2}{2\beta}-\frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}\right\}.\end{aligned} \tag{6.9}$$

We have the following system

$$\left\{\begin{aligned} c_{1,0}^{-1}\partial_\tau \varepsilon_1(\tau) &= \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\langle \partial_\tau \varepsilon, \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} + \tilde{K}_1,\\ -\partial_\tau \varepsilon_1(\tau) &= \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\langle \partial_\tau \varepsilon, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + \tilde{K}_0.\end{aligned}\right. \tag{6.10}$$

Since ε solves (2.17), we obtain

$$\langle \partial_\tau \varepsilon, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} = \langle \mathscr{L}_b \varepsilon, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + \langle B(\varepsilon), \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + \langle \Phi, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}},$$

which implies

$$\left\{\begin{aligned} c_{1,0}^{-1}\partial_\tau \varepsilon_1 &= \|\phi_{1,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\left\{\langle \mathscr{L}_b \varepsilon, \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} + \langle B(\varepsilon), \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} + \langle \Phi, \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}}\right\} + \tilde{K}_1\\ -\partial_\tau \varepsilon_1 &= \|\phi_{0,b,\beta}\|^{-2}_{L^2_{\rho_\beta}}\left\{\langle \mathscr{L}_b \varepsilon, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + \langle B(\varepsilon), \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}} + \langle \Phi, \phi_{0,b,\beta}\rangle_{L^2_{\rho_\beta}}\right\} + \tilde{K}_0\end{aligned}\right. . \tag{6.11}$$

We only estimate all terms of the first equation in (6.11), the rest is left to the reader.

- *For* $\langle \mathscr{L}_b\varepsilon, \phi_{j,b,\beta}\rangle_{L^2_{\rho_\beta}}$: Using the fact that $\mathscr{L}_b$ is self-adjoint and the special decomposition (7.6), we have

$$\langle \mathscr{L}_b\varepsilon, \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} = \langle \varepsilon, \mathscr{L}_b\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} = \lambda_{1,b,\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}\frac{\varepsilon_1}{c_{1,0}} \tag{6.12}$$

where

$$\lambda_{1,b,\beta} = 2\beta\left(\frac{\alpha}{2}-1\right) + \tilde{\lambda}_{1,b,\beta}, \text{ with } \left|\tilde{\lambda}_{1,b,\beta}\right| \lesssim b^{1-\frac{\epsilon}{2}}.$$

- *For* $\langle B(\varepsilon), \phi_{j,b,\beta}\rangle_{L^2_{\rho_\beta}}$: We recall $B(\varepsilon)$ (2.19) in the below

$$B(\varepsilon) = -3(d-2)(1+|y|^2Q_b)\varepsilon^2 - (d-2)|y|^2\varepsilon^3.$$

From (5.1) we have

$$\begin{aligned} \left|3(d-2)(1+y^2Q_b)\varepsilon^2\right| &\lesssim \varepsilon_1^2\left(\frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta}\right)^2 + \varepsilon_-^2, \\ \left|(d-2)|y|^2\varepsilon^3\right| &\lesssim y^2\left(|\varepsilon_\ell|^3\left|\frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta}\right|^3 + |\varepsilon_-|^3\right). \end{aligned}$$

Since $(\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau)$, for all $\tau \in [\tau_0, \tau_1]$ which ensures the pointwise estimates given in Proposition (4.2) to deduce that

$$\left|\langle B(\varepsilon), \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}}\right| \lesssim |\varepsilon_1|^2 + \int_{\mathbb{R}}\left[\varepsilon_-^2 + y^2\left|\varepsilon_-\right|^3\right]|\phi_{j,b,\beta}|\,\rho_\beta dy.$$

Lemma 5.2 yields

$$|\varepsilon_-(y,\tau)| \le CA^4 b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\frac{\langle y\rangle^{2\ell+2}}{y^\gamma}, \forall y \in \mathbb{R}_+^*, \tag{6.13}$$

which yields

$$\int_{\mathbb{R}_+}\left[\varepsilon_-^2 + y^2\left|\varepsilon_-\right|^3\right]|\phi_{j,b,\beta}|\,\rho_\beta dy \lesssim b^{\frac{\alpha}{2}+4\eta}(\tau).$$

Hence, we get

$$\langle B(\varepsilon), \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} \lesssim b^{\frac{\alpha}{2}+4\eta}(\tau). \tag{6.14}$$

- *For* $\langle \Phi(.,\tau), \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}}$: Following Φ's definition in (2.20), we have

$$\Phi(y,\tau) = \left[\frac{b'(\tau)}{b(\tau)} - 2\beta\right]\frac{1}{2b}\Lambda_y Q\left(\frac{y}{\sqrt{b}}\right) = \left[\frac{b'(\tau)}{b(\tau)} - 2\beta\right]\frac{1}{2b}\Lambda_\xi Q(\xi), \text{ with } \xi = \frac{y}{\sqrt{b}}.$$

Accordingly to $\phi_{0,b,\beta}$'s formula in Proposition 4.2, and the construction of T_0 in Lemma 10.1, we write Φ as follows

$$\Phi = \left[\frac{b'}{b} - 2\beta\right]m_0 b^{\frac{\alpha}{2}}\phi_{0,b,\beta} + \tilde{\Phi}, \tag{6.15}$$

for some $m_0 \neq 0$, and

$$\|\tilde{\Phi}\|_{L^2_{\rho_\beta}} \lesssim \left|\frac{b_\tau}{b} - 2\beta\right| b^{\frac{\alpha}{2}+1-\frac{\epsilon}{2}}.$$

This immediately implies

$$\langle \Phi(.,\tau), \phi_{j,b,\beta}\rangle_{L^2_{\rho_\beta}} = \left\{\begin{array}{lr} m_0 b^{\frac{\alpha}{2}}\left[\frac{b'(\tau)}{b(\tau)} - 2\beta\right]\|\phi_{0,b,\beta}\|^2_{L^2_{\rho_\beta}} + O\left(\left|\frac{b_\tau}{b} - 2\beta\right| b^{\frac{\alpha}{2}+1-\frac{\epsilon}{2}}\right) & \text{if} \quad j=0 \\ O\left(\left|\frac{b_\tau}{b} - 2\beta\right| b^{\frac{\alpha}{2}+1-\frac{\epsilon}{2}}\right) & \text{if} \quad j=1 \end{array}\right. . \tag{6.16}$$

- *For* $\tilde{K}_1$: Let us consider $\delta \ll \min\left(\frac{1}{2}, 1-\frac{\epsilon}{2}\right)$ with ϵ defined as in Proposition (4.2) and $\delta \gg \eta \gg \tilde{\eta}$. From $\tilde{K}_1$'s definition given in (6.9), we will prove the following bounds:

$$\langle \phi_{1,b,\beta}, b\partial_b \phi_{1,b,\beta}\rangle_{L^2_\rho} \lesssim b^\delta, \tag{6.17}$$

$$\langle \varepsilon, b\partial_b \phi_{1,b,\beta}\rangle_{L^2_\rho} \lesssim b^\delta\left(|\varepsilon_1| + \left\|y^\gamma \frac{\varepsilon_-(\cdot,\tau)}{1+y^4}\right\|_{L^\infty_{[0,b^{-\tilde{\eta}}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right), \tag{6.18}$$

$$\langle \phi_{1,b,\beta}, \partial_\beta \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} = \frac{1}{\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^\delta), \tag{6.19}$$

$$\begin{aligned}\left\langle \phi_{1,b,\beta}, \phi_{1,b,\beta}\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}} &= \left[\frac{d+2}{2\beta} - \frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right) + \frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right]\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}\\ &+ O(b^\delta),\end{aligned} \tag{6.20}$$

and with the equality (7.6)

$$\begin{aligned}\langle \varepsilon, \partial_\beta \phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} &= \frac{1}{\beta}\frac{\varepsilon_1}{c_{1,0}}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} - \frac{\varepsilon_1}{4\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} + O\left(\left\|y^\gamma \frac{\varepsilon_-}{1+y^4}\right\|_{L^\infty}\right) \qquad (6.21)\\ &+ O\left(b^\delta\left(|\varepsilon_\ell| + \left\|y^\gamma \frac{\varepsilon_-(\cdot,\tau)}{1+y^4}\right\|_{L^\infty_{[0,b^{-\tilde{\eta}}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right)\right), \qquad (6.22)\end{aligned}$$

and

$$\left\langle \varepsilon, \phi_{1,b,\beta}\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}} = \left\{\begin{array}{l} \varepsilon_1\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}}\left[\frac{1}{c_{1,0}}\left(\frac{d+2}{2\beta} - \frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right) + \frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right) + \frac{1}{4\beta}\right]\\ + \; O\left(b^\delta\left(|\varepsilon_1| + \left\|y^\gamma\frac{\varepsilon_-(\cdot,\tau)}{1+y^4}\right\|_{L^\infty_{[0,b^{-\tilde{\eta}}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right)\right),\end{array}\right. \tag{6.23}$$

and

$$\left\langle \varepsilon, \phi_{0,b,\beta}\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}} = \frac{\gamma\varepsilon_1}{\beta}\|\phi_{0,b,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^{\frac{\alpha}{2}+\delta}). \tag{6.24}$$

+ For (6.17): As we assumed $\delta \ll \frac{1}{2}$, then, for all $y \ge b^\delta$, we have

$$\xi = \frac{y}{\sqrt{b}} \to +\infty, \text{ as } b \to 0.$$

Hence, we have

$$\begin{aligned}\left|\partial_b \sum_{j=0}^{1} c_{1,j}(\sqrt{b})^{2j-\gamma}T_j(\xi)\right| &= \left|\frac{1}{2b}\sum_{j=0}^{1} c_{1,j}(2j-\gamma)(\sqrt{b})^{2j-\gamma}T_j(\xi) - \frac{1}{2b}\sum_{j=0}^{1} c_{1,j}(\sqrt{b})^{2j-\gamma}\xi\partial_\xi T_j(\xi)\right|\\ &\lesssim \sum_{j=0}^{1}\sqrt{b}^{2j-2-\gamma}\xi^{2j-\gamma-2}|\ln\xi|, \text{ as } \xi \to +\infty,\end{aligned}$$

and (10.6) ensures that

$$\xi\partial_\xi T_j(\xi) = (2j-\gamma)T_j(\xi) + O(\xi^{-\gamma+2j-2}\ln\xi), \text{ as } \xi \to +\infty.$$

Then

$$\left|b\partial_b \sum_{j=0}^{i} c_{i,j}(\sqrt{b})^{2j-\gamma}T_j(\xi)\right| \lesssim \sum_{j=0}^{i}\sqrt{b}^{2j-\gamma}\xi^{2j-\gamma-2}|\ln\xi|, \text{ as } \xi \to +\infty,$$

which implies, from (4.5) and the above inequalities, that

$$|b\partial_b\phi_{j,b,\beta}(y)| \lesssim b^{1-\frac{\epsilon}{2}} \frac{\langle y\rangle^2 |\ln y|}{y^{\gamma+2}}, \forall y \geq b^\delta \text{ and } j \leq 1. \tag{6.25}$$

Hence, we derive on the one hand

$$\left| \int_{y\geq b^\delta} \phi_{j,b,\beta} b\partial_b \phi_{i,b,\beta}\rho_\beta dy \right| \leq C b^\delta, \forall i,j \leq 1. \tag{6.26}$$

On the other hand, we apply the pointwise estimates given in Proposition 4.2 that yields

$$\left| \int_{y\leq b^\delta} \phi_{j,b,\beta} b\partial_b \phi_{i,b,\beta}\rho_\beta dy \right| \lesssim \int_{y\leq b^\delta} y^{d+1-2\gamma} e^{-\frac{2\beta y^2}{4}} dy \lesssim \int_{y\leq b^\delta} y^{d+1-2\gamma} dy \lesssim b^{\delta(d+2-2\gamma)}. \tag{6.27}$$

Combining (6.26) and (6.27), we obtain

$$\left| \langle \phi_{j,b,\beta}, b\partial_b\phi_{i,b,\beta}\rangle_{L^2_{\rho_\beta}} \right| \lesssim b^{1-\frac{\epsilon}{2}} + b^{\delta(d+2-\gamma)} \lesssim b^\delta, i,j \leq 1, \tag{6.28}$$

thus, (6.17) follows.

+ For (6.18): Using (7.6), we estimate as follows

$$|\langle \varepsilon, b\partial_b\phi_{1,b,\beta}\rangle| \lesssim \sum_{j=0}^{1} |\varepsilon_j| \left| \langle \phi_{j,b,\beta}, b\partial_b\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} \right| + \left| \langle \varepsilon_-, b\partial_b\phi_1\rangle_{L^2_{\rho_\beta}} \right|.$$

Using (6.28), we get

$$\sum_{j=0}^{1} |\varepsilon_j| \left| \langle \phi_{j,b,\beta}, b\partial_b\phi_1\rangle_{L^2_{\rho_\beta}} \right| \lesssim b^\delta \sum_{j=0}^{1} |\varepsilon_j|.$$

Next, we estimate the projection on $\partial_b\phi_1$ of ε_-. Indeed, we split the integral:

$$\begin{aligned} \left| \langle \varepsilon_-(\tau), b\partial_b\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} \right| &\leq \int_0^{b^\delta} |\varepsilon_-(\tau)| \, |b\partial_b\phi_{1,b,\beta}| \, \rho_\beta dy + \int_{b^\delta}^{b^{-\tilde\eta}} |\varepsilon_-(\tau)| \, |b\partial_b\phi_{1,b,\beta}| \, \rho_\beta dy \\ &+ \int_{b^{-\tilde\eta}}^{\infty} |\varepsilon_-(\tau)| \, |b\partial_b\phi_{1,b,\beta}| \, \rho_\beta dy. \end{aligned}$$

For the integral on $[0, b^\delta]$, we estimate as follows

$$\begin{aligned} \left| \int_0^{b^\delta} \varepsilon_-(\tau) b\partial_b\phi_{1,b,\beta}\rho_\beta dy \right| &\leq \left\| y^\gamma \frac{\varepsilon_-(\cdot,\tau)}{\langle y\rangle^4} \right\|_{L^\infty[0,b^{-\tilde\eta}]} \int_0^{b^\delta} y^{d+1-2\gamma} \\ \leq \left\| y^\gamma \frac{\varepsilon_-(\cdot,\tau)}{\langle y\rangle^4} \right\|_{L^\infty[0,b^{-\tilde\eta}]} b^{\delta(d+2-2\gamma)} &\leq \left\| y^\gamma \frac{\varepsilon_-(\cdot,\tau)}{\langle y\rangle^4} \right\|_{L^\infty[0,b^{-\tilde\eta}]} b^\delta. \end{aligned}$$

On the interval $[b^\delta, b^{-\tilde\eta}]$, we estimate

$$\begin{aligned} \left| \int_{b^\delta}^{b^{-\tilde\eta}} \varepsilon_-(\tau) b\partial_b\phi_{1,b,\beta}\rho_\beta dy \right| &\leq b^{1-\frac{\epsilon}{2}} \left\| y^\gamma \frac{\varepsilon_-(\cdot,\tau)}{1+y^4} \right\|_{L^\infty[0,b^{-\tilde\eta}]} \int_{b^\delta}^{b^{-\tilde\eta}} \frac{(1+y^4)(1+y^2)|\ln y|}{y^{2\gamma+2}} y^{d+1} e^{-\frac{2\beta y^2}{4}} dy \\ \lesssim\ & b^{1-\frac{\epsilon}{2}} \left\| y^\gamma \frac{\varepsilon_-(\cdot,\tau)}{1+y^4} \right\|_{L^\infty[0,b^{-\tilde\eta}]} \left\{ \int_{b^\delta}^1 y^{d-2\gamma-1} dy + \int_1^{+\infty} \frac{(1+y^4)(1+y^2)|\ln y|}{y^{2\gamma+2}} y^{d+1} e^{-\frac{2\beta y^2}{4}} dy \right\} \\ \lesssim\ & b^\delta \left\| y^\gamma \frac{\varepsilon_-(\cdot,\tau)}{1+y^4} \right\|_{L^\infty[0,b^{-\tilde\eta}]}. \end{aligned}$$

For the interval $[b^{-\tilde\eta}, +\infty)$, we use Cauchy-Schwarz inequality and (6.25) to arrive at

$$\left|\int_{b^{-\tilde\eta}}^{+\infty} |\varepsilon_-|\,|b\partial_b\phi_{1,b,\beta}|\rho_\beta\right| \lesssim \|\varepsilon_-\|_{L^2_{\rho_\beta}} \left(\int_{b^{-\tilde\eta}}^{\infty} |b\partial_b\phi_{1,b,\beta}|^2\rho_\beta\right)^{\frac{1}{2}} \lesssim b^{1-\frac{\epsilon}{2}}\|\varepsilon_-\|_{L^2_{\rho_\beta}} \lesssim b^{\delta}\|\varepsilon_-\|_{L^2_{\rho_\beta}},$$

which concludes (6.18) by adding all related terms.
+ For estimates (6.19) and (6.20): Indeed, from (5.31), we immediately deduce (6.19). In addition to that, by combining (5.34) and (5.35) one gets (6.20).
+ For (6.22): According to (7.6), we decompose ε as follows

$$\langle \varepsilon, \partial_\beta\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} = \varepsilon_1(\tau)\left\langle \frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta}, \partial_\beta\phi_{1,b,\beta}\right\rangle_{L^2_{\rho_\beta}} + \langle \varepsilon_-(\tau), \partial_\beta\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}}.$$

Similarly to (6.18), one can deduce that

$$\langle \varepsilon_-(y,\tau), \partial_\beta\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} \lesssim b^{\delta}\left(\left\| y^{\gamma}\frac{\varepsilon_-(\cdot,\tau)}{1+y^4}\right\|_{L^\infty_{[0,b^{-\tilde\eta}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right).$$

In addition to that, we derive from (5.32)

$$\left\langle \frac{\phi_{1,b,\beta}}{c_{1,0}}, \partial_\beta\phi_{1,b,\beta}\right\rangle_{L^2_{\rho_\beta}} = \frac{1}{c_{1,0}}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}}),$$

and from (5.36), we have

$$\langle \phi_{0,b,\beta}, \partial_\beta\phi_{1,b,\beta}\rangle_{L^2_{\rho_\beta}} = \frac{1}{4\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}}).$$

Finally, we use the above facts to get (6.22).
+ For (6.23): We firstly write as follows

$$\left\langle \varepsilon, \phi_{1,b,\beta}\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}} = \varepsilon_1(\tau)\left\langle \frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta}, \phi_{1,b,\beta}\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}$$
$$+\left\langle \varepsilon_-(y,\tau), \phi_{1,b,\beta}\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}.$$

On one hand, we have

$$\left\langle \varepsilon_-(y,\tau), \phi_{1,b,\beta}\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}$$
$$\lesssim \sum_{j=1}^{1-1}|\varepsilon_j| + b^{\delta}\left(\left\| y^{\gamma}\frac{\varepsilon_-(\cdot,\tau)}{1+y^{21+2}}\right\|_{L^\infty_{[0,b^{-\tilde\eta}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right).$$

For the rest, we obtain

$$\left\langle \frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta}, \phi_{1,b,\beta}\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}} = \frac{1}{c_{1,0}}\frac{d+2}{2\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} - \frac{1}{c_{1,0}}\left\langle \phi_{1,b,\beta}, \phi_{1,b,\beta}\frac{y^2}{2}\right\rangle_{L^2_{\rho_\beta}}$$
$$+ \left\langle \phi_{0,b,\beta}, \phi_{1,b,\beta}\frac{y^2}{2}\right\rangle_{L^2_{\rho_\beta}}.$$

Using (5.34), (5.35),(5.36) and (5.37), we have that

$$\left\langle \phi_{0,b,\beta}, \phi_{1,b,\beta}\frac{y^2}{2}\right\rangle_{L^2_{\rho_\beta}} = \frac{1}{4\beta}\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}})$$

and

$$\left\langle \phi_{1,b,\beta}, \phi_{1,b,\beta}\frac{y^2}{2}\right\rangle_{L^2_{\rho_\beta}} = \left(\frac{2}{\beta}\left(\frac{d}{2}-\gamma+2\right) - \frac{1}{\beta}\left(\frac{d}{2}-\gamma+1\right)\right)\|\phi_{1,b,\beta}\|^2_{L^2_{\rho_\beta}} + O(b^{1-\frac{\epsilon}{2}}),$$

which implies (6.23).

- For (6.24): The proof is similar to (6.23) which also follows from (5.34), (5.35),(5.36) and (5.37).

Now, combining (6.17) to (6.24), we derive

$$\begin{aligned}\tilde{K}_1 &= -\frac{\varepsilon_1\beta'}{c_{1,0}\beta} + O\left(\left|\frac{b'}{b}\right|+1\right)b^\delta\left(|\varepsilon_1| + \left\|y^\gamma\frac{\varepsilon_-(\cdot,\tau)}{1+y^4}\right\|_{L^\infty_{[0,b^{-\tilde{\eta}}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right)\\ &+ O\left(\beta' b^\delta\left(|\varepsilon_1| + \left\|y^\gamma\frac{\varepsilon_-(\cdot,\tau)}{1+y^4}\right\|_{L^\infty_{[0,b^{-\tilde{\eta}}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right)\right).\end{aligned}$$

In a convenient way, we denote

$$\begin{aligned}L &= \left(\left|\frac{b'}{b}\right|+1\right)b^\delta\left(|\varepsilon_1| + \left\|y^\gamma\frac{\varepsilon_-(\cdot,\tau)}{1+y^4}\right\|_{L^\infty_{[0,b^{-\tilde{\eta}}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right)\\ &+ \left(|\beta'|\left(b^\delta|\varepsilon_1| + \left\|y^\gamma\frac{\varepsilon_-(\cdot,\tau)}{1+y^4}\right\|_{L^\infty_{[0,b^{-\tilde{\eta}}]}} + \|\varepsilon_-\|_{L^2_{\rho_\beta}}\right)\right).\end{aligned}$$

Then, we have

$$\tilde{K}_1 = -\frac{\beta'\varepsilon_1}{c_{1,0}\beta} + O(L). \tag{6.29}$$

- Applying $\tilde{K}_1$'s process to $\tilde{K}_0$, we get

$$\tilde{K}_0 = -\frac{\|\phi_{1,b,\beta}\|^2\|\phi_{0,b,\beta}\|^{-2}\beta'}{4\beta}\frac{\varepsilon_1}{c_{1,0}} + O(L). \tag{6.30}$$

Now, we are ready to start to the proof of the Lemma.

- *Proof for (i):* We use system (6.11) combined with all of the previous estimates to derive

$$\left\{\begin{array}{lll}\partial_\tau\varepsilon_1 &=& 2\beta\left(\frac{\alpha}{2}-1\right) - \frac{\beta'}{\beta}\varepsilon_1 + O(L) + O\left(\left|\frac{b_\tau}{b}-2\beta\right|b^{\frac{\alpha}{2}+\delta}\right),\\ \partial_\tau\varepsilon_1 &=& 2\beta\frac{\alpha}{2}\varepsilon_1 - m_0\left[\frac{b'}{b}-2\beta\right]b^{\frac{\alpha}{2}} + O(L) + O\left(\left|\frac{b_\tau}{b}-2\beta\right|b^{\frac{\alpha}{2}+\delta}\right).\end{array}\right. \tag{6.31}$$

In particular, since $(\varepsilon, b, \beta)(\tau) \in V[A, \eta, \tilde{\eta}](\tau), \forall \tau \in [\tau_0, \tau_1]$, the pointwise estimates in Lemma 5.2, imply (6.1).

- *Proof for (ii):* The results immediately follows item (i). □

7. Control of the infinite dimensional part

In this part, we aim to give *a priori estimates* involving ε_- and ε_e

7.1. Energy estimate

In below, we will prove *a priori estimates* on $\|\varepsilon_-\|^2_{\rho_\beta}$.

Lemma 7.1 (A L^2_ρ-priori estimates on ε_-)**.** *For all $A \geq 1, \eta$, and $\tilde\eta$ satisfying $1 \ll \eta \ll \tilde\eta$, there exists $\tau_3(A,\eta,\tilde\eta)$ and τ^* such that for all $\tau_0 \geq \tau_4$ and the solution $(\varepsilon, b, \beta)(\tau) \in V_1[A,\eta,\tilde\eta](\tau^*), \forall \tau \in [\tau_0,\tau^*]$ and*

$$\|\varepsilon_-(\tau)\|_{L^2_{\rho_\beta}} \leq CAb^{\frac{\alpha}{2}+\eta}(\tau), \forall \tau \in [\tau_0, \tau^*]. \tag{7.1}$$

Proof. The result is mainly based on the *spectral gap* property. First, we claim that (7.1) follows from

$$\frac{1}{2}\frac{d}{d\tau}\|\varepsilon_-\|^2_{L^2_{\rho_\beta}} - \left(\frac{\alpha}{2} - 2\right)\|\varepsilon_-\|^2_{L^2_\rho} \leq CAb^{\alpha+3\eta}. \tag{7.2}$$

Indeed, let us assume (7.2) holds, we infer that

$$\frac{d}{d\tau}\left(e^{2\left(2-\frac{\alpha}{2}\right)\tau}\|\varepsilon_-(\tau)\|^2_{L^2_\rho}\right) \leq CAe^{2\left(2-\frac{\alpha}{2}\right)\tau}b^{\frac{\alpha}{2}+3\eta}, \forall \tau \in [\tau_0,\tau^*].$$

From the fact $(\varepsilon, b,\beta)(\tau) \in V_1[A,\eta,\tilde\eta](\tau)$, for all $\tau \in [\tau_0,\tau^*]$, we can apply Lemma 5.2 to deduce

$$\begin{aligned}\|\varepsilon_-\|^2_{L^2_\rho} &\leq e^{-2\left(2-\frac{\alpha}{2}\right)(\tau-\tau_0)}\|\varepsilon_-(\tau_0)\|^2_{L^2_\rho} + CAe^{-2\left(2-\frac{\alpha}{2}\right)\tau}\int_{\tau_0}^{\tau} e^{2\left(2-\frac{\alpha}{2}\right)\tau'}b^{\frac{\alpha}{2}+3\eta}(\tau')d\tau' \\ &\leq CAb^{\alpha+2\eta}(\tau), \forall \tau \in [\tau_0,\tau^*].\end{aligned}$$

Then, (7.1) follows. Now, it remains to give the proof of (7.2). Indeed, we multiply equation (2.17) by ε_- and integrate

$$\frac{1}{2}\frac{d}{d\tau}\|\varepsilon_-\|^2_{L^2_\rho} = \langle \partial_\tau\varepsilon_-, \varepsilon_-\rangle_{L^2_{\rho_\beta}} + \beta'\left\langle \varepsilon_-, \varepsilon_-\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}. \tag{7.3}$$

Next, we will prove that for all $\tau \in [\tau_0,\tau^*]$

$$\langle \partial_\tau\varepsilon_-, \varepsilon_-\rangle_{L^2_{\rho_\beta}} \leq \left(\frac{\alpha}{2} - 2\right)\|\varepsilon_-\|^2_{L^2_{\rho_\beta}} + O(b^{\alpha+3\eta})(\tau), \tag{7.4}$$

$$\left|\beta'\left\langle \varepsilon_-, \varepsilon_-\left(\frac{d+2}{2\beta} - \frac{y^2}{2}\right)\right\rangle_{L^2_{\rho_\beta}}\right| \lesssim b^{\alpha+3\eta}(\tau). \tag{7.5}$$

Let us start with (7.4). Indeed, from (2.17), and the decomposition

$$\varepsilon = \varepsilon_\ell\left(\frac{\phi_{\ell,b,\beta}}{c_{\ell,0}} - \phi_{0,b,\beta}\right) + \sum_{j=1}^{\ell-1}\varepsilon_j\phi_{j,b,\beta} + \varepsilon_- := \varepsilon_+ + \varepsilon_-, \tag{7.6}$$

ε_- solves

$$\partial_\tau\varepsilon_- = \mathscr{L}_b(\varepsilon_-) + B(\varepsilon_+ + \varepsilon_-) + \Phi - \partial_\tau\varepsilon_+ + \mathscr{L}_b\varepsilon_+.$$

Taking $L^2_{\rho_\beta}$ scalar product to the both sides of the above equation, we deduce

$$\langle \partial_\tau\varepsilon,\varepsilon_-\rangle_{\rho_\beta} = \langle \mathscr{L}_b\varepsilon_-, \varepsilon_-\rangle_{\rho_\beta} + \langle B(\varepsilon_+ + \varepsilon_-), \varepsilon_-\rangle_{\rho_\beta} + \langle \Phi - \partial_\tau\varepsilon_+, \varepsilon_-\rangle_{\rho_\beta},$$

since $\langle \mathscr{L}_b\varepsilon_+, \varepsilon_-\rangle_{L^2_\rho} = 0$.

+ Estimate to $\langle \mathscr{L}_b\varepsilon_-, \varepsilon_-\rangle_{L^2_\rho}$: Using the orthogonality

$$\langle \phi_{j,b,\beta}, \varepsilon_-(\tau)\rangle_{L^2_{\rho_\beta}} = 0, \text{for} j = 0 \text{ and } j = 1,$$

the spectral gap in Proposition 4.2 ensures

$$\langle \mathscr{L}_b\varepsilon_-, \varepsilon_-\rangle_{L^2_{\rho_\beta}} \leq \lambda_{2,b,\beta}\|\varepsilon_-\|^2_{L^2_{\rho_\beta}}.$$

In addition, we have

$$\lambda_{2,b,\beta} = \frac{\alpha}{2} - 2 + O(b^{1-\frac{\epsilon}{2}}),$$

which yields

$$\langle \mathscr{L}_b \varepsilon_-, \varepsilon_- \rangle_{L^2_{\rho_\beta}} \leq \left(\frac{\alpha}{2} - (\ell + 1) \right) \|\varepsilon_-\|^2_{L^2_\rho} + C b^{\alpha + 3\eta}.$$

+ Estimate for $\langle \Phi - \partial_\tau \varepsilon_+, \varepsilon_- \rangle_{L^2_{\rho_\beta}}$: Recall that

$$\varepsilon(\tau) = \varepsilon_1(\tau) \left(\frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta} \right) + \varepsilon_-(\tau) = \varepsilon_+ + \varepsilon_-.$$

We decompose

$$\langle \Phi - \partial_\tau \varepsilon_+, \varepsilon_- \rangle = \langle \Phi, \varepsilon_- \rangle - \langle \partial_\tau \varepsilon_+, \varepsilon_- \rangle. \tag{7.7}$$

For $\langle \Phi, \varepsilon_- \rangle_{L^2_{\rho_\beta}}$, we use (6.15) and by Cauchy-Schwarz inequality to deduce that

$$\left| \langle \Phi, \varepsilon \rangle_{L^2_{\rho_\beta}} \right| = \left| \langle \tilde{\Phi}, \varepsilon_- \rangle_{L^2_{\rho_\beta}} \right| \lesssim \|\tilde{\Phi}\|_{L^2_{\rho_\beta}} \|\varepsilon_-\|_{L^2_{\rho_\beta}} \leq C A^3 b^{\alpha + \tilde{\eta} + 1 - \frac{\epsilon}{2}} \leq b^{\alpha + 6\eta}(\tau), \forall \tau \in [\tau_0, \tau^*].$$

For the second term, we have

$$\begin{aligned} \partial_\tau \varepsilon_+ &= \varepsilon_1' \left[\frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta} \right] + \varepsilon_j \left[\frac{b'}{b} b \partial_b \phi_{j,b,\beta} + \beta' \partial_\beta \phi_{j,b,\beta} \right] \\ &+ \varepsilon_1 \left[\frac{\frac{b'}{b} b \partial_b \phi_{1,b,\beta} + \beta' \partial_\beta \phi_{1,b,\beta}}{c_{1,0}} - \left(\frac{b'}{b} b \partial_b \phi_{0,b,\beta} + \beta' \partial_\beta \phi_{0,b,\beta} \right) \right]. \end{aligned}$$

Note that

$$\langle \varepsilon_-, \phi_{j,b,\beta} \rangle_{L^2_{\rho_\beta}} = 0, \text{for} j = 0 \text{ and } j = 1, \tag{7.8}$$

combining this with (6.2), (6.3), the necessary bounds in $V_1[A, \eta, \tilde{\eta}](\tau)$, and Cauchy-Schwarz inequality, we infer

$$\left| \langle \partial_\tau \varepsilon_+, \varepsilon_- \rangle_{L^2_{\rho_\beta}} \right| \leq b^{\alpha + 3\eta}(\tau).$$

Finally, we give the following estimate

$$\left| \langle \Phi - \partial_\tau \varepsilon_+, \varepsilon_- \rangle_{L^2_{\rho_\beta}} \right| \leq b^{\alpha + 3\eta}(\tau), \forall \tau \in [\tau_0, \tau^*].$$

- For $\langle B(\varepsilon), \varepsilon_- \rangle_{L^2_{\rho_\beta}}$ with $\varepsilon = \varepsilon_+ + \varepsilon_-$. We explicitly write $B(\varepsilon)$ in (2.19) as follows

$$B(\varepsilon) = -3(n-2)(1 + |y|^2 Q_b)(\varepsilon_+^2 + 2\varepsilon_+ \varepsilon_- + \varepsilon_-^2) - (d-2)|y|^2 (\varepsilon_+^3 + 3\varepsilon_+^2 \varepsilon_- + 3\varepsilon_+ \varepsilon_-^2 + \varepsilon_-^3).$$

From γ's definition, we observe that once $d \geq 11$, one has

$$\gamma \leq 3.7 < 4.$$

In addition, from the fact that $(\varepsilon, b, \beta)(\tau) \in V_1[A, \eta, \tilde{\eta}](\tau), \forall \tau \in [\tau_0, \tau^*]$ and (6.13), we have

$$|\varepsilon_+(y)| \leq \frac{A b^{\frac{\alpha}{2} + \eta}(\tau) \langle y \rangle^4}{y^\gamma} + \frac{b^{\frac{\alpha}{2}} y^2 \langle y \rangle^2}{y^\gamma}, \text{ and } |\varepsilon_-(y)| \leq \frac{A^4 b^{\frac{\alpha}{2} + \tilde{\eta}}(\tau) \langle y \rangle^4}{y^\gamma},$$

which yields

$$\left| \langle B(\varepsilon), \varepsilon_- \rangle_{L^2_{\rho_\beta}} \right| \leq C b^{\alpha + 3\eta}.$$

Thus, we finish the proof of (7.4). In particular, using (6.2) and (6.13), we get

$$\left| \beta' \left\langle \varepsilon_-, \varepsilon_- \left(\frac{d+2}{2\beta} - \frac{y^2}{2} \right) \right\rangle_{L^2_{\rho_\beta}} \right| \leq b^{\alpha + 3\eta}, \forall \tau \in [\tau_0, \tau^*], \tag{7.9}$$

which implies (7.5). Finally, by combining (7.4) and (7.5) we deduce (7.2) and then the proof of the Lemma follows. □

7.2. L^∞ bounds

In order to handle the nonlinear term in the L^2_ρ-energy estimate, we used the control of a weighted L^∞-norm of ε_-. The rest of the section is devoted to it. In the next step, we aim to give *a priori* estimates to the infinite part, ε_-. More precisely, we have the following proposition:

Lemma 7.2 (Control of the infinite dimensional part)**.** *Then, there exists $A_4 \geq 1$ such that for $A \geq A_3, \delta \ll 1$, there exists $\eta_4(A,\delta) \ll 1$ such that for all $\eta \leq \eta_4$, there exists $\tilde{\eta}_4(A,\eta) \ll \eta$ such that for all $\tilde{\eta} \leq \tilde{\eta}_4$, there exists $\tau_4(A,\eta,\tilde{\eta}) \geq 1$, such that for all $\tau_0 \geq \tau_5$, the following holds: assume that initial data is defined as in (5.14) and the solution $(\varepsilon, b, \beta)(\tau) \in V_1[A,\eta,\tilde{\eta}](\tau), \forall \tau \in [\tau_0, \tau^*]$, for some $\tau^* \geq \tau_0$ then we have the following*

$$\left\| \frac{y^\gamma}{\langle y\rangle^4} \varepsilon_-(.,\tau) \right\|_{L^\infty[0,b^{-\tilde{\eta}}(\tau)]} \leq \frac{A^3}{2} I^{\frac{\alpha}{2}+\tilde{\eta}}(\tau), \forall \tau \in [\tau_0, \tau_1]. \tag{7.10}$$

Proof. The proof relies on the maximum principal for the control near the origin i.e. $\left[0, b^{\frac{\eta}{4}}\right]$, and pointwise estimates on $\left[b^{\frac{\eta}{4}}, b^{-\tilde{\eta}}\right]$.

a) Let us consider $y \in \left[0, b^{\frac{\eta}{4}}\right]$. We apply Proposition 11.1 to obtain

$$|\varepsilon(y,\tau)| \leq b^{-1}(\tau) H\left(\frac{y}{\sqrt{b(\tau)}}\right) \leq \frac{C b^{\frac{\alpha}{2}+\frac{\eta}{4}}(\tau)\langle y\rangle^4}{y^\gamma}, \text{ for all} y \in \left[0, b^{\frac{\eta}{4}}\right].$$

In addition, ε_+ can be estimated by

$$|\varepsilon_+(y,\tau)| \leq |\varepsilon_1(\tau)| \left| \frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta} \right|. \tag{7.11}$$

On the one hand, we use the pointwise estimates given in Proposition 4.2

$$\left| \frac{\phi_{1,b,\beta}}{c_{1,0}} - \phi_{0,b,\beta} \right| \leq \left| \frac{c_{1,1}}{c_{1,0}} (\sqrt{b})^{2-\gamma} T_1\left(\frac{y}{\sqrt{b}}\right) \right| + \left|\tilde{\phi}_{1,b,\beta}\right| + \left|\tilde{\phi}_{0,b,\beta}\right| \leq \frac{C b^{1-\frac{\epsilon}{2}}(\tau)\langle y\rangle^2}{y^\gamma}.$$

On the other hand, from the compatibility

$$\varepsilon_1(\tau) = -\frac{2}{\alpha} m_0 b^{\frac{\alpha}{2}}(\tau),$$

we deduce that

$$|\varepsilon_+(y,\tau)| \leq \frac{C b^{\frac{\alpha}{2}+\frac{\eta}{2}}(\tau)\langle y\rangle^4}{y^\gamma}.$$

Thus, we obtain

$$\sup_{y \in [0, b^{\frac{\eta}{4}}(\tau)]} \frac{y^\gamma}{\langle y\rangle^{2\ell+2}} |\varepsilon_-(y,\tau)| \leq C A b^{\frac{\alpha}{2}+\frac{\eta}{4}}(\tau) \leq \frac{A^3}{2} b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau), \tag{7.12}$$

provided that $A \geq A_4$.

b) Let us consider the control on $\left[b^{\frac{\eta}{4}}, b^{-\tilde{\eta}}(\tau)\right]$. On this domain, we are far the origin so we can not use the spectrum properties of $\mathscr{L}_\infty$. The idea is inspired from [4]. We are going to use pointwise estimates based on the semi-group. As for $\beta = \frac{1}{2}$, $\mathscr{L}_\infty$ has explicit structure. We introduce the basis of L^∞

$$\phi_{0,\infty} = \phi_{0,\infty,\frac{1}{2}} \text{ and } \phi_{1,\infty} = \phi_{1,\infty,\frac{1}{2}},$$

and for all $j \geq 2$ we renormalize as in [4, Lemma 3.4] that

$$\phi_{j,\infty}(y) = \mathcal{N}_j y^{-\gamma} L_j^{\left(\frac{d}{2}-\gamma\right)}\left(\frac{y^2}{4}\right),$$

where L_j^{ν} denoted by the generalized Laguerre polynomial, and the renormalisation constant $\mathcal{N}_j$ ensures that $\|\phi_{j,\infty}\|_{L^2_\rho} = 1$ and

$$\phi_{j,\infty}(y) = \begin{cases} \alpha_j y^{-\gamma}(1+o(1)) & \text{as} \quad y \to 0 \\ \beta_j y^{2j-\gamma}(1+o(1)) & \text{as} \quad y \to +\infty, \end{cases}$$

with α_j and β_j satisfies

$$\alpha_j \sim j^{\frac{\omega}{4}} \text{ and } \beta_n = \frac{j^{-\frac{\omega}{4}}}{4^j j!} \text{as } j \to +\infty.$$

The pointwise estimates given in Proposition 4.2 ensures that $\phi_{j,b,\beta}$ is very close to $\phi_{j,\infty,\beta}$ on this interval by the following

$$\left|\phi_{j,b(\tau),\beta}(y) - \phi_{j,\infty,\beta}(y)\right| \lesssim \frac{b^{\frac{\eta}{2}}(\tau)\langle y\rangle^4}{y^\gamma}, \; \forall y \geq b^{\frac{\eta}{4}}, j \leq 1. \tag{7.13}$$

In addition, the condition

$$\left|\beta(\tau) - \frac{1}{2}\right| \leq AI^\eta(\tau_0),$$

defined in the Shrinking set $V_1[A,\eta,\tilde{\eta}](\tau)$ shows that $\phi_{j,\infty,\beta}$ is close to $\phi_{j,\infty,\frac{1}{2}} := \phi_{j,\infty}$ since for all j

$$\begin{aligned} \left|\phi_{j,\infty,\beta}(y) - \phi_{j,\infty,\frac{1}{2}}(y)\right| &\lesssim \left|\beta(\tau) - \frac{1}{2}\right| \frac{\langle y\rangle^4}{y^\gamma}, && (7.14) \\ \left|e^{-\frac{(2\beta)y^2}{4}} - e^{-\frac{y^2}{4}}\right| &\lesssim \left|\beta(\tau) - \frac{1}{2}\right| \frac{y^2}{4} e^{-\frac{y^2}{8}}, && (7.15) \end{aligned}$$

since for all α, it holds $|e^\alpha - 1| \leq C\alpha e^\alpha$, we have

$$|\hat{\varepsilon}_1(\tau)| \lesssim b^{\frac{\alpha}{2}}(\tau), \tag{7.16}$$

where $\hat{\varepsilon}_j = \|\phi_{j,\infty}\|^{-2}_{L^2_\rho}\langle \varepsilon, \phi_{j,\infty}\rangle_{L^2_\rho}$ is the projection of ε on the basis $\{\phi_{j,\infty}, j \geq 0\}$. Hence, we use the semi-group pointwise estimates and we decompose ε on the basis $\phi_{j,0}$

$$\varepsilon = \hat{\varepsilon}_+ + \hat{\varepsilon}_-. \tag{7.17}$$

Thus, we will prove that

$$\sup_{y \in \left[b^{\frac{\eta}{4}}(\tau), b^{-\tilde{\eta}}(\tau)\right]} \left|\frac{y^\gamma}{\langle y\rangle^4}\hat{\varepsilon}_-(y,\tau)\right| \leq \frac{A^3}{4} b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau). \tag{7.18}$$

Since ε satisfies (2.17), $\hat{\varepsilon}_-$ solves

$$\partial_\tau \hat{\varepsilon}_- = \mathscr{L}_\infty \hat{\varepsilon}_- + \hat{B}(\hat{\varepsilon}_+ + \hat{\varepsilon}_-) + \left(\frac{1}{2} - \beta(\tau)\right)\Lambda_y(\hat{\varepsilon}_-), \tag{7.19}$$

where $\mathscr{L}_\infty$ was defined in (2.22) by taking $\beta = \frac{1}{2}$ and $\hat{B}$ is defined by

$$\begin{aligned} \hat{B} &= -3(d-2)\left[2Q_b + y^2 Q_b^2 + \frac{1}{y^2}\right](\hat{\varepsilon}_+ + \hat{\varepsilon}_-) + B(\hat{\varepsilon}_+ + \hat{\varepsilon}_-) && (7.20) \\ &+ \Phi(\tau) + \mathscr{L}^\beta_\infty(\hat{\varepsilon}_+) - \partial_\tau \hat{\varepsilon}_+, \end{aligned}$$

with B and Φ defined in (2.19) and (2.20), respectively. By using Duhamel's formula, we get

$$\hat{\varepsilon}_-(\tau) = e^{(\tau-\tau_0)\mathscr{L}_\infty}\hat{\varepsilon}_-(\tau_0) + \int_{\tau_0}^{\tau} e^{(\tau-\tau')\mathscr{L}_\infty}\left[\hat{B}(\hat{\varepsilon}_+ + \hat{\varepsilon}_-) + \left(\frac{1}{2} - \beta(\tau')\right)\Lambda_y \hat{\varepsilon}_-\right](\tau')d\tau'. \tag{7.21}$$

In addition, we denote f_- as the part of f which is orthogonal to $\phi_{0,\infty}$ and $\phi_{1,\infty}$. Then

$$f_-(y) = f - \sum_{j=0}^{1}\langle f, \phi_{j,\infty}\rangle_{L^2_\rho}\phi_{j,\infty}.$$

In particular, if the series

$$\sum_{j=0}^{\infty}\langle f, \phi_{j,\infty}\rangle_{L^2_\rho}\phi_{j,\infty}$$

is convergent and well defined, then we can define f_- pointwisely as

$$f_-(y) = \sum_{j=2}^{\infty}\langle f, \phi_{j,\infty}\rangle_{L^2_\rho}\phi_{j,\infty}.$$

Since ε_- is orthogonal to $\phi_{0,\infty}$ and $\phi_{1,\infty}$, we can write

$$\begin{aligned} \varepsilon_- &= \left(e^{(\tau-\tau_0)\mathcal{L}_\infty}(\varepsilon_-(\tau_0))\right)_- \\ &+ \int_{\tau_0}^{\tau}\left(e^{(\tau-\tau')\mathscr{L}_\infty}(\hat{B}(\tau'))\right)_- d\tau' + \int_{\tau_0}^{\tau}\left(\left[\frac{1}{2} - \beta(\tau')\right]\Lambda\hat{\varepsilon}_-(\tau')\right)_- d\tau'. \end{aligned} \tag{7.22}$$

We remark that (7.18) immediately follows from

$$\sup_{y\in\left[b^{\frac{\eta}{4}}(\tau), b^{-\tilde{\eta}}(\tau)\right]}\left|\frac{y^\gamma}{\langle y\rangle^4}\left(e^{(\tau-\tau_0)\mathscr{L}_\infty}\hat{\varepsilon}_-(\tau_0)\right)_-(y,\tau)\right| \le \frac{A^3}{16}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau), \tag{7.23}$$

$$\sup_{y\in\left[b^{\frac{\eta}{4}}(\tau), b^{-\tilde{\eta}}(\tau)\right]}\left|\frac{y^\gamma}{\langle y\rangle^4}\int_{\tau_0}^{\tau}\left(e^{(\tau-\tau')\mathscr{L}_\infty}\left[\hat{B}(\tau')\right]\right)_- d\tau'\right| \le \frac{A^3}{16}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau), \tag{7.24}$$

$$\sup_{y\in\left[b^{\frac{\eta}{4}}(\tau), b^{-\tilde{\eta}}(\tau)\right]}\left|\frac{y^\gamma}{\langle y\rangle^4}\int_{\tau_0}^{\tau}\left(e^{(\tau-\tau')\mathscr{L}_\infty}\left[\frac{1}{2} - \beta(\tau')\right]\Lambda_y(\hat{\varepsilon}_-)\right)_-(\tau')d\tau'\right| \le \frac{A^3}{16}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau). \tag{7.25}$$

To prove estimates ((7.23) -(7.25)), we need to consider different cases as

- The first case, we consider $\tau - \tau_0 \le \frac{\ln A}{K_0}$

- The second case, we consider $\tau - \tau_0 > \frac{\ln A}{K_0}$. In addition, the second will be divided again by two sub-cases that $\frac{1}{L_0}e^{\frac{\tau-\tau_0}{2}\left(1-\eta\left(\frac{2\ell}{\alpha}-1\right)\right)} \le b^{-\tilde{\eta}}(\tau)$ and $\frac{1}{L_0}e^{\frac{\tau-\tau_0}{2}\left(1-\eta\left(\frac{2\ell}{\alpha}-1\right)\right)} > b^{-\tilde{\eta}}(\tau)$ and in these sub-cases also includes some smaller case that there are some large constant L_0, K_0, R appear which are fixed at the end of the proof. Let us go to the details of the proof.

First case $\tau - \tau_0 \le \frac{\ln A}{K_0}$

- **Proof of** (7.23) : Note that $K_0 \gg 1$ will be fixed at the end of the proof. Now, we deduce from (5.14) in accordance with the decomposition (7.17), we arrive at

$$|\hat{\varepsilon}_-(\tau_0)y^\gamma| \le CAb^{\frac{\alpha}{2}+\eta}(\tau_0)\langle y\rangle^4, \tag{7.26}$$

where $b(\tau_0) = I(\tau_0)$ defined in (5.5). Since $\langle y\rangle^4$ is increasing, then, we apply Lemma C.1 and we obtain

$$\begin{aligned} \left|e^{(\tau-\tau_0)\mathscr{L}_\infty}\hat{\varepsilon}_-(\tau_0)(y,\tau)\right| &\leq y^{-\gamma}e^{\frac{\alpha(\tau-\tau_0)}{2}}M(\hat{\varepsilon}_-(\tau_0))(y) \\ &\leq CAe^{\frac{\alpha(\tau-\tau_0)}{2}}b^{\frac{\alpha}{2}+\eta}(\tau_0)y^{-\gamma}\frac{\int_y^\infty \langle y'\rangle^4(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'}{\int_y^\infty (y')^{1+\omega}e^{-\frac{(y')^2}{2}}dy'} \\ &\leq CAe^{\frac{\alpha}{2}(\tau-\tau_0)}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)b^{\frac{\alpha}{2}+\eta}(\tau_0)y^{-\gamma}\langle y\rangle^4. \end{aligned} \tag{7.27}$$

Using (5.11), we get

$$\begin{aligned} e^{\frac{\alpha}{2}(\tau-\tau_0)}b^{\frac{\alpha}{2}+\eta}(\tau_0)b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau) &\leq Ce^{\frac{\alpha}{2}(\tau-\tau_0)}e^{\left(1-\frac{2}{\alpha}\right)\left(\left(\frac{\alpha}{2}+\delta\right)\left(1-\frac{\tilde{\eta}}{10}\right)\tau_0-\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\tau\right)} \\ &\leq Ce^{-c(\eta)\tau_0}e^{\frac{\alpha}{2}(\tau-\tau_0)}e^{\left(1-\frac{2}{\alpha}\right)\left(\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\tau_0-\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\tau\right)} \\ &\leq Ce^{-c(\eta)\tau_0}e^{\left(\frac{\alpha}{2}+\left(\frac{2}{\alpha}-1\right)\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\right)(\tau-\tau_0)} \\ &\leq Ce^{-c(\eta)\tau_0}A^{\left(\frac{\alpha}{2}+\left(\frac{2}{\alpha}-1\right)\left(\frac{\alpha}{2}+\tilde{\eta}\right)(1+\tilde{\eta})\right)\frac{1}{K_0}}, \text{ for some } c(\eta)>0, \end{aligned} \tag{7.28}$$

which yields

$$\left|\frac{y^\gamma}{\langle y\rangle^4}e^{(\tau-\tau_0)\mathscr{L}_\infty}\hat{\varepsilon}_-(\tau_0)(y,\tau)\right| \leq \frac{A^3}{16}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau), \tag{7.29}$$

provided that $K_0 \geq K_4, A \geq A_4$. Finally, we conclude (7.23).

-Proof of (7.24): for $\tau' \in [\tau_0,\tau]$, we apply Lemma C.1 to get

$$\left|e^{(\tau-\tau')\mathscr{L}_\infty}[\hat{B}](\tau')\right| \leq Cy^{-\gamma}e^{\frac{\alpha(\tau-\tau')}{2}}\left\{M(\mathbb{1}_{(0,b^\delta(\tau')]}\hat{B}) + M(\mathbb{1}_{y\geq b^\delta(\tau')}\hat{B})\right\}. \tag{7.30}$$

To evaluate $M(\mathbb{1}_{(0,b^\delta(\tau')]}\hat{B})(\tau')$, we apply the result in Lemma A.1 to obtain

$$\begin{aligned} M(\mathbb{1}_{[0,b^\delta(\tau')]}\hat{B}(\tau')) &\leq C\Big[b^{\frac{\alpha}{2}}(\tau')M(\mathbb{1}_{(0,b^\delta(\tau')]}y^{-\gamma}) + A^3b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau')M(\mathbb{1}_{(0,b^\delta(\tau')]}y^{-\gamma-2}) \\ &+ A^8b^{2\left(\frac{\alpha}{2}+\tilde{\eta}\right)}(\tau')M(\mathbb{1}_{(0,b^\delta(\tau')]}y^{-2\gamma}) + A^{12}b^{3\left(\frac{\alpha}{2}+\tilde{\eta}\right)}(\tau')M(\mathbb{1}_{[0,b^\delta(\tau')]}y^{-3\gamma+2})\Big]. \end{aligned}$$

For the first term on the right hand side of the above inequality, we rewrite from (C.3)

$$M(\mathbb{1}_{[0,b^\delta(\tau')]}y^{-\gamma}) = \sup_{y\in\mathcal{I}}\frac{\int_{\mathcal{I}}|\mathbb{1}_{[0,b^\delta(\tau')]}|(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'}{\int_{\mathcal{I}}(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'},$$

since $\mathbb{1}_{[0,b^\delta(\tau')]}$ is non increasing, then, we apply the result in Lemma C.1 to get

$$M(\mathbb{1}_{(0,b^\delta(\tau')]}y^{-\gamma}) \leq \frac{\int_0^{b^\delta(\tau')}|\mathbb{1}_{[0,b^\delta(\tau')]}|(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'}{\int_0^y(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'} \lesssim \frac{(b^\delta(\tau'))^{2+\omega}}{\int_0^y(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'}.$$

Besides that, once $y \geq 1$, it follows that

$$\int_0^y(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy' \geq C,$$

which yields

$$\left(\int_0^y(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'\right)^{-1} \lesssim \frac{\langle y\rangle^{2+\omega}}{y^{2+\omega}}.$$

Otherwise, once $y \le 1$, we have

$$\left(\int_0^y (y')^{1+\omega} e^{-\frac{(y')^2}{4}} dy'\right)^{-1} \lesssim \left(\int_0^y (y')^{1+\omega} dy'\right)^{-1} \lesssim \frac{\langle y\rangle^{2+\omega}}{y^{2+\omega}}.$$

Then, we derive

$$M(\mathbb{1}_{(0,b^\delta(\tau')]} y^{-\gamma}) \quad \le \quad \frac{\int_0^{b^\delta(\tau')} |\mathbb{1}_{[0,b^\delta(\tau')]}|(y')^{1+\omega} e^{-\frac{(y')^2}{4}} dy'}{\int_0^y (y')^{1+\omega} e^{-\frac{(y')^2}{4}} dy'} \lesssim \frac{(b^\delta(\tau'))^{2+\omega}\langle y\rangle^{2+\omega}}{y^{2+\omega}}.$$

Similarly, we have

$$M(\mathbb{1}_{[0,b^\delta(\tau')]} y^{-\gamma-2}) \lesssim \frac{(b^\delta(\tau'))^{\omega}\langle y\rangle^{2+\omega}}{y^{2+\omega}}, \text{ and } M(\mathbb{1}_{(0,b^\delta(\tau')]} y^{-2\gamma}) \lesssim \frac{(I^\delta(\tau'))^{\omega+2-\gamma}\langle y\rangle^{2+\omega}}{y^{2+\omega}}$$

and

$$M(\mathbb{1}_{(0,b^\delta(\tau')]} y^{-3\gamma+2}) \lesssim \frac{(b^\delta(\tau'))^{\omega+4-2\gamma}\langle y\rangle^{2+\omega}}{y^{2+\omega}}.$$

Combining all the related terms with the condition that $y \in \left[b^{\frac{\eta}{4}}(\tau), b^{-\tilde\eta}\right]$, we deduce

$$\begin{aligned} \left| e^{(\tau-\tau')\mathscr{L}_\infty}\left[\mathbb{1}_{(0,b^\delta(\tau')]}\hat{B}\right](\tau')\right| \quad &\lesssim \quad y^{-\gamma} e^{\frac{\alpha}{2}(\tau-\tau')} b^{\delta(\omega+4-2\gamma)}(\tau') \frac{\langle y\rangle^{\omega+2}}{y^{\omega+2}} \lesssim y^{-\gamma} e^{\frac{\alpha}{2}(\tau-\tau')} b^{\delta(\omega+4-2\gamma)-\frac{\eta}{4}(\omega+2)}(\tau') \\ &\lesssim \quad \frac{b^{\frac{\alpha}{2}+\tilde\eta}(\tau)\langle y\rangle^4}{y^\gamma}\left[e^{\frac{\alpha}{2}(\tau-\tau')} b^{\delta(\omega+4-2\gamma)-\frac{\eta}{4}(\omega+2)}(\tau') b^{-\frac{\alpha}{2}-\tilde\eta}(\tau)\right]. \end{aligned} \tag{7.31}$$

In addition, by the same argument used in (7.28) and (7.29) and that fact that $\tau - \tau_0 \le \frac{\ln A}{K_0}$, we have

$$\int_{\tau_0}^{\tau}\left[e^{\frac{\alpha}{2}(\tau-\tau')} b^{\delta(\omega+4-2\gamma)-\frac{\eta}{4}(\omega+2)}(\tau') b^{-\frac{\alpha}{2}-\tilde\eta}(\tau)\right] d\tau' \lesssim A^{\frac{C(\delta,\eta,\tilde\eta)}{K_0}}.$$

Finally, we conclude

$$\sup_{y\in\left[b^{\frac{\eta}{4}}(\tau), b^{-\tilde\eta}(\tau)\right]} \left|\frac{y^\gamma}{\langle y\rangle^{2\ell+2}} \int_{\tau_0}^{\tau} e^{(\tau-\tau')\mathscr{L}_\infty}\left|\mathbb{1}_{(0,b^\delta(\tau')]}\hat{B}(\tau')\right| d\tau'\right| \le \frac{A^3}{16} b^{\frac{\alpha}{2}+\tilde\eta}(\tau),$$

provided that $K_0 \ge K_4$, $A \ge A_5$, and $\tau_0 \ge \tau_5(A, K_0, \delta, \eta, \tilde\eta)$.

It remains to evaluate $M(\mathbb{1}_{y\ge b^\delta(\tau')}\hat{B}(\tau'))$. Using Lemma A.1 with $\ell = 1$, we get

$$\begin{aligned} M(\mathbb{1}_{y\ge b^\delta(\tau')}\hat{B}(\tau')) \quad &\lesssim \quad b^{\frac{\alpha}{2}(\tau')+4\eta} M\left(\mathbb{1}_{y\ge b^\delta(\tau')}\langle y\rangle^4 y^{-\gamma}\right) + b^{\alpha+\delta(1-\gamma)} M\left(\mathbb{1}_{y\ge b^\delta(\tau')}\langle y\rangle^{12} y^{-\gamma}\right) \\ &+ \quad b^{\frac{3\alpha}{2}-2\delta\gamma}(\tau') M\left(\mathbb{1}_{y\ge b^\delta(\tau')}\langle y\rangle^{20} y^{-\gamma}\right). \end{aligned}$$

First, we observe that the function $\mathbb{1}_{y\ge b^\delta(\tau')}\langle y\rangle^4$ is non decreasing, we apply Lemma C.1 and we have

$$M\left(\mathbb{1}_{y\ge b^\delta(\tau')}\langle y\rangle^4 y^{-\gamma}\right) \lesssim \frac{\int_y^\infty \langle y'\rangle^4 (y)'^{1+\omega} e^{-\frac{(y')^2}{4}} dy'}{\int_y^\infty (y')^{1+\omega} e^{-\frac{(y')^2}{4}} dy'}.$$

From a standard result on Γ function, we have

$$\frac{\int_y^\infty \langle y'\rangle^4 (y')^{1+\omega} e^{-\frac{(y')^2}{4}} dy'}{\int_y^\infty (y')^{1+\omega} e^{-\frac{(y')^2}{4}} dy'} \quad \lesssim \quad \langle y\rangle^4,$$

which implies

$$M\left(\mathbb{1}_{y\geq b^{\delta}(\tau')}\langle y\rangle^{2\ell+2}y^{-\gamma}\right)\lesssim\langle y\rangle^{2\ell+2}.$$

Similarly, from Lemma A.1 and the fact $y\leq b^{-\tilde{\eta}}(\tau)$, we write

$$\begin{aligned} M\left(\mathbb{1}_{y\geq b^{\delta}(\tau')}\langle y\rangle^{4+8}y^{-\gamma}\right) &\lesssim \langle y\rangle^{4+8}\lesssim b^{-\tilde{\eta}(2+6)}(\tau)\langle y\rangle^{4},\\ M\left(\mathbb{1}_{y\geq b^{\delta}(\tau')}\langle y\rangle^{6+14}y^{-\gamma}\right) &\lesssim \langle y\rangle^{6+14}\lesssim b^{-\tilde{\eta}(4+12)}(\tau)\langle y\rangle^{4}. \end{aligned}$$

Thus, we derive for all $y\in\left[b^{\eta}(\tau),b^{-\tilde{\eta}}(\tau)\right]$

$$\begin{aligned} &\left|e^{(\tau-\tau')\mathscr{L}_\infty}\left[\mathbb{1}_{y\geq b^{\delta}(\tau')}\hat{B}\right](\tau')\right| \\ \lesssim\ & y^{-\gamma}\langle y\rangle^{2+2}e^{\frac{\alpha}{2}(\tau-\tau')}\left[b^{\frac{\alpha}{2}+4\eta}(\tau')+b^{\alpha+\delta(1-\gamma)}(\tau')b^{-\tilde{\eta}(2+6)}(\tau)+b^{\frac{3\alpha}{2}-2\delta\gamma}(\tau')b^{-\tilde{\eta}(4+12)}(\tau)\right], \end{aligned}\tag{7.32}$$

which implies

$$\begin{aligned} &\frac{y^{\gamma}}{\langle y\rangle^{2+2}}\int_{\tau_0}^{\tau}e^{(\tau-\tau')\mathscr{L}_\infty}M(\mathbb{1}_{y\geq b^{\delta}(\tau')}\hat{B}(\tau'))d\tau' \\ \lesssim\ & b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\int_{\tau_0}^{\tau}e^{\frac{\alpha}{2}(\tau-\tau')}b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)\left[b^{\frac{\alpha}{2}+4\eta}(\tau')+b^{\alpha+\delta(1-\gamma)}(\tau')b^{-\tilde{\eta}(2+6)}(\tau)+b^{\frac{3\alpha}{2}-2\delta\gamma}(\tau')b^{-\tilde{\eta}(4+12)}(\tau)\right]d\tau'. \end{aligned}\tag{7.33}$$

From the assumption $\tau-\tau_0\leq\frac{\ln A}{K_0}$ with K_0 large enough, $A\geq A_4$ and $\alpha\gg\delta\gg\eta\gg\tilde{\eta}$, and $\tau_0\geq\tau_4(A,K_0,\delta,\eta,\tilde{\eta})$, we proceed similarly as in (7.28) and (7.29) and we obtain

$$\int_{\tau_0}^{\tau}e^{\frac{\alpha}{2}(\tau-\tau')}b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)\left[b^{\frac{\alpha}{2}+4\eta}(\tau')+b^{\alpha+\delta(1-\gamma)}(\tau')b^{-\tilde{\eta}(2\ell+6)}(\tau)+b^{\frac{3\alpha}{2}-2\delta\gamma}(\tau')b^{-\tilde{\eta}(4\ell+12)}(\tau)\right]d\tau'\leq\frac{A^3}{32}.$$

Finally, we get

$$\frac{y^{\gamma}}{\langle y\rangle^{4}}\int_{\tau_0}^{\tau}e^{(\tau-\tau')\mathscr{L}_\infty}M(\mathbb{1}_{y\geq b^{\delta}(\tau')}\hat{B}(\tau'))d\tau'\leq\frac{A^3}{32}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau),$$

and (7.24) immediately follows.

- The proof of (7.25): We first recall the following identity

$$\varepsilon_++\varepsilon_-=\varepsilon=\hat{\varepsilon}_++\hat{\varepsilon}_-,$$

then, we get

$$\hat{\varepsilon}_-(\tau')=\varepsilon_+(\tau')+\varepsilon_-(\tau')+\hat{\varepsilon}_+(\tau').$$

Since $(\varepsilon,b,\beta)(\tau)\in V_1[A,\eta,\tilde{\eta}](\tau_1),\forall\tau\in[\tau_0,\tau_1]$, the pointwise estimates given in Lemma (5.2) and also (7.16) hold, so we get a rough estimate for all $\tau'\in[\tau_0,\tau],\tau\leq\tau_1$

$$\left|\left(\frac{1}{2}-\beta(\tau')\right)\hat{\varepsilon}_-(\tau')\right|\leq CA^6b^{\frac{\alpha}{2}}(\tau')I^{\eta}(\tau_0)\frac{\langle y\rangle^4}{y^{\gamma}}.\tag{7.34}$$

Now, we apply Lemma C.2 and we obtain

$$\left|e^{(\tau-\tau')\mathscr{L}_\infty}\left(\left(\frac{1}{2}-\beta(\tau')\right)\hat{\varepsilon}_-(\tau')\right)\right|\leq CA^6b^{\frac{\alpha}{2}}(\tau')I^{\eta}(\tau_0)\frac{\langle y\rangle^6}{y^{\gamma}},\forall\tau'\in[\tau_0,\tau),$$

which yields

$$\left|\int_{\tau_0}^{\tau}e^{(\tau-\tau')\mathscr{L}_\infty}\left(\left(\frac{1}{2}-\beta(\tau')\right)\hat{\varepsilon}_-(\tau')\right)d\tau'\right|\leq CA^6I^{\eta}(\tau_0)\frac{\langle y\rangle^6}{y^{\gamma}}\int_{\tau_0}^{\tau}b^{\frac{\alpha}{2}}(\tau')d\tau'.$$

Using Lemma 5.2 and a similar estimate as in (7.28), we derive

$$\left|A^6I^{\eta}(\tau_0)\frac{\langle y\rangle^6}{y^{\gamma}}\int_{\tau_0}^{\tau}b^{\frac{\alpha}{2}}(\tau')d\tau'\right|\leq\frac{A^3}{16}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau),$$

provided that $K_0 \geq K_4, A \geq A_4, \tau_0 \geq \tau_4(K_0, A, \eta, \tilde{\eta})$ and $\tau - \tau_0 \leq \frac{\ln A}{K_0}$. Finally, (7.25) follows.

The second case $\tau - \tau_0 \geq \frac{\ln A}{K_0}$

As we mentioned, this case will be divided into two sub-cases

$$\frac{1}{L_0} e^{\frac{\tau-\tau_0}{2}\left(1-\eta\left(\frac{2\ell}{\alpha}-1\right)\right)} \leq b^{-\tilde{\eta}}(\tau) \text{ and } \frac{1}{L_0} e^{\frac{\tau-\tau_0}{2}\left(1-\eta\left(\frac{2\ell}{\alpha}-1\right)\right)} > b^{-\tilde{\eta}}(\tau),$$

where L_0 is large enough.

The first subcase $\frac{1}{L_0} e^{\frac{\tau-\tau_0}{2}\left(1-\eta\left(\frac{2}{\alpha}-1\right)\right)} \leq b^{-\tilde{\eta}}(\tau)$

From (5.11) and $\frac{1}{L_0} e^{\frac{\tau-\tau_0}{2}\left(1-\eta\left(\frac{2}{\alpha}-1\right)\right)} \leq b^{-\tilde{\eta}}(\tau)$, we get

$$\tau < \frac{\tau_0\left(1-\eta\left(1-\frac{2}{\alpha}\right)\right)+2\ln L_0}{1-\left(\eta-2\tilde{\eta}\left(1+\frac{\tilde{\eta}}{10}\right)\right)\left(1-\frac{2}{\alpha}\right)}, \tag{7.35}$$

which yields

$$\frac{\ln A}{K_0} \leq \tau - \tau_0 \leq \frac{2\tilde{\eta}\tau_0\left(\frac{2}{\alpha}-1\right)\left(1+\frac{\tilde{\eta}}{10}\right)+2\ln L_0}{1-\left(\eta-2\tilde{\eta}\left(1+\frac{\tilde{\eta}}{10}\right)\right)\left(\frac{2}{\alpha}-1\right)}. \tag{7.36}$$

for L_0 large enough. According to (7.36), we see that this the present sub-case can be handled similarly as the fist one, since τ is not too far from τ_0.

- The proof of (7.23): From (7.27), we have

$$\left| e^{(\tau-\tau_0)\mathscr{L}_\infty}\hat{\varepsilon}_-(\tau_0)(y,\tau)\right| \leq CAb^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\frac{\langle y\rangle^4}{y^\gamma}\left[e^{\frac{\alpha}{2}(\tau-\tau_0)}b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau_0)b^{\frac{\alpha}{2}+\eta}(\tau_0)\right].$$

The same process used for (7.28) yields

$$e^{\frac{\alpha}{2}(\tau-\tau_0)}b^{\frac{\alpha}{2}+\eta}(\tau_0)b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau) \leq e^{-c(\eta)\tau_0}e^{X(\tau-\tau_0)}, \text{ with } X = \left(\frac{\alpha}{2}+\left(\frac{2}{\alpha}-1\right)\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\right).$$

From (7.36), we can prove that there exists $c(\eta) > 0$ such that

$$\begin{aligned} & e^{-c(\eta)\tau_0}e^{X(\tau-\tau_0)}, \text{ with } X = \left(\frac{\alpha}{2}+\left(\frac{2}{\alpha}-1\right)\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\right) \\ \lesssim \quad & e^{-c(\eta)\tau_0+X\left(\frac{2\tilde{\eta}\tau_0\left(\frac{2}{\alpha}-1\right)\left(1+\frac{\tilde{\eta}}{10}\right)+2\ln L_0}{1-\left(\eta-2\tilde{\eta}\left(1+\frac{\tilde{\eta}}{10}\right)\right)\left(\frac{2}{\alpha}-1\right)}\right)} \leq 1, \end{aligned}$$

provided that $\tilde{\eta} \leq \tilde{\eta}_4(\eta, L_0)$ and this gives (7.23).

- The proof of (7.24): we use (7.31) and (7.32) to get

$$\begin{aligned} & \left|\frac{y^\gamma}{\langle y\rangle^{2+2}}\int_{\tau_0}^{\tau} e^{(\tau-\tau')\mathscr{L}_\infty}\left[\hat{B}(\hat{\varepsilon}_{\beta,+},\hat{\varepsilon}_{\beta,-})\right](\tau')d\tau'\right| \\ \leq \quad & Cb^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\left\{\int_{\tau_0}^{\tau}\left[e^{\frac{\alpha}{2}(\tau-\tau')}b^{\delta(\omega+4-2\gamma)-\frac{\eta}{4}(\omega+2)}(\tau')b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)\right]d\tau'\right. \\ + \quad & \left.\int_{\tau_0}^{\tau} e^{\frac{\alpha}{2}(\tau-\tau')}b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)\left[b^{\frac{\alpha}{2}+4\eta}(\tau')+b^{\alpha+\delta(1-\gamma)}(\tau')b^{-\tilde{\eta}(2+6)}(\tau)+b^{\frac{3\alpha}{2}-2\delta\gamma}(\tau')b^{-\tilde{\eta}(16)}(\tau)\right]d\tau'\right\}. \end{aligned}$$

From (7.28) and (7.36), we have

$$\begin{aligned}&\int_{\tau_0}^{\tau}\left[e^{\frac{\alpha}{2}(\tau-\tau')}b^{\delta(\omega+4-2\gamma)-\frac{\eta}{4}(\omega+2)}(\tau')b^{-\frac{\alpha}{2}-\tilde\eta}(\tau)\right]d\tau'\\ +&\int_{\tau_0}^{\tau}e^{\frac{\alpha}{2}(\tau-\tau')}b^{-\frac{\alpha}{2}-\tilde\eta}(\tau)\left[b^{\frac{\alpha}{2}+4\eta}(\tau')+b^{\alpha+\delta(1-\gamma)}(\tau')b^{-\tilde\eta(8)}(\tau)+b^{\frac{3\alpha}{2}-2\delta\gamma}(\tau')b^{-\tilde\eta(16)}(\tau)\right]d\tau'\\ \le\ & C,\end{aligned}$$

provided that $\tilde\eta\le\tilde\eta_4(\eta,\delta,L_0)$, $A\ge A_4$ and $\tau_0\ge\tau_4(A,\eta,\tilde\eta,L_0)$. Then, we infer

$$\left|\frac{y^\gamma}{\langle y\rangle^4}\int_{\tau_0}^{\tau}e^{(\tau-\tau')\mathscr{L}_\infty}\left[\hat B(\hat\varepsilon_{\beta,+},\hat\varepsilon_{\beta,-})\right](\tau')d\tau'\right|\le\frac{A^3}{16}b^{\frac{\alpha}{2}+\tilde\eta}(\tau),\forall y\in\left[b^\eta(\tau),b^{-\tilde\eta}(\tau)\right],$$

which implies (7.24).

- Proof of (7.25): it is similar to the that of (7.24) in the first case.

The second sub-case $\frac{1}{L_0}e^{\frac{\tau-\tau_0}{2}\left(1-\eta\left(\frac{2\ell}{\alpha}-1\right)\right)}>b^{-\tilde\eta}(\tau)$

We introduce R large, to be fixed later, and we decompose

$$\left[b^\eta(\tau),b^{-\tilde\eta}(\tau)\right]=\left[b^\eta(\tau),R\right]\cup\left[R,b^{-\tilde\eta}(\tau)\right].$$

Recall that for each $f\in L^2_\rho(\mathbb{R}^+)$, for each $\nu>0$, there exists $Y(v)>0$ satisfying $Y(\nu)\to+\infty$ as $\nu\to+\infty$ and such that $\forall y\in[Y^{-1}(\nu),Y(\nu)]$

$$e^{\nu\mathscr{L}_\infty}f(y)=\sum_{j=0}^{\infty}e^{\left(\frac{\alpha}{2}-j\right)\nu}\langle f,\phi_{j,\infty}\rangle_{L^2_\rho}\phi_{j,\infty}(y)\text{ pointwisely }.\tag{7.37}$$

Following the above remarks, the expression (7.21) and the fact that $\hat\varepsilon_-$ is orthogonal to $\phi_{0,\infty}$ and $\phi_{1,\infty}$ we are led to

$$\begin{aligned}\hat\varepsilon_-(y,\tau)\ &=\ \left(e^{(\tau-\tau_0)\mathcal{L}_\infty}(\varepsilon_-(\tau_0))\right)_-+\int_{\tau_0}^{\tau}\left(e^{(\tau-\tau')\mathscr{L}_\infty}(\hat B(\tau'))\right)_-d\tau'+\int_{\tau_0}^{\tau}\left(\left[\frac12-\beta(\tau')\right]\Lambda\hat\varepsilon_-(\tau')\right)_-d\tau'\\ &=\ \sum_{j=2}^{\infty}e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau_0)}\langle\hat\varepsilon_-(\tau_0),\phi_{j,\infty}\rangle_{L^2_\rho}\phi_{j,\infty}(y)\\ &+\ \int_{\tau_0}^{\tau-L_0}\sum_{j=2}^{\infty}e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau')}\left\langle\hat B(\tau'),\phi_{j,\infty}\right\rangle_{L^2_\rho}\phi_{j,\infty}(y)d\tau'+\int_{\tau-L_0}^{\tau}\left(e^{(\tau-\tau')\mathscr{L}_\infty}(\hat B(\tau'))\right)_-d\tau'\\ &+\ \int_{\tau_0}^{\tau-L_0}\sum_{j=2}^{\infty}e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau')}\left\langle\left(\left[\frac12-\beta(\tau')\right]\Lambda\hat\varepsilon_-(\tau')\right)d\tau',\phi_{j,\infty}\right\rangle_{L^2_\rho}\phi_{j,\infty}(y)d\tau'\\ &+\ \int_{\tau-L_0}^{\tau}\left(\left[\frac12-\beta(\tau')\right]\Lambda\hat\varepsilon_-(\tau')\right)_-d\tau'\end{aligned}\tag{7.38}$$

on $[Y^{-1}(\tau),Y(\tau)]$ ($Y(\tau)\to+\infty$ as $\tau\to+\infty$). Let us consider $y\in[b^\eta(\tau),R]$. We consider the initial data $\varepsilon(\tau_0)$ defined in (5.14), we have

$$\left|\langle\hat\varepsilon_-(y,\tau_0),\phi_{j,\infty}\rangle_{L^2_\rho}\right|\le Cb^{\frac{\alpha}{2}+\delta},\forall j\ge2.$$

Note that $\delta \gg \eta \gg \tilde{\eta}$ and we are in the case $\tau - \tau_0 > \frac{\ln A}{K_0}$, we have the estimate

$$\begin{aligned}
&\left|\sum_{j=2}^{\infty} e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau_0)}\langle \hat{\varepsilon}_-(\tau_0), \phi_{j,\infty}\rangle_{L^2_\rho}\phi_{j,\infty}(y)\right| \\
\leq\ & Cb^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\frac{\langle y\rangle^4}{y^\gamma}\sum_{j=2}^{\infty} e^{(\frac{\alpha}{2}-j)(\tau-\tau_0)}b^{\frac{\alpha}{2}+\delta}(\tau_0)b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)\alpha_j(1+y)^{2(j-1)} \\
\leq\ & CCb^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\frac{\langle y\rangle^4}{y^\gamma}\sum_{j=2}^{\infty} j^{\frac{\omega}{4}}e^{(\frac{\alpha}{2}-j)(\tau-\tau_0)}b^{\frac{\alpha}{2}+\delta}(\tau_0)b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)(1+y)^{2(j-1)}.
\end{aligned}$$

Similarly to the technique given in (7.28), we have

$$\begin{aligned}
e^{(\frac{\alpha}{2}-j)(\tau-\tau_0)}b^{\frac{\alpha}{2}+\delta}(\tau_0)b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau) &\lesssim Ce^{(\frac{\alpha}{2}-j)(\tau-\tau_0)}e^{\left(\frac{2}{\alpha}-1\right)\left(\frac{\alpha}{2}+\tilde{\eta}\right)(1+\tilde{\eta})(\tau-\tau_0)} \\
&\lesssim Ce^{(\tau-\tau_0)(1-j+c(\tilde{\eta}))} \text{ with } c(\tilde{\eta}) \lesssim \tilde{\eta}, j \geq 2,
\end{aligned}$$

which yields to

$$\begin{aligned}
&\left|\sum_{j=2}^{\infty} e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau_0)}\langle \hat{\varepsilon}_-(\tau_0), \phi_{j,\infty}\rangle_{L^2_\rho}\phi_{j,\infty}(y)\right| \\
\leq\ & Cb^{\frac{\alpha}{2}+\tilde{\eta}}\frac{\langle y\rangle^4}{y^\gamma}\sum_{j=2}^{\infty} j^{\frac{\omega}{4}}R^{2(j-1)}\left(A^{\frac{1}{K_0}}\right)^{1-j+c(\tilde{\eta})} \leq \frac{A^3}{32}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\frac{\langle y\rangle^4}{y^\gamma}, \qquad (7.39)
\end{aligned}$$

provided that $K_0 \geq K_4, A \geq A_4(R, K_0)$, and $\tilde{\eta} \leq \tilde{\eta}_4(R, K_0, A, \eta, \tilde{\eta}, L_0)$.

Next, observe that $\tau - (\tau - L_0) = L_0 \leq \frac{\ln A}{K_0}$ if $A \geq A_4(L_0)$, so we go back to the first case. Indeed, we argue similarly as in (7.31) and (7.33) to get

$$\begin{aligned}
\left|\int_{\tau-L_0}^{\tau} e^{(\tau-\tau')\mathscr{L}_\infty}(\hat{B})(\tau')d\tau'\right| &\leq \frac{\langle y\rangle^4}{y^\gamma}\int_{\tau-L_0}^{\tau} e^{\frac{\alpha}{2}(\tau-\tau')}b^{\frac{\alpha}{2}+4\eta}(\tau')d\tau \\
&\leq Cb^{\frac{\alpha}{2}+3\eta}(\tau)\frac{\langle y\rangle^4}{y^\gamma},
\end{aligned}$$

provided that $s_0 \geq s_4(\eta, L_0)$. This yields

$$\left|\int_{\tau-L_0}^{\tau}\left(e^{(\tau-\tau')\mathscr{L}_\infty}(\hat{B})(\tau')\right)_- d\tau'\right| \leq Cb^{\frac{\alpha}{2}+3\eta}(\tau)\frac{\langle y\rangle^4}{y^\gamma}.$$

For the integral $\int_{\tau_0}^{\tau-L_0}\sum_{j=2}^{\infty} e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau')}\left\langle \hat{B}(\tau'), \phi_{j,\infty}\right\rangle_{L^2_\rho}\phi_{j,\infty}(y)d\tau'$, we deduce from Lemma A.1 that

$$\left|\langle \hat{B}(\tau'), \phi_{j,\infty}\rangle_{L^2_\rho}\right| \lesssim b^{\frac{\alpha}{2}+4\eta}(\tau').$$

Then, for all $y \in [b^\eta(\tau), R]$, we have

$$\begin{aligned}
&\left|\int_{\tau_0}^{\tau-L_0}\left(\sum_{j=2}^{\infty} e^{(\frac{\alpha}{2}-j)(\tau-\tau')}\langle \hat{B}(\tau'), \phi_{j,\infty}\rangle_{L^2_\rho}\phi_{j,\infty}\right)d\tau'\right| \\
\lesssim\ & \frac{\langle y\rangle^4 b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)}{y^\gamma}\sum_{j=2}^{\infty}|\alpha_j|R^{2(j-1)}\int_{\tau_0}^{\tau-L_0} e^{(\frac{\alpha}{2}-j)(\tau-\tau')}b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)b^{\frac{\alpha}{2}+4\eta}(\tau')d\tau'.
\end{aligned}$$

We repeat the techniques given in (7.28) and (7.29) to obtain

$$\begin{aligned} b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau')}b^{\frac{\alpha}{2}+\eta}(\tau') &\lesssim b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau')e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau')} \\ &\lesssim e^{(1-j+c(\tilde{\eta}))(\tau-\tau')}, \text{ with } c(\tilde{\eta}) \lesssim \tilde{\eta}. \end{aligned}$$

It follows that for all $j \geq 2$, we have

$$\int_{\tau_0}^{\tau-L_0} b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)e^{(\frac{\alpha}{2}-j)(\tau-\tau')}b^{\frac{\alpha}{2}+\eta}(\tau')d\tau' \lesssim \exp\left([1-j+c(\tilde{\eta})]\,(L_0)\right), \text{ and } 1-j+c(\tilde{\eta})<0.$$

By taking $L_0 \gg R$, we get

$$\begin{aligned} &\sum_{j=2}^{\infty} |\alpha_j| R^{2(j-1)} \int_{\tau_0}^{\tau-L_0} e^{(\frac{\alpha}{2}-j)(\tau-\tau')} b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau) b^{\frac{\alpha}{2}+4\eta}(\tau')d\tau' \\ \lesssim\ &\sum_{j=2}^{\infty} j^{\frac{\omega}{4}} R^{2(j-1)} \exp\left([1-j+c(\tilde{\eta})]\,(L_0)\right) \lesssim 1, \end{aligned}$$

provided that $L_0 \gg R$and $\tau_0 \geq \tau_4(A, L_0, R, \delta, \eta, \tilde{\eta})$. Then

$$\left| \int_{\tau_0}^{\tau-L_0} \sum_{j=2}^{\infty} e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau')} \left\langle \hat{B}(\tau'), \phi_{j,\infty} \right\rangle_{L^2_\rho} \phi_{j,\infty}(y) d\tau' \right| \leq C b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau) \frac{\langle y \rangle^4}{y^\gamma}, y \in [b^{\frac{\eta}{4}}, R].$$

Recall that

$$\left|\beta(\tau)-\frac{1}{2}\right| \leq CAI^{\eta}(\tau_0),$$

so by repeating the above arguments to handle the remaining terms in (7.38), the following result holds

$$\begin{aligned} &\left| \int_{\tau_0}^{\tau-L_0} \sum_{j=2}^{\infty} e^{\left(\frac{\alpha}{2}-j\right)(\tau-\tau')} \left\langle \left(\left[\frac{1}{2}-\beta(\tau')\right] \Lambda \hat{\varepsilon}_-(\tau') \right) d\tau', \phi_{j,\infty} \right\rangle_{L^2_\rho} \phi_{j,\infty}(y) d\tau' \right| \\ + &\left| \int_{\tau-L_0}^{\tau} \left(\left[\frac{1}{2}-\beta(\tau')\right] \Lambda \hat{\varepsilon}_-(\tau') \right)_- d\tau' \right| \leq C b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau) \frac{\langle y \rangle^4}{y^\gamma}. \end{aligned}$$

We conclude that

$$\left| \int_{\tau_0}^{\tau} e^{(\tau-\tau')\mathscr{L}_\infty} \left[\frac{1}{2}-\beta(\tau')\right] \Lambda_y(\hat{\varepsilon}_{\beta,-})(\tau') d\tau' \right| \leq \frac{A^3}{16} b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau) \langle y \rangle^4 y^{-\gamma}, \forall y \in [b^{\frac{\eta}{4}}, R]. \tag{7.40}$$

- For the case $y \in \left[R, b^{-\tilde{\eta}}(\tau)\right]$. First, we observe that once $y \leq b^{-\tilde{\eta}}(\tau) \leq \frac{1}{L_0} e^{\frac{\tau-\tau_0}{2}\left(1-\eta\left(\frac{2}{\alpha}-1\right)\right)}$, then, there exists $\bar{\tau} \in [\tau_0, \tau-1]$ such that $y \in \left[\frac{1}{2L_0} e^{\frac{\tau-\bar{\tau}}{2}\left(1-\eta\left(\frac{2}{\alpha}-1\right)\right)}, \frac{1}{L_0} e^{\frac{\tau-\bar{\tau}}{2}\left(1-\eta\left(\frac{2}{\alpha}-1\right)\right)}\right]$. Since $y \geq R$ we have $\tau - \bar{\tau} \geq C(R) \to +\infty$ as $R \to +\infty$. Now, write the integral equation at the initial data $\bar{\tau}$

$$\hat{\varepsilon}_-(\tau) = e^{(\tau-\bar{\tau})\mathscr{L}_\infty} \hat{\varepsilon}_-(\bar{\tau}) + \int_{\bar{\tau}}^{\tau} e^{(\tau-\tau')\mathscr{L}_\infty} \left[\hat{B}(\tau') + \left(\frac{1}{2}-\beta(\tau')\right) \Lambda_y \hat{\varepsilon}_- \right](\tau') d\tau'. \tag{7.41}$$

Note that for all $j \geq 2$, we have

$$\langle \hat{\varepsilon}_-, \phi_{j,\infty} \rangle_{L^2_\rho} = \langle \varepsilon_+, \phi_{j,\infty} \rangle_{L^2_\rho} + \langle \varepsilon_-, \phi_{j,\infty} \rangle.$$

Using the fact that ε_+ is orthogonal to $\phi_{j,b,\beta}$, estimate (4.7), $\phi_{j,\infty,\beta}$'s definition given in Proposition 4.1 and $\phi_{i,\infty} = \phi_{j,\infty,\frac{1}{2}}$, we derive that

$$\left| \langle \varepsilon_+, \phi_{j,\infty} \rangle_{L^2_\rho} \right| \leq C b^{\frac{\alpha}{2}+\eta}(\bar{\tau}), \forall j \geq 2.$$

Moreover, using the definition of ρ_β given in (2.25) and $\rho = \rho_{\frac{1}{2}}$, we apply Cauchy Schwarz inequality and estimate (7.1) to derive

$$\begin{aligned}\left|\langle \varepsilon_-, \phi_{j,\infty}\rangle_{L^2_\rho}\right| &= \left|\int_{\mathbb{R}_+} \varepsilon_- \phi_{j,\infty}\rho dy\right| \le C\int_{\mathbb{R}_+} |\varepsilon_-|\sqrt{\rho_\beta}|\phi_{j,\infty}|\frac{\rho}{\sqrt{\rho_\beta}}dy \\ &\le C\|\varepsilon_-\|_{L^2_\rho}\sqrt{\int_{\mathbb{R}_+}|\phi_{j,\infty}|^2\frac{\rho^2}{\rho_\beta}dy} \le Cb^{\frac{\alpha}{2}+\eta}.\end{aligned}$$

Hence, we have

$$\left|\langle \hat{\varepsilon}_-(\bar{\tau}), \phi_{j,\infty}\rangle_{L^2_\rho}\right| \le Cb^{\frac{\alpha}{2}+\eta}(\bar{\tau}).$$

Now, we use formula (7.37) to get

$$\begin{aligned}\left|e^{(\tau-\bar{\tau})\mathscr{L}_\infty}\hat{\varepsilon}_-(\bar{\tau})(y,\tau)\right| &\lesssim \sum_{j=2}^{\infty} e^{(\frac{\alpha}{2}-j)(\tau-\bar{\tau})}\left|\langle\hat{\varepsilon}_-(\bar{\tau})\phi_{j,\infty}\rangle_{L^2_\rho}\right||\phi_{j,\infty}(y)| \\ &\lesssim \sum_{j=2}^{\infty} 4^j j! e^{(\frac{\alpha}{2}-j)(\tau-\bar{\tau})}b^{\frac{\alpha}{2}+\eta}(\bar{\tau})\,|\phi_{j,\infty}(y)| \\ &\lesssim \langle y\rangle^4 y^{-\gamma} b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\sum_{j=2}^{\infty}|\beta_j|\, e^{(\frac{\alpha}{2}-j)(\tau-\bar{\tau})}b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau)b^{\frac{\alpha}{2}+\eta}(\bar{\tau})y^{2(j-1)}.\end{aligned}$$

Using (5.11), we have for all $j \ge 2$

$$e^{(\frac{\alpha}{2}-j)(\tau-\bar{\tau})}b^{\frac{\alpha}{2}+\eta}(\bar{\tau})b^{-\frac{\alpha}{2}-\tilde{\eta}}(\tau) \lesssim e^{(\frac{\alpha}{2}-j)(\tau-\bar{\tau})}e^{\left(1-\frac{2}{\alpha}\right)\left[\left(\frac{\alpha}{2}+\eta\right)\left(1-\frac{\tilde{\eta}}{10}\right)\bar{\tau}-\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\tau\right]}.$$

In addition, since $\tilde{\eta} \ll \eta$, and $\left(1-\frac{2}{\alpha}\right) < 0$, it follows the following

$$\begin{aligned}&\left(1-\frac{2}{\alpha}\right)\left[\left(\frac{\alpha}{2}+\eta\right)\left(1-\frac{\tilde{\eta}}{10}\right)\bar{\tau}-\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\tau\right] \\ \le\ &\left(1-\frac{2}{\alpha}\right)\left[\left(\frac{\alpha}{2}+\frac{\eta}{2}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\bar{\tau}-\left(\frac{\alpha}{2}+\tilde{\eta}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\tau\right] \\ =\ &\left(1-\frac{2}{\alpha}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\left[\frac{\alpha}{2}(\bar{\tau}-\tau)+\frac{\eta}{2}\bar{\tau}-\tilde{\eta}\tau\right] \\ \le\ &\left(1-\frac{2}{\alpha}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\left[\frac{\alpha}{2}(\bar{\tau}-\tau)+\frac{\eta}{2}(\bar{\tau}-\tau)\right],\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{\alpha}{2}-j\right)(\tau-\bar{\tau})+\left(1-\frac{2}{\alpha}\right)\left(1+\frac{\tilde{\eta}}{10}\right)\left[\frac{\alpha}{2}(\bar{\tau}-\tau)+\frac{\eta}{2}(\bar{\tau}-\tau)\right] \\ \le\ &\left[1-j+\frac{\tilde{\eta}}{10}\left(\frac{2}{\alpha}-1\right)\frac{\alpha}{2}+\frac{\eta}{2}\left(\frac{2}{\alpha}-1\right)\left(1+\frac{\tilde{\eta}}{10}\right)\right](\tau-\bar{\tau}) \\ \le\ &\left[1-j+\eta\left(\frac{2}{\alpha}-1\right)\right](\tau-\bar{\tau}),\end{aligned}$$

and $|\beta_j| \le C\frac{j^{-\frac{\omega}{4}}}{4^j j!}$. Hence, we obtain

$$\begin{aligned}\left|e^{(\tau-\bar{\tau})\mathscr{L}_\infty}\hat{\varepsilon}_-(\bar{\tau})(y,\tau)\right| &\lesssim Ab^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\sum_{j=2}^{\infty}\frac{j^{-\frac{\omega}{4}}}{4^j j!}\left(ye^{-\frac{\tau-\bar{\tau}}{2}\left(1-\eta\left(\frac{2\ell}{\alpha}-1\right)\right)}\right)^{2(j-1)} \\ &\lesssim Ab^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\sum_{j=2}^{\infty}\frac{j^{-\frac{\omega}{4}}}{4^j j!}(L_0^{-1})^{2(j-\ell)} \le \frac{A^3}{16}b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau),\end{aligned}$$

when L_0 is large and $A \geq A_4$.

From the restriction $\frac{1}{2L_0} e^{\frac{\tau-\bar{\tau}}{2}\left(1-\eta\left(\frac{2\ell}{\alpha}-1\right)\right)} \leq y \leq b^{-\tilde{\eta}}(\tau)$, it follows

$$\tau < \frac{\bar{\tau}\left(1-\eta\left(1-\frac{2\ell}{\alpha}\right)\right)+2\ln L_0}{1-\left(\eta-2\tilde{\eta}\left(1+\frac{\tilde{\eta}}{10}\right)\right)\left(1-\frac{2\ell}{\alpha}\right)},$$

which yields

$$\tau-\bar{\tau} \leq \frac{2\tilde{\eta}\bar{\tau}\left(\frac{2\ell}{\alpha}-1\right)\left(1+\frac{\tilde{\eta}}{10}\right)+2\ln L_0}{1-\left(\eta-2\tilde{\eta}\left(1+\frac{\tilde{\eta}}{10}\right)\right)\left(\frac{2\ell}{\alpha}-1\right)}. \tag{7.42}$$

Hence, it turns out that the second case is obtained by replacing τ_0 by $\bar{\tau}$

$$\left|\int_{\bar{\tau}}^{\tau} e^{(\tau-\tau')\mathscr{L}_\infty}\left[\hat{B}(\hat{\varepsilon}_{\beta,+}+\hat{\varepsilon}_{\beta,-})\right](\tau')d\tau'\right| \leq \frac{A^3}{16} b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\langle y\rangle^{2\ell+2} y^{-\gamma}, \tag{7.43}$$

$$\left|\int_{\bar{\tau}}^{\tau} e^{(\tau-\tau')\mathscr{L}_\infty}\left(\frac{1}{2}-\beta(\tau')\right)\Lambda_y\hat{\varepsilon}_{\beta,-}(\tau')d\tau'\right| \leq \frac{A^3}{16} b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)\langle y\rangle^{2\ell+2} y^{-\gamma}. \tag{7.44}$$

At final, by adding all the terms, we get

$$|\varepsilon_{\beta,-}(\tau)| \leq \frac{A^3}{2} b^{\frac{\alpha}{2}+\tilde{\eta}} \frac{\langle y\rangle^{2\ell+2}}{y^\gamma}, \forall y \in \left[R, b^{-\tilde{\eta}}(\tau)\right],$$

which concludes the third case. □

The final step is to focus on the *a priori estimates* on the exterior part ε_e. More precisely, we have the following Lemma

Lemma 7.3 (A priori estimates on the exterior part)**.** *Let us consider ε to satisfy equation (2.17) with initial data given in (5.16), and $\varepsilon(\tau) \in V[A,\eta,\tilde{\eta}](\tau)$ for all $\tau \in [\tau_0,\tau_1]$, for some $\tau_1 > \tau_0$. Then, we have the following estimate*

$$\||y|\varepsilon_e(\cdot,\tau)\|_{L^\infty} \leq \frac{A^4}{2} b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tau).$$

Proof. To get the conclusion of the Lemma, we consider the natural $(d+2)$-dimensional extension as follows

$$\varepsilon_{ext}(z,\tau) = \varepsilon(y,\tau), |z| = y \text{ and } z \in \mathbb{R}^{d+2}, \tag{7.45}$$

and we introduce

$$\varepsilon_{ext,e}(z,\tau) = |z|\left(1-\chi_0\left(\frac{8}{3}|z|b^{\tilde{\eta}}(\tau)\right)\right)\varepsilon_{ext}(z,\tau),$$

where χ_0 was defined in (5.10) and γ defined was in (B). Note that u's extension defined in (7.45) is $C^2(\mathbb{R}^{d+2})$, thanks to the parabolic regularity of the semi-group $e^{t\Delta_{d+2}}$ and so is ε_{ext} and we derive from (2.17) that ε_{ext} satisfies

$$\begin{aligned}
\partial_\tau \varepsilon_{ext} &= \Delta\varepsilon_{ext} - \beta(\tau) y \cdot \nabla\varepsilon_{ext} - 2\beta(\tau)\varepsilon_{ext} \\
&- 3(d-2)\left[2Q_b(|z|) + |z|^2 Q_b^2(|z|)\right]\varepsilon_{ext} + B(\varepsilon_{ext},|z|) + \Phi(|z|,\tau),
\end{aligned}$$

where Λ_z is similarly defined as in (2.14), B and Φ were defined in (2.19) and (2.20), respectively. Now, we introduce

$$\varepsilon_{ext,e}(z,\tau) = |z|\left(1-\chi_{\tilde{\eta}}\right)\varepsilon_{ext}, \text{ and } \chi_{\tilde{\eta}}(z,\tau) = \chi_0\left(\frac{8}{3}|z|b^{\tilde{\eta}}(\tau)\right), \tag{7.46}$$

where χ_0 was defined in (5.10) and we have the following facts

$$\text{supp}(1-\chi_{\tilde{\eta}}) \subset \left\{ z \in \mathbb{R}^n \text{ such that } |z| \geq \frac{3}{8} b^{-\tilde{\eta}}(\tau) \right\}, \tag{7.47}$$

and $\varepsilon_{ext,e} = \varepsilon_{ext}$ for all $|z| \geq \frac{3}{4} b^{-\tilde{\eta}}(\tau)$. We here mention that the conclusion of the Lemma immediately follows from

$$\|\varepsilon_{ext,e}(\cdot,\tau)\|_{L^\infty[b^{-\tilde{\eta}}(\tau),+\infty)} \leq \frac{A^4}{2} b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tau), \forall \tau \in [\tau_0,\tau_1].$$

By using ε_{ext}'s equation above, we that $\varepsilon_{ext,e}$ exactly solves

$$\partial_\tau \varepsilon_{ext,e} = \mathscr{L}_\beta(\varepsilon_{ext,e}) + \mathcal{C}(\varepsilon_{ext}) + \mathcal{N}(\varepsilon_{ext}), \tag{7.48}$$

where $\mathcal{L}_\beta$ is defined by

$$\mathcal{L}_\beta = \Delta - \beta(\tau) z \cdot \nabla - \beta(\tau) Id \tag{7.49}$$

and the terms $\mathcal{C}(\varepsilon_{ext})$ and $\mathcal{N}(\varepsilon_{ext})$ are respectively defined by

$$\begin{aligned} \mathcal{C}(\varepsilon_{ext}) &= -2\text{div}(\varepsilon_{ext}\nabla(|z|(1-\chi_{\tilde{\eta}}))) - 3(d-2)\left[2Q_b(|z|) + |z|^2 Q_b^2(|z|)\right]\varepsilon_{ext,e} \\ &+ \varepsilon_{ext}\left[\partial_\tau(1-\chi_{\tilde{\eta}})|z| + \Delta(|z|(1-\chi_{\tilde{\eta}})) + \beta(\tau) z \cdot \nabla(1-\chi_{\tilde{\eta}})|z|\right], \end{aligned}$$

and

$$\mathcal{N}(\varepsilon_{ext}) = |z|(1-\chi_{\tilde{\eta}})\left(B(\varepsilon_{ext}) + \Phi(\cdot,\tau)\right).$$

- *The semi-group of $\mathcal{L}_\beta$:* Let us define $\mathcal{K}_\beta(\tau',\tau), \tau > \tau' \geq \tau_0$; the semi-group associated to $\mathcal{L}_\beta$ with

$$\mathcal{K}_\beta(\tau,\tau') f = \int_{\mathbb{R}^{d+2}} \mathcal{K}_\beta(\tau,\tau',z,z') f(z') dz'$$

and the Kernel $\mathcal{K}_\beta(\tau,\tau',z,z')$

$$\mathcal{K}_\beta(\tau,\tau',z,z') = \frac{\zeta(\tau,\tau')}{\left[4\pi\int_{\tau'}^{\tau}\zeta^2(\tilde{\tau},\tau')d\tilde{\tau}\right]^{\frac{d+2}{2}}} \exp\left(-\frac{(z'-z\zeta(\tau,\tau'))^2}{4\int_{\tau'}^{\tau}\zeta^2(\tilde{\tau},\tau')d\tilde{\tau}}\right),$$

$$\zeta(\tau,\tau') = e^{-\int_{\tau'}^{\tau}\beta(\tilde{\tau})d\tilde{\tau}}, \tau > \tau'.$$

In particular, when $\beta \equiv \frac{1}{2}$, our situation is the same as the operator considered in [26, Lemma A.1] since

$$\zeta(\tau,\tau') = e^{-\frac{\tau-\tau'}{2}}, \text{ and } \int_{\tau'}^{\tau}\zeta^2(\tilde{\tau},\tau')d\tilde{\tau} = 1 - e^{-(\tau-\tau')}.$$

Since our operator has the same structure as the one in [26, Lemma A.1], we can apply the arguments there to get

$$\|\mathcal{K}_\beta(\tau',\tau)(\varphi)\|_{L^\infty} \leq \zeta(\tau,\tau')\|\varphi\|_{L^\infty} \leq e^{-\frac{(\tau-\tau')}{4}}\|\varphi\|_{L^\infty}, \tag{7.50}$$

$$\|\mathcal{K}_\beta(\tau,\tau')\text{div}(\varphi)\|_{L^\infty} \leq C\frac{\zeta(\tau,\tau')}{\sqrt{\int_{\tau'}^{\tau}\zeta^2(\tilde{\tau},\tau')d\tilde{\tau}}}\|\varphi\|_{L^\infty} \leq C\frac{e^{-\frac{\tau-\tau'}{4}}}{\sqrt{\tau-\tau'}}, \tag{7.51}$$

since (5.4) holds for all $\tau \in [\tau_0,\tau_1]$. By Duhamel's formula, we now write equation (7.48) as follows

$$\varepsilon_{ext,e}(\tau) = \mathcal{K}_\beta(\tau,\tau')\varepsilon_{ext,e}(\tau') + \int_{\tau'}^{\tau}\mathcal{K}_\beta(\tau,\tilde{\tau})\left[\mathcal{C}(\varepsilon_{ext}) + \mathcal{N}(\varepsilon_{ext})\right](\tilde{\tau})d\tilde{\tau}. \tag{7.52}$$

We now aim to estimate the terms involving $\mathcal{N}(\varepsilon_{ext}(\tau))$ and $\mathcal{C}(\varepsilon_{ext})$.

- For $\mathcal{N}$, since (7.47) holds, we only consider $|z| \geq \frac{3}{8}b^{-\tilde{\eta}}(\tau)$ that will be divided into two small cases $|z| \in \left[\frac{3}{8}b^{-\tilde{\eta}}(\tau), b^{-\tilde{\eta}}(\tau)\right]$ and $|z| \geq b^{-\tilde{\eta}}(\tau)$. Since $\varepsilon(\tau) \in V_1[A,\eta,\tilde{\eta}](\tau), \forall \tau \in [\tau_0,\tau_1]$, we get the following

$$||z|\varepsilon_{ext}(z,\tau)| \leq A^3 b^{\frac{\alpha}{2}+\gamma\tilde{\eta}}\langle |z|\rangle^4 \leq CA^3 b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tau), \forall |z| \in \left[\frac{3}{8}b^{-\tilde{\eta}}(\tau), b^{-\tilde{\eta}}(\tau)\right],$$

and

$$||z|\varepsilon_{ext}(z,\tau)| \leq A^4 b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tau), \forall |z| \geq b^{-\tilde{\eta}}(\tau)$$

which yields

$$\|\mathcal{N}(\varepsilon_{ext})\|_{L^\infty} \leq CA^3 b^{\frac{\alpha}{2}+10\tilde{\eta}}(\tau), \forall \tau \in [\tau_0,\tau_1],$$

provided that $\tilde{\eta} \leq \tilde{\eta}_5(\alpha), \tau_0 \geq \tau_5(A,\tilde{\eta})$. Using (7.50), we deduce

$$\|\mathcal{K}_\beta(\tau,\tilde{\tau})\mathcal{N}(\tilde{\tau}))\|_{L^\infty} \leq CA^3 e^{-\frac{\tilde{\tau}-\tilde{\tau}}{4}} b^{\frac{\alpha}{2}+10\tilde{\eta}}(\tilde{\tau}).$$

- For $\mathcal{C}(\varepsilon_{ext})$: From (7.51), we have

$$\begin{aligned}
\|\mathcal{K}_\beta(\tau,\tilde{\tau})\text{div}(\varepsilon_{ext}\nabla[(1-\chi_{\tilde{\eta}})|z|])(\tilde{\tau})\|_{L^\infty} &\leq C\frac{e^{-\frac{\tau-\tilde{\tau}}{4}}}{\sqrt{\tau-\tilde{\tau}}}\|(\varepsilon_{ext}\nabla[(1-\chi_{\tilde{\eta}})|z|])(\tilde{\tau})\|_{L^\infty} \\
&\leq CA^3\frac{e^{-\frac{\tau-\tilde{\tau}}{4}}}{\sqrt{\tau-\tilde{\tau}}} b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tilde{\tau}).
\end{aligned}$$

Similarly, we have

$$\left\|\mathcal{K}_\beta(\tau,\tilde{\tau})[3(d-2)(2Q_b+|z|^2Q_b^2)\varepsilon_{ext,e}](\tilde{\tau})\right\|_{L^\infty} \leq CA^3 e^{-\frac{\tau-\tilde{\tau}}{4}} b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tilde{\tau}).$$

Besides that, we derive from (7.46) that

$$\|\varepsilon_{ext}(\tilde{\tau})\partial_\tau(1-\chi_{\tilde{\eta}})|z|\|_{L^\infty} \leq C\tilde{\eta}A^4 b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tilde{\eta}) \leq CA^3 b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tilde{\tau}).$$

Hence, we get

$$\|\mathcal{K}_\beta(\tau,\tilde{\tau})\varepsilon_{ext}(\tilde{\tau})\partial_\tau(1-\chi_{\tilde{\eta}})|z|\|_{L^\infty} \leq CA^3 e^{-\frac{\tau-\tilde{\tau}}{4}} b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tilde{\tau}). \tag{7.53}$$

By the same technique, we can establish the following

$$\|\mathcal{K}_\beta(\tau,\tilde{\tau})\varepsilon_{ext}(\tilde{\tau})\left[\Delta(|z|(1-\chi_{\tilde{\eta}}))+\beta(\tau)z\cdot\nabla(1-\chi_{\tilde{\eta}})|z|\right]\|_{L^\infty} \leq CA^3 e^{-\frac{\tau-\tilde{\tau}}{4}} b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tilde{\tau}).$$

Taking L^∞-estimate on both sides of (7.52), we get

$$\begin{aligned}
\|\varepsilon_{ext,e}(\tau)\|_{L^\infty} &\leq e^{-\frac{\tau-\tau'}{4}}\|\varepsilon_{ext,e}(\tau')\|_{L^\infty} + CA^3\int_{\tau'}^{\tau} e^{-\frac{\tau-\tilde{\tau}}{4}} b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tilde{\tau})\left[\frac{1}{\sqrt{\tau-\tilde{\tau}}}+1\right]d\tilde{\tau} \\
&\leq e^{-\frac{\tau-\tau'}{4}}\|\varepsilon_{ext,e}(\tau')\|_{L^\infty} + CA^3 b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tau')\left(\sqrt{\tau-\tau'}+(\tau-\tau')\right).
\end{aligned}$$

We now apply the technique used in [26, Proposition 4.5]. Our the choice of the initial data in (5.14) allows us to improve the bound on $\varepsilon_{ext,e}$ by

$$\|\varepsilon_{ext,e}(\tau)\|_{L^\infty} \leq \frac{A^4}{2} b^{\frac{\alpha}{2}+(\gamma-4)\tilde{\eta}}(\tau), \forall \tau \in [\tau_0,\tau_1],$$

provided that $A \geq A_5$ and $\tilde{\eta} \leq \tilde{\eta}_5(A)$ and $\tau_0 \geq \tau_5(A,\tilde{\eta})$. Finally, the conclusion of the lemma follows. ∎

8. The existence of unstable blowup solutions

In this Section, we aim to give a sketch of the proof to the existence of unstable blowup solutions to equation (1.3) with blowup speeds

$$\lambda_\ell(t) = C(u_0)(T-t)^{\frac{2\ell}{\alpha}} \text{ as } t \to T.$$

The general strategy of the proof is the same as in the stable setting, except to some technical modifications. For that reason, we only give main changes which lead to the unstable existence. Let us consider the similarity variable (2.11) with $\mu(\tau) = T - t$ and $\tau = -\ln(T-t)$ then w satisfies (2.12) with $\beta \equiv \frac{1}{2}$. We linearize around $Q_{b(\tau)}$ given in (2.15) by $\varepsilon = w - Q_{b(\tau)}$ and it holds that ε satisfies (2.17). Note that all terms in the equation remain the same with $\beta \equiv \frac{1}{2}$. In particular, the spectral analysis of $\mathscr{L}_b$ is valid without the appearance of β. Now, we consider the composition (7.6) with $\ell \geq 2$.

8.1. Shrinking set for $\ell \geq 2$

In this part, we modify a little bit the set in Definition 5.1 to be compatible with the new setting

Definition 8.1 (Shrinking set). Let $A, \eta, \tilde{\eta}$ be positive constants satisfying $A \gg 1$ and $1 \gg \eta \gg \tilde{\eta}$, we define $V_\ell[A, \eta, \tilde{\eta}](\tau)$ as the set of all $(\varepsilon, b) \in L^\infty \times \mathbb{R}$ satisfying:

(i) The dominating mode ε_ℓ and b satisfy

$$\left| \varepsilon_\ell + \frac{2}{\alpha} m_0 b^{\frac{\alpha}{2}} \right| \leq A b^{\frac{\alpha}{2}+\eta}, \tag{8.1}$$

and

$$\frac{1}{2} \leq b I_\ell^{-1}(\tau) \leq 2, \tag{8.2}$$

where

$$I_\ell(\tau) = e^{\left(\frac{2\ell}{\alpha}-1\right)\tau}. \tag{8.3}$$

In addition the others modes $\varepsilon_j \in j \in \{1, ..., \ell-1\}$ satisfy

$$|\varepsilon_j| \leq A b^{\frac{\alpha}{2}+\eta}. \tag{8.4}$$

(iii) L^2_ρ-decay: The part ε_- satisfies the following:

$$\|\varepsilon_-(.)\|_{L^2_{\rho_\beta}} \leq A^2 b^{\frac{\alpha}{2}+\eta}(\tau).$$

(iv) The remainders ε_- given in (7.6), and ε_e satisfy

$$\begin{aligned} \left\| y^\gamma \frac{\varepsilon_-(.,\tau)}{\langle y \rangle^{2\ell+2}} \right\|_{L^\infty[0, b^{-\tilde{\eta}}(\tau)]} &\leq A^3 b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau), \\ \| |y| \varepsilon_e(.,\tau)\|_{L^\infty} &\leq A^4 b^{\frac{\alpha}{2}+(\gamma-(2\ell+2))\tilde{\eta}}(\tau), \end{aligned}$$

where ε_e defined as in (5.9).

8.2. Preparing initial data

In this part, we aim to construct a class of initial data corresponding to the Shrinking set $V_\ell[A, \eta, \tilde{\eta}]$. Let us consider $A \geq 1$, and $0 < \tilde{\eta} \ll \eta \ll \delta \leq 1$, α, m_0 and $c_{\ell,0}$ defined as in (2.27), (6.15) and (2.28), respectively.

$$\psi(\ell, \tau_0) = \chi\left(y b^{\frac{\delta}{2}}(\tau_0)\right)\left(1 - \chi\left(\frac{y}{b^{\frac{\delta}{2}}(\tau_0)}\right)\right) \tag{8.5}$$

$$\times \left\{ \sum_{j=1}^{\ell-1} A d_j b^{\frac{\alpha}{2}+\eta}(\tau_0)\phi_{j,b(\tau_0),\beta(\tau_0)} - \frac{2}{\alpha} m_0 b^{\frac{\alpha}{2}}(\tau_0)\left[1 + d_\ell A b^{\frac{\alpha}{2}+\eta}(\tau_0)\right]\left(\frac{\phi_{\ell,b(\tau_0),\beta(\tau_0))}}{c_{\ell,0}} - \phi_{0,b(\tau_0),\beta(\tau_0))}\right) \right\}$$

In particular, the class of initial data implies the following result

Lemma 8.2 (Preparing the initial data). *There exists $A_6 \geq 1$, such that for all $A \geq A_6$, there exist $\eta_6(A) > 0$ such that for all $\delta \leq \delta_6$ there exists $\eta_6(A,\delta) > 0$ such that for all $\eta \leq \eta_6$ there exists $\tilde{\eta}_6(A,\delta,\eta)$ such that for all $\tilde{\eta} \leq \tilde{\eta}_6$ there exists $\tau_6(A,\delta,\eta,\tilde{\eta}) > 1$ such that for all $\tau_0 \geq \tau_6$, there exists $\mathcal{D}_A \subset [-2,2]^\ell$ such that the following properties are valid*

(i) *the mapping*

$$\begin{array}{rcl} \Gamma : \mathbb{R}^\ell & \to & \mathbb{R}^\ell \\ (d_1,...,d_{\ell-1}) & \mapsto & (\psi_1,...,\psi_\ell)(\tau_0), \end{array}$$

is affine and one to one from $\mathcal{D}_A$ to $\hat{V}[A,\eta](\tau_0)$, where

$$\hat{V}[A,\eta](\tau) = \left[-Ab^{\frac{\alpha}{2}+\eta}(\tau), Ab^{\frac{\alpha}{2}+\eta}(\tau)\right]^\ell. \tag{8.6}$$

In particular, we have the following property

$$\Gamma \mid_{\partial\mathcal{D}_A} \in \partial\hat{V}_A(\tau_0),$$

and $deg(\Gamma \mid_{\partial\mathcal{D}_A}) \neq 0$.

(ii) *for all $(d_1,...,d_\ell) \in \mathcal{D}_A$, it follows that $\psi_\ell(\tau_0) \in V_\ell[A,\tau,\tilde{\eta}](\tau_0)$ with strictly improved bounds as follows*

$$\begin{array}{rcl} \left|\psi_\ell(\tau_0) + \dfrac{2}{\alpha}m_0 b^{\frac{\alpha}{2}}(\tau_0)\right| & \leq & b^{\frac{\alpha}{2}+\eta}(\tau_0), \\ \|\psi_-(\tau_0)\|_{L^2_\rho} & \leq & b^{\frac{\alpha}{2}+\eta}(\tau_0), \\ \left\|\dfrac{y^\gamma}{\langle y\rangle^{2\ell+2}}\psi_-(.,\tau_0)\right\|_{L^\infty[0,b^{-\frac{\tilde{\eta}}{2}}(\tau_0)]} & \leq & b^{\frac{\alpha}{2}+\eta}(\tau_0), \\ \|\psi_e(\cdot,\tau)\|_{L^\infty} & \leq & b^{\frac{\alpha}{2}+\eta}(\tau_0), \end{array}$$

and $b(\tau_0)I_\ell^{-1}(\tau_0) \in \left[\frac{1}{4},\frac{3}{2}\right]$.

Proof. The bounds in item (ii) immediately follow from the explicit form of $\psi(\ell,\tau_0)$ in (8.5). In addition, the existence of $\mathcal{D}_A$ and mapping Γ follows from the concentration of modes $\psi_j, j \in \{0,1,....,\ell\}$ of $\psi(\ell,\tau_0)$ and the argument is quite the same as in [32, Proposition 4.5]. We kindly refer the reader to check the details of this result. □

8.3. Finite dimensional reduction

Since the shrinking set $V_\ell[A,\eta,\tilde{\eta}](\tau)$ has special properties, the conclusion of Theorem 1.2 immediately the following

$$(\varepsilon,b)(\tau) \in V_\ell[A,\eta,\tilde{\eta}](\tau)\forall\tau \in [\tau_0,+\infty), \text{ for some } \tau_0 \text{ large enough.}$$

In particular, we prove in this part that this control is reduced to a finite dimensional problem on $(\varepsilon_j)_{j\in\{1,2,...,\ell\}}$

Proposition 8.3 (Finite dimensional reduction). *There exists $A_7 \geq 1$, such that for all $A \geq A_7$, there exists $\delta_7(A)$ such that for all $\delta \leq \delta_7$ there exists $\eta_7(A,\delta)$ such that for all $\eta \leq \eta_7$ there exists $\tilde{\eta}_7(A,\delta,\eta)$ such that for all $\tilde{\eta} \leq \tilde{\eta}_7$ there exists $\tau_7(A,\delta,\eta,\tilde{\eta})$ such that for all $\tau_0 \geq \tau_7$, the following property is valid: If (ε,b) is the solution to the coupled system (2.17-3.4) with initial data (8.5), $(\varepsilon,b)(\tau) \in V_\ell[A,\eta,\tilde{\eta}](\tau)$ for all $\tau \in [\tau_0,\tau^*]$ for some $\tau^* > \tau_0$ and $(\varepsilon,b)(\tau^*) \in \partial V_\ell[A,\eta,\tilde{\eta}](\tau^*)$, then we have the following:*

(i) *It holds that $(\varepsilon_1,...,\varepsilon_\ell)(\tau^*) \in \partial\hat{V}[A,\eta](\tau^*)$ defined as in (8.6).*

(ii) Transversality: There exists $\nu_0 > 0$ such that

$$(\varepsilon_1, ..., \varepsilon_\ell)(\tau^* + \nu) \notin \partial\hat{V}[A, \eta](\tau^* + \nu), \forall \nu \in (0, \nu_0)$$

which implies

$$(\varepsilon, b)(\tau^* + \nu) \notin V[A, \eta, \tilde{\eta}](\tau^* + \nu), \forall \nu \in (0, \nu_0).$$

Proof. The proof mainly relies on the *priori estimate* which is the same as in Lemmas [6.1-7.3].

- *Proof of item* (i): Let us consider $A \geq A_7$, $\delta \leq \delta_7(A)$, $\eta \leq \eta_7(A, \delta), \tilde{\eta} \leq \tilde{\eta}_7(A, \delta, \eta)$, $\tau_0 \geq \tau_7(A, \delta, \eta, \tilde{\eta})$ and $(\varepsilon, b)(\tau) \in V_\ell[A, \eta, \tilde{\eta}](\tau)\forall \tau \in [\tau_0, \tau^*]$ such that Lemmas 7.1, 7.2, and 7.3 remain true (the technique of the proof is exactly the same and we kindly refer the reader to check the details). So, it immediately follows that the bounds of ε_- and ε_e given in Definition 8.1 for $V_\ell[A, \eta, \tilde{\eta}]$ are always improved by $\frac{1}{2}$-factor. In addition, we completely reproduce the argument of Lemma 6.1 to establish the following results: For all $\tau \in [\tau_0, \tau^*]$, we have

$$\left|\varepsilon_j'(\tau) - \left(\frac{\alpha}{2} - j\right)\varepsilon_j(\tau)\right| \leq C b^{\frac{\alpha}{2}+4\eta}(\tau), \forall \tau \in [\tau_0, \tau_1], \tag{8.7}$$

and

$$\begin{cases} \partial_\tau \varepsilon_\ell - \left(\frac{\alpha}{2} - \ell\right)\varepsilon_\ell = O\left(b^{\frac{\alpha}{2}+4\eta}(\tau)\right), \\ \partial_\tau \varepsilon_\ell - \left(\frac{\alpha}{2}\right)\varepsilon_\ell + m_0\left(\frac{b_\tau}{b} - 1\right)b^{\frac{\alpha}{2}} = O(b^{\frac{\alpha}{2}+4\eta}), \end{cases} \tag{8.8}$$

and

$$\left|\frac{b'(\tau)}{b(\tau)} - \left(1 - \frac{2\ell}{\alpha}\right)\right| \leq CAb^{4\eta}(\tau). \tag{8.9}$$

From (8.9), we derive that

$$\frac{3}{4} < b(\tau)I_\ell^{-1}(\tau) \leq \frac{3}{2}\forall \tau \in [\tau_0, \tau^*],$$

provided that $\tau_0 \geq \tau_7(A, \eta, \tilde{\eta})$. Thus, we derive from the fact that $(\varepsilon, b)(\tau) \in \partial V_\ell[A, \eta, \tilde{\eta}](\tau^*)$ the following

$$(\varepsilon_1, \varepsilon_2, ..., \varepsilon_\ell)(\tau^*) \in \partial\hat{V}[A, \eta](\tau^*)$$

which concludes item (i).

- *Proof of item* (ii): it is sufficient to prove that there exists ν_0 small enough such that
- Either there exists $j \in \{1, ..., \ell - 1\}$, such that

$$|\varepsilon_j(\tau^* + \nu)| > Ab^{\frac{\alpha}{2}+\eta}(\tau_1 + \nu), \forall \nu \in (0, \nu_0), \tag{8.10}$$

- or the following holds

$$\left|\varepsilon_\ell(\tau^* + \nu) + \frac{2}{\alpha}m_0 b^{\frac{\alpha}{2}}(\tau_1 + \nu)\right| > Ab^{\frac{\alpha}{2}+\eta}(\tau_1 + \nu), \forall \nu \in (0, \nu_0), \tag{8.11}$$

As we proved in item (i), one of the following two cases holds

- *Case 1:* There exists $j \in \{1, ..., \ell - 1\}$ such that

$$\varepsilon_j(\tau^*) = \sigma_j A b^{\frac{\alpha}{2}+\eta}(\tau^*),$$

- *Case 2:*

$$\varepsilon_\ell(\tau_1) + \frac{2}{\alpha}m_0 b^{\frac{\alpha}{2}}(\tau_1) = \sigma_\ell A b^{\frac{\alpha}{2}+\eta}(\tau_1),$$

where $\sigma_j = \pm 1$. The goal is to prove that the first case implies (8.10) and the second one concludes (8.11). Indeed, we have

+ Assume that case 1 occurs. Without loss of generality, we can assume $\sigma_j = 1$ and introduce

$$B_j(\tau) = \varepsilon_j(\tau) - Ab^{\frac{\alpha}{2}+\eta}(\tau).$$

It is obvious that $B(\tau^*) = 0$ and we also get from (8.7) and (8.9),

$$\begin{aligned} B_j'(\tau^*) &= \left(\frac{\alpha}{2} - j\right)\varepsilon_j(\tau^*) - A\left(\frac{\alpha}{2}+\eta\right)\frac{b'(\tau^*)}{b(\tau^*)} b^{\frac{\alpha}{2}+\eta}(\tau^*) + O(b^{\frac{\alpha}{2}+4\eta}(\tau^*)) \\ &= Ab^{\frac{\alpha}{2}+\eta}(\tau^*)\left((\ell - j) - \eta\left(1 - \frac{2\ell}{\alpha}\right)\right) + O\left(b^{\frac{\alpha}{2}+4\eta}(\tau^*)\right) > 0, \end{aligned}$$

provided that $\ell - j \geq 1$ and $\eta \leq \eta_7(A, \eta, \tilde{\eta})$ and $\tau^* \geq \tau_0 \geq \tau_7(A)$. Then, $B_j(\tau^* + \nu) > 0$ for all $\nu \in (0, \nu_0)$ for some ν_0 small enough. Thus, (8.10) follows.

+ If the case 2 occurs. We also assume $\sigma = 1$ (the opposite sign is the same), we then define

$$B_\ell(\tau) = \varepsilon_\ell(\tau) + \frac{2}{\alpha} m_0 b^{\frac{\alpha}{2}}(\tau) - Ab^{\frac{\alpha}{2}+\eta}(\tau),$$

and it holds that $B_\ell(\tau^*) = 0$. and we derive from (8.8) and (8.9) that

$$\begin{aligned} B_\ell'(\tau^*) &= \left(\frac{\alpha}{2} - \ell\right)\varepsilon_\ell(\tau^*) + \frac{2}{\alpha} m_0 \frac{\alpha}{2}\frac{b'}{b} b^{\frac{\alpha}{2}}(\tau^*) - A\left(\frac{\alpha}{2}+\eta\right)\frac{b'}{b} b^{\frac{\alpha}{2}+\eta}(\tau^*) + O(b^{\frac{\alpha}{2}+4\eta}(\tau_1)) \\ &= Ab^{\frac{\alpha}{2}+\eta}(\tau^*)\left[\eta\left(\frac{2\ell}{\alpha} - 1\right)\right] + O(b^{\frac{\alpha}{2}+4\eta}(\tau^*)) > 0, \end{aligned}$$

since $\eta\left(\frac{2\ell}{\alpha} - 1\right) > 0$ and $\tau^* \geq \tau_0 \geq \tau_7(A, \eta, \tilde{\eta})$. Thus, we conclude that $B_\ell(\tau^* + \nu) > 0$ for all $\nu \in (0, \nu_0)$, and (8.11) follows. This concludes the proof of the Proposition. □

8.4. Topological argument

In this part, we aim to prove the existence of an initial datum $(\varepsilon, b)(\tau_0)$ that leads to the global existence of $(\varepsilon, b)(\tau) \in V_\ell[A, \eta, \tilde{\eta}](\tau), \forall \tau \in [\tau_0, +\infty)$. The following is our result:

Proposition 8.4. *There exist $A, \eta, \tilde{\eta}$ and $\delta \ll 1$ satisfying $A \gg 1, 1 \gg \delta \gg \eta \gg \tilde{\eta} > 0, \tilde{\eta}$ and $\tau_0(A, \eta, \tilde{\eta}, \delta) \gg 1$ such that there exists $\tilde{d} = (d_1, ..., d_\ell) \in \mathcal{D}_A$ defined in Lemma 8.2 such that with initial data $\varepsilon(\ell, \tau_0)$ defined as in (8.5), the solution (ε, b) to the coupled problem (2.17-3.4), globally exists and the following holds*

$$(\varepsilon, b)(\tau) \in V_\ell[A, \eta, \tilde{\eta}](\tau), \forall \tau \geq \tau_0.$$

Proof. The proof follows from the topological argument which was used in [3] and [25]. Let us assume $A, \eta, \tilde{\eta}$ and δ are suitably chosen such that Lemma 8.2 and Proposition 8.3 hold true. We now proceed to the proof by contradiction and we assume that for all $\tilde{d} = (d_1, ..., d_\ell) \in \mathcal{D}_A$, there exists $\tau(\tilde{d}) \in [\tau_0, +\infty)$ such that $(\varepsilon, b)(\tau(\tilde{d})) \notin V_\ell[A, \eta, \tilde{\eta}](\tau(\tilde{d}))$. Then, we can define for each $\tilde{d} \in \mathcal{D}_A$ the maximum time

$$\tau^*(\tilde{d}) = \sup\{\tau_1 \text{ such that } (\varepsilon, b)(\tau) \in V_\ell[A, \eta, \tilde{\eta}](\tau), \forall \tau \in [\tau_0, \tau_1], \} \tag{8.12}$$

Now, let the mapping Π be defined by

$$\Pi : \mathcal{D}_{A,\tau_0} \to \partial\,[-1, 1]^\ell \tag{8.13}$$

$$\tilde{d} = (d_1, ..., d_\ell) \mapsto \Pi(\tilde{d}), \tag{8.14}$$

where

$$\Pi(\tilde{d}) = (Ab^{\frac{\alpha}{2}+\eta}(\tau^*(\tilde{d})))^{-1}\left(\varepsilon_1, ..., \varepsilon_\ell + \frac{2}{\alpha} m_0 b^{\frac{\alpha}{2}}\right)(\tau^*(\tilde{d})).$$

In particular, the following properties hold:

(*i*) Π is continuous from $\mathcal{D}_A$ to $\partial\,[-1, 1]^\ell$. Indeed, since $\tau^*(\tilde{d})$'s definition implies

$$(\varepsilon, b)(\tau^*(d)) \in \partial V_\ell[A, \eta, \tilde{\eta}](\tau^*(d)),$$

and item (*ii*) of Proposition 8.3 immediately yields the result.

(ii) $\mathrm{Deg}(\Pi|_{\partial\mathcal{D}_A}) \neq 0$. The result follows from item (i) of Lemma 8.2.

Thus, such a mapping Π can not exist, since it contradicts the index theory and the conclusion of the Proposition follows. □

9. Existence of ground state

We show in this part the asymptotic behavior of the ground state solution to (2.1). Let us introduce Q to be the function satisfying

$$Q''(\xi) + \frac{d+1}{\xi}Q'_\xi - 3(d-2)Q^2 - (d-2)\xi^2 Q^3 = 0, \quad Q(0) = -1 \text{ and } Q'(0) = 0. \tag{9.1}$$

We have the following result:

Lemma 9.1. *Let $d \geq 10$, then there exists a unique solution Q to equation (9.1) satisfying the following:*

(i) *Asymptotic behavior of Q:*

$$Q(\xi) = -1 + \sum_{i=1}^{k} a_i \xi^{2i} + O(\xi^{2k+2}) \text{ as } \xi \to 0, \tag{9.2}$$

$$Q(\xi) = -\frac{1}{\xi^2} + q_0 \xi^{-\gamma} + O(\xi^{-3\gamma-4}) \text{ as } \xi \to +\infty, \tag{9.3}$$

and $q_0 > 0$.

(ii) *In particular, when $d = 10$, the ground state is explicitly given by*

$$Q_{10}(\xi) = -\frac{1}{\xi^2+1}$$

(iii) *Asymptotic of $\Lambda Q = 2Q + \xi \cdot \partial_\xi Q$:*

$$\Lambda Q(\xi) < 0, \text{ and } \Lambda Q = \begin{cases} -2 + 4\left(\frac{3d-6}{3d+6}\right)\xi^2 + \sum_{i=2}^{k} a'_i \xi^{2i} + O(\xi^{2k+2}) \text{ as } \xi \to 0, \\ a_0 \xi^{-\gamma} + O(\xi^{-\gamma-g}) \text{ as } \xi \to +\infty, \end{cases} \tag{9.4}$$

for some $a_0 < 0$.

We note that ΛQ's asymptotic at infinity is stable under ∂_ξ^k for all $k \in \mathbb{N}$, more precisely

$$\partial_\xi^k \Lambda Q = \partial_\xi^k \left(a_0 \xi^{-\gamma}\right) + O(\xi^{-\gamma-g-k}) \text{ as } \xi \to +\infty. \tag{9.5}$$

Proof. - The proof of item (i): Following [18], we reformulate the ground state equation as an autonomous ODE. Indeed, let

$$Z(\xi) = -\xi^2 Q(\xi),$$

then

$$Z'' + \frac{d-3}{\xi}Z' - \frac{(d-2)}{\xi^2}Z(Z-1)(Z-2) = 0. \tag{9.6}$$

Again, apply the change of function

$$Z(\xi) = v(x) \text{ where } \xi = e^x,$$

to get

$$v''(x) + (d-4)v'(x) - (d-2)v(v-1)(v-2) = 0, x \in (-\infty, +\infty). \tag{9.7}$$

To prove global existence and asymptotic behavior of the solution, we employ the phase portrait analysis that used in [4] for the ground-state of the heat flow maps. First observe that (9.7) has three critical points $v = 0, 1, 2$. We choose $v = 1$ to start our analysis (for $v = 0, 2$, the linear operator will have complex or positive eigenvalues). According to our initial condition $Q(0) = -1, Q'(0) = 0$,

Q locally exists which in turn implies v's existence on $(-\infty, x_0)$ for some $x_0 < 0$ and $|x_0|$ large enough. In particular, we have the boundary condition at $-\infty$:

$$v(x) = e^{2x} + \sum_{i=2}^{k} c_i e^{2ix} + O(e^{(2k+2)x}) \text{ as } x \to -\infty.$$

This allows us to consider $v(x)$ as the flow starting at $(0,0)$ and ending at $(1,0)$. Following [2], we linearize around 1, i.e.

$$\epsilon = v - 1,$$

then ϵ solves

$$\epsilon'' + (d-4)\epsilon' - (d-2)\epsilon(\epsilon+1)(\epsilon-1) = 0. \tag{9.8}$$

The linearisation is given by

$$\epsilon'' + (d-4)\epsilon' + (d-2)\epsilon = (d-2)\epsilon^3,$$

We write the above equation in matrix form

$$\begin{pmatrix} \epsilon' \\ \epsilon \end{pmatrix}' = \begin{pmatrix} -(d-4) & -(d-2) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon' \\ \epsilon \end{pmatrix} + \begin{pmatrix} (d-2)\epsilon^3 \\ 0 \end{pmatrix}.$$

The eigenvalues are

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(\sqrt{d^2 - 12d + 24} - d + 4) \\ \lambda_2 &= \frac{1}{2}(-\sqrt{d^2 - 12d + 24} - d + 4), \end{aligned} \tag{9.9}$$

provided that

$$d^2 - 12d + 24 \geq 0,$$

otherwise, the solution is spiral at $+\infty$. We see that $\lambda_{1,2}(d) < 0$ for all $d \geq 10$.

- *Construction of no-escape region:* Let us define

$$F(\epsilon, \epsilon') = (\epsilon', -(d-4)\epsilon' + (d-2)(\epsilon^3 - \epsilon)).$$

We introduce a trapping region

$$\mathcal{S} = \{(\epsilon, \epsilon') | \quad \epsilon^3 - \epsilon \leq \epsilon' \leq 2(\epsilon^3 - \epsilon), \epsilon \in (-1, 0)\}.$$

- The lower boundary curve $\epsilon' = (\epsilon^3 - \epsilon)$. In the phase portrait space (ϵ, ϵ'), we define the normal vector ν_{in} which points inward $\mathcal{S}$

  $$\nu_{in} = (-(3\epsilon^2 - 1), 1).$$

  We easily check that

  $$F(\epsilon, \epsilon') \cdot \nu_{in} = (\epsilon^3 - \epsilon)3(1 - \epsilon^2) > 0, \forall \epsilon \in (-1, 0).$$

- The upper boundary curve $\epsilon' = 2(\epsilon^3 - \epsilon)$. In the phase portrait space (ϵ, ϵ'), we define the normal vector ν_{in} which points inward $\mathcal{S}$

  $$\nu_{in} = (2(3\epsilon^2 - 1), -1).$$

  Then,

  $$F(\epsilon, \epsilon') \cdot \nu_{in} = (\epsilon^3 - \epsilon)(12\epsilon^2 + d - 10) > 0, \forall \epsilon \in (-1, 0) \text{ and } m \geq 10.$$

  The vector field F points always inward on the whole boundary of $\mathcal{S}$ (excluding the stationary points $(-1, 0)$ and $(0, 0)$). This implies that the integral curve of F starting in $\mathcal{S}$ must stay in $\mathcal{S}$.

- *The boundary conditions trapped in $\mathcal{S}$:* Note that with initial data $Q(0) = -1, Q'(0) = 0$, Q locally exists, this leads that ϵ exists locally, i.e., it exists on $(-\infty, x_0)$, for some x_0 near $-\infty$. In particular, we have $Q \in C^\infty$. Using Taylor expansion together with equation (9.1), we see that $Q(y)$ behaves as follows

$$Q(\xi) = -1 + \left(\frac{3d-6}{2d+6}\right)\xi^2 + \left(\frac{1}{4}.\frac{21d^2-74d+64}{(3d+4)(d+4)}\right)\xi^4 + O(\xi^6) \text{ as } \xi \to 0.$$

- Asymptotic of the trapped solution in $\mathcal{S}$: Let us discuss the boundary condition at $-\infty$: we have

$$\epsilon(x) = -1 + e^{2x} - \frac{3d-6}{3d+6}e^{4x} - \frac{1}{4}\frac{21d^2-74d+64}{(3d+4)(d+4)}e^{6x} + O(e^{8x}), \text{ as } x \to -\infty.$$

Using this asymptotic, the solution can't end up at $(-1, 0)$. In addition, at $(0, 0)$, it gives the following general asymptotic of ϵ:

$$\epsilon(x) = h_+ e^{\lambda_1 x}(1 + O(e^{-2x})) + h_- e^{\lambda_2 x}(1 + O(e^{-2x})),$$

where

$$\lambda_1 = \frac{1}{2}(\sqrt{d^2 - 12d + 24} - d + 4) \text{ and } \lambda_2 = \frac{1}{2}(-\sqrt{d^2 - 12d + 24} - d + 4).$$

Assuming that $h_+ = 0$, we derive from the shrinking set $\mathcal{S}$ that

$$-\epsilon < \epsilon' < -2\epsilon \forall$$

Then

$$h_-(1 + \lambda_2) > 0 \text{ and } h_-(2 + \lambda_2) < 0,$$

this contradicts the formula of λ_2.

So, $h_+ \neq 0$.

In addition to that, we require the same condition as h_-

$$h_+(1 + \lambda_1) > 0 \text{ and } h_+(2 + \lambda_1) < 0.$$

since

$$\lambda_1 + 1 < 0 \text{ and } \lambda_1 + 2 > 0$$

we see that $h_+ < 0$ and we get the conclusion.

- The proof of item (ii): can be done in straightforward way, we omit the details.
- The proof of item (iii): We reformulate $Q(\xi)$ by

$$Q(\xi) = -\frac{(\epsilon(x) + 1)}{e^{2x}}, \xi = e^x. \tag{9.10}$$

Computation yields

$$\xi Q'_\xi = -\frac{\epsilon'_x(x)}{e^{2x}} - 2Q.$$

Thus,

$$\Lambda Q = 2Q + y\partial_y Q = -\frac{\epsilon'_x(x)}{e^{2x}} < 0 \quad \forall x \in (-\infty, \infty). \tag{9.11}$$

Now, we aim to find the higher derivative of ϵ, i.e., $\partial_x^k \epsilon$ for all $k \geq 1$. In fact, ϵ satisfies the following integral equation

$$\epsilon(x) = h_+ e^{\lambda_1 x} + h_- e^{\lambda_2 x} - \frac{1}{\lambda_1 - \lambda_2}\int_x^\infty \left(e^{\lambda_1(x-x')} - e^{\lambda_2(x-x')}\right) g(\epsilon(x'))dx', \tag{9.12}$$

where $g(z) = (d-2)z^3$. This gives us

$$\epsilon(x) = h_+ e^{\lambda_1 x} + O\left(e^{3\lambda_1 x}\right),$$

as $x \to +\infty$. In particular, applying ∂_x^k to the right hand side of equation (9.12) , we derive the following

$$\partial_x^k \epsilon(x) = \partial_x^k (h_+ e^{\lambda_1 x}) + O(e^{3\lambda_1 x}), \text{ as } x \to +\infty. \tag{9.13}$$

Let us remark that, it remains to prove (9.5). Indeed, we have the following

$$\begin{aligned} \Lambda Q &= -\epsilon'(x) e^{-2x}, \\ \partial_y \Lambda Q &= -\epsilon''(x) e^{-3x} + 2\epsilon'(x) e^{-3x}, \\ \partial_y^2 \Lambda Q &= -\epsilon'''(x) e^{-4x} + 3\epsilon'' e^{-4x} + 2\epsilon''(x) e^{-4x} - 6\epsilon'(x) e^{-4x}, \\ &= (-\epsilon''' + 5\epsilon'' - 6\epsilon') e^{-4x}, \\ \partial_y^3 \Lambda Q &= (-\epsilon^{(4)} + 5\epsilon''' - 6\epsilon'') e^{-5x} - 4(-\epsilon''' + 5\epsilon'' - 6\epsilon') e^{-5x}. \end{aligned}$$

By induction, we can prove that

$$\partial_y^k \Lambda Q = -\Pi_{j=0}^{k-1} (\partial_x - 2 - j) \partial_x \epsilon(x) e^{(-2-k)x}, \forall k \geq 1.$$

Using (9.13) and the fact that $\xi = e^x$, we get

$$\begin{aligned} \Lambda_\xi Q(\xi) &= a_0 \xi^{-\gamma} + O(\xi^{-3\gamma - 4}), \\ \partial_\xi \Lambda Q(\xi) &= a_0 (\lambda_1 - 2) \xi^{-\gamma - 1} + O(\xi^{-3\gamma - 5}), \\ \partial_\xi^2 \Lambda Q(\xi) &= a_0 (\lambda^2 - 5\lambda_1 + 6) \xi^{-\gamma - 2} + O(\xi^{-3\gamma - 6}). \end{aligned}$$

In particular, we have the general case as follows: for all $k \geq 1$

$$\begin{aligned} \partial_\xi^k \Lambda Q &= a_0 \Pi_{j=0}^{k-1} (\lambda_1 - 2 - j) \xi^{\lambda_1 - 2 - k} + O(\xi^{-\lambda_1 - 2 - k}) \\ &= a_0 (-\gamma) ... (-\gamma - (k-1)) \xi^{-\gamma - k} + O(\xi^{-3\gamma - 4 - k}), \end{aligned}$$

where $\gamma = 2 - \lambda_1, a_0 = -\lambda_1 h_+$. Thus, (9.5) directly follows. This finishes the proof of the Lemma. □

10. Diagonalisation of $\mathscr{L}_b$

The goal of this section is to give a detailed proof of Proposition 4.2 which is the same as the route map established in the Section 2 of [9].

10.1. Interior problem

In the sequel, we construct eigenfunctions for $\mathscr{L}_b$ in the region $0 \leq y \leq y_0 \ll 1$. First, we introduce

$$w(y, \tau) = v(\xi, \tau), \quad \text{with } \xi = \frac{y}{\sqrt{b}}. \tag{10.1}$$

The interior zone can be written in terms of the blow-up variable ξ as

$$0 \leq \xi \leq \xi_0 := \frac{y_0}{\sqrt{b}}.$$

Recall the definition of $\mathscr{L}_b$

$$\mathscr{L}_b w(y) = \frac{1}{b} \left(H_\xi - \beta b \Lambda_\xi \right) v.$$

where the Shrödinger type operator H_ξ defined by

$$H = \partial_\xi^2 + \frac{d+1}{\xi} \partial_\xi - 3(d-2) \left(2Q(\xi) + \xi^2 Q^2(\xi) \right). \tag{10.2}$$

Lemma 10.1 (Generators of the Kernel of H)**.** *There exists a family $\{T_i\}_{i\geq 0}$ with initial element $T_0 = a_0^{-1}\Lambda_\xi Q$ belonging to the kernel of H_ξ and such that for all $i \in \mathbb{N}$*

$$H\left(T_{i+1}\right) = T_i. \tag{10.3}$$

Moreover, T_i admits the expansion

$$T_i(\xi) = \begin{cases} \sum_{l=0}^{q} t_{i,l}\xi^{2i+2l} + O(\xi^{2i+2q+2}), \forall q \in \mathbb{N}, \text{ as } \xi \to 0, \\ C_i \xi^{-\gamma+2i}\left(1 + O\left(\frac{\ln \xi}{\xi^2}\right)\right), \text{ as } \xi \to +\infty, \end{cases} \tag{10.4}$$

and the derivatives, up to order $k = 3$, are such that

$$\partial_\xi^k T_i(\xi) = \partial_\xi^k\left(C_i\xi^{-\gamma+2i}\right) + O\left(\xi^{-\gamma+2i-2-k}\ln\xi\right), \text{ as } \xi \to +\infty. \tag{10.5}$$

Here γ and C_j were defined in (2.26), and (2.29), respectively.
Let

$$\Theta_i = \Lambda T_i - (2i-\alpha)T_i.$$

then, for all $k \in \{0, 1, 2\}$

$$\partial_\xi^k \Theta_i(\xi) = O\left(\xi^{-\gamma+2i-k-2}\ln\xi\right) \text{ as } \xi \to +\infty. \tag{10.6}$$

Proof. A detailed proof is to be presented in Appendix D. □

The generators of the kernel of H_ξ are at hand, we are in position to perform the construction of the eigenvalues and the eigenfunctions in the interior zone. More precisely, our result reads

Proposition 10.2 (Inner eigenfunctions)**.** *Let $\ell \in \mathbb{N}, \ell \geq 1$, $i \in \{0, ..., \ell\}$ and $\beta \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Then, there exists $\epsilon_0(\beta) > 0$ small enough such that for all $\epsilon \in (0, \epsilon_0)$ such that there exists $y^*(\epsilon) \ll 1$ such that for all $0 < y_0 \leq y^*$ there exist $b^*(y_0)$ and $\tilde{\lambda}^*(y_0)$ such that for all $0 < b < b^*(y_0)$ and $|\tilde{\lambda}| \leq \tilde{\lambda}^*$ there exists $\phi_{i,int} \in C^\infty\left(\left[0, \frac{y_0}{\sqrt{b}}\right], \mathbb{R}\right)$ such that the following hold:*

$$\left(H - b\beta\Lambda\right)\phi_{i,int,\beta} = 2\beta b\left(\frac{\alpha}{2} - i + \tilde{\lambda}\right)\phi_{i,int,\beta}, \tag{10.7}$$

where $\phi_{i,int,\beta}$ has the following decomposition

$$\phi_{i,int,\beta}(\xi) = \sum_{j=0}^{i} c_{i,j}(2\beta)^j b^j T_j(\xi) + \tilde{\lambda}\sum_{j=0}^{i} b^{j+1}\left(c_{i,j}(2\beta)^{j+1}T_{j+1}(\xi) + S_j(\xi)\right) + bR_i(\xi), \tag{10.8}$$

where the correction R_i and S_j satisfy the following estimates

$$\|S_j\|_{X_{\xi_0}^{2j+2-\gamma}} \leq Cy_0^2, \|\partial_b S_j\|_{X_{\xi_0}^{2j+4-\gamma}} \leq C, \|\partial_{\tilde{\lambda}} S_j\|_{X_{\xi_0}^{2j+2-\gamma}} \leq Cy_0^2, \text{ and } \|\partial_\beta S_j\|_{X_{\xi_0}^{2j+2-\gamma}} \leq Cy_0^2,$$
$$\|R_i\|_{X_{\xi_0}^{-\gamma+\epsilon}} \leq C(\epsilon), \|\partial_b R_i\|_{X_{\xi_0}^{2-\gamma+\epsilon}} \leq C(\epsilon), \|\partial_{\tilde{\lambda}} R_i\|_{X_{\xi_0}^{2-\gamma+\epsilon}} \leq C(\epsilon)b, \text{ and } \|\partial_\beta R_i\|_{X_{\xi_0}^{2-\gamma+\epsilon}} \leq C(\epsilon).$$

Proof. Due to the lengthy proof, we aim to put the details in Appendix E. □

10.2. Exterior problem

In this part, we aim to construct the eigenfunctions of $\mathscr{L}_b$ on $[y_0, +\infty)$, for some $y_0 \ll 1$. The following is our result

Proposition 10.3 (Outer eigenfunctions)**.** *Let $\ell \in \mathbb{N}, \ell \geq 1$, $i \in \{0, ..., \ell\}$ and $\beta \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Then, there exists $y^*(\beta) \ll 1$ such that for all $y_0 \leq y^*$, there exist $b^*(y_0, \beta)$ and $\tilde{\lambda}^*(y_0, \beta, b^*)$ such that for all $b \in (0, b^*)$ and $\tilde{\lambda} \in (-\tilde{\lambda}^*, \tilde{\lambda}^*)$, there exits a $C^\infty[y_0, +\infty)$ function $\phi_{i,out,\beta}$ satisfying*

$$\mathscr{L}_b \phi_{i,out,\beta} = \left(2\beta\left(\frac{\alpha}{2} - i\right) + \tilde{\lambda}\right)\phi_{i,out,\beta},$$

and having the following decomposition

$$\phi_{i,out,\beta} = \phi_{i,\infty,\beta} + \tilde{\lambda}(\tilde{\phi}_{i,\beta} + \tilde{R}_{i,1}) + \tilde{R}_{i,2},$$

where $\tilde{\phi}_{i,\beta}$ *satisfies*

$$\left(\mathscr{L}_\infty^\beta - 2\beta\left(\frac{\alpha}{2} - i\right)\right)\tilde{\phi}_{i,\beta} = \phi_{i,\infty,\beta} \text{ with } \phi_{i,\infty,\beta} \text{ defined as in (4.2)},$$

and $\tilde{R}_{i,1}$ *and* $\tilde{R}_{i,2}$ *fulfil the following estimates*

$$\|\tilde{R}_{i,1}\|_{X_{y_0}^{\gamma-d,2i-\gamma+2}} \le C|\tilde{\lambda}|, \quad \partial_b \tilde{R}_{i,1} = 0, \|\partial_{\tilde{\lambda}}\tilde{R}_{i,1}\|_{X_{y_0}^{\gamma-d,2i-\gamma+2}} \le C, \|\partial_\beta \tilde{R}_{i,1}\|_{X_{y_0}^{\gamma-d,2i-\gamma+2}} \le C,$$

and

$$\|\tilde{R}_{i,2}\|_{X_{y_0}^{-d,a'}} \le Cb^{\frac{\alpha}{2}}, \quad \|\partial_b\tilde{R}_{i,2}\|_{X_{y_0}^{-d,a'}} \le Cb^{\frac{\alpha}{2}-1}, \|\partial_{\tilde{\lambda}}\tilde{R}_{i,2}\|_{X_{y_0}^{-d,a'}} \le Cb^{\frac{\alpha}{2}}, \|\partial_\beta\tilde{R}_{i,2}\|_{X_{y_0}^{-d,a'}} \le Cb^{\frac{\alpha}{2}},$$

for $a' = 2i + 2 - \gamma$ *and* $X_{y_0}^{a,a'}$ *is a Banach space generated by the norm*

$$\|f\|_{X_{y_0}^{a,a'}} = \sup_{y\in[y_0,1]} y^{-a}\left\{\sum_{i=0}^{2} y^i \left|\partial_y^i f(y)\right|\right\} + \sup_{y\in[1,+\infty)} y^{-a'}\left\{\sum_{i=0}^{2} y^i \left|\partial_y^i f(y)\right|\right\}. \tag{10.9}$$

Proof. See Appendix F. □

10.3. Matching asymptotic

This part is devoted to conclude the proof of the diagonalisation on $\mathscr{L}_b$.

Proof of Proposition 4.2: Let $i \in \{0, 1, ..., \ell\}$ where $\ell \in \mathbb{N}, \ell \ge 2, \beta \in \left[\frac{1}{4}, \frac{3}{4}\right]$, $y_0 \le y_1^*, b \le b_1^*$ such that Propositions (10.2- 10.3) hold and $\phi_{i,int}$ and $\phi_{i,out,\beta}$ are defined in there.

A) The proof of item (I): We define

$$\phi_{i,b,\beta}(y) = \begin{cases} b^{-\frac{\gamma}{2}}\phi_{i,int,\beta}\left(\frac{y}{\sqrt{b}}\right) \text{ if } y \in [0, y_0], \\ \frac{b^{-\frac{\gamma}{2}}\phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right)}{\phi_{i,out,\beta}(y_0)}\phi_{i,out,\beta}(y) \text{ if } y \in [y_0, +\infty). \end{cases} \tag{10.10}$$

The main goal is to prove that there exists $y_0 \in (0, 1)$ small enough and $b^*(y_0) > 0$ such that, for all $b \in (0, b^*)$, there exists a unique $\tilde{\lambda}_i(b, \beta) = \tilde{\lambda}$ satisfying

$$\mathscr{L}_b\phi_{i,b,\beta} = \left(2\beta\left(\frac{\alpha}{2} - i\right) + \tilde{\lambda}\right)\phi_{i,b,\beta}, \tag{10.11}$$

and $\tilde{\lambda}$ satisfies (4.6).

First, we observe that $\phi_{i,int,\beta} \in C^\infty\left(\left[0, \frac{y_0}{\sqrt{b}}\right]\right)$ and $\phi_{i,out,\beta} \in C^\infty[y_0, +\infty)$ and they solve the regular second order differential equations, so $\phi_{i,b,\beta} \in C^\infty[0, +\infty)$ if and only if

$$b^{-\frac{\gamma}{2}-\frac{1}{2}}\partial_\xi\phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right) = b^{-\frac{\gamma}{2}}\frac{\phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right)}{\phi_{i,out,\beta}(y_0)}\partial_y\phi_{i,out,\beta}(y_0), \tag{10.12}$$

this condition ensures $\phi_{i,b,\beta}$'s differential is continuous at y_0. In particular, it is equivalent to

$$b^{-\frac{1}{2}}\partial_\xi\phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right)\phi_{i,out,\beta}(y_0) - \phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right)\partial_y\phi_{i,out,\beta}(y_0) = 0. \tag{10.13}$$

Here we use the implicit function theorem by applying it to the function $\tilde{F}[y_0](\tilde{\lambda}, b, \beta)$ defined by

$$\tilde{F}[y_0](\tilde{\lambda}, b, \beta) = b^{-\frac{1}{2}}\partial_\xi\phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right)\phi_{i,ext,\beta}(y_0) - \phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right)\partial_y\phi_{i,out,\beta}(y_0).$$

We firstly prove the following expansions:

$$\begin{aligned}\tilde F[y_0](\tilde\lambda,b,\beta) &= \tilde\lambda K_0 a_{i,0}(\tilde\gamma-\gamma)y_0^{-\gamma-\tilde\gamma-1}(1+O(y_0^2)+O(|\tilde\lambda|))\\ &+ O(b^{1-\frac{\epsilon}{2}}), \quad (10.14)\\ \partial_b\tilde F[y_0](\tilde\lambda,b,\beta) &= O(y_0^{-2\gamma-1+\epsilon}b^{-\frac{\epsilon}{2}})+O(|\tilde\lambda|b^{-1}y_0^{-2\gamma+3}), \quad (10.15)\\ \partial_{\tilde\lambda}\tilde F[y_0](\tilde\lambda,b,\beta) &= (\tilde\gamma-\gamma)K_0 a_{i,0}y_0^{-\gamma-\tilde\gamma-1}(1+O(y_0^2)+O(|\tilde\lambda|)), \quad (10.16)\end{aligned}$$

and for β-derivative

$$(10.17)$$

$$\partial_\beta\tilde F[y_0](\tilde\lambda,b,\beta) = \begin{cases}\tilde\lambda 2a_{i,1}K_0(2+\tilde\gamma-\gamma)y_0^{-\tilde\gamma-\gamma+1}(1+O(|y_0|^2)+O(|\tilde\lambda|)) & \text{if } i\geq 1,\\ \qquad +O(b^{1-\frac{\epsilon}{2}}) & (10.18)\\ O(|\tilde\lambda|y_0^{-\tilde\gamma-\gamma+3})+O(|\tilde\lambda^2|)+O(b^{1-\frac{\epsilon}{2}}) & \text{if } i=0.\end{cases}$$

Since the proof of asymptotic expansions (10.14-10.18) are technical and long, we complete them when we finish the proof of Proposition 4.2. Assume that the asymptotic expansions hold for all $b\in(0,b^*(y_0))$, $y_0\leq y_0^*$, and $\beta\in\left[\frac14,\frac34\right]$. We mention that the expansions are uniform in $\beta,\tilde\lambda$ and β. So, the argument from the implicit function theorem yields that $\forall b\in(0,b^*(y_0))$ and $\beta\in\left[\frac14,\frac34\right]$, there exists a unique $\tilde\lambda=\tilde\lambda(b,\beta)$ such that

$$\tilde F[y_0](\tilde\lambda,b,\beta)=0.$$

In particular, (10.14) ensures that $\tilde\lambda(b,\beta)=O\left(b^{1-\frac{\epsilon}{2}}\right)$ and expansions (10.15-10.18) imply

$$\left|b\partial_b\tilde\lambda(b,\beta)\right|\lesssim_{y_0} b^{1-\frac{\epsilon}{2}} \text{ and } \left|\partial_\beta\tilde\lambda(b,\beta)\right|\lesssim_{y_0} 1$$

yielding (4.6). Next, we decompose $\phi_{i,b,\beta}$ as follows

$$\phi_{i,b,\beta}(y)=\sum_{j=0}^{i}c_{i,j}(2\beta)^j(\sqrt b)^{2j-\gamma}T_j\left(\frac{y}{\sqrt b}\right)+\tilde\phi_{i,b}(y), \quad (10.19)$$

and we aim to prove

$$\|\tilde\phi_{i,b}\|_{H^1_\rho}\leq Cb^{1-\frac{\epsilon}{2}}. \quad (10.20)$$

In particular, we can specify it by

$$\tilde\phi_{i,b,\beta}(y)=\begin{cases} b^{-\frac{\gamma}{2}}\phi_{i,int,\beta}\left(\frac{y}{\sqrt b}\right)-\sum_{j=0}^i c_{i,j}\left(\sqrt b\right)^{2j-\gamma}T_j\left(\frac{y}{\sqrt b}\right) & \text{if } y\in[0,y_0],\\ \frac{b^{-\frac{\gamma}{2}}\phi_{i,int,\beta}\left(\frac{y_0}{\sqrt b}\right)}{\phi_{i,out,\beta}(y_0)}\phi_{i,out,\beta}(y)-\sum_{j=0}^i c_{i,j}\left(\sqrt b\right)^{2j-\gamma}T_j\left(\frac{y}{\sqrt b}\right) & \text{if } y\in[y_0,+\infty).\end{cases}$$

Now, we aim to prove that

$$|\partial_y^k\tilde\phi_{i,b,\beta}(y)|\leq C\left(y^{-\gamma+2-k}I_{y\in[0,y_0]}+y^{-\gamma+2i+2-k}I_{y\in[y_0,+\infty)}\right)b^{1-\frac{\epsilon}{2}},\quad y\in\mathbb R \text{ and } k=0,1. \quad (10.21)$$

Since the proofs for $k=0$ and $k=1$ are the same, we only give the proof of (10.21) for the case $k=0$ and we kindly refer the reader to check the details. Let us start the proof by considering two cases, namely, $y\in[0,y_0]$ and $y\in[y_0,+\infty)$.

1. $\underline{y\in[0,y_0]}$: write $\tilde\phi_{i,b}$ as

$$\tilde\phi_{i,b}(y)=\tilde\lambda\sum_{j=0}^{i}b^{j+1-\frac{\gamma}{2}}\left[c_{i,j}T_{j+1}\left(\frac{y}{\sqrt b}\right)+S_j\left(\frac{y}{\sqrt b}\right)\right]+b^{1-\frac{\gamma}{2}}R_i\left(\frac{y}{\sqrt b}\right).$$

According to Lemma 10.1, we have

$$|T_{j+1}(\xi)|\leq C\xi^{-\gamma+2j+2},\quad \forall\xi\in\mathbb R^+,$$

so that

$$\left|\tilde{\lambda} b^{1+j-\frac{\gamma}{2}} c_{i,j} T_{j+1}\left(\frac{y}{\sqrt{b}}\right)\right| \le C|\tilde{\lambda}| y^{2j+2-\gamma}.$$

Proposition 10.2 yields

$$\left|\tilde{\lambda} b^{1+j-\frac{\gamma}{2}} S_j\left(\frac{y}{\sqrt{b}}\right)\right| \le C\left|\tilde{\lambda}\right| y^{2j+2-\gamma}, \quad \textit{and} \quad b^{1-\frac{\gamma}{2}}\left|R_i\left(\frac{y}{\sqrt{b}}\right)\right| \le C b^{1-\frac{\epsilon}{2}} y^{-\gamma+\epsilon}.$$

The above three estimates allows us to infer that, for $y \in [0, y_0]$

$$|\tilde{\phi}_{i,b}(y)| \le C b^{1-\frac{\epsilon}{2}} y^{2-\gamma}.$$

2. $y \in [y_0, +\infty]$: write $\tilde{\phi}_{i,b}$ as follows

$$\begin{aligned} \tilde{\phi}_{i,b}(y) &= \frac{b^{-\frac{\gamma}{2}} \phi_{i,int}\left(\frac{y_0}{\sqrt{b}}\right)}{\phi_{i,ext}(y_0)} \phi_{i,ext}(y) - \sum_{j=0}^{i} c_{i,j}\left(\sqrt{b}\right)^{2j-\gamma} T_j\left(\frac{y}{\sqrt{b}}\right) \\ &= \underbrace{\phi_{i,ext}(y) - \sum_{j=0}^{i} c_{i,j}(\sqrt{b})^{2j-\gamma} T_j\left(\frac{y}{\sqrt{b}}\right)}_{=I} + \underbrace{\left[\frac{b^{-\frac{\gamma}{2}} \phi_{i,int}\left(\frac{y_0}{\sqrt{b}}\right)}{\phi_{i,ext}(y_0)} - 1\right] \phi_{i,ext}(y)}_{=II}. \end{aligned}$$

Let

$$\tilde{T}_j(\xi) = T_j(\xi) - C_j \xi^{2j-\gamma}, \tag{10.22}$$

and we decompose $\phi_{i,ext,\beta}$ as follows

$$\begin{aligned} I &= \phi_{i,ext}(y) - \sum_{j=0}^{i} c_{i,j}(\sqrt{b})^{2j-\gamma} C_j\left(\frac{y}{\sqrt{b}}\right)^{2j-\gamma} - \sum_{j=0}^{i} c_{i,j}(\sqrt{b})^{2j-\gamma} \tilde{T}_j\left(\frac{y}{\sqrt{b}}\right) \\ &= -\sum_{j=0}^{i} c_{i,j}(\sqrt{b})^{2j-\gamma} \tilde{T}_j\left(\frac{y}{\sqrt{b}}\right) + \tilde{\lambda}(\tilde{\phi}_i + \tilde{R}_{i,1}) + \tilde{R}_{i,2} \end{aligned}$$

Lemma 10.1 gives, for $y \ge y_0$

$$\begin{aligned} \left|(\sqrt{b})^{2j-\gamma} \tilde{T}_j\left(\frac{y}{\sqrt{b}}\right)\right| &\le C y^{-\gamma+2j-2} |\ln y| b| \ln b| \\ &\le C y^{-\gamma+2j-2} |\ln y| b^{1-\frac{\epsilon}{2}}. \end{aligned}$$

From Proposition 10.3, we deduce that

$$\begin{aligned} \left|\tilde{\phi}_i(y)\right| &\le C y^{2i-\gamma} |\ln y|, \\ \left|\tilde{R}_{i,1}(y)\right| &\le C\left(y^{-\tilde{\gamma}} + y^{2i-\gamma+2}\right) |\tilde{\lambda}|, \\ \left|\tilde{R}_{i,2}(y)\right| &\le C(y^{-\tilde{\gamma}-2-\alpha} + y^{2i+2-\gamma}) b^{\alpha}, \end{aligned}$$

for all $y \ge y_0$. Since $\alpha \ge 1 - \frac{\epsilon}{2}$, we obtain for all $y \ge y_0$

$$\left|\phi_{i,out,\beta}(y) - \sum_{j=0}^{i} c_{i,j}(\sqrt{b})^{2j-\gamma} T_j\left(\frac{y}{\sqrt{b}}\right)\right| \le C y^{2i+2-\gamma} |\ln y| b^{1-\frac{\epsilon}{2}}. \tag{10.23}$$

For the term II, we use the estimate obtained for I at $y = y_0$ to get

$$|\mathrm{II}| = \left|\frac{b^{-\frac{\gamma}{2}} \phi_{i,int}\left(\frac{y_0}{\sqrt{b}}\right)}{\phi_{i,ext}(y_0)} - 1\right| \le C(y_0) b^{1-\frac{\epsilon}{2}}.$$

Putting together the estimates for I and II yields for all $y \geq y_0$

$$\left|\tilde{\phi}_{i,b}(y)\right| \leq C(y_0) y^{-\gamma+2i+2} b^{1-\frac{\epsilon}{2}}.$$

Similarly for $\partial_y \tilde{\phi}_{i,b}$, we establish

$$\left|\partial_y \tilde{\phi}_{i,b}(y)\right| \leq C(y_0) y^{-\gamma+2i+1} b^{1-\frac{\epsilon}{2}}.$$

- Now, we have

$$\|\phi_{i,b} - \phi_{i,\infty}\|_{H^1_\rho} \leq \left\| \sum_{j=0}^{i} c_{i,j} (\sqrt{b})^{2j-\gamma} T_j\left(\frac{y}{\sqrt{b}}\right) - \phi_{i,\infty} \right\|_{H^1_\rho} + \|\tilde{\phi}_{i,b}\|_{H^1_\rho}.$$

Taking into account, (10.20), it is sufficient to establish

$$\left\| \sum_{j=0}^{i} c_{i,j} (\sqrt{b})^{2j-\gamma} T_j\left(\frac{y}{\sqrt{b}}\right) - \phi_{i,\infty} \right\|_{H^1_\rho} \leq C b^{1-\frac{\epsilon}{2}}.$$

As above, we have

$$\sum_{j=0}^{i} c_{i,j} (\sqrt{b})^{2j-\gamma} T_j\left(\frac{y}{\sqrt{b}}\right) - \phi_{i,\infty}(y) \quad = \quad \sum_{j=0}^{i} c_{i,j} (\sqrt{b})^{2j-\gamma} \tilde{T}_j\left(\frac{y}{\sqrt{b}}\right).$$

which yields, after splitting the integral in two regions $\{y \leq \sqrt{b}\}$ and $\{y \geq \sqrt{b}\}$ then using Lemma 10.1, to

$$\left\| \sum_{j=0}^{i} c_{i,j} (\sqrt{b})^{2j-\gamma} \tilde{T}_j\left(\frac{y}{\sqrt{b}}\right) \right\|_{H^1_\rho} \leq C b^{1-\frac{\epsilon}{2}}.$$

- Now, we move to the proof of item (iii) in Proposition 4.2. We distinguish two regions:

- $\underline{y \in [0, y_0]}$: from definition (10.10), we have

$$\begin{aligned}
\phi_{i,b}(y) = b^{-\frac{\gamma}{2}} \phi_{i,int}\left(\frac{y}{\sqrt{b}}\right) \quad &= \quad \sum_{j=0}^{i} c_{i,j} (\sqrt{b})^{2j-\gamma} T_j\left(\frac{y}{\sqrt{b}}\right) \\
&+ \quad \tilde{\lambda} \sum_{j=0}^{i} b^{j+1-\frac{\gamma}{2}} \left[c_{i,j} T_{j+1}\left(\frac{y}{\sqrt{b}}\right) + S_j\left(\frac{y}{\sqrt{b}}\right) \right] + b^{1-\frac{\gamma}{2}} R_i\left(\frac{y}{\sqrt{b}}\right).
\end{aligned}$$

Since, for all $\xi \in \mathbb{R}$

$$|T_j(\xi)| \leq C \frac{\xi^{2j}}{1+\xi^\gamma},$$

we have, by Lemma 10.1

$$\left| \sum_{j=0}^{i} b^{j-\frac{\gamma}{2}} T_j\left(\frac{y}{\sqrt{b}}\right) \right| \leq C(i+1) \frac{\langle y \rangle^{2i}}{(\sqrt{b}+y)^\gamma}.$$

and

$$\left| \sum_{j=0}^{i} b^{j+1-\frac{\gamma}{2}} T_{j+1}\left(\frac{y}{\sqrt{b}}\right) \right| \leq C(i+1) \frac{\langle y \rangle^{2i+2}}{(\sqrt{b}+y)^\gamma}.$$

For S_j, we have

$$\left|\tilde{\lambda}\sum_{j=0}^{i} b^{1+j-\frac{\gamma}{2}} S_j\left(\frac{y}{\sqrt{b}}\right)\right| \le C\frac{\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma}},$$

while for R_i

$$\left|b^{1-\frac{\gamma}{2}} R_i\left(\frac{y}{\sqrt{b}}\right)\right| \le C\frac{\langle y\rangle^{\epsilon}}{(\sqrt{b}+y)^{\gamma}}.$$

The above allows one to conclude that

$$|\phi_{i,b}(y)| \le C\frac{\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma}}, \forall y \in [0, y_0].$$

- $\underline{y \in [y_0, +\infty)}$: from the definition of ϕ_b on this region and the fact that

$$\left|b^{-\frac{\gamma}{2}}\frac{\phi_{i,int}(\frac{y_0}{\sqrt{b}})}{\phi_{i,ext}(y_0)}\right| \le C(y_0),$$

it is sufficient to estimate $\phi_{i,ext}$. Recall that

$$\phi_{i,ext}(y) = \phi_{i,\infty}(y) + \tilde{\lambda}(\tilde{\phi}_i(y) + \tilde{R}_{i,1}) + \tilde{R}_{i,2}.$$

The asymptotic behavior of $\tilde{\phi}_i$ yields, for all $y \ge y_0$

$$\left|\tilde{\phi}_i(y)\right| \le C\frac{y^{2i}|\ln y|}{y^{\gamma}} \le C(y_0)\frac{\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma}}.$$

Moreover, we have the following facts: for all $y \ge y_0$

$$\begin{aligned}
\left|\tilde{R}_{i,1}(y)\right| &\le C(y_0)|\tilde{\lambda}|\frac{\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma}},\\
\left|\tilde{R}_{i,2}(y)\right| &\le C(y_0)b^{\alpha}\frac{\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma}}.
\end{aligned}$$

Putting together the above estimates, one gets

$$|\partial_y\phi_{i,b}(y)| \le C\frac{\langle y\rangle^{2i+2}}{(\sqrt{b}+y)^{\gamma}}, \forall y \in [y_0, \infty).$$

A similar reasoning allows us to obtain the rest of the estimates, we omit the details.

- *Proof of* (10.14) *:*

First, we decompose $\phi_{i,int,\beta}$ and $\phi_{i,out,\beta}$ by

$$\begin{aligned}
\phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right) &= b^{\frac{\gamma}{2}}\left\{\phi_{i,\infty,\beta}(y_0) + \tilde{\lambda}\sum_{j=0}^{i} c_{i,j}C_{j+1}(2\beta)^{j+1}y_0^{-\gamma+2j+2} + A_{i,1}(\tilde{\lambda}, y_0, b, \beta)\right\},\\
b^{-\frac{1}{2}}\partial_\xi\phi_{i,int,\beta}\left(\frac{y_0}{\sqrt{b}}\right) &= b^{\frac{\gamma}{2}}\left\{\partial_y\phi_{i,\infty,\beta}(y_0) + \tilde{\lambda}\sum_{j=0}^{i} c_{i,j}C_{j+1}(2\beta)^{j+1}(-\gamma+2j+2)y_0^{-\gamma+2j+1}\right.\\
&+ \left.A_{i,2}(\tilde{\lambda}, y_0, b, \beta)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\phi_{i,out,\beta}(y_0) &= \phi_{i,\infty,\beta}(y_0) + \tilde{\lambda}K_0 y_0^{-\tilde{\gamma}} + B_{i,1}(\tilde{\lambda}, y_0, b, \beta),\\
\partial_y\phi_{i,out,\beta}(y_0) &= \partial_y\phi_{i,\infty,\beta}(y_0) - \tilde{\lambda}K_0\tilde{\gamma}y_0^{-\tilde{\gamma}-1} + B_{i,2}(\tilde{\lambda}, y_0, b, \beta).
\end{aligned}$$

where

$$\begin{aligned} A_{i,1} &= \sum_{j=0}^{i} c_{i,j}(2\beta)^j b^{j-\frac{\gamma}{2}} \tilde{T}_j\left(\frac{y_0}{\sqrt{b}}\right) + \tilde{\lambda}\sum_{j=0}^{i} c_{i,j}(2\beta)^{j+1} b^{j+1-\frac{\gamma}{2}} \tilde{T}_{j+1}\left(\frac{y_0}{\sqrt{b}}\right) \\ &+ \tilde{\lambda}\sum_{j=0}^{i} b^{j+1-\frac{\gamma}{2}} S_j\left(\frac{y_0}{\sqrt{b}}\right) + b^{1-\frac{\gamma}{2}} R_i\left(\frac{y_0}{\sqrt{b}}\right), \\ A_{i,2} &= \sum_{j=0}^{i} c_{i,j}(2\beta)^j b^{j-\frac{1}{2}-\frac{\gamma}{2}} \partial_\xi \tilde{T}_j\left(\frac{y_0}{\sqrt{b}}\right) + \tilde{\lambda}\sum_{j=0}^{i} c_{i,j}(2\beta)^{j+1} b^{j+\frac{1}{2}-\frac{\gamma}{2}} \partial_\xi \tilde{T}_{j+1}\left(\frac{y_0}{\sqrt{b}}\right) \\ &+ \tilde{\lambda}\sum_{j=0}^{i} b^{j+\frac{1}{2}-\frac{\gamma}{2}} \partial_\xi S_j\left(\frac{y_0}{\sqrt{b}}\right) + b^{\frac{1}{2}-\frac{\gamma}{2}} \partial_\xi R_i\left(\frac{y_0}{\sqrt{b}}\right), \end{aligned}$$

and

$$\begin{aligned} B_{i,1} &= \tilde{\lambda}(\tilde{\phi}_{i,\beta} - K_0 y_0^{-\tilde{\gamma}}) + \tilde{\lambda}\tilde{R}_{i,1} + \tilde{R}_{i,2}, \\ B_{i,2} &= \tilde{\lambda}\partial_y\left(\tilde{\phi}_{i,\beta} - K_0 y_0^{-\tilde{\gamma}}\right) + \tilde{\lambda}\partial_y \tilde{R}_{i,1} + \partial_y \tilde{R}_{i,2}, \end{aligned}$$

and $\tilde{T}_j$ and C_j defined as in (10.22) and (2.29), respectively.

We aim to estimate A_i and B_i by using the results of Propositions 10.2 and 10.3.

- estimate on $A_{i,1}$: From Lemma 10.1, we use T_j's expansion at ∞, to obtain the following

$$|\tilde{T}_j(\xi)| \le C\xi^{-\gamma+2j-2}\ln\xi,$$

for all ξ large enough, i.e $\xi \ge \xi_0 > 1$. Applying the above for $\xi_0 = \frac{y_0}{\sqrt{b}}$, we derive the following

$$\left|\tilde{T}_j\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C\left(\frac{y_0}{\sqrt{b}}\right)^{-\gamma+2j-2}(|\ln y_0| + |\ln b|), \forall j \ge 1,$$

and for $j = 0$, we have

$$\left|\tilde{T}_0\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C\left(\frac{y_0}{\sqrt{b}}\right)^{-\gamma-2}.$$

This yields

$$\left|\sum_{j=0}^{i} c_{i,j} b^{-\frac{\gamma}{2}+j}\tilde{T}_j\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C y_0^{-\gamma-2} b|\ln b|.$$

For the second term of $A_{i,1}$, the same process as above gives

$$\left|\sum_{j=0}^{i} c_{i,j} b^{j+1-\frac{\gamma}{2}}\tilde{T}_{j+1}\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C y_0^{-\gamma-2} b|\ln b|.$$

We now estimate to S_j. Accordingly to Proposition 10.2, and the definition of $X_{\xi_0}^{2j+2-\gamma}$, we have

$$\left|S_j\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C y_0^2\left(\frac{y_0}{\sqrt{b}}\right)^{2j+2-\gamma},$$

so that

$$\left|\tilde{\lambda}\sum_{j=0}^{i} b^{j+1-\frac{\gamma}{2}} S_j\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C\tilde{\lambda} y_0^{4-\gamma}.$$

The last term in $A_1$is to be estimated again via Proposition 10.2, where we have

$$\left|R_i\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C\left(\frac{y_0}{\sqrt{b}}\right)^{-\gamma+\epsilon}, \text{ with } \epsilon \ll 1.$$

Hence

$$\left|b^{1-\frac{\gamma}{2}}R_i\left(\frac{y_0}{\sqrt{b}}\right)\right| \le Cy_0^{-\gamma+\epsilon}b^{1-\frac{\epsilon}{2}}.$$

Finally, we get

$$A_1 = O(\tilde{\lambda}y_0^{4-\gamma}) + O\left(y_0^{-\gamma-2}b^{1-\frac{\epsilon}{2}}\right).$$

- <u>For $A_{i,2}$</u>: First, by Lemma 10.1 we get

$$\left|\partial_\xi \tilde{T}_j(\xi_0)\right| \le C\xi_0^{-\gamma+2j-3}|\ln \xi_0|, \forall j \ge 1,$$

and

$$\left|\partial_\xi \tilde{T}_0(\xi_0)\right| \le C|\xi_0|^{-\gamma-g-1}.$$

Then

$$\begin{aligned}\left|\sum_{j=0}^{i} c_{i,j}b^{-\frac{\gamma}{2}-\frac{1}{2}+j}\partial_\xi\tilde{T}_j(\xi_0)\right| &\le C\sum_{j=1}^{i} b^{j-\frac{\gamma}{2}-\frac{1}{2}}|\xi_0|^{-\gamma+2j-3}|\ln\xi_0| + Cb^{-\frac{\gamma}{2}-\frac{1}{2}}|\xi_0|^{-\gamma-g-1}\\ &\le Cy_0^{-\gamma-1}|\ln y_0|b|\ln b| + Cy_0^{-\gamma-g-1}b^{\frac{g}{2}}.\end{aligned}$$

Next, we estimate the second term in $A_{i,2}$:

$$\left|\tilde{\lambda}\sum_{j=0}^{i} c_{i,j}b^{j+\frac{1}{2}-\frac{\gamma}{2}}\partial_\xi\tilde{T}_{j+1}\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C|\tilde{\lambda}|y_0^{-\gamma-1}|\ln y_0|b^{\frac{3}{2}}|\ln b|.$$

Using Proposition 10.2 for S_j

$$|\partial_\xi S_j(\xi_0)| \le C|\xi_0|^{a-1} \le C|\xi_0|^{2j+1-\gamma},$$

we obtain

$$\left|\tilde{\lambda}\sum_{j=0}^{i} b^{j+\frac{1}{2}-\frac{\gamma}{2}}\partial_\xi S_j\left(\frac{y_0}{\sqrt{b}}\right)\right| \le C|\tilde{\lambda}|y_0^{-\gamma+1}.$$

The last term in $A_{i,2}$ is estimated similarly and we have

$$|b^{1-\frac{\gamma}{2}}\partial_\xi R_i(\xi_0)| \le Cy_0^{-\gamma+\epsilon-1}b^{\frac{3}{2}-\frac{\epsilon}{2}}.$$

Hence, the expansion of $A_{i,2}$ is

$$A_{i,2} = O(|\tilde{\lambda}|y_0^{-\gamma+1}) + O(y_0^{-\gamma-1}|\ln y_0|b|\ln b|).$$

- <u>For $B_{i,1}$</u> Using Lemma F.1 we have

$$\left|\tilde{\lambda}(\tilde{\phi}_i(y_0) - K_0y_0^{-\tilde{\gamma}})\right| \le C|\tilde{\lambda}|y_0^{-\tilde{\gamma}+2}.$$

For the second and third term, we use Proposition 10.3

$$|\tilde{R}_{i,1}(y_0)| \le C|\tilde{\lambda}|y_0^{-\tilde{\gamma}}$$

and

$$\left|\tilde{R}_{i,2}(y_0)\right| \le Cy_0^{-\tilde{\gamma}-2-\alpha}b^{\alpha}.$$

Then, $B_{i,1}$ reads as follows

$$B_{i,1}(y_0) = O\left(|\tilde{\lambda}|y_0^{-\tilde{\gamma}+2}\right) + O(|\tilde{\lambda}|^2y_0^{-\tilde{\gamma}}) + O(y_0^{-\tilde{\gamma}-2-\alpha}b^{\alpha}).$$

- For $B_{i,2}$: a similar reasoning gives

$$B_{i,2}(y_0) = O\left(|\tilde{\lambda}| y_0^{-\tilde{\gamma}+1}\right) + O(|\tilde{\lambda}|^2 y_0^{-\tilde{\gamma}-1}) + O(y_0^{-\tilde{\gamma}-3-\alpha} b^{\alpha}).$$

Putting the above expansions together, we derive for $\tilde{F}[y_0](\tilde{\lambda}, b)$

$$\begin{aligned}
\tilde{F}[y_0](\tilde{\lambda}, b, \beta) &= (\partial_y \phi_{i,\infty,\beta}(y_0) + \tilde{\lambda}\sum_{j=0}^{i} c_{i,j}(2\beta)^{j+1} C_{j+1}(-\gamma+2j+2) y_0^{-\gamma+2j+1} + A_2(\tilde{\lambda}, y_0, b, \beta)) \\
&\times (\phi_{i,\infty,\beta}(y_0) + \tilde{\lambda} K_0 y_0^{-\tilde{\gamma}} + B_1(\tilde{\lambda}, y_0, b, \beta)) \\
&- (\partial_y \phi_{i,\infty,\beta}(y_0) - \tilde{\lambda} K_0 \tilde{\gamma} y_0^{-\tilde{\gamma}-1} + B_2(\tilde{\lambda}, y_0, b, \beta)) \\
&\times (\phi_{i,\infty,\beta}(y_0) + \tilde{\lambda}\sum_{j=0}^{i} c_{i,j} C_{j+1}(2\beta)^{j+1} y_0^{-\gamma+2j+2} + A_1(\tilde{\lambda}, y_0, b, \beta)) \\
&= \tilde{\lambda} K_0 a_{i,0}(\tilde{\gamma}-\gamma) y^{-\gamma-\tilde{\gamma}-1}\left(1 + O(y_0^2) + O(|\tilde{\lambda}|)\right) \\
&+ O\left(y_0^{-2\gamma-2} b^{1-\frac{\epsilon}{2}}\right) + O\left(y_0^{-\gamma-\tilde{\gamma}-3-\alpha} b^{\alpha}\right).
\end{aligned}$$

The proofs of (10.15) and (10.16) follow the same outline.

□

11. Maximum principal

The main goal in this section is to use Maximum principal to construct the sub solution and the super solution to (2.17) on the interval $\left[0, b^{\frac{\eta}{4}}(\tau)\right]$ that leads to suitable estimates for ε.

Proposition 11.1 (Sub and super solutions)**.** *Let us consider $\eta, \tilde{\eta}$ be positive constants such that $1 \gg \eta \gg \tilde{\eta}$, $A \geq 1$. We assume furthermore that ε is the solution to (2.17) on $[\tau_0, \tau_1]$ with initial data given in (5.16) and the flow $(b, \beta)(\tau) \in (C^1(\tau_0, \tau_1])^2$ satisfy $(\varepsilon, b, \beta)(\tau) \in V[A, \eta, \tilde{\eta}](\tau)$ for all $\tau \in [\tau_0, \tau_1]$. Then, there exists $H(\xi)$ satisfying*

$$|H(\xi)| \leq C(\eta)\left[b(\tau)\frac{\xi^2}{1+\xi^{\gamma}} + \frac{b^{\frac{\eta}{4}}(\tau)}{1+\xi^{\gamma}}\right], \text{ for all } \xi \in \mathbb{R}_+,$$

such that

$$|\varepsilon(y,\tau)| \leq b^{-1}(\tau) H\left(\frac{y}{\sqrt{b(\tau)}}\right), \forall y \in [0, b^{\frac{\eta}{4}}(\tau)), \tag{11.1}$$

where Q_b defined as in (2.15). In other words, (11.2) remains true with τ lager than τ_1 as long as w exists and b satisfies the hypothesis of the Proposition.

Proof. First, we claim that the following

$$\left|w(y,\tau) - Q_{b(\tau)}(y)\right| \leq b^{-1}(\tau) H\left(\frac{y}{\sqrt{b(\tau)}}\right), \forall y \in [0, b^{\frac{\eta}{4}}(\tau)) \tag{11.2}$$

implies (11.1). We also mention that the proof is similar to the one in [5] where the authors constructed sub-solution and super-solution to equation (2.12) on the small interval $[0, b^{\frac{\eta}{4}}(\tau)]$. Let us consider the blowup variable (ξ, τ) and

$$w(y,\tau) = \frac{1}{b(\tau)}\omega\left(\frac{y}{\sqrt{b(\tau)}}, \tau\right) = \frac{1}{b(\tau)}\omega(\xi, \tau).$$

Introducing $v = \omega - Q(\xi)$ and v reads

$$b(\tau)\partial_\tau v = \partial_\xi^2 v + \frac{d+1}{\xi}\partial_\xi v - 3(d-2)(2Q + \xi^2 Q^2)v + B(v) + \theta(\tau)\Lambda_\xi Q + \theta(\tau)\Lambda_\xi v, \tag{11.3}$$

where B defined as in (2.19) and $\theta(\tau)$ defined by

$$\theta(\tau) = \beta(\tau)\left(b'(\tau) - b(\tau)\right). \tag{11.4}$$

We also introduce the operator $\mathcal{P}$ as follows

$$\mathcal{P}(v) := \partial_\xi^2 v + \frac{d+1}{\xi}\partial_\xi v - 3(d-2)(2Q + \xi^2 Q^2)v + \bar{B}(v) + \theta(\tau)\Lambda_\xi Q + \theta(\tau)\Lambda_\xi v - b(\tau)\partial_\tau v, \tag{11.5}$$

In order to construct the sub-solution and the super-solution, we need to construct two functions as follows: let Q be the ground state satisfying (9.1), $Q(0) = -1, Q'(0) = 0$, and we introduce

$$Q_\sigma = \frac{1}{\sigma} Q\left(\frac{\xi}{\sqrt{\sigma}}\right), \sigma > 0,$$

exactly solves (9.1) thanks to the scaling (1.4). Next, define

$$H_0(\xi) = \Lambda_\xi Q_\sigma(\xi), \tag{11.6}$$

satisfying

$$H_0'' + \frac{d+1}{\xi}H_0' - 3(d-2)(2Q_\sigma + \xi^2 Q_\sigma^2)H_0 = 0, \tag{11.7}$$

and let $H_1(\xi)$ solve the following

$$H_1'' + \frac{d+1}{\xi}H_1' - 3(d-2)(2Q_\sigma + \xi^2 Q_\sigma^2)H_1 = T(\xi), \tag{11.8}$$

where $T(\xi) = -\Lambda_\xi Q$. In particular, H_1 is explicitly given by

$$H_1(\xi) = H_0 \int_0^\xi \frac{\mathscr{L}(T)(\xi')}{H_0(\xi')} d\xi', \tag{11.9}$$

where $\mathcal{L}$ was defined in (D.3) Using (11.6) and (11.9), H_0 and H_1 have the following asymptotics:

$$H_0(\xi) = \begin{cases} -\frac{2}{\sigma} & \text{as} \quad \xi \to 0, \\ a_0 \sigma^\alpha \xi^{-\gamma} & \text{as} \quad \xi \to \infty, \end{cases} \tag{11.10}$$

where $a_0 < 0$ and $\alpha = 2 - \gamma$ and

$$H_1(\xi) = \begin{cases} \frac{\xi^2}{d+2} & \text{as} \quad \xi \to 0, \\ \frac{-a_0}{2(d+2-\gamma)}\xi^{2-\gamma} & \text{as} \quad \xi \to \infty. \end{cases} \tag{11.11}$$

Inspired by [5], we define

$$v^+(\xi,\tau) = \theta^+(\tau)H_1(\xi) - M(\eta)b^{\frac{\eta}{4}}(\tau)H_0(\xi) \text{ and } v^- = \theta^-(\tau)H_1(\xi) + M(\eta)b^{\frac{\eta}{4}}(\tau)H_0(\xi), \tag{11.12}$$

where

$$\theta^+(\tau) = b(\tau)\left[\beta(2\beta-1) - 4\beta^2\frac{\ell}{\alpha} - b^{\frac{\eta}{8}}(\tau)\right] \text{ and } \theta^-(\tau) = b(\tau)\left[\beta(2\beta-1) - 4\beta^2\frac{\ell}{\alpha} + b^{\frac{\eta}{8}}(\tau)\right] \tag{11.13}$$

Note that $H_0(\xi) = \Lambda_\xi Q_\sigma < 0$ see more (9.4). In particular, our aim is to prove

$$w^+(y,\tau) = Q_{b(\tau)}(y) + \frac{1}{b(\tau)}v^+\left(\frac{y}{\sqrt{b(\tau)}}\right) \text{ and } w^-(y,\tau) = Q_{b(\tau)}(y) + \frac{1}{b(\tau)}v^-\left(\frac{y}{\sqrt{b(\tau)}}\right)$$

are respectively the super-solution and the sub-solution to (2.12) which immediately implies (11.2). Following [5], it is sufficient to check that

(i) $\mathcal{P}(v^+) < 0 \quad (\mathcal{P}(v^-) > 0), \forall \tau \in [\tau_0, \tau_1]$ and $\xi \le b^{\frac{\eta}{4}-\frac{1}{2}}(\tau)$.

(*ii*) Initial estimate: $\frac{1}{b(\tau_0)}v^-\left(\frac{y}{\sqrt{b(\tau_0)}},\tau_0\right) < w(y,\tau_0) - Q_{b(\tau_0)}(\xi) < \frac{1}{b(\tau_0)}v^+\left(\frac{y}{\sqrt{b(\tau_0)}},\tau_0\right), \forall y \leq b^{\frac{\eta}{4}}(\tau_0)$.

We remark that the proof of the estimates on v^- are quite the same as for v^+. Thus, we only handle the latter.

- Proof of (i): plugging v^+ into (11.5), we get

$$\begin{aligned}
\mathcal{P}(v^+) &= \theta^+(\tau)\partial_\xi^2 H_1 - Mb^{\frac{\eta}{4}}(\tau)\partial_\xi^2 H_0 + \theta^+(\tau)\left(\frac{d+1}{\xi}\partial_\xi H_1\right) - M\frac{\eta}{4}(\tau)\frac{d+1}{\xi}\partial_\xi H_0\\
&- 3(d-2)\left[2Q+\xi^2Q^2\right]\left(\mu^+(\tau)H_1 - Mb^{\frac{\eta}{4}}(\tau)H_0(\xi)\right) + \theta(\tau)\Lambda_\xi Q + \bar{B}(v^+)\\
&+ \theta(\tau)\left[\theta^+\Lambda_\xi H_1 - Mb^{\frac{\eta}{4}}\Lambda_\xi H_0\right] - b(\tau)\left[\partial_\tau\theta^+(\tau)H_1 - M\partial_\tau b^{\frac{\eta}{4}}(\tau)H_0(\xi)\right]\\
&= -3(d-2)\left[2Q+\xi^2Q^2-(2Q_\sigma+\xi^2Q_\sigma^2)\right]\left(\theta^+(\tau)H_1(\xi) - Mb^{\eta}(\tau)H_0(\xi)\right)\\
&+ \bar{B}(v^+) + \left[\theta-\theta^+\right]\Lambda_\xi Q + \theta(\tau)\left[\theta^+\Lambda_\xi H_1 - Mb^{\frac{\eta}{4}}\Lambda_\xi H_0\right] - b(\tau)\left[\partial_\tau\theta^+ H_1 - M\partial_\tau b^{\frac{\eta}{4}}H_0\right],
\end{aligned}$$

where the simplification comes from the facts that H_0 and H_1 solve (11.7) and (11.8), respectively. Since $\xi \leq b^{\frac{\eta}{4}-\frac{1}{2}}(\tau)$ with $\eta \ll 1$ and $b(\tau)\to 0$, the range of ξ will be large, and we should divide it into two cases $\xi = O(1)$ and $\xi \gg 1$.

+ For the case $\xi \gg 1$, we derive on the one hand, from the definitions of θ, θ^+ in (11.4) (11.13), and (6.3), that

$$\theta(\tau) - \theta^+(\tau) = b^{1+\frac{\eta}{8}} + O(b^{1+4\eta}).$$

Thus we derive from (9.4) that

$$\left[\theta(\tau)-\theta^+\right]\Lambda_\xi Q = a_0 b^{1+\frac{\theta}{8}}(\tau)\xi^{-\gamma} + O(\xi^{-\gamma}b^{1+4\eta}) + O(b^{1+\frac{\eta}{8}}\xi^{-\gamma-2}),$$

where $a_0 < 0$. On the other hand, by a similar argument, we have

$$-b(\tau)\left[\partial_\tau\theta^+ H_1 - M\partial_\tau b^{\frac{\eta}{4}}H_0\right] = O\left(b^{1+\frac{\eta}{4}}\xi^{-\gamma}\right),$$

$$\theta(\tau)\left[\theta^+ H_1 - Mb^{\frac{\eta}{4}}H_0\right] = 0\left(b^{1+\frac{\eta}{4}}\xi^{-\gamma}\right) \text{ as } \xi\to\infty.$$

Thus, we conclude the following inequality

$$\mathcal{P}(z^+) \leq -3(d-2)\left[2Q+\xi^2Q^2-(2Q_\sigma+\xi^2Q_\sigma^2)\right]v^+ + \bar{B}(v^+),$$

provided that b,η small enough.

Next, note that $\xi \leq b^{\frac{\eta}{4}-\frac{1}{2}}(\tau)$ implies the following

$$|b(\tau)\xi^2| \leq b^{\frac{\eta}{2}}(\tau).$$

Hence, by using v^+'s definition in (11.12), along with (11.10) and (11.11), we obtain

$$v^+(\xi,\tau) = -Ma_0\sigma^\alpha b^{\frac{\eta}{4}}\xi^{-\gamma} + O(b^\eta\xi^{-\gamma-2}) + O(b^{2\beta}\xi^{-\gamma}), \text{ as } \xi\to+\infty.$$

In addition, recall that

$$\begin{aligned}
2Q(\xi)+\xi^2Q^2(\xi) &\sim -\frac{1}{\xi^2} + q_0^2\xi^{-2\gamma+2} + o(\xi^{-2\gamma+2}),\\
2Q_\sigma(\xi)+\xi^2Q_\sigma^2(\xi) &\sim -\frac{1}{\xi^2} + q_0^2\sigma^{\gamma-2}\xi^{-2\gamma+2} + o(\xi^{-2\gamma+2}),
\end{aligned}$$

as $\xi\to+\infty$. Then, fixing σ less that 1, we derive

$$2Q(\xi)+\xi^2Q^2(\xi) - \left[2Q_\beta(\xi)+\xi^2Q_\beta^2(\xi)\right] = q_0^2\left(1-\sigma^{\gamma-2}\right)\xi^{-3\gamma+2} + o(\xi^{-2\gamma+2}).$$

Thus,

$$-3(d-2)\left[2Q+\xi^2Q^2-(2Q_\sigma+\xi^2Q_\sigma^2)\right]v^+ = m_0 b^{\frac{\eta}{4}}(\tau)\xi^{-3\gamma+2}+o(\xi^{-3\gamma+2}),\ \text{as } \xi\to+\infty. \quad (11.14)$$

where $m_0 = 3(d-2)q_0^2(1-\sigma^{\gamma-2})Ma_0\sigma^\gamma < 0$. Next, we study $\bar{B}(v^+)$ defined by

$$\begin{aligned}\bar{B}(v^+) &= -3(d-2)(1+\xi^2Q(\xi))(v^+)^2-(d-2)\xi^2(v^+)^3\\ &= \left(m_1 b^{\frac{\eta}{2}}+m_2 b^{\frac{3\eta}{4}}\right)\xi^{-3\gamma+2}+o(\xi^{-3\gamma+2}),\ \text{as } \xi\to+\infty,\end{aligned}$$

where

$$m_1 = -3(d-2)q_0M^2a_0^2\sigma^{2\alpha} \text{ and } m_1 = -(d-2)M^3a_0^3\sigma^{3\alpha}.$$

Finally, we derive

$$\mathcal{P}(v^+) < 0,$$

provided that $\xi \gg 1$, $b \le b_1 \ll 1$ and $\eta \ll 1$.

\+ For $\xi = O(1)$, i.e., $\xi\in[0,K]$ for some $K>0$ large enough: thanks to the smallness of b, the dominating term in v^+ is $-Mb^{\frac{\eta}{4}}(\tau)H_0(\xi) > 0$. Besides that, we have

$$\partial_\sigma\left(2Q_\sigma+\xi^2Q_\sigma^2\right) = -\frac{1}{\sigma}\Lambda_\xi Q_\sigma - \xi^2\frac{Q_\sigma}{\sigma}\Lambda_\xi Q_\sigma = -\frac{\Lambda_\xi Q_\sigma}{\sigma}\left(1+\xi^2 Q_\sigma(\xi)\right).$$

Note that the construction of Q (see more in (9.10)) ensures $\xi^2 Q_\sigma(\xi) > -1$ and we derive

$$\partial_\sigma\left(2Q_\sigma+\xi^2Q_\sigma^2\right) > 0,$$

from which we infer the existence of $m_3(\sigma,K) > 0$ such that

$$2Q+\xi^2Q^2-(2Q_\alpha+\xi^2Q_\alpha^2) \ge m_3 \text{ with } \sigma < 1.$$

Since $[0,K]$ is compact, we get

$$v^+ = \theta^+H_1(\xi) - Mb^{\frac{\eta}{4}}H_0 \ge m_4(\sigma,K)b^{\frac{\eta}{4}} \text{ with } m_4 > 0.$$

Thus, we get

$$-3(d-2)\left[2Q+\xi^2Q^2-(2Q_\alpha+\xi^2Q_\alpha^2)\right]v^+ \le -3(d-2)m_3m_4b^{\frac{\eta}{4}}.$$

This concludes $\mathcal{P}(z^+) < 0$ for the case $\xi = O(1)$.

\- Proof of (ii): Notice that for $M(\eta)$ large enough and σ small, we have

$$\frac{v^+\left(\frac{y}{\sqrt{b(\tau_0)}}\right)}{b(\tau_0)} > 0,$$

and from $\varepsilon(\tau_0)$'s definition in (5.16), we see that it vanishes when $y \le b(\tau_0)$ and it is sufficient to check it for $y\in[b^{\frac{\delta}{2}}(\tau_0), b^{\frac{\eta}{4}}(\tau_0)]$ giving

$$\frac{y}{\sqrt{b(\tau_0)}} \to +\infty.$$

Thus, using (11.10) and (11.11), we can find a c_0 such that

$$\frac{v^+\left(\frac{y}{\sqrt{b(\tau_0)}}\right)}{b(\tau_0)} \ge c_0 b^{\frac{\alpha}{2}+\frac{\eta}{4}}(\tau_0)y^{-\gamma}.$$

On the other hand, ε's formula and the fact that $y \le b^{\frac{\eta}{4}}(\tau_0)$ imply

$$\left|\frac{\phi_{\ell,b(\tau_0),\beta(\tau_0)}}{c_{\ell,0}} - \phi_{0,b(\tau_0),\beta(\tau_0)}\right| \lesssim b^{\frac{\eta}{4}}(\tau_0).$$

This yields

$$|\varepsilon(\tau_0)| \lesssim b^{\frac{\alpha}{2}+\frac{\eta}{2}}(\tau_0) y^{-\gamma}.$$

□

A. Details on pointwise estimates

In the sequel, we give details to some pointwise estimates used in our paper.

Lemma A.1. *Let us consider $\hat{B}$ defined as in (7.20), $(\varepsilon, b, \beta)(\tau) \in V[A, \eta, \tilde{\eta}](\tau)$, for all $\tau \in [\tau_0, \tau^*]$ for some $\tau^* \geq \tau_0$, and $\delta > \eta > \tilde{\eta}$. Then, there exists $\tau_9(A, \delta, \eta, \tilde{\eta}) \geq 1$, such that for all $\tau_0 \geq \tau_9$, the following holds*

$$\begin{aligned} \left|\mathbb{1}_{y\in(0,b^\delta]} \hat{B}(y,\tau)\right| &\leq \frac{Cb^{\frac{\alpha}{2}}}{y^\gamma} + \frac{CA^3 b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau)}{y^{\gamma+2}} + \frac{CA^6 b^{2\left(\frac{\alpha}{2}+\tilde{\eta}\right)}(\tau)}{y^{2\gamma}} + \frac{CA^9 b^{3\left(\frac{\alpha}{2}+\tilde{\eta}\right)}(\tau)}{y^{3\gamma-2}}, \\ \left|\mathbb{1}_{y\geq b^\delta} \hat{B}(y,\tau)\right| &\leq C\frac{b^{\frac{\alpha}{2}+4\eta}\langle y\rangle^{2\ell+2}}{y^\gamma} + \frac{Cb^{\alpha+\delta(1-\gamma)}\langle y\rangle^{4\ell+8}}{y^\gamma} + \frac{Cb^{\frac{3\alpha}{2}-2\gamma\delta}\langle y\rangle^{6\ell+14}}{y^\gamma}, \forall \tau \in [\tau_0, \tau^*]. \end{aligned}$$

Proof. Let us consider $\delta \gg \eta \gg \tilde{\eta}$. First, since $(\varepsilon, b, \beta)(\tau) \in V_\ell[A, \eta, \tilde{\eta}](\tau)$ and by applying Lemma 6.1 we have

$$|\beta'(\tau)| \lesssim Ab^{4\eta}(\tau) \text{ and } \left|\frac{b'}{b} - 2\beta\left(1 - \frac{2\ell}{\alpha}\right)\right| \lesssim Ab^{4\eta}(\tau).$$

Write

$$\hat{\varepsilon}_{\beta,j} = \|\phi_{j,\infty,\beta}\|^{-2}_{L^2_{\rho_\beta}} \langle \varepsilon, \phi_{j,\infty,\beta}\rangle_{L^2_{\rho_\beta}}. \tag{A.1}$$

We observe that even though $\phi_{j,b,\beta}$ is not orthogonal to $\phi_{k,\infty,\beta}, k \neq j$, we have

$$\left|\langle \phi_{j,b,\beta}, \phi_{k,\infty,\beta}\rangle_{L^2_{\rho_\beta}}\right| \leq \int_{y\leq b^\delta} |\phi_{j,b,\beta}\phi_{k,\infty,\beta}| \rho_\beta dy + \int_{y\geq b^\delta} |\phi_{j,b,\beta}\phi_{k,\infty,\beta}| \rho_\beta dy \lesssim b^\delta.$$

Thus, we use pointwise estimates in Lemma 5.2 to obtain

$$|\hat{\varepsilon}_{\beta,j}(\tau)| \leq CAb^{\frac{\alpha}{2}+\tilde{\eta}}, \forall j < \ell, \quad \hat{\varepsilon}_{\beta,\ell} = \frac{\varepsilon_\ell}{c_{\ell,0}} + O(b^{\frac{\alpha}{2}+4\eta}), \text{ and } \hat{\varepsilon}_{\beta,0} = -\varepsilon_\ell + O(b^{\frac{\alpha}{2}+4\eta}). \tag{A.2}$$

In particular, repeating the technique in Lemma 6.1 we get

$$\left\{\begin{array}{rcl} \partial_\tau \hat{\varepsilon}_{\beta,\ell} &=& 2\beta\left(\frac{\alpha}{2} - \ell\right)\hat{\varepsilon}_{\beta,\ell} + O(b^{\frac{\alpha}{2}+4\eta}), \\ \partial_\tau \hat{\varepsilon}_{\beta,0} &=& 2\beta\frac{\alpha}{2}\hat{\varepsilon}_{\beta,0} + \left[\frac{b'}{b} - 2\beta\right] m_0 b^{\frac{\alpha}{2}} + O(b^{\frac{\alpha}{2}+4\eta}). \end{array}\right. \tag{A.3}$$

- The first case: $y \in \left(0, b^\delta(\tau)\right]$. From (7.20), we have

$$\begin{aligned} &\left|\hat{B}(\hat{\varepsilon}_{\beta,+} + \hat{\varepsilon}_{\beta,-})\right| \\ \leq\ &\left|3(d-2)\left(2Q_b + y^2 Q_b^2 + \frac{1}{y^2}\right)(\hat{\varepsilon}_{\beta,+} + \hat{\varepsilon}_{\beta,-})\right| + |B(\hat{\varepsilon}_{\beta,+} + \hat{\varepsilon}_{\beta,-})| + \left|\Phi + \mathscr{L}^\beta_\infty \hat{\varepsilon}_{\beta,+} - \partial_\tau \hat{\varepsilon}_{\beta,+}\right|. \end{aligned}$$

From Q's asymptotic given in Lemma 9.1 and (2.15), we get

$$\left|2Q_b + y^2 Q_b + \frac{1}{y^2}\right| \lesssim y^{-2}.$$

Besides that, since $\hat{\varepsilon}_{\beta,+} + \hat{\varepsilon}_{\beta,-} = \varepsilon = \varepsilon_+ + \varepsilon_-$, and the pointwise estimates in Lemma 5.2, we obtain

$$|\hat{\varepsilon}_{\beta,+} + \hat{\varepsilon}_{\beta,-}| \leq C\left(A^4 b^{\frac{\alpha}{2}+\tilde{\eta}}(\tau) y^{-\gamma} + b^{\frac{\alpha}{2}} y^{2-\gamma}\right),$$

Thus, the following is valid

$$\left|3(d-2)\left(2Q_b+y^2Q_b^2+\frac{1}{y^2}\right)(\hat{\varepsilon}_{\beta,+}+\hat{\varepsilon}_{\beta,-})\right|\le C\left(A^4b^{\frac{\alpha}{2}+\tilde{\eta}}y^{-2-\gamma}+b^{\frac{\alpha}{2}}(\tau)y^{-\gamma}\right).$$

Similarly, we have

$$|B(\hat{\varepsilon}_{\beta,+}+\hat{\varepsilon}_{\beta,-})|\le C\left(|\varepsilon|^2+y^2|\varepsilon|^3\right)\le C\left(A^8b^{\alpha+2\tilde{\eta}}y^{-2\gamma}+A^{12}b^{\frac{3}{2}\alpha+3\tilde{\eta}}y^{2-3\gamma}\right).$$

For the last term, we immediately deduce from (A.2) and (A.3) that

$$\left|\Phi+\mathscr{L}_\infty^\beta\hat{\varepsilon}_{\beta,+}-\partial_\tau\hat{\varepsilon}_{\beta,+}\right|\le Cb^{\frac{\alpha}{2}}y^{-\gamma}.$$

By adding all related terms, we conclude the estimate on $\mathbb{1}_{y\in(0,b^\delta]}\hat{B}$.

- The second case: $y\in\left[b^\delta(\tau)+\infty\right)$. Regarding (6.15), we can improve it as follows

$$\Phi=\left[\frac{b'}{b}-2\beta\right]m_0b^{\frac{\alpha}{2}}\phi_{0,\infty,\beta}+\tilde{\Phi}(y,\tau),$$

where

$$\left|\tilde{\Phi}(y,\tau)\right|\le Cb^{\frac{\alpha}{2}+\delta}y^{-\gamma}\langle y\rangle^{2\ell+2},\forall y\ge b^\delta(\tau).$$

In addition to that, we have

$$\left|\mathscr{L}_\infty^\beta\hat{\varepsilon}_{\beta,+}-\sum_{j=0}^{\ell}\left(\frac{\alpha}{2}-j\right)\hat{\varepsilon}_{\beta,j}\phi_{j,\infty,\beta}\right|\lesssim b^{\frac{\alpha}{2}+4\eta}\langle y\rangle^{2\ell+2}y^{-\gamma},$$

$$\left|\partial_\tau\hat{\varepsilon}_{\beta,+}-\sum_{j=0}^{\ell}\left(\frac{\alpha}{2}-j\right)\hat{\varepsilon}_{\beta,j}\phi_{j,\infty,\beta}\right|\lesssim b^{\frac{\alpha}{2}+4\eta}\langle y\rangle^{2\ell+2}y^{-\gamma}.$$

Finally, we conclude

$$\left|\Phi+\mathscr{L}_\infty^\beta\hat{\varepsilon}_{\beta,+}-\partial_\tau\hat{\varepsilon}_{\beta,+}\right|\le\frac{CAb^{\frac{\alpha}{2}+4\eta}(\tau)\langle y\rangle^{2\ell+2}}{y^\gamma}.$$

Next, we study the estimate involving Q_b. Using (9.3) and the facts that $y\ge b^\delta,\delta\ll1$, we get

$$\xi=\frac{y}{\sqrt{b}}\gg1,$$

then, it follows

$$Q_b(y)=-\frac{1}{y^2}\left(1+\tilde{Q}\right),$$

where

$$\left|\tilde{Q}(y)\right|\le C(\delta)b^{\frac{\alpha}{2}}y^{2-\gamma}\le b^{\frac{\alpha}{2}-\gamma\delta}\le Cb^\delta(\tau),y\ge b^\delta.$$

Hence, we have

$$\left|2Q_b(y)+y^2Q_b^2(y)+\frac{1}{y^2}\right|\le Cb^\delta,$$

which implies

$$\left|\left(2Q_b(y)+y^2Q_b^2(y)+\frac{1}{y^2}\right)(\hat{\varepsilon}_{\beta,+}+\hat{\varepsilon}_{\beta,-})\right|\le Cb^{\frac{\alpha}{2}+\delta}\langle y\rangle^{2\ell+2}y^{-\gamma}.$$

Similarly, since

$$\left|1+y^2Q_b(y)\right|\le Cb^{\frac{\alpha}{2}}(\tau)y^{2-\gamma}\le Cb^\delta,\forall y\ge b^\delta(\tau),$$

we deduce

$$\begin{aligned}|B(\hat\varepsilon_{\beta,+}+\hat\varepsilon_{\beta,-})| &\lesssim b^\delta\left(\frac{b^\alpha y^4\langle y\rangle^{4\ell+4}}{y^{2\gamma}}+\frac{A^8 b^{\alpha+2\tilde\eta}\langle y\rangle^{4\ell+4}}{y^{2\gamma}}\right)+y^2\left(\frac{b^{\frac{3\alpha}{2}}y^6\langle y\rangle^{6\ell+6}}{y^{3\gamma}}+\frac{A^{12}b^{\frac{3\alpha}{2}+3\tilde\eta}\langle y\rangle^{6\ell+6}}{y^{3\gamma}}\right)\\ &\lesssim b^{\alpha+\delta}\langle y\rangle^{2\ell+8}y^{-2\gamma}+b^{\frac{3\alpha}{2}}\langle y\rangle^{6\ell+14}y^{-3\gamma}.\end{aligned}$$

In particular, once $y\geq b^\delta(\tau')$, it follows $\frac{1}{y^\gamma}\lesssim b^{-\gamma\delta}$. By adding the related bounds, we conclude the estimate on $\mathbb{1}_{y\geq b^\delta}\hat B$. This achieves the proof of the Lemma. □

B. Detail on spectral analysis computation of $\mathscr{L}_\infty$

In this part, we aim to give a complete computation to formulate constant in Proposition 4.1. Let us consider the following quadratic equation

$$\gamma^2-d\gamma+3(d-2)=0. \tag{B.1}$$

The equation has two distinct solutions

$$\begin{cases}\gamma_1 &= \frac12(d-\sqrt{d^2-12d+24}),\\ \gamma_2 &= \frac12(d+\sqrt{d^2-12d+24}).\end{cases}$$

We remark that $\frac{1}{r^{\gamma_2}}$ does not belong to H^1_ρ, but $\frac{1}{r^{\gamma_1}}$ does. In addition, we also define

$$\gamma=\gamma_1=\frac12(d-\sqrt{d^2-12d+24}),\ \text{and}\ \tilde\gamma=\gamma_2=\frac12(d+\sqrt{d^2-12d+24}).$$

From γ's formula above, we can get the first eigenfunction and eigenvalue as follows

$$\phi_{0,\beta,\infty}(r)=\frac{1}{r^\gamma}\ \text{and}\ \lambda_{0,\beta,\infty}=2\beta\left(\frac12(\gamma-2)\right):=2\beta\left(\frac{\alpha}{2}\right).$$

Following [9] (also [8], and [7]), we search the eigenfunctions and eigenvalues in the following forms

$$\phi_{i,\beta,\infty}(r)=\sum_{j=0}^i a_{i,j}(2\beta)^j r^{2j-\gamma},\ \text{and}\ a_{i,i}=1,\ \text{and}\ \lambda_{i,\beta,\infty}=2\beta\left(\frac{\alpha}{2}-i\right),\forall i\geq 0. \tag{B.2}$$

Plugging the form (B.2) into the following relation

$$\mathscr{L}^\beta_\infty\phi_{i,\beta,\infty}=2\beta\left(\frac{\alpha}{2}-i\right)\phi_{i,\beta,\infty}, \tag{B.3}$$

we get

$$\begin{aligned}\mathscr{L}^\beta_\infty\phi_{i,\beta,\infty} &= \sum_{j=0}^i a_{i,j}(2\beta)^j A_j r^{2(j-1)-\gamma}+\sum_{j=0}^i a_{i,j}(2\beta)^j(-2\beta-\beta(2j-\gamma))r^{2j-\gamma}\\ &= 2\beta\left(\frac{\alpha}{2}-i\right)\sum_{j=0}^i a_{i,j}(2\beta)^j r^{2j-\gamma},\end{aligned} \tag{B.4}$$

where $A_j=(2j-\gamma)(2j-\gamma-1)+(d+1)(2j-\gamma)+3(d-2)$. Fix $0\leq j\leq i-1$.

+ For $j=i$: We choose $a_{i,i}=1$, then, we get

$$(-2\beta-\beta(2i-\gamma))=2\beta\left(\frac{\alpha}{2}-i\right),$$

then, (B.4) is satisfied.

+ For all $j\leq i-1$: (B.4) yields

$$a_{i,j+1}(2\beta)^{j+1}A_{j+1}+a_{i,j}(2\beta)^j(-2\beta-\beta(2j-\gamma))=2\beta\left(\frac{\alpha}{2}-i\right)a_{i,j}(2\beta)^j,$$

which yields

$$a_{i,j+1}A_{j+1} = (j-i)a_{i,j}. \tag{B.5}$$

By a simple recurrence, we obtain

$$a_{i,j} = (-1)^{i-j}\Pi_{k=j}^{i-1}\frac{A_{k+1}}{i-k} = \frac{(-1)^{i-j}}{(i-j)!}\Pi_{k=j+1}^{i}A_k.$$

Thus

$$a_{i,j} = \frac{(-1)^j}{(i-j)!} \times \Pi_{k=j+1}^{i-1}\left((2k-\gamma)(2k-\gamma-1)+(d+1)(2k-\gamma)+3(d-2)\right).$$

In particular, in (B.4), the order $r^{-\gamma-2}$ remains. However, its coefficient is equal to 0, since γ solves (B.1). Finally, (B.3) is completely satisfied by the choice of $a_{i,j}$ above. Next, we aim to decompose $a_{i,j}$ as follows: First, we deduce from (B.1) that

$$\begin{aligned} & (2k-\gamma)(2k-\gamma-1)+(d+1)(2k-\gamma)+3(d-2) \\ = \;& 2k.2k+2kd-4k\gamma = 2k(2k+d-2\gamma) = 4k\left(\frac{d}{2}-\gamma+k\right). \end{aligned}$$

Then, $a_{i,j}$ is decomposed as

$$a_{i,j} = \frac{(-1)^{i-j}}{(i-j)!}\Pi_{k=j+1}^{i}4k\left(\frac{d}{2}-\gamma+k\right) = \frac{(-1)^{i-j}}{(i-j)!}4^{i-j}\frac{i!}{j!}\frac{(\frac{d}{2}-\gamma)_i}{(\frac{d}{2}-\gamma)_j} = c_{i,j}C_j,$$

where $c_{i,j} = \frac{(-1)^{i-j}4^i i!\left(\frac{d}{2}-\gamma\right)_i!}{(i-j)!}$, $C_j = \frac{1}{4^j j!\left(\frac{d}{2}-\gamma\right)_j!}$, and

$$\left(\frac{d}{2}-\gamma\right)_i! = \left(\frac{d}{2}-\gamma+1\right)\left(\frac{d}{2}-\gamma+2\right)...\left(\frac{d}{2}-\gamma+i\right) \text{ and } \left(\frac{d}{2}-\gamma\right)_0! = 1.$$

C. Poisson kernel for Laguerre expansions

In this part, we aim to provide some pointwise estimates involving semi-group $e^{\tau\mathscr{L}_\infty}$ with $\mathscr{L}_\infty$ defined as in (2.22). Recall that for $f \in L^1(\mathbb{R}_+, x^\omega e^{-x}dx)$, we have the following presentation

$$\left[e^{(\tau-\tau_0)\mathcal{L}_\infty}\right]f(y,\tau) = 2^{\omega+1}y^{-\gamma}e^{\frac{\alpha}{2}(\tau-\tau_0)}\int_0^\infty P_{\frac{\omega}{2}}\left(\frac{y^2}{4},\frac{x^2}{4},e^{-(\tau-\tau_0)}\right)[f(x)x^\gamma]x^{\omega+1}e^{-\frac{x^2}{4}}dx. \tag{C.1}$$

where P_ζ is defined by

$$\begin{aligned} P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right) &= e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\frac{(-r\frac{x^2\cdot y^2}{16})^{-\frac{\zeta}{2}}}{1-r}J_\zeta\left(\frac{2(-r\frac{x^2\cdot y^2}{16})^{\frac{1}{2}}}{1-r}\right), \\ &= \frac{(\sqrt{r\frac{y^2}{4}\frac{x^2}{4}})^{-\zeta}}{1-r}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}i^\zeta J_\zeta\left(\frac{2r^{\frac{1}{2}}\frac{y}{2}\frac{x}{2}}{1-r}i\right) \\ &= \frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\mathbf{I}_\zeta\left(\frac{r^{\frac{1}{2}}yx}{2(1-r)}\right), \end{aligned}$$

and $\mathbf{I}_\zeta$ is the function of imaginary argument corresponding to

$$\mathbf{I}_\zeta(z) = \frac{\left(\frac{1}{2}z\right)^\zeta}{\Gamma(\zeta+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi \cosh\left(z\cos\theta\right)\sin^{2\zeta}(\theta)d\theta, \tag{C.2}$$

provided that $\mathrm{Re}(\zeta+\frac{1}{2}) > 0$, the reader can check the formula at page 79, formula (2) in [34]. We have the following result

Lemma C.1 (Maximal estimate, [4]). *Let us consider $f \in L^2_\rho$ with $\rho = \rho_{\frac{1}{2}}$ defined in (2.25), then,*

$$\left|e^{(\tau-\tau_0)\mathscr{L}_\infty} f(y)\right| \le C y^{-\gamma} e^{\frac{\alpha}{2}(\tau-\tau_0)}[Mf](y), \forall y > 0, \tau > \tau_0, \tag{C.3}$$

where α and γ were defined in (2.27) and (2.26), respectively, and Mf is given by

$$Mf(y) = \sup_{y\in\mathcal{I}} \frac{\int_{\mathcal{I}} |f(y')(y')^\gamma|(y')^{1+\omega} e^{-\frac{(y')^2}{4}} dy'}{\int_{\mathcal{I}} (y')^{1+\omega} e^{-\frac{(y')^2}{4}} dy'}, \omega = \sqrt{d^2 - 12d + 24}, \tag{C.4}$$

here the supremum is taken over all sub-intervals $\mathcal{I}$ containing y. In particular, if $|f(y)y^\gamma|$ is a non decreasing function, then the supremum in (C.4) is attained by $\mathcal{I} = [y, +\infty)$. Otherwise, if $|f(y)y^\gamma|$ is a non increasing, then the supremum is attained by $\mathcal{I} = [0, y]$.

Proof. The proof is quite the same as for Lemma $VI.2$ in [4]. □

Next, we will estimate the growth of the action $e^{(\tau-\tau')\mathscr{L}_\infty}$ to Λ:

Lemma C.2. *Let us consider $f \in L^2_\rho$ and $\Lambda f \in L^2_\rho$ where Λ was defined in (2.14). Assume further more that*

$$|f(y)| \le \mathcal{B}\frac{\langle y\rangle^{2\ell+2}}{y^\gamma}, \text{ for some } \mathcal{B} \in \mathbb{R}^*_+. \tag{C.5}$$

Then, it holds that for $\tau > \tau'$

$$\left|e^{(\tau-\tau')\mathscr{L}_\infty}(\Lambda f)\right| \lesssim e^{\tau-\tau'}\mathcal{B}\langle y\rangle^{2\ell+3} y^{-\gamma}. \tag{C.6}$$

Proof. Recall from (2.14) that $\Lambda f = y\partial_y f + 2f$ and apply (C.1) in deriving

$$\left[e^{(\tau-\tau')\mathcal{L}_\infty}\right]\Lambda f(y) = 2^{\omega+1} y^{-\gamma} e^{\frac{\alpha}{2}(\tau-\tau')}\int_0^\infty P_\zeta\left(\frac{y^2}{4}, \frac{x^2}{4}, r\right)(x\partial_x f + 2f)\, x^{\omega+1+\gamma} e^{-\frac{x^2}{4}} dx,$$

where $r = e^{-(\tau-\tau')}$ and $\zeta = \frac{\omega}{2}$. First, Lemma C.1 results in

$$\left|2^{\omega+1} y^{-\gamma} e^{\frac{\alpha}{2}(\tau-\tau')}\int_0^\infty P_\zeta\left(\frac{y^2}{4}, \frac{x^2}{4}, r\right) f(x) x^{\omega+1+\gamma} e^{-\frac{x^2}{4}} dx\right| \lesssim \mathcal{B} e^{\frac{\alpha}{2}(\tau-\tau')}\langle y\rangle^{2\ell+2} y^{-\gamma}.$$

Then, it is sufficient to prove that

$$|I(y)| \lesssim \mathcal{B}\langle y\rangle^{2\ell+3} \text{ where } I = \int_0^\infty P_\zeta\left(\frac{y^2}{4}, \frac{x^2}{4}, r\right)(x\partial_x f(x))\, x^{\omega+1+\gamma} e^{-\frac{x^2}{4}} dx.$$

Using the integration by parts provided that the functions go to 0 at $+\infty$ and 0, we get

$$\begin{aligned} I &= -\int_0^\infty f(x)\partial_x\left(P_\zeta\left(\frac{y^2}{4}, \frac{x^2}{4}, r\right) x^{\omega+2+\gamma} e^{-\frac{x^2}{4}}\right) dx \\ &= -\int_0^\infty f(x)\partial_x\left(P_\zeta\left(\frac{y^2}{4}, \frac{x^2}{4}, r\right)\right) x^{\omega+2+\gamma} e^{-\frac{x^2}{4}} dx. \\ &- (\omega+2+\gamma)\int_0^\infty f(x) P_\zeta\left(\frac{y^2}{4}, \frac{x^2}{4}, r\right) x^{\omega+1+\gamma} e^{-\frac{x^2}{4}} dx \\ &+ \frac{1}{2}\int_0^\infty f(x) P_\zeta\left(\frac{y^2}{4}, \frac{x^2}{4}, r\right) x^{\omega+3+\gamma} e^{-\frac{x^2}{4}} dx. \end{aligned}$$

We now explicitly compute

$$\begin{aligned}
\partial_x\left(P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right) &= \partial_x\left(\frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\mathbf{I}_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)\right)\\
&= \partial_x\left(\frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\right)\mathbf{I}_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)\\
&+ \frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\partial_x\left(\mathbf{I}_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)\right)\\
&= \partial_x\left(\frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\right)\mathbf{I}_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)\\
&+ \frac{4^\zeta}{x\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\frac{r^{\frac12}yx}{2(1-r)}\mathbf{I}'_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right),
\end{aligned}$$

from equality (3) at page 79 in [34]

$$\frac{r^{\frac12}yx}{2(1-r)}\mathbf{I}'_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)=\zeta\mathbf{I}_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)+\frac{r^{\frac12}yx}{2(1-r)}\mathbf{I}_{\zeta+1}\left(\frac{r^{\frac12}yx}{2(1-r)}\right),$$

we infer

$$\begin{aligned}
\partial_x\left(P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right) &= \partial_x\left(\frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\right)\mathbf{I}_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)\\
&+ \frac{4^\zeta}{x\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\zeta\mathbf{I}_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)\\
&+ \frac{4^\zeta}{x\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\frac{r^{\frac12}yx}{2(1-r)}\mathbf{I}_{\zeta+1}\left(\frac{r^{\frac12}yx}{2(1-r)}\right).
\end{aligned}$$

Besides, we have

$$\begin{aligned}
\partial_x\left(\frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\right) &= \frac{4^\zeta(-\zeta)}{\sqrt{r}^\zeta(1-r)(yx)^\zeta x}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\\
&- \frac{r}{1-r}\frac{x}{2}\frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}.
\end{aligned}$$

At final, we arrive to

$$\begin{aligned}
x\partial_x\left(P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right) &= -\frac{r}{1-r}\frac{x^2}{2}\frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\mathbf{I}_\zeta\left(\frac{r^{\frac12}yx}{2(1-r)}\right)\\
&+ \frac{4^\zeta}{\sqrt{r}^\zeta(1-r)(yx)^\zeta}\frac{r^{\frac12}yx}{2(1-r)}e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}\mathbf{I}_{\zeta+1}\left(\frac{r^{\frac12}yx}{2(1-r)}\right)\\
&= -\frac{rx^2}{2(1-r)}P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)+\frac{ry^2x^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right).
\end{aligned}$$

Plugging into I's formula, we get

$$\begin{aligned}
I &= \frac{r}{2(1-r)}\int_0^\infty f(x)P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}\\
&- \frac{ry^2}{8(1-r)}\int_0^\infty f(x)P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}\\
&- (\omega+2+\gamma)\int_0^\infty f(x)P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)x^{\omega+1+\gamma}e^{-\frac{x^2}{4}}\\
&+ \frac{1}{2}\int_0^\infty f(x)P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}\\
&= \int_0^\infty\left[\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right]f(x)x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}\,dx \qquad (C.7)\\
&- (\omega+2+\gamma)\int_0^\infty f(x)P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)x^{\omega+1+\gamma}e^{-\frac{x^2}{4}}\,dx.
\end{aligned}$$

It also follows from Lemma C.1 that

$$\left|\int_0^\infty f(x)P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)x^{\omega+1+\gamma}e^{-\frac{x^2}{4}}\,dx\right|\lesssim M(f),$$

and from (C.5), we have

$$\begin{aligned}
M(f)(y) &= \sup_{y\in\mathcal{I}}\frac{\int_{\mathcal{I}}|f(y')(y')^\gamma|(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'}{\int_{\mathcal{I}}(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'}\lesssim\mathcal{B}\frac{\int_y^\infty\langle y'\rangle^{2\ell+2}(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'}{\int_y^\infty(y')^{1+\omega}e^{-\frac{(y')^2}{4}}dy'} \qquad (C.8)\\
&\lesssim \mathcal{B}\langle y\rangle^{2\ell+2}. \qquad (C.9)
\end{aligned}$$

Now, it remains to prove the following:

$$|I_1(y)|\lesssim\frac{\mathcal{B}}{\sqrt{1-r}}\langle y\rangle^{2\ell+3}, \qquad (C.10)$$

where

$$I_1(y)=\int_0^\infty\left[\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right]f(x)x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}\,dx$$

We recall formulae (2) at page 77 and formula (2) at page 203 in [34] regarding the function I_ζ of imaginary argument

$$\begin{aligned}
cz^\zeta\le\mathbf{I}_\zeta(z)\le Cz^\zeta &\quad \text{if } z\in[0,1],\\
cz^{-\frac{3}{2}}e^z\le\left|\mathbf{I}_\zeta(z)-\frac{e^z}{\sqrt{2\pi z}}\right|\le Cz^{-\frac{3}{2}}e^z &\quad \text{if } z\in[1,+\infty),
\end{aligned} \qquad (C.11)$$

In particular, (C.11) implies

$$C^{-1}H_\zeta(y,x,r)\le P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\le CH_\zeta(y,x,r), \qquad (C.12)$$

where

$$H_\zeta(y,x,r)=\begin{cases}(1-r)^{-\zeta-1}e^{-\frac{r(\frac{y^2}{4}+\frac{x^2}{4})}{1-r}} & \text{if } x\in\left[0,\frac{2(1-r)}{\sqrt{r}y}\right]\\ \frac{(4r)^{-\frac{\zeta}{2}-\frac{1}{4}}(yx)^{-\zeta-\frac{1}{2}}}{e(\sqrt{1-r})}e^{\frac{-r\frac{y^2}{4}+\frac{1}{2}(r)^{\frac{1}{2}}(yx)-r\frac{x^2}{4}}{1-r}} & \text{if } z\in\left[\frac{2(1-r)}{\sqrt{r}y},+\infty\right)\end{cases}. \qquad (C.13)$$

Using (C.5), we estimate

$$\begin{aligned}
&\left|I_1(y,\tau,\tau')\right| \\
\lesssim\ & \mathcal{B}\left(\int_0^{\frac{2(1-r)}{\sqrt{r}y}}\left|\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right|(1+x^{2\ell+2})x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}dx\right. \\
+\ & \left.\int_0^{\frac{2(1-r)}{\sqrt{r}y}}\left|\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right|(1+x^{2\ell+2})x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}dx\right).
\end{aligned}$$

+ For the integral on $\left[0,\frac{2(1-r)}{\sqrt{r}y}\right]$, we use the first asymptotic in (C.13) to obtain

$$\begin{aligned}
&\int_0^{\frac{2(1-r)}{\sqrt{r}y}}\left|\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right|\left(1+x^{2\ell+2}\right)x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}dx \\
\leq\ & C(1-r)^{-\zeta-2}\int_0^{\frac{2(1-r)}{\sqrt{r}y}}\left(1+\frac{ry^2}{(1-r)}\right)e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}(1+x^{2\ell+2})x^{\omega+3}e^{-\frac{x^2}{4}}dx.
\end{aligned}$$

On the one hand, once $r\in(0,\frac{1}{4})$, it immediately follows that

$$\begin{aligned}
&(1-r)^{-\zeta-2}\int_0^{\frac{2(1-r)}{\sqrt{r}y}}\left(1+\frac{ry^2}{(1-r)}\right)e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}(1+x^{2\ell+2})x^{\omega+3}e^{-\frac{x^2}{4}}dx \\
\leq\ & C\int_0^{\frac{2(1-r)}{\sqrt{r}y}}(1+x^{2\ell+2})x^{\omega+3}e^{-\frac{x^2}{4}-\frac{rx^2}{4(1-r)}}dx\leq C\int_0^\infty(1+x^{2\ell+2})x^{\omega+3}e^{-\frac{x^2}{4}}dx\leq C,
\end{aligned}$$

where C is independent of r. On the other hand, once $r\geq\frac{1}{4}$ and by a change of variable $z=\frac{\sqrt{r}}{2\sqrt{1-r}}x$ we obtain

$$\begin{aligned}
&\int_0^{\frac{2(1-r)}{\sqrt{r}y}}\left(1+\frac{ry^2}{(1-r)}\right)e^{-\frac{r}{1-r}\left(\frac{y^2}{4}+\frac{x^2}{4}\right)}(1+x^{2\ell+2})x^{\omega+3}e^{-\frac{x^2}{4}}dx \\
\leq\ & Ce^{-\frac{ry^2}{4(1-r)}}\int_0^{\frac{\sqrt{1-r}}{y}}\left(1+\frac{ry^2}{(1-r)}\right)e^{-z^2\left(1+\frac{1-r}{r}\right)}\left(1+\frac{(1-r)^{\ell+1}}{r^{\ell+1}}z^{2\ell+2}\right)\frac{(1-r)^{\frac{\omega}{2}+2}}{r^{\frac{\omega}{2}+2}}z^{\omega+3}dz \\
\leq\ & C(1-r)^{\frac{\omega}{2}+2}e^{-\frac{r}{4(1-r)}y^2}\left(1+\frac{ry^2}{(1-r)}\right)\int_0^{\frac{\sqrt{1-r}}{y}}e^{-z^2\left(1+\frac{1-r}{r}\right)}(1+(1-r)^{\ell+1}z^{2\ell+2})z^{\omega+3}dz \\
\leq\ & (1-r)^{\zeta+2}e^{-\frac{r}{4(1-r)}y^2}\int_0^\infty e^{-z^2}(1+z^{2\ell+2+\omega+3})dz\leq C(1-r)^{\zeta+2},\ \text{with }\zeta=\frac{\omega}{2}.
\end{aligned}$$

Finally, we obtain

$$\left|\int_0^{\frac{2(1-r)}{\sqrt{r}y}}\left|\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right|\left(1+x^{2\ell+2}\right)x^{\omega+3+\gamma}e^{-\frac{x^2}{4}}dx\right|\lesssim\frac{\langle y\rangle^{2\ell+3}}{\sqrt{1-r}}.$$

- For the integral on $\left[\frac{2(1-r)}{\sqrt{r}y},+\infty\right)$, we apply (C.11) and (C.13) and noticing that $z=\frac{\sqrt{r}yx}{2(1-r)}\geq 1$

$$\frac{x}{2(1-r)}P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)=\frac{4^\zeta}{2\sqrt{r}^\zeta(1-r)(yx)^\zeta}\left(\frac{x}{2(1-r)}\right)e^{-\frac{r}{4(1-r)}(y^2+x^2)}\left(\frac{e^z}{\sqrt{2\pi z}}+O(e^zz^{-\frac{3}{2}})\right),$$

and

$$\frac{ry^2x}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)=\frac{ry^2x}{8(1-r)}\frac{4^{\zeta+1}}{\sqrt{r}^{\zeta+1}(1-r)(yx)^{\zeta+1}}e^{-\frac{r}{4(1-r)}(y^2+x^2)}\left(\frac{e^z}{\sqrt{2\pi z}}+O(e^z z^{-\frac{3}{2}})\right).$$

Then

$$\begin{aligned}
&\left|x\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-x\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right|\\
\le\ & C\left|\frac{x-\sqrt{r}y}{(1-r)}\right|\frac{4^\zeta}{2\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{4(1-r)}(y^2+x^2)}\frac{e^z}{\sqrt{2\pi z}}\\
+\ & C\frac{x}{(1-r)}\frac{4^\zeta}{2\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{4(1-r)}(y^2+x^2)}\frac{e^z}{z^{\frac{3}{2}}}\\
+\ & C\frac{\sqrt{r}y}{(1-r)}\frac{4^\zeta}{2\sqrt{r}^\zeta(1-r)(yx)^\zeta}e^{-\frac{r}{4(1-r)}(y^2+x^2)}\frac{e^z}{z^{\frac{3}{2}}}\\
\le\ & C\frac{r^{-\frac{\zeta}{2}-\frac{1}{4}}(yx)^{-\zeta-\frac{1}{2}}}{\sqrt{1-r}}\left|\frac{x-\sqrt{r}y}{1-r}\right|e^{\frac{-\frac{ry^2}{4}+\frac{1}{2}\sqrt{r}yx-\frac{rx^2}{4}}{1-r}}\\
+\ & Cr^{-\frac{\zeta}{2}-\frac{3}{4}}(yx)^{-\zeta-\frac{3}{2}}\left|\frac{x+\sqrt{r}y}{\sqrt{1-r}}\right|e^{\frac{-\frac{ry^2}{4}+\frac{1}{2}\sqrt{r}yx-\frac{rx^2}{4}}{1-r}}
\end{aligned}$$

Notice that we have the following identity

$$e^{\frac{-r\frac{y^2}{4}+\frac{1}{2}(r)^{\frac{1}{2}}(yx)-r\frac{x^2}{4}}{1-r}}e^{-\frac{x^2}{4}}=e^{-\frac{1}{4}\left(\frac{x-\sqrt{r}y}{\sqrt{1-r}}\right)^2},$$

and by a change of variable $z=\frac{x-\sqrt{r}y}{\sqrt{1-r}}$, we get

$$\begin{aligned}
&\int_{\frac{2(1-r)}{\sqrt{r}y}}^\infty\left|x\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-x\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right|(1+x^{2\ell+2})x^{\omega+2}e^{-\frac{x^2}{4}}\,dx\\
\le\ & C\frac{y^{-\zeta-\frac{1}{2}}r^{-\frac{\zeta}{4}-\frac{1}{4}}}{\sqrt{1-r}}\int_{\frac{2(1-r)}{\sqrt{r}y}}^\infty\left|\frac{x-\sqrt{r}y}{1-r}\right|(1+x^{2\ell+2})x^{\zeta+\frac{3}{2}}e^{-\frac{1}{4}\left(\frac{x-\sqrt{r}y}{\sqrt{1-r}}\right)^2}dx\\
+\ & C\frac{y^{-\zeta-\frac{3}{2}}r^{-\frac{\zeta}{2}-\frac{3}{4}}}{\sqrt{1-r}}\int_{\frac{2(1-r)}{\sqrt{r}y}}^\infty\left|\frac{x+\sqrt{r}y}{\sqrt{1-r}}\right|(1+x^{2\ell+2})x^{\zeta+\frac{1}{2}}e^{-\frac{1}{4}\left(\frac{x-\sqrt{r}y}{\sqrt{1-r}}\right)^2}dx\\
\le\ & C\frac{(\sqrt{r}y)^{-\zeta-\frac{1}{2}}}{\sqrt{1-r}}\int_{\frac{\frac{2(1-r)}{\sqrt{r}y}-\sqrt{r}y}{\sqrt{1-r}}}^\infty|z|(1+(z\sqrt{1-r}+\sqrt{r}y)^{2\ell+2})\left|z\sqrt{1-r}+\sqrt{r}y\right|^{\zeta+\frac{3}{2}}e^{-\frac{z^2}{4}}dz\\
+\ & C\frac{(\sqrt{r}y)^{-\zeta-\frac{3}{2}}}{\sqrt{1-r}}\int_{\frac{\frac{2(1-r)}{\sqrt{r}y}-\sqrt{r}y}{\sqrt{1-r}}}^\infty(1+(z\sqrt{1-r}+\sqrt{r}y)^{2\ell+2})\left|z\sqrt{1-r}+\sqrt{r}y\right|^{\zeta+\frac{3}{2}}e^{-\frac{z^2}{4}}dz.
\end{aligned}$$

Now, we have

$$|z|(1+(z\sqrt{1-r}+\sqrt{r}y)^{2\ell+2})\left|z\sqrt{1-r}+\sqrt{r}y\right|^{\zeta+\frac{3}{2}}\lesssim\langle y\rangle^{2\ell+2}\langle z\rangle^{2\ell+3}\left[|z\sqrt{1-r}|^{\zeta+\frac{3}{2}}+|\sqrt{r}y|^{\zeta+\frac{3}{2}}\right]$$

and

$$\begin{aligned}
&(\sqrt{r}y)^{-\zeta-\frac{1}{2}}\int_{\frac{\frac{2(1-r)}{\sqrt{r}y}-\sqrt{r}y}{\sqrt{1-r}}}^{\infty}\langle z\rangle^{2\ell+3}|z\sqrt{1-r}|^{\zeta+\frac{3}{2}}e^{-\frac{z^2}{4}}dz\\
\lesssim\ &(\sqrt{r}y)^{-\zeta-\frac{1}{2}}\sqrt{1-r}^{\zeta+\frac{3}{2}}\int_{\frac{\frac{2(1-r)}{\sqrt{r}y}-\sqrt{r}y}{\sqrt{1-r}}}^{\infty}\langle z\rangle^{2\ell+3+\zeta+\frac{3}{2}}e^{-\frac{z^2}{4}}dz\\
\lesssim\ &X^{\zeta+\frac{1}{2}}\int_{2X-\frac{1}{X}}^{\infty}\langle z\rangle^{2\ell+3+\zeta+\frac{3}{2}}e^{-\frac{z^2}{4}}dz\lesssim 1,\ \text{with } X=\frac{\sqrt{1-r}}{\sqrt{r}y}>0,
\end{aligned}$$

yielding

$$\begin{aligned}
&\frac{(\sqrt{r}y)^{-\zeta-\frac{1}{2}}}{\sqrt{1-r}}\int_{\frac{\frac{2(1-r)}{\sqrt{r}y}-\sqrt{r}y}{\sqrt{1-r}}}^{\infty}|z|(1+(z\sqrt{1-r}+\sqrt{r}y)^{2\ell+2})\left|z\sqrt{1-r}+\sqrt{r}y\right|^{\zeta+\frac{3}{2}}e^{-\frac{z^2}{4}}dz\\
\lesssim\ &\frac{\langle y\rangle^{2\ell+3}}{\sqrt{1-r}}.
\end{aligned}$$

Similarly we have

$$\frac{(\sqrt{r}y)^{-\zeta-\frac{3}{2}}}{\sqrt{1-r}}\int_{\frac{\frac{2(1-r)}{\sqrt{r}y}-\sqrt{r}y}{\sqrt{1-r}}}^{\infty}(1+(z\sqrt{1-r}+\sqrt{r}y)^{2\ell+2})\left|z\sqrt{1-r}+\sqrt{r}y\right|^{\zeta+\frac{3}{2}}e^{-\frac{z^2}{4}}dz\lesssim\frac{\langle y\rangle^{2\ell+3}}{\sqrt{1-r}}.$$

Thus, we obtain

$$\int_{\frac{2(1-r)}{\sqrt{r}y}}^{\infty}\left|x\frac{P_\zeta\left(\frac{y^2}{4},\frac{x^2}{4},r\right)}{2(1-r)}-x\frac{ry^2}{8(1-r)}P_{\zeta+1}\left(\frac{y^2}{4},\frac{x^2}{4},r\right)\right|(1+x^{2\ell+2})x^{\omega+2}e^{-\frac{x^2}{4}}dx\lesssim\frac{\langle y\rangle^{2\ell+3}}{\sqrt{1-r}}.$$

By adding all related terms, we conclude (C.10) and with it the proof of the Lemma is accomplished. □

D. Generators of the Kernel of H

We construct the family $\{T_{i+1}\}$ via the recursive formula

$$T_{i+1}=H^{-1}T_i,\quad \text{ and } T_0=c\Lambda_\xi Q \tag{D.1}$$

here c is some constant which will be chosen later. In other words, we have for $i\geq 1$

$$T_i=H^{-i}(T_0),\quad i\geq 1.$$

The operator H^{-1} is explicitly given by

$$H^{-1}f(\xi)=\Lambda Q(\xi)\int_0^\xi\frac{Lf(\xi')}{\Lambda Q(\xi')}d\xi', \tag{D.2}$$

with L

$$Lf(\xi)=\frac{1}{\xi^{d+1}\Lambda Q(\xi)}\int_0^\xi f(\xi')\Lambda Q(\xi')(\xi')^{d+1}d\xi'. \tag{D.3}$$

We start now our induction argument. For $i=0$, we have by assumption

$$T_0=c\Lambda Q(\xi).$$

The asymptotic (9.4) yields

$$T_0(\xi)=\begin{cases}-2c+\sum_{i=1}^k a_i''\xi^{2i}+O(\xi^{2k+2}) \text{ as } \xi\to 0,\\ a_0c\xi^{-\gamma}+O\left(\xi^{-\gamma-2\alpha}\right)\text{ as } \xi\to\infty.\end{cases} \tag{D.4}$$

This proves (10.4) for the case $i=0$. Now, let us suppose that (10.4) is true for some k and let us prove that it holds true for $k+1$.

- At ∞: we have

$$\mathscr{L}(T_k) = \frac{C_k}{-2\gamma+d+2+2k}\xi^{-\gamma+1+2k}\left(1+O\left(\frac{\ln\xi}{\xi^2}\right)\right).$$

Hence

$$T_{k+1} = H^{-1}(T_k) = \frac{C_k}{4(k+1)(\frac{d}{2}-\gamma+k+1)}\xi^{-\gamma+2(k+1)}\left(1+O\left(\frac{\ln\xi}{\xi^2}\right)\right)$$

The above expansion yields

$$C_{k+1} = \frac{C_k}{4(k+1)(\frac{d}{2}-\gamma+k+1)},$$

so that

$$C_k = \frac{ca_0}{4^k k!(\frac{d}{2}-\gamma)_k!}.$$

Choosing $c=\frac{1}{a_0}$ concludes the first part of the proof.
In a similar fashion and upon using that $\partial_\xi^k T_{i+1} = H^{-1}T_i$, one can establish the mentioned result regarding the asymptotic behavior of the derivatives of T_i. We omit the details.

E. Inner eigenfunctions computation

Now we prove Proposition 10.2. Taking into account (10.8), the eigenvalue problem (10.7) reads

$$\begin{aligned}
0 &= \left\{H-b\beta\Lambda-2\beta b\left(\frac{\alpha}{2}-i+\tilde{\lambda}\right)\right\}\phi_{i,\text{int}} = \sum_{j=0}^{i} c_{i,j}(2\beta)^j b^j\left\{H-b\beta\Lambda-2\beta b\left(\frac{\alpha}{2}-i+\tilde{\lambda}\right)\right\}T_j \\
&+ \tilde{\lambda}\sum_{j=0}^{i} c_{i,j}(2\beta)^{j+1}b^{j+1}\left\{H-b\beta\Lambda-2\beta b\left(\frac{\alpha}{2}-i+\tilde{\lambda}\right)\right\}T_{j+1} \\
&+ \tilde{\lambda}\sum_{j=0}^{i} b^{j+1}\left\{H-b\beta\Lambda-2\beta b\left(\frac{\alpha}{2}-i+\tilde{\lambda}\right)\right\}S_j \\
&+ b\left\{H-\beta b\Lambda-2\beta b\left(\frac{\alpha}{2}-i+\tilde{\lambda}\right)\right\}R_i.
\end{aligned}$$

Since $T_{i+1}=H^{-1}T_i$ and $H(T_0)=0$, we get

$$\begin{aligned}
&\sum_{j=0}^{i} c_{i,j}(2\beta)^j b^j\left\{H-\beta b\Lambda-2\beta b\left(\frac{\alpha}{2}-i+\tilde{\lambda}\right)\right\}T_j \\
&= \sum_{j=0}^{i} c_{i,j}(2\beta)^j b^j\left\{HT_j-\beta b((2j-\alpha)T_j+\Theta_j)-2\beta b\left(\frac{\alpha}{2}-i\right)T_j-2\beta b\tilde{\lambda}T_j\right\} \\
&= -\frac{1}{2}\sum_{j=0}^{i} c_{i,j}(2\beta)^{2j}b^{j+1}\Theta_j-\tilde{\lambda}\sum_{j=0}^{i} c_{i,j}(2\beta)^{j+1}b^{j+1}T_j,
\end{aligned}$$

where we used the fact that $c_{i,j}$ defined in (2.28) satisfies $c_{i,j+1} + c_{i,j}(i-j) = 0$. Thus, the construction of S_j and R_i reduces the following equations (for all $j \le i$)

$$HS_j = 2\beta b\left\{\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)\left(S_j + c_{i,j}(2\beta)^{j+1}T_{j+1}\right)\right\}, \tag{E.1}$$

$$HR_i = 2\beta b\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)R_i + \frac{1}{2}\sum_{j=0}^{i} c_{i,j}(2\beta)^{j+1}b^j\Theta_j. \tag{E.2}$$

We note that the construction of S_j and R_i follows [9] (see also [8]) which relies on the Banach fixed point theorem in the functional space $X^a_{\xi_0}$ for some $a \in \mathbb{R}$ and $\xi_0 > 0$ and where the norm is given by

$$\|f\|_{X^a_{\xi_0}} = \sup_{\xi\in[0,\xi_0]}\sum_{i=0}^{2}\frac{\left|(\xi\partial_\xi)^i f(\xi)\right|}{\langle\xi\rangle^a}, \tag{E.3}$$

with $\langle\xi\rangle = \sqrt{1+\xi^2}$. For sake of shortness and since the determination of both S_j and R_i follows the same reasoning, we only consider S_j in the sequel.

Step 1: Construction of S_j

Identity (E.1) can be put in the form

$$S_j = 2\beta b H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)\left(S_j + c_{i,j}(2\beta)^{j+1}T_{j+1}\right)\right]. \tag{E.4}$$

Now, write S_j as

$$S_j = L(S_j) = L(0) + DL(S_j), \tag{E.5}$$

where

$$L(0) = bc_{i,j}(2\beta)^{j+2}H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)T_{j+1}\right],$$

$$DL(S_j) = b(2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)S_j\right].$$

Our goal is to prove that for all $j \le i \le \ell$

$$\|L(0)\|_{X^{2j-\gamma+2}_{\xi_0}} \le Cy_0^2, \tag{E.6}$$

$$\|DL(S_j)\|_{X^{2j-\gamma+2}_{\xi_0}} \le Cy_0^2\|S_j\|_{X^{2j-\gamma+2}_{\xi_0}}. \tag{E.7}$$

Once these estimates are established, we apply the Banach fixed point theorem to $L(S_j)$ mapping the ball $B(0, 2Cy_0^2)$ into itself with $y_0 \le \frac{1}{2\sqrt{C}}$. This yields the existence and uniqueness of S_j satisfying

$$\|S_j\|_{X^{2j-\gamma+2}_{\xi_0}} \le 2Cy_0^2. \tag{E.8}$$

We are now in position to prove (E.6) and (E.7):

- *Proof of* (E.6)*:* take $a = 2j + 2 - \gamma$, we get

$$\|L(0)\|_{X^a_{\xi_0}} \le |c_{i,j}|b(2\beta)^{j+2}\left(\left|\frac{\alpha}{2} - i + \tilde\lambda\right|\|H^{-1}(T_{j+1})\|_{X^a_{\xi_0}} + \frac{1}{2}\|H^{-1}(\Lambda T_{j+1})\|_{X^a_{\xi_0}}\right).$$

Lemma E.1 yields

$$\|L(0)\|_{X^a_{\xi_0}} \le C\|T_{j+1}\|_{X^{a-2}_{\xi_0}} \le Cb\xi_0^2\|T_{j+1}\|_{X^a_{\xi_0}}.$$

From lemma 10.1 and $X^a_{\xi_0}$'s definition, one infers

$$\|T_{j+1}\|_{X^a_{\xi_0}} \le C.$$

Plugging $\xi_0 = \frac{y_0}{\sqrt{b}}$, we get

$$\|L(0)\|_{X^a_{\xi_0}} \le C y_0^2,$$

which concludes (E.6).

- *Proof of* (E.7)*:* we argue as in the proof of (E.6). Indeed, we apply Lemma E.1 so that

$$\begin{aligned}\|DL(S_j)\|_{X^a_{\xi_0}} &\le Cb(2\beta)(\|H^{-1}(S_j)\|_{X^a_{\xi_0}} + \|H^{-1}(\Lambda S_j)\|_{X^a_{\xi_0}})\\ &\le Cy_0^2\|S_j\|_{X^a_{\xi_0}},\end{aligned}$$

We conclude (E.8) as above.

Our task now is to establish the desired estimates for $\partial_{\tilde{\lambda}} S_j$, $\partial_b S_j$ and $\partial_\beta S_j$. Since the proofs are quite the same we only estimate $\partial_b S_j$. Apply ∂_b to both sides of (E.5) to get

$$\begin{aligned}\partial_b S_j &= c_{i,j}(2\beta)^{j+2}H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde{\lambda} + \frac{1}{2}\Lambda\right)T_{j+1}\right] + (2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde{\lambda} + \frac{1}{2}\Lambda\right)S_j\right]\\ &+ b(2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde{\lambda} + \frac{1}{2}\Lambda\right)\partial_b S_j\right].\end{aligned}$$

Using Lemma E.1 with $\tilde{a} = 2j + 4 - \gamma$, we derive

$$\begin{aligned}\|\partial_b S_j\|_{X^{\tilde{a}}_{\xi_0}} &\le C\left(\|T_{j+1}\|_{X^{\tilde{a}-2}_{\xi_0}} + \|S_j\|_{X^{\tilde{a}-2}_{\xi_0}} + b\|\partial_b S_j\|_{X^{\tilde{a}-2}_{\xi_0}}\right)\\ &\le C(1+y_0^2) + Cb\xi_0^2\|\partial_b S_j\|_{X^{\tilde{a}}_{\xi_0}} \le C(1+y_0^2) + Cy_0^2\|\partial_b S_j\|_{X^a_{\xi_0}}),\end{aligned}$$

this implies that $\|\partial_b S_j\|_{X^{2j+4-\gamma}_{\xi_0}} \le C$, provided $y_0 \le y_0^*$ is small enough.

+ For $\partial_\beta S_j$: Applying ∂_β to (E.5), we get

$$\begin{aligned}\partial_\beta S_j &= b2(j+2)(2\beta)^{j+1}c_{i,j}H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde{\lambda} + \frac{1}{2}\Lambda\right)T_{j+1}\right] + 2bH^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde{\lambda} + \frac{1}{2}\Lambda\right)S_j\right]\\ &+ b(2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde{\lambda} + \frac{1}{2}\Lambda\right)\partial_\beta S_j\right].\end{aligned}$$

using the boundedness of j, we get (with $a = 2j + 2 - \gamma$)

$$\begin{aligned}\|\partial_\beta S_j\|_{X^a_{\xi_0}} &\le Cb\left(\|T_{j+1}\|_{X^{a-2}_{\xi_0}} + \|S_j\|_{X^{a-2}_{\xi_0}} + \|\partial_\beta S_j\|_{X^{a-2}_{\xi_0}}\right)\\ &\le Cb\xi_0^2\left(\|T_{j+1}\|_{X^a_{\xi_0}} + \|S_j\|_{X^a_{\xi_0}} + \|\partial_\beta S_j\|_{X^a_{\xi_0}}\right)\\ &\le Cy_0^2\left(1 + \|\partial_\beta S_j\|_{X^a_{\xi_0}}\right) \le Cy_0^2.\end{aligned}$$

provided that $y_0 \le y_0^*$ small enough.

Step 2: construction of R_i

Taking H^{-1} to (E.2), we get

$$\begin{aligned}R_i &= b(2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde{\lambda} + \frac{1}{2}\Lambda\right)R_i\right] + \frac{1}{2}\sum_{j=0}^{i} c_{i,j}(2\beta)^{j+1}b^j H^{-1}(\Theta_j) \qquad \text{(E.9)}\\ &= J(R_i).\end{aligned}$$

Let us consider $a = -\gamma + \epsilon > -\gamma$, then, we apply Lemma E.1

$$\begin{aligned}\|J(R_i)\|_{X^a_{\xi_0}} &\le C\left(b(2\beta)\|H^{-1}R_i\|_{X^a_{\xi_0}} + \sum_{j=0}^{i}(2\beta)^{j+1}b^j\|H^{-1}(\Theta_j)\|_{X^a_{\xi_0}}\right)\\ &\le C\left(b\|R_i\|_{X^{a-2}_{\xi_0}} + \sum_{j=0}^{i} b^j\|\Theta_j\|_{X^{a-2}_{\xi_0}}\right).\end{aligned}$$

Now, we derive from (10.6) that

$$\|\Theta_j\|_{X^{-2-\gamma+\epsilon}_{\xi_0}} \le C(\epsilon)\xi_0^{2j}.$$

Thus, we derive

$$\|J(R_i)\|_{X^a_{\xi_0}} \le Cy_0^2\|R_i\|_{X^a_{\xi_0}} + C(\epsilon). \tag{E.10}$$

Taking y_0 small enough, J maps the ball $B(0, 2C(\epsilon))$ into itself. In addition to that, it is similar to prove J is a contraction. Hence, by using Banach fixed point theorem, we imply the existence and the uniqueness of R_i satisfying

$$\|R_i\|_{X^{-\gamma+\epsilon}_{\xi_0}} \le 2C(\epsilon).$$

Similarly for S_j, we can respectively take ∂_b, $\partial_{\tilde\lambda}$, and ∂_β to (E.9) by using $R_i \in B(0, 2C(\epsilon))$, and we get

$$\begin{aligned}\|\partial_b R_i\|_{X^{-\gamma+2+\epsilon}_{\xi_0}} &\le C(\epsilon),\\ \|\partial_{\tilde\lambda} R_i\|_{X^{-\gamma+2+\epsilon}_{\xi_0}} &\le C(\epsilon)b,\\ \|\partial_\beta R_i\|_{X^{-\gamma+2+\epsilon}_{\xi_0}} &\le C(\epsilon),\end{aligned}$$

where the constant $C(\epsilon)$ is universal. Finally, we conclude the proof of the Proposition 10.2 . □

$$\begin{aligned}\partial_b R_i &= (2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)R_i\right] + b(2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)\partial_b R_i\right]\\ &+ \frac{1}{2}\sum_{j=0}^{i} c_{i,j}j(2\beta)^{j+1}b^{j-1}H^{-1}\Theta_j,\end{aligned}$$

and

$$\begin{aligned}\partial_\beta R_i &= 2bH^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)R_i\right] + b(2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)\partial_\beta R_i\right]\\ &+ \sum_{j=0}^{i} c_{i,j}(j+1)(2\beta)^j b^j H^{-1}\Theta_j,\end{aligned}$$

and

$$\partial_{\tilde\lambda} R_i = (2\beta)bH^{-1}R_i + b(2\beta)H^{-1}\left[\left(\frac{\alpha}{2} - i + \tilde\lambda + \frac{1}{2}\Lambda\right)\partial_{\tilde\lambda} R_i\right].$$

The rest of this part is devoted to the results which are used to complete the proof of Proposition 10.2.

Lemma E.1 (Continuity of H^{-1} in $X^a_{\xi_0}$)**.** *For all $a \geq -\gamma$, we have*

$$\|H^{-1} f\|_{X^a_{\xi_0}} \leq C \sup_{\xi \in [0,\xi_0]} \langle \xi \rangle^{2-a} |f(\xi)|, \tag{E.11}$$

and

$$\|H^{-1}\left(\Lambda f\right)\|_{X^a_{\xi_0}} \leq C(a) \sup_{\xi \in [0,\xi_0]} \left[\langle \xi \rangle^{2-a} |f(\xi)| + \langle \xi \rangle^{3-a} |\partial_\xi f(\xi)|\right]. \tag{E.12}$$

In particular

$$\|H^{-1} f\|_{X^a_{\xi_0}} \leq C \|f\|_{X^{a-2}_{\xi_0}}, \tag{E.13}$$

and

$$\|H^{-1}\left(\Lambda f\right)\|_{X^a_{\xi_0}} \leq C \|f\|_{X^{a-2}_{\xi_0}}. \tag{E.14}$$

Using the fact that, for $\xi_0 \geq 1$, we have

$$\|f\|_{X^{a-2}_{\xi_0}} \leq 2\xi_0^2 \|f\|_{X^a_{\xi_0}},$$

Formulae (E.11) *and* (E.12) *read, for $\xi_0 \geq 1$*

$$\|H^{-1} f\|_{X^a_{\xi_0}} \leq C\xi_0^2 \|f\|_{X^a_{\xi_0}}, \tag{E.15}$$

and

$$\|H^{-1}\left(\Lambda f\right)\|_{X^a_{\xi_0}} \leq C\xi_0^2 \|f\|_{X^a_{\xi_0}}. \tag{E.16}$$

Proof. Regarding (E.3), we have

$$\sum_{j=0}^{2} \frac{\left|(\xi\partial_\xi)^i g(\xi)\right|}{\langle \xi \rangle^a} \quad \leq \quad \langle \xi \rangle^{-a} |g(\xi)| + 2\langle \xi \rangle^{1-a} |\partial_\xi g| + \langle \xi \rangle^{2-a} |\partial_\xi^2 g|.$$

In addition to that, we derive from (D.2) that

$$\begin{aligned} \partial_\xi \left(H^{-1} f\right) &= \partial_\xi \Lambda Q \int_0^\xi \frac{\mathscr{L} f}{\Lambda Q}(\xi') d\xi' + \mathscr{L}(f)(\xi), \\ \partial_\xi^2 \left(H^{-1} f\right) &= \partial_\xi^2 \Lambda Q \int_0^\xi \frac{\mathscr{L} f}{\Lambda Q}(\xi') d\xi' + \frac{\partial_\xi \Lambda Q}{\Lambda Q} \mathscr{L}(f) + \partial_\xi(\mathscr{L}(f)), \end{aligned}$$

where $\mathscr{L}$ defined as in (D.3). We remark that (E.11) follows from: for all $\xi \in [0, \xi_0]$:

$$\langle \xi \rangle^{-a} \left|H^{-1} f(\xi)\right| \quad \leq \quad \frac{C}{|a|} \sup_{\xi \in [0,\xi_0]} \xi^{2-a} (1 - \xi^\gamma) |f(\xi)|, \tag{E.17}$$

$$\langle \xi \rangle^{1-a} \left|\partial_\xi (H^{-1} f(\xi))\right| \quad \leq \quad \frac{C}{|a|} \sup_{\xi \in [0,\xi_0]} \xi^{2-a} (1 - \xi^\gamma) |f(\xi)|, \tag{E.18}$$

$$\langle \xi \rangle^{2-a} \left|\partial_\xi^2 (H^{-1} f(\xi))\right| \quad \leq \quad \frac{C}{|a|} \sup_{\xi \in [0,\xi_0]} \xi^{2-a} (1 - \xi^\gamma) |f(\xi)|, \tag{E.19}$$

where C does not depend on ξ_0. Let us start with the proof of these estimates:

- *The proof of* (E.17)*:* From $\mathscr{L}$'s formula in (D.3), we have

$$\begin{aligned} |\mathscr{L}(f)| &\leq \frac{1}{|\xi|^{d+1} |\Lambda Q|} \int_0^\xi \langle \xi' \rangle^{2-a} |f(\xi')| \langle \xi' \rangle^{a-2} |\Lambda Q(\xi')| (\xi')^{d+1} d\xi' \\ &\leq \sup_{\xi \in [0,\xi_0]} \left\{\langle \xi \rangle^{2-a} |f(\xi)|\right\} \frac{1}{|\xi|^{d+1} |\Lambda Q|} \int_0^\xi \langle \xi' \rangle^{a-2} |\Lambda Q(\xi')| (\xi')^{d+1} d\xi' \\ &= \sup_{\xi \in [0,\xi_0]} \left\{\langle \xi \rangle^{2-a} |f(\xi)|\right\} \tilde{L}(\xi). \end{aligned}$$

Plugging this estimate to $H^{-1}f$, we obtain

$$\left|H^{-1}f(\xi)\right| \le \left[\sup_{\xi\in[0,\xi_0]} \langle\xi\rangle^{2-a}|f(\xi)|\right] |\Lambda Q| \int_0^\xi \frac{\tilde{L}(\xi')}{|\Lambda Q|} d\xi'.$$

Hence, it is sufficient to prove

$$|\Lambda Q| \int_0^\xi \frac{\tilde{L}(\xi')}{|\Lambda Q|} d\xi' \le C\langle\xi\rangle^a,$$

where C does not depend on ξ_0. Indeed, we consider two cases where $\xi_0 \ll 1$ and $\xi_0 \gg 1$:

+ The case $\xi_0 \ll 1$: We have the following for all $\xi \in [0, \xi_0]$

$$\frac{1}{C} \le |\Lambda Q| \le C,$$

and

$$\tilde{L}(\xi) = \frac{1}{|\xi|^{d+1}|\Lambda Q|} \int_0^\xi \langle\xi'\rangle^{a-2}|\Lambda Q|(\xi')^{d+1} d\xi' \le C\xi,$$

this yields

$$|\Lambda Q| \int_0^\xi \frac{\tilde{L}(\xi')}{|\Lambda Q|} d\xi' \le C\xi^2 \le 2C.$$

which concludes the case $\xi_0 \ll 1$.

+ The case $\xi_0 \gg 1$: We observe that there exists $M > 0$ such that for all $\xi \in [M, \xi_0]$

$$\frac{1}{C}\xi^{-\gamma} \le |\Lambda Q| \le C\xi^{-\gamma}.$$

Then, we have

$$\begin{aligned}
\int_0^\xi \frac{\tilde{L}(\xi')}{|\Lambda Q|} d\xi' &= \int_0^M \frac{\tilde{L}(\xi')}{|\Lambda Q|} d\xi' + \int_M^\xi \frac{\tilde{L}(\xi')}{|\Lambda Q|} d\xi' \\
&\le C(M) + C\int_M^\xi \tilde{L}(\xi')(\xi')^\gamma d\xi'.
\end{aligned}$$

Besides that, we estimate $\tilde{L}(\xi')$, for all $\xi' \in [M, \xi_0]$ as follows

$$\begin{aligned}
\tilde{L}(\xi) &= \frac{1}{\xi^{d+1}\Lambda Q}\left(\int_0^M \langle\xi'\rangle^{a-2}|\Lambda Q|(\xi')^{d+1} d\xi' + \int_M^\xi \langle\xi'\rangle^{a-2}|\Lambda Q|(\xi')^{d+1} d\xi'\right) \\
&\le C(M)\left(\xi^{-d-1+\gamma} + \xi^{a-1}\right),
\end{aligned}$$

it follows that

$$\int_M^\xi \tilde{L}(\xi')(\xi')^\gamma d\xi' \le C(M)(1 + \xi^{-d+2\gamma} + \xi^{a+\gamma}).$$

Thus, we derive

$$|\Lambda Q| \int_0^\xi \frac{\tilde{L}(\xi')}{|\Lambda Q|} d\xi' \le C(M,a)\left(\xi^{-\gamma} + \xi^a\right).$$

Finally, we have

$$|H^{-1}f(\xi)| \le C(a)\langle\xi\rangle^a,$$

provided that

$$a > -\gamma.$$

- The proofs of (E.18) and (E.19) are the same.

Now, it remains to prove that if $f \in X^a_{\xi_0}$, then we have

$$\|f\|_{X^{a-2}_{\xi_0}} \leq C(a)\xi_0^2 \|f\|_{X^a_{\xi_0}},$$

provided that $\xi_0 \geq 1$. Indeed, this comes from

$$\langle \xi \rangle^{2-a} = \langle \xi \rangle^{-a} \langle \xi \rangle^2 \leq 2\xi_0^2 \langle \xi \rangle^{-a},$$

provided that $\xi_0 \geq 1$. Thus, we have

$$\|f\|_{X^{a-2}_{\xi_0}} = \sup_{\xi \in [0,\xi_0]} \sum_{j=0}^{2} \left| \frac{(\langle \xi \rangle \partial_\xi)^i f}{\langle \xi \rangle^{a-2}} \right| \leq \sup_{\xi \in [0,\xi_0]} \sum_{j=0}^{2} 2\xi_0^2 \left| \frac{(\langle \xi \rangle \partial_\xi)^i f}{\langle \xi \rangle^{a}} \right| \leq 2\xi_0^2 \|f\|_{X^a_{\xi_0}}.$$

This concludes the proof of the Lemma. □

In the following Lemma, we aim to estimate $\partial^3_\xi S_j$ and $\partial^3_\xi R_i$:

Lemma E.2 (Higher estimates for $\partial^3_\xi S_j$ and $\partial^3_\xi R_i$)**.** *Let us consider S_j and R_i which satisfying* (E.4) *and* (E.9)*. Furthermore, we assume that the following estimates hold*

$$\|S_j\|_{X^{2j+2-\gamma}_{\xi_0}} \leq C y_0^2 \text{ and } \|R_i\|_{X^{\epsilon-\gamma}_{\xi_0}} \leq C, \text{ with } \xi_0 = \frac{y_0}{\sqrt{b}}.$$

Then, the following holds: for all $\xi \in [0, \xi_0]$

$$\left|\partial^3_\xi S_j(\xi)\right| \leq C\langle \xi \rangle^{2j-\gamma-1}, \tag{E.20}$$

$$\left|\partial^3_\xi R_i(\xi)\right| \leq C\langle \xi \rangle^{\epsilon-\gamma-3}. \tag{E.21}$$

Proof. - The proof of (E.20): We remark that when $b \to 0$, $\xi_0 \to +\infty$. Then, we will consider two situations, namely, $\xi \ll 1$ and $\xi \gg 1$. Recall the inverse formula

$$S_j = bH^{-1}(f), f = \left(\frac{\alpha}{2} - i + \tilde{\lambda} + \frac{1}{2}\Lambda\right)(S_j + c_{i,j}T_{j+1})$$

and write $\partial^3_\xi S_j$ as follows

$$\begin{aligned} b^{-1}\partial^3_\xi S_j(\xi) &= \partial^3_\xi \Lambda Q \int_0^\xi \frac{\mathcal{L}(f)}{\Lambda Q} d\xi' + \frac{\partial^2_\xi \Lambda Q}{\Lambda Q}\mathcal{L}(f) + \partial_\xi f - \frac{(d+1)(d+2)}{\xi^2}\mathcal{L}(f) - \frac{(d+1)\partial_\xi \Lambda Q}{\xi \Lambda Q}\mathcal{L}(f) \\ &- \frac{(d+1)}{\xi} f. \end{aligned}$$

+ The case $\xi \in [0,1]$: We observe that when $\xi \leq 1$, it follows that $|f(\xi)| \leq C$, then, plugging to (E.4), we obtain

$$|S_j(\xi)| = |H^{-1} f| \leq C\xi^2,$$

Hence, we refine the behavior near 0 as follows

$$|f(\xi)| \leq C\xi^2,$$
$$|\mathcal{L}(f)(\xi)| \leq C\xi^3,$$

continuing this process we enhance the behavior to

$$|f(\xi)| \leq C\xi^{2j+2},$$
$$|\mathcal{L}(f)(\xi)| \leq C\xi^{2j+3},$$

(we can get a precise behavior for S_j at 0 by $S_j(\xi) = O(\xi^{2j+4})$). Then, it is easy to derive

$$b^{-1}\partial^3_\xi S_j(\xi) = O(\xi^{2j+1}) \text{ as } \xi \to 0.$$

+ The case $\xi \geq 1$: we have the following fact

$$|S_j(\xi)| \leq C\langle \xi \rangle^{2j+2-\gamma} \text{ and } |T_{j+1}(\xi)| \leq C\langle \xi \rangle^{2j+2-\gamma},$$

which yields

$$|f(\xi)| \le C\langle\xi\rangle^{2j+2-\gamma}.$$

Since

$$|\mathcal{L}(f)(\xi)| \le C\langle\xi\rangle^{2j+3-\gamma}, \text{ and } |S_j(\xi)| \quad \le \quad C\langle\xi\rangle^{2j+2-\gamma},$$

we get

$$\left|\partial_\xi^3 S_j(\xi)\right| \quad \le \quad Cb\langle\xi\rangle^{2j+1-\gamma} \le C\langle\xi\rangle^{2j-\gamma-1},$$

due to the fact that

$$b\xi^2 = y_0 \le 1.$$

Thus, we conclude the proof of (E.20). By the same technique, we derive (E.21). This finalizes the proof of the Lemma. □

F. Outer eigenfunctions construction

This paragraph is devoted to give the complete proof to Proposition 10.3:

Proof. Now, let $\phi_{i,out,\beta}$ be of the form

$$\phi_{i,out,\beta}(y) = \phi_{i,\infty,\beta}(y) + \tilde\lambda(\tilde\phi_{i,\beta}(y) + R_{i,1}(y)) + R_{i,2}(y), \tag{F.1}$$

where $R_{i,1}$ and $R_{i,2}$ are to be constructed. Rewrite (2.18) as

$$\mathscr{L}_b = \mathscr{L}_\infty^\beta - 3(d-2)\left(\frac{1}{y^2} + 2Q_b + Q_b^2 y^2\right),$$

where $\mathscr{L}_\infty^\beta$ was defined in (2.22) and $\mathscr{L}_{i,ext}^\beta = \mathscr{L}_\infty^\beta - 2\beta\left(\frac{\alpha}{2} - i\right)$. Plugging (F.1) into

$$\left[\mathscr{L}_b - 2\beta\left(\frac{\alpha}{2} - i\right) - \tilde\lambda\right](\phi_{i,out,\beta}) = 0,$$

which yields

$$\begin{aligned} &\mathscr{L}_{i,ext}^\beta \phi_{i,out,\beta} - \tilde\lambda\phi_{i,out,\beta} - 3(d-2)\left(\frac{1}{y^2} + 2Q_b + Q_b^2y^2\right)\phi_{i,out,\beta} = \left(\mathscr{L}_{i,ext}^\beta R_{i,1} - \tilde\lambda(\tilde\phi_{i,\beta} + R_{i,1})\right) \\ + &\left(\mathscr{L}_{i,ext}^\beta R_{i,2} - 3(d-2)\left(\frac{1}{y^2} + 2Q_b + Q_b^2y^2\right)\phi_{i,out} - \tilde\lambda R_{i,2}\right) = 0, \end{aligned}$$

where we used $\mathscr{L}_{i,ext}^\beta \phi_{i,\infty} = 0$ as well as $\mathscr{L}_{i,ext}^\beta(\tilde\phi_{i,\beta}) = \phi_{i,\infty,\beta}$. Thus, it is sufficient to construct $R_{i,1}$ and $R_{i,2}$ satisfying

$$\begin{aligned} \mathscr{L}_{i,ext}^\beta(R_{i,1}) &= \tilde\lambda R_{i,1} + \tilde\lambda\tilde\phi_{i,\beta} \\ \mathscr{L}_{i,ext}^\beta(R_{i,2}) &= \left(\tilde\lambda + 3(d-2)\left(\frac{1}{y^2} + 2Q_b + Q_b^2y^2\right)\right)R_{i,2} \\ &+ 3(d-2)\left(\frac{1}{y^2} + 2Q_b + Q_b^2y^2\right)(\phi_{i,\infty,\beta} + \tilde\lambda(\tilde\phi_{i,\beta} + R_{i.1})), \end{aligned}$$

or equivalently

$$R_{i,1} = \tilde\lambda\mathscr{L}_{i,ext}^{-1}(R_{i,1}) + \tilde\lambda\phi_{i,\infty,\beta}, \tag{F.2}$$

$$R_{i,2} = \mathscr{L}_{i,ext}^{-1}(H_1R_{i,2}) + \mathscr{L}_{i,ext}^{-1}(H_2). \tag{F.3}$$

Here

$$\begin{aligned} H_1(y) &= \left(\tilde\lambda + 3(d-2)\left(\frac{1}{y^2} + 2Q_b + Q_b^2 y^2\right)\right) \\ H_2(y) &= 3(d-2)\left(\frac{1}{y^2} + 2Q_b + Q_b^2 y^2\right)(\phi_{i,\infty,\beta} + \tilde\lambda(\tilde\phi_{i,\beta} + R_{i,1})). \end{aligned}$$

Step 1: Construction of $R_{i,1}$: The construction is based on Banach fixed point theorem on Banach space $X_{y_0}^{a,a'}$ equipped with the norm introduced in (10.9). Now, denote the right hand side of (F.2) by $K(R_{i,1})$, and apply Lemma F.2 with $a = \gamma - d$ and $a' = 2i + 2 - \gamma$

$$\|K(R_{i,1})\|_{X_{y_0}^{a,a'}} \le C_1(a,a')|\tilde\lambda| \left(\|R_{i,1}\|_{X_{y_0}^{a,a'}} + \|\phi_{i,\infty,\beta}\|_{X_{y_0}^{a,a'}}\right) \text{ and } \|\phi_{i,\infty,\beta}\|_{X_{y_0}^{a,a'}} \le C_2.$$

Then, K maps the ball $B(0, 2C_1C_2|\tilde\lambda|)$ into itself provided that $|\tilde\lambda| \le \min\left(\frac{1}{2}, \frac{1}{2C_1}\right)$. In addition to that, for all $X_1, X_2 \in B(0, 2C_1C_2|\tilde\lambda|)$, we have

$$\|K(X_1) - K(X_2)\|_{X_{y_0}^{a,a'}} \lesssim |\tilde\lambda|\,\|X_1 - X_2\|_{X_{y_0}^{a,a'}}.$$

Hence, for $\tilde\lambda$ small enough, K is a contraction and the existence of $R_{i,1}$ follows with the bound

$$\|R_{i,1}\|_{X_{y_0}^{a,a'}} \le 2C_1C_2|\tilde\lambda|.$$

Now, we establish the estimates for $\partial_b R_{i,1}, \partial_b R_{i,1}$ and $\partial_\beta R_{i,1}$.

- *For $\partial_b R_{i,1}$:* From (F.2), we see that $R_{i,1}$ is independent of b so $\partial_b R_{i,1} = 0$.
- *For $\partial_{\tilde\lambda} R_{i,1}$:* Applying $\partial_{\tilde\lambda}$ to (F.2), we obtain

$$\partial_{\tilde\lambda} R_{i,1} = \left(\mathscr{L}^\beta_{i,ext}\right)^{-1}(R_{i,1}) + \tilde\lambda\left(\mathscr{L}^\beta_{i,ext}\right)^{-1}(\partial_{\tilde\lambda} R_{i,1}) + \phi_{i,\infty,\beta}.$$

Lemma F.2 implies

$$\|\partial_{\tilde\lambda} R_{i,1}\|_{X_{y_0}^{a,a'}} \le C_1\|R_{i,1}\|_{X_{y_0}^{a,a'}} + C_1|\tilde\lambda|\|\partial_{\tilde\lambda} R_{i,1}\|_{X_{y_0}^{a,a'}} + \|\phi_{i,\infty,\beta}\|_{X_{y_0}^{a,a'}},$$

hence

$$\|\partial_{\tilde\lambda} R_{i,1}\|_{X_{y_0}^{a,a'}} \le C.$$

- *For $\partial_\beta R_{i,1}$:* Applying ∂_β to (F.2), we obtain

$$\partial_\beta R_{i,1} = \tilde\lambda\partial_\beta\left(\left(\mathscr{L}^\beta_{i,ext}\right)^{-1}\right)(R_{i,1}) + \tilde\lambda\left(\mathscr{L}^\beta_{i,ext}\right)^{-1}(\partial_\beta R_{i,1}) + \tilde\lambda\partial_\beta\phi_{i,\infty,\beta}.$$

Applying $X_{y_0}^{a,a'}$ norm to the above equality and Lemma F.2, we deduce

$$\|\partial_\beta R_{i,1}\|_{X_{y_0}^{a,a'}} \le C_1|\tilde\lambda|\|R_{i,1}\|_{X_{y_0}^{a,a'}} + C_2|\tilde\lambda|\|\partial_\beta R_{i,1}\|_{X_{y_0}^{a,a'}} + |\tilde\lambda|\|\partial_\beta\phi_{i,\infty,\beta}\|_{X_{y_0}^{a,a'}}.$$

On the one hand, we have

$$\|R_{i,1}\|_{X_{y_0}^{a,a'}} + \|\partial_\beta\phi_{i,\infty,\beta}\|_{X_{y_0}^{a,a'}} \le C.$$

Finally, we get

$$\|\partial_\beta R_{i,1}\|_{X_{y_0}^{a,a'}} \le C|\tilde\lambda|.$$

The construction and estimates on $R_{i,2}$ are very similar to those established above and are left to the reader.

- *Step 2: Construction of $R_{i,2}$:*

The construction is also based on the Banach fixed point theorem on the Banach space $X_{y_0}^{a,a'}$ with $a = -\tilde{\gamma} - 2 - \alpha$, and $a' = 2i + 2 - \gamma$. First, we define the right hand side of (F.3) to be $J(R_{i,2})$. Using Lemma F.2, we have the following

$$\|J(R_{2,i})\|_{X_{y_0}^{a,a'}} \le C_1 \left(\|H_1 R_{i,2}\|_{X_{y_0}^{a,a'}} + \|H_2\|_{X_{y_0}^{a,a'}} \right).$$

We now aim to prove the following estimates

$$\|H_1 R_{i,2}\|_{X_{y_0}^{a,a'}} \le C(y_0) b^\alpha \|R_{i,2}\|_{X_{y_0}^{a,a'}}, \tag{F.4}$$

$$\|H_2\|_{X_{y_0}^{a,a'}} \le C(y_0) b^\alpha. \tag{F.5}$$

- *The proof of* (F.4): From Lemma 9.1, we have

$$Q_b(y) = \frac{1}{b} Q\left(\frac{y}{\sqrt{b}}\right) = \frac{1}{b}\left(-\frac{1}{\left(\frac{y}{\sqrt{b}}\right)^2} + q_0 \left(\frac{y}{\sqrt{b}}\right)^{-\gamma} + O_{b\to 0}\left(\frac{y}{\sqrt{b}}\right)^{-\gamma-g} \right),$$

since $y \ge y_0$, $\frac{y}{\sqrt{b}} \to +\infty$. This implies that for all $y \ge y_0$ and $b \in (0, b^*(y_0))$

$$\left| 2Q_b(y) - \left(-\frac{2}{y^2} + 2q_0 y^{-\gamma} b^{\frac{\alpha}{2}} \right) \right| \le C(y_0) b^\alpha,$$

$$\left| y^2 Q_b^2(y) - \left(\frac{1}{y^2} - 2q_0 y^{-\gamma} b^{\frac{\alpha}{2}} \right) \right| \le C(y_0) b^\alpha,$$

since $g = -2\lambda_1 = 2(\gamma - 2)$. Thus, for all $y \ge y_0$, we have

$$\left| 2Q_b(y) + Q_b^2(y) y^2 + \frac{1}{y^2} \right| \le C(y_0) b^\alpha \lesssim b^{\frac{\alpha}{2}}. \tag{F.6}$$

- *Proof of* (F.5)*:* Recall that

$$\|R_{i,1}\|_{X_{y_0}^{-\tilde{\gamma},a'}} \le C|\tilde{\lambda}|,$$

which implies

$$\|R_{i,1}\|_{X_{y_0}^{a,a'}} \le C(y_0)|\tilde{\lambda}| \text{ with } a = -\tilde{\gamma} - 2 - \alpha.$$

Similarly, we also have

$$\|\phi_{i,\infty,\beta}\|_{X_{y_0}^{a,a'}} + \|\tilde{\phi}_{i,\beta}\|_{X_{y_0}^{a,a'}} \le C(y_0).$$

Thus, the above estimates and (F.6) immediately conclude (F.5).

Using estimates (F.4) and (F.5), we get

$$\|J(R_{2,i})\|_{X_{y_0}^{a,a'}} \le C_1(y_0) b^\alpha \left(\|R_{i,2}\|_{X_{y_0}^{a,a'}} + 1 \right).$$

Consequently, once $b \ll 1$, J becomes a contraction from the ball $B(C_1 b^\alpha, 0)$ to itself. Thus, it follows Banach fixed point theorem the existence of $R_{i,2}$ satisfying

$$\|R_{i,2}\|_{X_{y_0}^{a,a'}} \le C b^{\frac{\alpha}{2}}. \tag{F.7}$$

Next, we focus on evaluating $\partial_{\tilde{\lambda}} R_{i,2}$, $\partial_b R_{i,2}$ and $\partial_\beta R_{i,2}$:

+ For $\partial_{\tilde{\lambda}} R_{i,2}$: We have

$$\partial_{\tilde{\lambda}} R_{i,2} = \mathscr{L}_{i,ext}^{-1}(R_{i,2}) + \mathscr{L}_{i,ext}^{-1}(H_1 \partial_{\tilde{\lambda}} R_{i,2}) + \mathscr{L}_{i,ext}^{-1}(\partial_{\tilde{\lambda}} H_2)$$

then

$$\begin{aligned}\|\partial_{\tilde{\lambda}} R_{i,2}\|_{X_{y_0}^{a,a'}} &\leq C_1\|R_{i,2}\|_{X_{y_0}^{a,a'}} + C_2\|H_1\partial_{\tilde{\lambda}} R_{i,2}\|_{X_{y_0}^{a,a'}} + C_3\|\partial_{\tilde{\lambda}} H_2\|_{X_{y_0}^{a,a'}}\\ &\leq C_1 b^{\frac{\alpha}{2}} + C_2 b^{\frac{\alpha}{2}}\|\partial_{\tilde{\lambda}} R_{i,2}\|_{X_{y_0}^{a,a'}} + C_3 b^{\frac{\alpha}{2}}\end{aligned}$$

which yields to

$$\|\partial_{\tilde{\lambda}} R_{i,2}\|_{X_{y_0}^{a,a'}} \leq C b^{\frac{\alpha}{2}}.$$

- For $\partial_b R_{i,2}$: We have

$$\partial_b R_{i,2} = \mathscr{L}_{i,ext}^{-1}(\partial_b H_1 R_{i,2}) + \mathscr{L}_{i,ext}^{-1}(H_1\partial_b R_{i,2}) + \mathscr{L}_{i,ext}^{-1}(\partial_b H_2)$$

then

$$\begin{aligned}\|\partial_b R_{i,2}\|_{X_{y_0}^{a,a'}} &\leq C_1\|\partial_b H_1 R_{i,2}\|_{X_{y_0}^{a,a'}} + C_2\|\Theta_1\partial_b R_{i,2}\|_{X_{y_0}^{a,a'}} + C_3\|\partial_b H_2\|_{X_{y_0}^{a,a'}}\\ &\leq C_1 b^{\frac{\alpha}{2}-1} + C_2 b^{\frac{\alpha}{2}}\|\partial_{\tilde{\lambda}} R_{i,2}\|_{X_{y_0}^{a,a'}} + C_3 b^{\frac{\alpha}{2}-1} \leq C b^{\frac{\alpha}{2}-1}.\end{aligned}$$

- For $\partial_\beta R_{i,2}$: We have

$$\partial_\beta R_{i,2} = \partial_\beta(\mathscr{L}_{i,ext}^{-1}(\Theta_1 R_{i,2})) + \mathscr{L}_{i,ext}^{-1}(\Theta_1\partial_\beta R_{i,2}) + \partial_\beta(\mathscr{L}_{i,ext}^{-1}(\Theta_2)),$$

then we use Lemma F.2 to get

$$\|\partial_\beta R_{i,2}\|_{X_{y_0}^{a,a'}} \leq C(y_0)\left(b^{\frac{\alpha}{2}}\|R_{i,2}\|_{X_{y_0}^{a,a'}} + b^{\frac{\alpha}{1}}\|\partial_\beta R_{i,2}\|_{X_{y_0}^{a,a'}} + b^{\alpha}\right) \leq C b^{\frac{\alpha}{2}}.$$

Finally, we conclude the proof of the Proposition. □

In the sequel, we aim to complete the results used in the proof of Proposition 10.3. To be begin with, we need the following result on the resonance of $\mathscr{L}_{i,ext}^{\beta}$. For sake of shortness we set

$$\mathscr{L}_{i,ext}^{\beta} u = \left(\mathscr{L}_{\infty}^{\beta} - 2\beta\left(\frac{\alpha}{2} - i\right)\right)u. \tag{F.8}$$

We have

Lemma F.1 (Resonance of $\mathscr{L}_{i,ext}^{\beta}$). *We consider $i \in \mathbb{N}$, then, there exists $\tilde{\psi}_{i,\beta}$ such that it solves $\mathscr{L}_{i,ext}^{\beta}\tilde{\psi}_{i,\beta} = 0$. Moreover, we have*

$$Ker(\mathscr{L}_{i,ext}^{\beta}) = Span\{\phi_{i,\infty,\beta}, \tilde{\psi}_{i,\beta}\},$$

where $\phi_{i,\infty,\beta}$ is the i-th eigenfunction of $\mathscr{L}_{\infty}^{\beta}$, given in Proposition 4.1; and $\tilde{\psi}_{i,\beta}$ has the following asymptotic:

$$\tilde{\psi}_{i,\beta}(y) = \begin{cases} \frac{y^{\gamma-d}}{a_{i,0}(d-2\gamma)}(1+O(y^2)) \text{ as } y \to 0,\\ -\frac{2}{a_{i,i}(2\beta)^i} y^{-2i+\gamma-(d+2)} e^{2\beta\frac{y^2}{4}}\left[1+O(y^{-2})\right] \text{ as } y \to +\infty\end{cases} \tag{F.9}$$

where γ and $a_{i,j}$ were defined in (2.26) and (2.28) . In particular, there exists a solution $\tilde{\phi}_{i,\beta}$ to $\mathscr{L}_{i,ext}^{\beta}\tilde{\phi}_i = \phi_{i,\infty,\beta}$, satisfying the following asymptotic

$$\tilde{\phi}_{i,\beta}(y) = \begin{cases} K_0 y^{-\tilde{\gamma}}(1+O(y^2)) \text{ as } y \to 0,\\ K_\infty y^{2i-\gamma}(\ln y + O(1)) \text{ as } y \to +\infty,\end{cases} \tag{F.10}$$

where $K_0 = \frac{1}{(2\gamma-d)(d+2-2\gamma)}, K_\infty = 2(2\beta)^i$ and $\tilde{\gamma}$ is defined by

$$\tilde{\gamma} = \frac{1}{2}(d + \sqrt{d^2 - 12d + 24}). \tag{F.11}$$

Proof. Recall that $\phi_{i,\infty,\beta}$ solves $\mathscr{L}^\beta_{i,ext}\phi_{i,\infty,\beta} = 0$, this follows from the fact that $\phi_{i,\infty}$ is the i-th eigenfunction of $\mathscr{L}_\infty$. According to $\mathscr{L}^\beta_{i,ext}$, the Wronskian is given by

$$W(y) = y^{-(d+1)}e^{2\beta\frac{y^2}{4}}. \tag{F.12}$$

Then, we can formulate an independent linear solution $\tilde{\psi}_{i,\beta}$ to $\mathscr{L}^\beta_{i,ext}\tilde{\psi}_{i,\beta} = 0$

$$\tilde{\psi}_{i,\beta}(y) = -\phi_{i,\infty,\beta}\int_1^y \frac{(y')^{-(d+1)}e^{2\beta\frac{(y')^2}{4}}}{\phi^2_{i,\infty,\beta}(y')}dy'. \tag{F.13}$$

From (4.2), we derive

$$\phi_{i,\infty,\beta}(y) = \begin{cases} a_{i,0}y^{-\gamma}\left(1+O(y^2)\right) \text{ as } y\to 0,\\ a_{i,i}(2\beta)^i y^{2i-\gamma}\left(1+O(y^{-2})\right) \text{ as } y\to+\infty, \end{cases} \tag{F.14}$$

where $a_{i,j}$'s general formula given in (2.28), and we plug the above fact into (F.13) to get

$$\tilde{\psi}_i(y) = \begin{cases} \frac{y^{\gamma-d}}{a_{i,0}(2\gamma-d)}\left(1+O(y^2)\right) \text{ as } \xi\to 0,\\ -\frac{2}{a_{i,i}(2\beta)^i}y^{-2i+\gamma-d-2}e^{2\beta\frac{y^2}{4}}\left(1+O(y^{-2})\right) \text{ as } \xi\to+\infty. \end{cases} \tag{F.15}$$

In particular, it is easy to see that

$$\text{Ker}(\mathscr{L}^\beta_{i,ext}) = \text{Span}\{\phi_{i,\infty,\beta},\tilde{\psi}_{i,\beta}\},$$

which leads to

$$\left(\mathscr{L}^\beta_{i,ext}\right)^{-1}f(y) = -\phi_{i,\infty,\beta}\int_1^y f(y')\frac{\tilde{\psi}_{i,\beta}(y')}{W(y')}d\xi' + \tilde{\psi}_{i,\beta}\int_y^{+\infty} f(y')\frac{\phi_{i,\beta,\infty}(y')}{W(y')}dy' + c_1\phi_{i,\beta,\infty} + c_2\tilde{\psi}_{i,\beta},$$

and since we need to construct a special solution with explicit asymptotics, we choose $c_1 = c_2 = 0$, then, we can omit the generality and write

$$\left(\mathscr{L}^\beta_{i,ext}\right)^{-1}f(y) = -\phi_{i,\infty,\beta}\int_1^y f(y')\frac{\tilde{\psi}_{i,\beta}(y')}{W(y')}dy' + \tilde{\psi}_{i,\beta}\int_y^{+\infty} f(y)\frac{\phi_{i,\beta,\infty}(y')}{W(y')}dy'. \tag{F.16}$$

Thus, the solution $\tilde{\phi}_{i,\beta}$ to $\mathscr{L}^\beta_{i,ext}\tilde{\phi}_{i,\beta} = \phi_{i,\infty,\beta}$ can be written

$$\tilde{\phi}_{i,\beta} = \left(\mathscr{L}^\beta_{i,ext}\right)^{-1}(\phi_{i,\infty,\beta}).$$

- *Behavior at* 0: From (F.14) and (F.15), we have

$$-\phi_{i,\infty,\beta}\int_1^y \frac{\phi_{i,\infty,\beta}(\xi')\tilde{\psi}_{i,\infty,\beta}(\xi')}{W(\xi')}d\xi' = \frac{a_{i,0}}{2(2\gamma-d)}y^{-\gamma+2}(1+O(\xi^2)) \text{ as } y\to 0$$

and also

$$\tilde{\psi}_{i,\beta}\int_y^\infty \frac{\phi^2_{i,\infty,\beta}(\xi')}{W(\xi')}d\xi' = \frac{a_{i,i}}{(2\gamma-d)(-2\gamma+d+2)}y^{\gamma-d}(1+O(\xi^2)) \text{ as } y\to 0,$$

noting that $\gamma - d = -\frac{1}{2}(d+\sqrt{d^2-12d+24}) = -\tilde{\gamma}$, we get

$$\tilde{\phi}_{i,\beta}(y) = K_0 y^{-\tilde{\gamma}}(1+y^2) \text{ as } y\to 0,$$

where $K_0 = \frac{1}{(2\gamma-d)(d+2-2\gamma)}$.
- *Behavior at* $+\infty$: Using (F.14) and (F.15) again, we obtain

$$-\phi_{i,\infty,\beta}\int_1^y \frac{\phi_{i,\infty,\beta}(\xi')\tilde{\psi}_{i,\infty,\beta}(\xi')}{W(\xi')}d\xi' = 2a_{i,i}(2\beta)^i y^{2i-\gamma}\left(\ln y + O(1)\right) \text{ as } y\to+\infty,$$

and for the second one

$$\tilde{\psi}_{i,\beta}\int_y^\infty \frac{\phi^2_{i,\infty,\beta}(\xi')}{W(\xi')}d\xi' = -\frac{2}{a_{i,i}(2\beta)^i}y^{-2i+\gamma-d-2}e^{\frac{2\beta y^2}{4}}O(y^{4i-2\gamma+d}e^{\frac{-2\beta y^2}{4}}) \text{ as } \xi\to+\infty.$$

Thus, we have

$$\tilde{\phi}_{i,\beta}(y) = 2(2\beta)^i y^{2i-\gamma}\left(\ln\xi + O(1)\right) \text{ as} \xi\to+\infty,$$

and this concludes the proof of the Lemma. □

It remains only to prove the continuity of $\mathscr{L}^{-1}_{i,ext}$ in the Banach space $X^{a,a'}$, a result that we have used above.

Lemma F.2 (Continuity of $\mathscr{L}^{-1}_{i,ext}$)**.** *For all $a\le\gamma-d$, $a\neq -d-2$ and $a'>2i-\gamma$ with $y_0\in(0,1)$ and $\beta\in\left(\frac14,\frac34\right)$, we have the following estimate*

$$\left\|\left(\mathscr{L}^{\beta}_{i,ext}\right)^{-1}f\right\|_{X^{a,a'}_{y_0}} \le C(a,a',\beta)\|f\|_{X^{a,a'}_{y_0}} \text{ and } \left\|\partial_\beta\left(\mathscr{L}^{\beta}_{i,ext}\right)^{-1}f\right\|_{X^{a,a'}_{y_0}} \le C(a,a',\beta)\|f\|_{X^{a,a'}_{y_0}}.$$

Proof. Define $g=\left(\mathscr{L}^{\beta}_{i,ext}\right)^{-1}f$ and recall from (F.16) that

$$g(y) = \left(\mathscr{L}^{\beta}_{i,ext}\right)^{-1}f(y) = -\phi_{i,\infty,\beta}\int_1^y f(\xi')\frac{\tilde{\psi}_{i,\beta}(\xi')}{W(\xi')}d\xi' + \tilde{\psi}_{i,\beta}\int_y^{+\infty} f(\xi)\frac{\phi_{i,\beta,\infty}(\xi')}{W(\xi')}d\xi'.$$

Then

$$\begin{aligned}\partial_y g(y) &= -\partial_y\phi_{i,\infty,\beta}\int_1^y f(\xi')\frac{\tilde{\psi}_{i,\beta}(\xi')}{W(\xi')}d\xi' + \partial_y\tilde{\psi}_{i,\beta}\int_y^{+\infty} f(\xi)\frac{\phi_{i,\beta,\infty}(\xi')}{W(\xi')}d\xi'\phi_{i,\infty,\beta}\\ &\quad -2\phi_{i,\infty,\beta}f(y)\frac{\tilde{\psi}_{i,\beta}(y)}{W(y)}\end{aligned}$$

and

$$\begin{aligned}\partial^2_y g(y) &= -\partial^2_y\phi_{i,\infty,\beta}\int_1^y f(\xi')\frac{\tilde{\psi}_{i,\beta}(\xi')}{W(\xi')}d\xi' + \partial^2_y\tilde{\psi}_{i,\beta}\int_y^{+\infty} f(\xi)\frac{\phi_{i,\beta,\infty}(\xi')}{W(\xi')}d\xi'\phi_{i,\infty,\beta}\\ &\quad -3\partial_y\phi_{i,\infty,\beta}f(y)\frac{\tilde{\psi}_{i,\beta}(y)}{W(y)} - 3\partial_y\tilde{\psi}_{i,\beta}f(y)\frac{\phi_{i,\beta,\infty}(y)}{W(y)}\\ &\quad -2\phi_{i,\infty,\beta}\partial_y f(y)\frac{\tilde{\psi}_{i,\beta}(y)}{W(y)} + 2\phi_{i,\infty,\beta}f(y)\frac{\tilde{\psi}_{i,\beta}(y)\partial_y W(y)}{W^2(y)}.\end{aligned}$$

In order to establish the desired estimate, we will only need to control higher order derivatives, namely, $y^{2-a}\partial^2_y g(y)$ for $y\in[y_0,1]$ and $y^{2-a'}\partial^2_y g(y)$ for $y\in[1,\infty)$.

- $\underline{y \in [y_0, 1]}$, we have

$$\begin{aligned}
\left|y^{2-a}\partial_y^2 g(y)\right| &\lesssim y^{2-a}|\partial_y^2\phi_{i,\infty,\beta}|\int_y^1 (\xi')^{-a}|f(\xi')|\frac{|(\xi')^a||\tilde{\psi}_{i,\beta}(\xi')|}{W(\xi')}d\xi' \\
&+ y^{2-a}|\partial_y^2\tilde{\psi}_{i,\beta}|\left(\int_y^1 (\xi')^{-a}|f(\xi')|\frac{|(\xi')^a||\phi_{i,\beta,\infty}(y)|}{W(\xi')}d\xi' + \int_1^\infty (\xi')^{-a'}|f(\xi')|\frac{|(\xi')^{a'}||\phi_{i,\beta,\infty}(y)|}{W(\xi')}d\xi'\right) \\
&+ y^{2-a}|f(y)|\left(|\partial_y\phi_{i,\infty,\beta}|\frac{|\tilde{\psi}_{i,\beta}(y)|}{W(y)} + |\partial_y\tilde{\psi}_{i,\beta}|\frac{|\phi_{i,\beta,\infty}(y)|}{W(y)} + |\phi_{i,\infty,\beta}|\frac{|\tilde{\psi}_{i,\beta}(y)\partial_y W(y)|}{W^2(y)}\right) \\
&+ y^{1-a}|\partial_y f(y)|\left(y|\phi_{i,\infty,\beta}|\frac{|\tilde{\psi}_{i,\beta}(y)|}{W(y)}\right) \\
&\lesssim \sup_{y\in[y_0,1]}|y^{-a}f(y)| + \sup_{y\in[y_0,1]}|y^{1-a}\partial_y f(y)| + \sup_{y\in[1,+\infty)}|y^{-a'}f(y)| \lesssim \|f\|_{X_{y_0}^{a,a'}},
\end{aligned}$$

provided that $a \le \gamma - d$ and $a' \ge 2i - \gamma$.

- $\underline{y \in [1, +\infty)}$, we have

$$\begin{aligned}
\left|y^{2-a'}\partial_y^2 g(y)\right| &\lesssim y^{2-a'}|\partial_y^2\phi_{i,\infty,\beta}|\int_1^\infty (\xi')^{-a'}|f(\xi')|\frac{|(\xi')^{a'}||\tilde{\psi}_{i,\beta}(\xi')|}{W(\xi')}d\xi' \\
&+ y^{2-a'}|\partial_y^2\tilde{\psi}_{i,\beta}|\int_1^\infty (\xi')^{-a'}|f(\xi')|\frac{|(\xi')^{a'}||\phi_{i,\beta,\infty}(y)|}{W(\xi')}d\xi' \\
&+ y^{-a'}|f(y)|\left(y^2|\partial_y\phi_{i,\infty,\beta}|\frac{|\tilde{\psi}_{i,\beta}(y)|}{W(y)} + y^2|\partial_y\tilde{\psi}_{i,\beta}|\frac{|\phi_{i,\beta,\infty}(y)|}{W(y)} + y^2|\phi_{i,\infty,\beta}|\frac{|\tilde{\psi}_{i,\beta}(y)\partial_y W(y)|}{W^2(y)}\right) \\
&+ y^{1-a'}|\partial_y f(y)|\left(y|\phi_{i,\infty,\beta}|\frac{|\tilde{\psi}_{i,\beta}(y)|}{W(y)}\right) \\
&\lesssim \sup_{y\in[1,+\infty)}|y^{-a'}f(y)| + \sup_{y\in[1,+\infty)}|y^{1-a'}\partial_y f(y)| \lesssim \|f\|_{X_{y_0}^{a,a'}}
\end{aligned}$$

which yields

$$\|g\|_{X_{y_0}^{a,a'}} = \|\left(\mathscr{L}_{i,ext}^{\beta}\right)^{-1} f\|_{X_{y_0}^{a,a'}} \le C(a,a')\|f\|_{X_{y_0}^{a,a'}}$$

as claimed. Finally, we finish the proof of the Lemma. □

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