QUASI-NORMAL FAMILY OF MEROMORPHIC FUNCTIONS

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Abstract. Let $\mathcal{F}$ be a family of meromorphic functions on a domain D such that for each $f \in \mathcal{F}$, its first derivative is bounded on the set of zeros of f. For all $f \in \mathcal{F}$ and $z \in D$, if there is a holomorphic function φ such that $f'(z) \neq \varphi'(z)$ then $\mathcal{F}$ is quasi-normal of order 1 on D. Moreover, $f' - R$ has infinitely many zeros, with $R \not\equiv 0$ is a rational function. This result is a generalization of a result of Jianming Chang [4] and a result of Pang et al. [15].

1. Introduction.

A family $\mathcal{F}$ of meromorphic functions on a domain D is said to be normal on D (in the sense of Montel) if for each sequence $\{f_n\} \subset \mathcal{F}$ there is a subsequence which converges spherically locally uniformly in D. $\mathcal{F}$ is said to be quasi-normal on D if for each sequence $\{f_n\} \subset \mathcal{F}$ there is a subsequence and a set E (can depend on the subsequence) has no accumulation point in D such that the subsequence converges spherically locally uniformly in $D \setminus E$. When the cardinality of E at most ν points, we say that $\mathcal{F}$ is quasi-normal of order ν on D (see [5, 6, 17]).

In 2005, Nevo, Pang and Zalcman [14] proved that the family $\mathcal{F}$ is quasi-normal on D if for any $h \in \mathcal{F}$, all of its zeros are multiple and $h'(z) \neq 1$, for all $z \in D$. Two years later, in [13] they improved to the case, for all $h \in \mathcal{F}$, if all zeros of h have multiplicity at least $k+1$. and there exists a univalent analytic function φ on D such that $h^{(k)}(z) \neq \varphi'(z)$ for all $z \in D$ then $\mathcal{F}$ is quasi-normal on D. In [10], [9], [19] the authors gave conditions such that a family $\mathcal{F}$ is normal if that all zeros of meromorphic functions $h \in \mathcal{F}$ are of multiplicity at least 3, and all zeros of $h^{(k)}$ are of multiplicity at least 2.

Jianming Chang in [4, Theorem 3] received the same conclusion when he replaced the condition "all zeros of h are multiple" by a weaker condition that the set

$$M_h = h'(h^{-1}(0)) = \{h'(z) : h(z) = 0\}$$

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is bounded and keep the condition that $h'(z) \neq 1$ for all $z \in D$.

In this paper, we will extend Chang's result by replacing the constant by an holomorphic function. Our results are stated as following.

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions on the plane domain D. Suppose that for each $f \in \mathcal{F}$, M_f is bounded. Assume that for all $f \in \mathcal{F}$ and $z \in D$, there is a holomorphic function φ univalent on D such that $f'(z) \neq \varphi'(z)$. Then $\mathcal{F}$ is quasi-normal of order 1 on D.*

Our proof is quite similar to the proofs of [4, Theorem 3]. However, we also need some new techniques to deal for rational cases.

Let f be a transcendental meromorphic function. Following the results about normal and quasi-normal families, in [4], Chang proved that if M_f is bounded, then f' takes each finite nonzero value infinitely many times. In [2] Bergweiler discussed a Yik-Man Chiang's question whether $(f^2)' - \alpha$ has infinitely many zeros if α a small function respected to f (i.e. it is a meromorphic function satisfies $T(r;\alpha) = o(T(r;f))$ as $r \to \infty$. Here $T(r;f)$ denotes the Nevanlinna characteristic of f. In that paper, Bergweiler gave positive answer for a special case when α is a polynomial and f has finite order. It was shown in [3] that if all zeros and poles of f are multiple, except possibly finitely many, and $R \not\equiv 0$ is a rational function, then $f' - R$ has infinitely many zeros. In 2008, Pang et al. [15] extended above result by removing the restriction on the poles of f. They shown that if all zeros are multiple and $R \not\equiv 0$ is a rational function, then $f' - R$ has infinitely many zeros. In the following theorem, we can remove the condition that all zeros and poles of f are multiple. We also generalize the result in [2] by replaced a polynomial α by a rational function R as follows.

Theorem 1.2. *Let f be a transcendental meromorphic function satisfying the set*

$$M_f := f'(f^{-1}(0)) = \{f'(z) : f(z) = 0\}$$

is bounded. Assume that R is a non-zero rational function. Then $f' - R$ has infinitely many zeros.

Thus, the above theorem is a generalization of a result of Jianming Chang [4, Theorem 1], where instead of the constant case, we consider a univalent holomorphic function, and we replace the condition result of Pang et al. [15] that all zeros of f are multiple by a weaker condition. We use the ideas as in the proof of Theorem 1 in [15], with some new ideas when we replace the condition that all zeros of f are multiples by the weaker condition that the set M_f is bounded and the constant is replaced by the rational function.

2. Notation and recall results

Let $\{f_n\}$ be a sequence of functions defined on a domain D. If for any compact subset E in D, there is $N \in \mathbb{N}$ such that for all $n > N$, f_n is holomorphic function on E, then $\{f_n\}$ is said to be locally uniformly holomorphic on D.

For each holomorphic function f and for each closed subset $E \subset \mathbb{C}$, we denote the spherical derivative by

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2},$$

$$S(E, f) := \frac{1}{\pi} \iint_E (f^{\#}(z))^2 d\sigma.$$

We write $S(\overline{D}_{z_0}(r), f) = S(t, f)$, where $\overline{D}_{z_0}(r) := \{z : |z - z_0| \leq r\}$.

The order of f on $\mathbb{C}$ is defined as

$$\rho(f) = \limsup_{r \to \infty} \frac{\log^+ T_0(r, f)}{\log r},$$

where $T_0(r, f) == \int_0^r \frac{S(t,f)}{t} dt$ is the Ahlfors - Shimizu characteristic of f [8, p. 12]. It is easy that if $f^{\#}$ is bounded on $\mathbb{C}$ then f has order at most 2.

We will use the notation $f_n \overset{\chi}{\Longrightarrow} f$ if $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and by $f_n \Longrightarrow f$ if it converges in the Euclidean metric. To prove our result, we first recall some of the following lemmas.

Lemma 2.1. [16, Lemma 2] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. Suppose that there exists $R \geq 1$ such that $M_f \subset \overline{D}_0(R)$ for each $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at z_0, then for each $0 \leq \alpha \leq 1$, there exists points $z_n \in D$ with $z_n \to z_0$, functions $f_n \in \mathcal{F}$ and positive numbers $\rho_n \to 0$ and a nonconstant meromorphic function g on $\mathbb{C}$ such that*

$$\rho_n^{-\alpha} f_n(z_n + \rho_n z) \overset{\chi}{\Longrightarrow} g(z)$$

on $\mathbb{C}$, and

$$M_g \subset \overline{D}_0(R), \quad g^{\#}(z) \leq g^{\#}(0) = R + 1.$$

In particular, g has order at most 2.

Lemma 2.2. [1, Lemma 5] *Let f be a meromorphic function of finite order on $\mathbb{C}$ such that $f' \neq 1$. If there exists $R > 0$ such that $M_f \subset \overline{D}_0(R)$, then f is rational of a form either*

$$f(z) = z + a + \frac{b}{(z + c)^m} \tag{2.1}$$

or $f(z) = \alpha z + \beta$ with with $a, b, c, \alpha, \beta \in \mathbb{C}, b \neq 0,$ and $m \in \mathbb{N}$.

Remark 2.1. The case $f(z) = \alpha z + \beta$ in Lemma 2.2 can be omitted if

$$f^{\#}(z) \leq f^{\#}(0) \leq R + 1.$$

Lemma 2.3. [4, Lemma 4] *Assume that the rational function f defined in* (2.1) *has two zeros $\pm\frac{1}{2}$ and $M_f \subset \overline{D}_0(R)$, then there is a positive constant K which depends only on R such that*

$$\sup_{\overline{D}_0(1)} f^{\#}(z) \leq K.$$

Denote by $n(D, f)$ the number of poles of f in D (counting multiplicity). We recall the following lemma.

Lemma 2.4. [18, Lemma 2.5] *Let $\{f_n\}$ be a family of meromorphic functions in $D_{z_0}(r)$. suppose that*

(a) *$f_n \overset{\chi}{\Longrightarrow} f$ in $D_{z_0}(r) \setminus \{z_0\}$, where $f(\not\equiv 0)$ may be ∞ identically, and*

(b) *there exists $M_0 > 0$ such that $n\big(D_{z_0}(r), \frac{1}{f_n}\big) \leq M_0$ for sufficiently large n.*

Then, there exists $M > 0$ such that $S(D_{z_0}(r/4), f_n) < M$ for sufficiently large n.

Lemma 2.5. [4, Lemma 10] *Let $\alpha \neq 0$ be a complex number and f be a meromorphic function on $\mathbb{C}$ of infinite order such that $f'(z) \neq \alpha$. If there exists an $M \geq 1$ such that $M_f \subset \overline{D}_0(M)$, then f has infinitely many pairs of distinct zeros $(z_{n,1}, z_{n,2})$ such that $z_{n,1} - z_{n,2} \to 0$ and*

$$\sup_{\overline{D}_0(1)} F_n^{\#}(z) \longrightarrow \infty, \text{ where } F_n(z) := \frac{f((z_{n,1} + z_{n,2})/2 + (z_{n,1} - z_{n,2})z)}{z_{n,1} - z_{n,2}}.$$

3. Lemmas

Next, we prove the following lemmas

Lemma 3.1. *Let $\{\psi_n\}$ be a sequence of holomorphic functions converged to ψ in the Euclidean metric on a domain D. Assume that $\psi(z) \neq 0, \infty$ on D. Let $\{f_n\}$ be a sequence of meromorphic functions on D such that for each n, $f_n'(z) \neq \psi_n(z)$ for all $z \in D$. Assume that $M_{f_n} \subset \overline{D}_0(M)$ for some $M \geq 1$. For each $z_0 \in D$ and each n, assume that f_n has at most one single pole in D and tending to z_0 as $n \to \infty$. Then $\{f_n\}$ is normal on $D \setminus \{z_0\}$.*

Proof. We will prove by counter argument, that there is $z_1 \in D \setminus \{z_0\}$ such that the sequence $\{f_n\}$ is not normal at z_0. From Lemma 2.1, we can find points $z_n \to z_1$ and positive numbers $\rho_n \to 0$ and a subsequence of $\{f_n\}$ (which is still call by $\{f_n\}$), and a nonconstant meromorphic function g on $\mathbb{C}$ such that

$$\rho_n^{-1} f_n(z_n + \rho_n z) \overset{\chi}{\Longrightarrow} g(z), \tag{3.1}$$

$$M_g \subset \overline{D}_0(M) \text{ and } g^{\#}(z) \le g^{\#}(0) = M + 1.$$

Put $g_n(z) := \rho_n^{-1} f_n(z_n + \rho_n z)$, we have $g_n'(z) \Longrightarrow g'(z)$ and

$$g_n'(z) = f_n'(z_n + \rho_n z) \ne \psi_n(z_n + \rho_n z) \Longrightarrow \psi(z_1).$$

Applying Hurwitz's Theorem, it follows either $g'(z) \ne \psi(z_1)$ or $g'(z) \equiv \psi(z_1)$. If $g'(z) \equiv \psi(z_1)$. Since $|g'(z)| \le M$ whenever $g(z) = 0$, hence $|\psi(z_1)| \le M$. Therefore,

$$M + 1 = g^{\#} = \frac{|g'(0)|}{1 + |g(0)|^2} = \frac{|\psi(z_1)|}{1 + |g(0)|^2} \le M,$$

which is a contradiction. Thus $g'(z) \ne \psi(z_1)$ on $\mathbb{C}$. So by Lemma 2.2 and its remark, we have

$$g(z) = \psi(z_1)\big(z + a + \frac{b}{(z+c)^m}\big) = \psi(z_1)\Big(\frac{(z+a)(z+c)^m + b}{(z+c)^m}\Big) \tag{3.2}$$

with $a, b, c \in \mathbb{C}, b \ne 0$ and $m \in \mathbb{N}$.

By (3.1) and (3.2), there exists a sequence $\zeta_{n,\infty} \to -c$ such that $g_n(\zeta_{n,\infty}) = \infty$ for sufficiently large n. Thus, writting $z_{n,\infty} = z_n + \rho_n \zeta_{n,\infty}$, we have $z_{n,\infty} \to z_1$ and $f_n(z_{n,\infty}) = \infty$ for sufficiently large n. Since each f_n has at most one single pole in D and tending to z_0 as $n \to \infty$, we get $z_1 = z_0$. This is impossible. □

The following two lemmas are extensions of [4, Lemma 7, Lemma 8], where instead of the condition $f_n'(z) \ne 1$ as in Lemmas of Jianming Chang, we consider the condition $f_n'(z) \ne \psi_n(z)$ with $\{\psi_n\}$ be a sequence of holomorphic functions. The proof of this lemmas is step-by-step the same as the proof of Lemma 7 and Lemma 8 in [4], but we need to modify the calculations when using the condition in the case of the holomorphic function. In the proof of these lemmas, we omit similar proofs. Interested readers can see in the proof of [4, Lemma 7, Lemma 8]. In this paper, we only mention the changes compared with the proof of Jianming Chang.

Lemma 3.2. *Let $\{\psi_n\}$ be a sequence of holomorphic functions converged to ψ in the Euclidean metric on a domain D. Assume that $\psi(z) \ne 0, \infty$ on D. Let $\{f_n\}$ be a sequence of meromorphic functions on D such that $f_n'(z) \ne \psi_n(z)$ for all n and all $z \in D$. Assume that $M_{f_n} \subset \overline{D}_0(M)$ for some $M \ge 1$. Assume that there is $z_0 \in D$ such that:*

(i) *All of subsequence of $\{f_n\}$ is not normal at z_0;*
(ii) *Each f_n has at most one single pole tending to z_0 as $n \to \infty$; and*
(iii) *$\{f_n\}$ is normal on $D \setminus \{z_0\}$.*

Put $f(z) = \int_{z_0}^{z} \psi(\zeta) d\zeta$, then

(1) *there exists a subsequence of* $\{f_n\}$, *(we still denote by* $\{f_n\}$*), converges to* f *in the spherical metric on* $D \setminus \{z_0\}$, *and*
(2) *there is* $r > 0$ *and* $K > 0$ *such that* $D_{z_0}(r) \subset D$ *and* $S(D_{z_0}(r), f_n) < K$.

Proof. Without loss of generality,we may assume that $z_0 = 0$ and $\psi(0) = 1$. By the same proof as in the proof of [4, Lemma 7], we deduce that f_n has $m+1$ zeros $z_{n,0}^{(i)} = z_n + \rho_n \zeta_{n,0}^{(i)}$ and a pole $z_{n,\infty} = z_n + \rho_n \zeta_{n,\infty}$ with multiplicity m and $\{f_n^*\}$ is normal on $D \setminus \{0\}$, where

$$f_n^*(z) = \frac{f_n(z)(z - z_{n,\infty})^m}{\prod_{i=1}^{m+1}(z - z_{n,0}^{(i)})} \tag{3.3}$$

Now, we will prove $\{f_n^*\}$ is normal on D. It suffices to show that $\{f_n^*\}$ is normal at 0. Indeed, by the condition (ii), there exists $\delta > 0$ such that f_n^* are holomorphic on $D_0(\delta) \subset D$.

First, we will prove that f_n^* has no zeros tending to 0 as $n \to \infty$. Otherwise, choose $z_{n,0}^* \to 0$ to be the zeros of f_n^* such that f_n^* has no zeros in $D_{z_n}(|z_{n,0}^* - z_n|)$. Write $z_{n,0}^* = z_n + \rho_n \zeta_{n,0}^*$. By

$$f_n^*(z_n + \rho_n \zeta) \Longrightarrow 1 \tag{3.4}$$

on $\mathbb{C}$, we get $\zeta_{n,0}^* \to \infty$. Let $0 < \delta_1 < \delta$ and

$$F_n^*(z) = f_n^*\big(z_n + \frac{z_{n,0}^* - z_n}{\delta_1} z\big).$$

Then F_n^* are locally uniformly holomorphic on $\mathbb{C}$ such that $F_n^*(\delta_1) = 0$ and $F_n^*(z) \neq 0$ on $D_0(\delta_1)$. Let

$$L_n(z) = \frac{\delta_1 R_n\big(z_n + \frac{z_{n,0}^* - z_n}{\delta_1} z\big)}{z_{n,0}^* - z_n} = \frac{\prod_{i=1}^{m+1}(z - \delta_1 \zeta_{n,0}^{(i)}/\zeta_{n,0}^*)}{(z - \delta_1 \zeta_{n,\infty}/\zeta_{n,0}^*)^m} \Longrightarrow z \quad \text{on } \mathbb{C}^* \tag{3.5}$$

and

$$F_n(z) := L_n(z) F_n^*(z) = \frac{\delta_1 f_n\big(z_n + \frac{z_{n,0}^* - z_n}{\delta_1} z\big)}{z_{n,0}^* - z_n}. \tag{3.6}$$

We have

$$F_n'(z) = f_n'\big(z_n + \frac{z_{n,0}^* - z_n}{\delta_1} z\big) \neq \psi_n\big(z_n + \frac{z_{n,0}^* - z_n}{\delta_1} z\big)$$

for all n and $z \in D_0(\delta)$, $M_{F_n} \subset \overline{D_0(\delta_1)}$ and each F_n has at most one single pole in $D_0(\delta)$ tending to 0 as $n \to \infty$. By Lemma 3.1, $\{F_n\}$ is normal on $D_0(\delta) \setminus \{0\}$ and hence $\{F_n^*\}$ is also normal on $D_0(\delta) \setminus \{0\}$. As $F_n^*(\delta_1) = 0$, we may assume that $F_n^* \Longrightarrow F^*$ on $D_0(\delta) \setminus \{0\}$ with $F^*(\delta_1) = 0$.

If $F^* \equiv 0$ then $F_n \Longrightarrow 0, F_n' \Longrightarrow 0$ and $F_n'' \Longrightarrow 0$ on $D_0(\delta)\setminus\{0\}$. Thus, we have

$$\Big|n\big(\delta_1, F_n'(z)-\psi_n\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)\big)-n\Big(\delta_1, \frac{1}{F_n'(z)-\psi_n\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)}\Big)\Big|$$

$$=\frac{1}{2\pi}\Big|\int_{|z|=\delta_1}\frac{F_n''(z)-\psi_n'\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)\big(\frac{z_{n,0}^*-z_n}{\delta_1}\big)}{F_n'(z)-\psi_n\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)}dz\Big| \Longrightarrow 0 \quad \text{as } n\to\infty,$$

which implies

$$n\big(\delta_1, F_n'(z)-\psi_n\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)\big)=n\Big(\delta_1, \frac{1}{F_n'(z)-\psi_n\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)}\Big)=0$$

for n big enough. Hence, F_n has no pole on $D_0(\delta_1)$. This is a contradiction as $\zeta_{n,\infty}/\zeta_{n,0}^* \to 0$ is a pole of F_n. Therefore $F^* \not\equiv 0$. By $F_n^*(z)\neq 0$ on $D_0(\delta_1)$ and by maximum modulus principle, we obtain $F_n^* \Longrightarrow F^*$ on $D_0(\delta)$. Since $F_n^*(0) = f_n^*(z_n) = g_n^*(0) \to 1 \quad$ as $n\to\infty$, we obtain $F^*(0)=1$, which implies F^* is non-constant.

On the other hand, we have

$$F_n'(z)-\psi_n\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)\Longrightarrow (zF^*)'-1$$

on $D_0(\delta)\setminus\{0\}$. If $(zF^*)'\equiv 1$ then there is a constant α such that $zF^*\equiv z+\alpha$. By $F^*(0)=1$, we get $\alpha=0$. Hence $F^*\equiv 1$. This is impossible as $F^*(\delta_1)=0$. Thus $(zF^*)'-1\not\equiv 0$.

By maximum modulus principle and $F_n'(z)-\psi_n\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)\neq 0$, we have

$$F_n'(z)-\psi_n\big(z_n+\frac{z_{n,0}^*-z_n}{\delta_1}z\big)\Longrightarrow (zF^*)'-1$$

on $D_0(\delta)$. By Hurwitz's theorem, we obtain $(zF^*)'-1\neq 0$. On the other hand, we have $((zF^*)'-1)|_{z=0}=F^*(0)-1=0$. This is a contradiction.

Therefor f_n^* has no zeros tending to 0, which implies f_n^* is non-zero holomorphic in some neighbourhood $D_0(\eta)$ for some $0<\eta<\delta$. By maximum modulus principle and $\{f_n^*\}$ is normal on $D\setminus\{0\}$, we deduce that $\{f_n^*\}$ is normal at 0. Therefore, it is normal on D. As $f_n^*(z_n)\to 1$, let $f_n^*\to f^*$ on D and $f^*(0)=1$. we have

$$f_n \stackrel{\chi}{\Longrightarrow} f = zf^* \quad \text{on } D\setminus\{0\}.$$

Since $f_n'(z)-\psi_n(z)\neq 0$ and $f_n'(z)-\psi_n(z)\Longrightarrow f'(z)-\psi(z)$ on $D\setminus\{0\}$. From Hurwitz's theorem, we have $f'(z)-\psi(z)\equiv 0$ or $f'(z)-\psi(z)\neq 0$. If $f'(z)-\psi(z)\neq 0$, by maximum modulus principle, we get $f_n'(z)-\psi_n(z)\Longrightarrow f'(z)-\psi(z)$ on $D_0(\delta)\subset D$. By Hurwitz's theorem again, we have $f'(z)-\psi(z)\neq 0$ on $\Delta(0,\delta)$. This is impossible, as $(f'(z)-\psi(z))|_{z=0}=((zf^*)'-\psi(z))|_{z=0}=f^*(0)-1=0$.

Thus, $f'(z) \equiv \psi(z)$ on $D \setminus \{0\}$. It follows that f' has no poles on $D \setminus \{0\}$ and $f'(z)$ has a removable singularity at $z = 0$ and hence $f(z)$ also has a removable singularity at $z = 0$. From maximum modulus principle, we obtain $f'(z) \equiv \psi(z)$ on D. Since $zf^* = f$ and $f^*(0) = 1$, we have $f(0) = 0$. Thus,

$$f(z) = \int_0^z f'(\zeta)d\zeta = \int_0^z \psi(\zeta)d\zeta.$$

We continue to prove Assertion (2). Since $f_n^* \neq 0$ on disk $D_0(\eta)$, by (3.3) we get $n\big(\eta, \frac{1}{f_n}\big) \leq m + 1$. By Lemma 2.4, there is a positive number K such that $S\big(D_0(\frac{\eta}{4}), f_n\big) < K$.

We are done for proof of Lemma 3.2. $\square$

Lemma 3.3. *Let $\{\psi_n\}$ be a sequence of holomorphic functions converged to ψ in the Euclidean metric on a domain D. Assume that $\psi(z) \neq 0$ and ∞ on D. For all n and all $z \in D$, let $\{f_n\}$ be a sequence of meromorphic functions on D such that $f_n'(z) \neq \psi_n(z)$. Assume that $f_n'(f_n^{-1}(0)) \subset \overline{D_0(M)}$ for some $M \geq 1$. Suppose that at some point $z_0 \in D$,*

(i) *All subsequence of $\{f_n\}$ is not normal at z_0; and*
(ii) *Each f_n has at least two distinct poles tending to z_0 as $n \to \infty$.*

Then there is a subsequence of $\{f_n\}$, (we denote by $\{f_n\}$ again), such that each f_n has distinct zeros a_n and b_n tending to z_0 as $n \to \infty$ such that

$$\sup_{\overline{D_0(1)}} h_n^{\#}(z) \longrightarrow \infty,$$

where

$$h_n(z) := \frac{f_n(d_n + (a_n - b_n)z)}{a_n - b_n}, \quad \text{and } d_n = \frac{a_n + b_n}{2}.$$

Proof. Without loss of generality, we may assume that $z_0 = 0$ and $\psi(0) = 1$. As in the proof of [4, Lemma 7], we get $z_{n,\infty} = z_n + \rho_n\zeta_{n,\infty}$ is a pole of f_n with exact multiplicity m. Let

$$f_n^*(z) = \frac{f_n(z)(z - z_{n,\infty})^m}{\prod_{i=1}^{m+1}(z - z_{n,0}^{(i)})} \tag{3.7}$$

Now, we show that f_n^* has at least one zero tending to 0. Indeed, suppose not, then there exists $\delta > 0$ such that $f_n^* \neq 0$ in $D_0(\delta) \subset D$. We have $f_n' = (R_n f_n^*)' \neq \psi_n$ and $R_n(z) \Longrightarrow z$ on $\mathbb{C}^*$. Applying Lemma 3.1, we deduce that $\{f_n^*\}$ is normal on $D_0(\delta) \setminus \{0\}$. Hence, we may say that $f_n^* \Longrightarrow f^*$ on $D_0(\delta) \setminus \{0\}$.

If $f^* \equiv 0$ then $f_n \Longrightarrow 0, f'_n \Longrightarrow 0$ and $f''_n \Longrightarrow 0$ on $D_0(\delta) \setminus \{0\}$. Applying the argument principle, we have

$$\big|n\big(\frac{\delta}{2}, f'_n(z) - \psi_n(z)\big) - n\big(\frac{\delta}{2}, \frac{1}{f'_n(z) - \psi_n(z)}\big)\big| = \frac{1}{2\pi}\Big|\int_{|z|=\frac{\delta}{2}} \frac{f''_n(z) - \psi'_n(z)}{f'_n(z) - \psi_n(z)}dz\Big|$$

$$\Longrightarrow \frac{1}{2\pi}\Big|\int_{|z|=\frac{\delta}{2}} \frac{\psi'(z)}{\psi(z)}dz\Big| = \big|n\big(\frac{\delta}{2}, \psi(z)\big) - n\big(\frac{\delta}{2}, \frac{1}{\psi(z)}\big)\big| = 0,$$

and hence

$$n\big(\frac{\delta}{2}, f'_n(z) - \psi_n(z)\big) = n\big(\frac{\delta}{2}, \frac{1}{f'_n(z) - \psi_n(z)}\big) = 0$$

for sufficiently large n. Hence, f_n has no pole on $D_0(\frac{\delta}{2})$, which is impossible as f_n has a pole $z_{n,\infty} \to 0$. So, $f^* \not\equiv 0$. Since $f^*_n \neq 0$ on disk $D_0(\delta)$, by maximum modulus principle, we deduce that $f^*_n \Longrightarrow f^*$ on $D_0(\delta)$. Since $f^*_n(z_n) = g^*(0) \to 1$, we obtain $f^*(0) = 1$. Therefore f^*_n has no poles tending to 0, which is a contradiction to (ii). Therefore f^*_n contains at least one zero $z^*_{n,0}$ tending to 0. From

$$f^*_n(z_n + \rho_n\zeta) \Longrightarrow 1$$

on $\mathbb{C}$, we have

$$\zeta^*_{n,0} = \frac{z^*_{n,0} - z_n}{\rho_n} \to \infty.$$

Set

$$h_n(z) = \frac{f_n(z^*_{n,0} + z^{(1)}_{n,0})/2 + (z^{(1)}_{n,0} - z^*_{n,0})z)}{z^{(1)}_{n,0} - z^*_{n,0}}.$$

Then we have

$$h_n\Big(\frac{1}{2}\Big) = 0 \quad \text{and} \quad h_n\Big(\frac{z_{n,\infty} - (z^*_{n,0} + z^{(1)}_{n,0})/2}{z^{(1)}_{n,0} - z^*_{n,0}}\Big) = \infty.$$

Since

$$\frac{z_{n,\infty} - (z^*_{n,0} + z^{(1)}_{n,0})/2}{z^{(1)}_{n,0} - z^*_{n,0}} = \frac{2\zeta_{n,\infty} - (\zeta^*_{n,0} + \zeta^{(1)}_{n,0})}{2(\zeta^{(1)}_{n,0} - \zeta^*_{n,0})} \to \frac{1}{2},$$

each subsequence of $\{h_n\}$ is not equi-continuous in a neibourhood of $z = \frac{1}{2}$ and hence it is not normal at $\frac{1}{2}$. The conclusion follows from Marty's theorem. We are done for a proof of Lemma 3.3. $\square$

4. Proof of Theorem 1.1

The idea of proof is following from [4], but we replace the condition $f'(z) \neq 1$ by $f'(z) \neq \varphi'(z)$ for a holomorphic function φ.

Proof of Theorem 1.1. Asume that there is $\{f_n\} \subset \mathcal{F}$ and $E \subset D$ such that $\{f_n\}$ is not normal.

Case 1. For any $z_0 \in E$, assume that f_n has at most only one single pole that tends to z_0.

In this case, by Lemma 3.1, we have $\{f_n\}$ is normal on $D_{z_0}(\delta) \setminus \{z_0\}$ for some $\delta > 0$. Therefore E does not have any accumulation point in D.

Assume that there are $z_1 \neq z_2 \in E$ and a subsequence of $\{f_n\}$, (we keep calling by $\{f_n\}$), such that $\{f_n\}$ is not normal at z_1 and z_2. Aplying Lemma 3.2 to φ', there is a subsequence of $\{f_n\}$, (we keep calling by $\{f_n\}$), such that f_n converges to $\varphi(z) - \varphi(z_1)$ and $\varphi(z) - \varphi(z_2)$ in the spherical metric on $D \setminus E$, so that $\varphi(z_1) = \varphi(z_2)$. By the hypothesis φ is univalent, we have $z_1 = z_2$. This is a contradiction. Therefore, $\{f_n\}$ has at most one non-normal point, which means $\mathcal{F}$ is quasinormal of order 1.

Case 2. There exists $z_0 \in E$, and a subsequence of $\{f_n\}$, (we keep calling by $\{f_n\}$), such that each f_n has at least two distinct poles tending to z_0.

In this case, for each $0 < \delta < 1$, there exists $N(\delta)$ such that for $n \geq N(\delta)$, the function f_n has at least two distinct zeros in $D_{z_0}(\delta)$. Lemma 3.3 implies that one can chose a subsequence of $\{f_n\}$, (we keep calling by $\{f_n\}$), and a constant K such that each f_n has zeros $a_n \neq b_n \in D_{z_0}(\delta)$ tending to z_0 and

$$\sup_{D_0(1)} h_n^{\#}(z) > K + 1, \tag{4.1}$$

for n big enough, where

$$h_n(z) := \frac{f_n(d_n + (a_n - b_n)z)}{a_n - b_n}, \quad \text{and } d_n = \frac{a_n + b_n}{2}.$$

Fix $\delta > 0$, we can choose zeros of f_n in $D_{z_0}(\delta)$ such that $a_n \neq b_n$ satisfying (4.1) and

$$\sigma_n := \frac{|a_n - b_n|}{\delta - |d_n - z_0|} \quad \text{is minimal.} \tag{4.2}$$

Therefore, $\sigma_n \to 0$.

Taking a subsequence if necessary, we may assume that $d_n \to d$. Clearly, we have $M_h = h_n'(h_n^{-1}(0)) \subset \overline{D_0(M)}$ and

$$h_n'(z) \neq \psi_n(z) := \varphi'(d_n + (a_n - b_n)z) \Longrightarrow \varphi'(d)$$

on $\mathbb{C}$.

We will prove that that all of subsequence of $\{h_n\}$ is not normal on $\mathbb{C}$. Otherwise, assume that $h_n \overset{\chi}{\Longrightarrow} h$ on $\mathbb{C}$. Since $h_n\big(\pm \frac{1}{2}\big) = 0$, $M_h \subset \overline{D_0(M)}$ and $h_n'(z) \neq \psi_n(z)$, we have $h\big(\pm \frac{1}{2}\big) = 0$, $M_h \subset \overline{D_0(M)}$ and either $h'(z) \neq \varphi'(d)$ or

$h'(z) \equiv \varphi'(d)$. If the latter case occurs, then $h(z) = \varphi'(d)z + \lambda$ for some constant λ. Since $h\big(\pm\frac{1}{2}\big)$, we have $\varphi'(d) = 0$, which is impossible as φ is univalent. Hence, we have $h'(z) \neq \varphi'(d)$. By (4.1), we have

$$\sup_{D_0(1)} h_n^{\#}(z) \geq K + 1.$$

Hence, applying Lemma 2.2 and Lemma 2.3 to the function h, we deduce that h is of infinite order. Therefore, by Lemma 2.5, there is infinitely many pairs of distinct zeros (α_l, β_l) of h such that $\alpha_l - \beta_l \to 0$ and

$$\sup_{D_0(1)} F_l^{\#}(z) \longrightarrow \infty$$

as $l \to \infty$, where $F_l(z) := \dfrac{h((\alpha_l + \beta_l)/2 + (\alpha_l - \beta_l)z)}{\alpha_l - \beta_l}$.

Fix l such that $|\alpha_l - \beta_l| < 1$ and

$$\sup_{D_0(1)} F_l^{\#}(z) \geq K + 1. \tag{4.3}$$

Since $h_n \overset{\chi}{\Longrightarrow} h$ on $\mathbb{C}$, there exist points $\alpha_{n,l} \to \alpha_l$ and $\beta_{n,l} \to \beta_l$ such that $h_n(\alpha_{n,l}) = h_n(\beta_{n,l}) = 0$. By (4.3), for n sufficiently large, we have

$$\sup_{D_0(1)} F_{n,l}^{\#}(z) > K + 1, \tag{4.4}$$

where $F_{n,l}(z) := \dfrac{h_n((\alpha_{n,l} + \beta_{n,l})/2 + (\alpha_{n,l} - \beta_{n,l})z)}{\alpha_{n,l} - \beta_{n,l}}$.

Put

$$a_{n,l}^* = d_n + (a_n - b_n)\alpha_{n,l}, \quad b_{n,l}^* = d_n + (a_n - b_n)\beta_{n,l}. \tag{4.5}$$

Then $f_n(a_{n,l}^*) = f_n(b_{n,l}^*) = 0$. Since $\sigma_n \to 0$ and $\alpha_{n,l} \to \alpha_l$, we have

$$|a_{n,l}^* - z_0| \leq |d_n - z_0| + |a_n - b_n||\alpha_{n,l}| = \delta - \Big(\frac{1}{\sigma_n} - |\alpha_{n,l}|\Big)|a_n - b_n| < \delta,$$

for sufficiently large n. Thus $a_{n,l}^* \in D_{z_0}(\delta)$. By similar arguments, we also have $b_{n,l}^* \in D_{z_0}(\delta)$.

Let

$$L_n(z) = \frac{f_n(d_{n,l}^* + (a_{n,l}^* - b_{n,l}^*)z)}{a_{n,l}^* - b_{n,l}^*}, \quad \text{where } d_{n,l}^* = \frac{a_{n,l}^* + b_{n,l}^*}{2}.$$

It is easy that

$$L_n(z) = \frac{h_n((\alpha_{n,l} + \beta_{n,l})/2 + (\alpha_{n,l} - \beta_{n,l})z)}{\alpha_{n,l} - \beta_{n,l}} = F_{n,l}(z).$$

Hence, we get

$$\sup_{D_0(1)} L_n^{\#}(z) > K + 1.$$

However, we have

$$\frac{|a^*_{n,l}-b^*_{n,l}|}{\delta-|d^*_{n,l}-z_0|}=\frac{|(a_n-b_n)(\alpha_{n,l}-\beta_{n,l})|}{\delta-|d_n-z_0+(a_n-b_n)(\alpha_{n,l}+\beta_{n,l})/2|}$$
$$=\frac{|a_n-b_n|}{\delta-|d_n-z_0|}\frac{|\alpha_{n,l}-\beta_{n,l}|(\delta-|d_n-z_0|)}{\delta-|d_n-z_0+(a_n-b_n)(\alpha_{n,l}+\beta_{n,l})/2|}. \quad (4.6)$$

Given $\epsilon>0$. Since $\sigma_n\to 0$, we have

$$|d_n-z_0+(a_n-b_n)\frac{\alpha_{n,l}+\beta_{n,l}}{2}|<|d_n|+\epsilon(\delta-|d_n-z_0|)\frac{|\alpha_{n,l}+\beta_{n,l}|}{2}$$

for n big enough. Choose $0<\epsilon_0<1$ such that $|\alpha_l-\beta_l|<\epsilon_0$. Since $\alpha_{n,l}-\beta_{n,l}\to\alpha_l-\beta_l$, we have $\alpha_{n,l}-\beta_{n,l}<\epsilon_0$ for n big enough. Therefore, by combining with (4.6), we have

$$\frac{|a^*_{n,l}-b^*_{n,l}|}{\delta-|d^*_{n,l}-z_0|}<\frac{|a_n-b_n|}{\delta-|d_n-z_0|}\frac{\epsilon_0}{1-\epsilon|\alpha_{n,l}+\beta_{n,l}|/2}<\sigma_n, \quad (4.7)$$

which contradicts that σ_n is the smallest. So, all of subsequence in $\{h_n\}$ is not normal on $\mathbb{C}$.

We denote by Λ the set of points such that $\{h_n\}$ is not normal. For each $\zeta_0\in\Lambda$, assume that h_n has at most one pole tending to ζ_0 as $n\to\infty$. Similar as in Case 1, we get $\{h_n\}$ is quasi-normal of order 1 and there is only $\zeta_0\in\Lambda$ such that

$$h_n(\zeta)\overset{\chi}{\Longrightarrow}\varphi'(d)(\zeta-\zeta_0) \text{ on } \mathbb{C}\setminus\{\zeta_0\},$$

which is contradiction $h_n\big(\pm\frac{1}{2}\big)=0$. Hence, by Lemma 3.3, we only consider the case there exists a subsequence of $\{h_n\}$, (which we continue to call $\{h_n\}$), such that each h_n has zeros $\alpha^*_n\neq\beta^*_n$ tending to ζ_0 as $n\to\infty$ and

$$\sup_{D_0(1)} H^{\#}_n(z)>K+1$$

where K is the constant defined in Lemma 2.3 and

$$H_n(z)=\frac{h_n(d^*_n+(\alpha^*_n-\beta^*_n)z)}{\alpha^*_n-\beta^*_n}, \text{with } d^*_n=\frac{\alpha^*_n+\beta^*_n}{2}.$$

Let

$$\widehat{\alpha}_n=d_n+(\alpha_n-\beta_n)\alpha^*_n,\quad \widehat{\beta}_n=d_n+(\alpha_n-\beta_n)\beta^*_n.$$

Using the same argument as the previous one, we have $\widehat{\alpha}_n$ and $\widehat{\beta}_n$ are two zeros of f_n in $\Delta(z_0,\delta)$. Let

$$\widehat{h}_n(z)=\frac{f_n(\widehat{d}_n+(\widehat{\alpha}_n-\widehat{\beta}_n)z)}{\widehat{\alpha}_n-\widehat{\beta}_n},\quad \text{where}\quad \widehat{d}_n=\frac{\widehat{\alpha}_n+\widehat{\beta}_n}{2}.$$

It is easy that $\widehat{h}_n(z) = H_n(z)$ and hence

$$\sup_{\overline{D_0(1)}} \widehat{h}_n^{\#}(z) > K + 1.$$

Similar as in the argument of (4.7), we have

$$\frac{|\widehat{\alpha}_n - \widehat{\beta}_n|}{\delta - |\widehat{d}_n - z_0|} < \sigma_n$$

for n big enough. We get a contradiction that σ_n is minimal. The proof is completed. □

5. Proof of Theorem 1.2

As a sequence of Theorem 1.1, we have following lemma.

Lemma 5.1. *Let $\{\psi_n\}$ be a sequence of holomorphic functions on the plane domain D such that $\psi_n \Longrightarrow \psi = \varphi'$ on D, where φ is univalent on D. Let $\{f_n\}$ be a sequence of meromorphic functions on D such that $f_n'(z) \neq \psi_n(z)$ for all n and all $z \in D$. Assume that there exists an $M \geq 1$ such that $f_n'(f_n^{-1}(0)) \subset \overline{D_0(M)}$. If all subsequence of $\{f_n\}$ is not normal at some $z_0 \in D$, then $f_n \Longrightarrow \varphi - \varphi(z_0)$ on $D \setminus \{z_0\}$ and there exist $\delta > 0$ and $K > 0$ such that $D_{z_0}(\delta) \subset D$ and $S(D_{z_0}(\delta), f_n) < K$ for sufficiently large n.*

We will leave a proof of Theorem 1.2 since one can step-by-step do similarly as in [15, Theorem 1] by replacing Theorem 1.1 and Lemma 5.1 instead of Theorem A as in [15].

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