A NON-ARCHIMEDEAN SECOND MAIN THEOREM FOR SMALL FUNCTIONS AND APPLICATIONS

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Abstract. We establish a slowly moving target second main theorem for meromorphic functions on a non-Archimedean field, with counting functions truncated to level 1. As an application, we show that two meromorphic functions on a non-Archimedean field must coincide if they share $q\,(q \geq 5)$ distinct small functions, ignoring multiplicities. Thus, our work improves the results in [2].

1. Introduction and main results

As a consequence of the Truncated Nevanlinna Second Main Theorem, R. Nevanlinna [5] himself proved that for two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, they cannot have the same inverse images for five distinct values. Then, some authors (Yuhua and Jianyong [10], Yao [8], Thai and Tan [6], for example) have generalized the result where distinct values are replaced by small functions. Here, a meromorphic function a is called a small function with respect to f if $T(r, a) = o(T(r, f))$ for $r \to \infty$, where $T(r, f)$ is the Nevanlinna characteristic function of f. In 2002, Yi [9] extended the five values theorem to the case of sharing five distinct small functions. The proofs of the above results are based straightforwardly on Cartan's auxiliary functions. In 2004, Yamanoi gave a sharp moving targets second main theorem with truncated counting functions, and as its direct consequence, one can obtain Yi's result.

Nevanlinna theory in complex analysis is so beautiful that one would naturally be interested in determining how such a theory would look in $\mathbf{K}$, an algebraically closed field of characteristic zero, complete with respect to a non-Archimedean absolute value $|.|$. Adams and Straus [1] (see also [3]) proved the above Nevanlinna result about five distinct values in the complex case can be replaced with

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4 four distinct values in the p-adic case. To date, we do not have a sharp non-Archimedean analog of the Yamanoi theorem. Therefore, one question is created: what is the fewest number of shared slowly moving targets that uniquely determines a non-constant non-Archimedean meromorphic function?

Recently, A. Escassut and C. C. Yang [2] gave a truncated slowly moving target second main theorem for non-Archimedean meromorphic functions. Their proof makes use of different techniques than the theorem of Yamanoi in complex analysis. As a consequence, they showed that two non-constant non-Archimedean meromorphic functions sharing 7 slowly moving targets must be equal. However, the number 7 is not sharp.

In this work we are able to increase the coefficient $\frac{q}{3}$ in front of the characteristic function in Theorem 2 in [2] to $\frac{2q}{5}$. This allows us to lower the number of slowly moving targets in the uniqueness result from seven to five. The problem of whether a non-constant non-Archimedean meromorphic function is determined by four slowly moving targets, as is the case with constant values as in Adams and Straus's work, remains open.

Our first result is as follows.

Theorem 1. *Let f be a nonconstant meromorphic function on $\mathbf{K}$. Let $a_1, \ldots, a_q$ be q distinct small functions with respect to f. Then, we have*

$$\frac{2q}{5}T(r,f) \leq \sum_{i=1}^{q} \overline{N}\big(r, \frac{1}{f-a_i}\big) + S(r,f).$$

Let k be a positive integer or ∞, we denote by $\overline{E}(a,k,f)$ the set of distinct zeros of $f-a$ with multiplicities at most k, where a zero of $f-\infty$ means a pole of f.

Remark. If $k=\infty$, then the set $\overline{E}(a,\infty,f)$ is just the set of distinct zeros of $f-a$ and was denoted by $\overline{E}(a,f)$ as usually.

Let f and g be nonconstant non-Archimedean meromorphic functions. Then, $\overline{E}(a,k,f)=\overline{E}(a,k,f)$ means that z_0 is a zero of $f-a$ with multiplicity $m \leq k$ if and only if it is a zero of $g-a$ with multiplicity $n \leq k$, where m is not necessarily equal to n, and if z_0 is a zero of $f-a$ with multiplicity $p > k$ then it does not need to be a zero of $g-a$.

In the special case $k=\infty$, the condition $\overline{E}(a,f)=\overline{E}(a,g)$ means f and g share the function a, ignoring multiplicities, as usual.

As an application of Theorem 1, we get a uniqueness theorem for the meromorphic functions sharing a few small functions as follows.

Theorem 2. *Let f and g be two nonconstant meromorphic functions on $\mathbf{K}$. Let $a_1, \dots, a_q \quad (q \geq 5)$ be q distinct small functions with respect to f and g. Let $k_1, \dots, k_q$ be q positive integers or $+\infty$ with*

$$\sum_{j=1}^{q} \frac{1}{k_j + 1} < \frac{2q(q-4)}{5(q+4)}.$$

If

$$\overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \quad (j = 1, \dots, q),$$

then $f \equiv g$.

In the case $k_1 = \cdots = k_q = k$, we can get the result with slightly smaller multiples as follows.

Theorem 3. *Let f and g be two nonconstant meromorphic functions on $\mathbf{K}$. Let $a_1, \dots, a_q \quad (q \geq 5)$ be q distinct small functions with respect to f and g. Let k be a positive integer or $+\infty$ with $k > \frac{3(q+4)}{2(q-4)}$. If*

$$\overline{E}(a_j, k, f) = \overline{E}(a_j, k, g) \quad (j = 1, \dots, q),$$

then $f \equiv g$.

By Theorem 3, we obtain the following corollary, which is a uniqueness theorem for non-Archimedean meromorphic functions sharing 5 small functions ignoring multiplicities.

Corollary 1. *Let f and g be two nonconstant meromorphic functions on $\mathbf{K}$. Let $a_1, \dots, a_5$ be 5 distinct small functions with respect to f and g. If f and g share a_j ignoring multiplicities $(j = 1, \dots, 5,)$ then $f \equiv g$.*

Note that this Corollary 1 improves a result of A. Escassut and C. C. Yang [2, Theorem 3], where the number of small functions is reduced to 5.

2. Preliminary on Nevanlinna Theory for non-Archimedean meromorphic functions

We recall the following definitions and results (cf. [4]). Let $\mathbf{K}$ be an algebraically closed field of arbitrary characteristic, complete with respect to a non-Archimedean absolute value $|.|$. Let f be a meromorphic function. We denote by $n(r, \frac{1}{f})$ the number of zeros of f in $\{z$ with $|z| < r\}$, counting multiplicity. Define the *counting function* of f by

$$N(r, \frac{1}{f}) = \int_0^r \frac{n(r, \frac{1}{f}) - n(0, \frac{1}{f})}{t} dt + n(0, \frac{1}{f}) \log r,$$

where $n\big(0, \frac{1}{f}\big)$ is the order of zero of f at $z = 0$.

We denote by $\overline{N}_{k)}\big(r, \frac{1}{f-a}\big)$ the counting function of zeros of $f - a$ with multiplicities at most k, by $\overline{N}_{(k+1}\big(r, \frac{1}{f-a}\big)$ the counting function of zeros of $f - a$ with multiplicities at least $k + 1$, where each multiple zero in these counting functions counted only once.

We define the *compensation function* by

$$m(r, f) = \log^+ |f|_r = \max\{0, \log |f|_r\},$$

and the *characteristic function*

$$T(r, f) = m(r, f) + N(r, f).$$

The logarithmic derivative lemma can be stated as follows (see [4]).

Lemma 1 (Logarithmic Derivative Lemma)**.** *Let f be a non-constant meromorphic function on $\mathbf{K}$. Then for any integer $k > 0$, we have*

$$m\big(r, \frac{f^{(k)}}{f}\big) = O(1)$$

as $r \to \infty$.

We state the first and second fundamental theorem in Nevanlinna theory (see e.g. [4]):

Theorem 4 (The First Main Theorem)**.** *Let $f(z)$ be a non-Archimedean meromorphic function and $c \in \mathbf{K}$. Then*

$$T(r, \frac{1}{f - c}) = T(r, f) + O(1).$$

Theorem 5 (Second fundamental theorem)**.** *Let $a_1, \cdots, a_q$ be a set of distinct numbers of $\mathbf{K}$. Let f be a non-constant meromorphic function on $\mathbf{K}$. Then, the inequality*

$$(q - 2)T(r, f) \le \sum_{j=1}^{q} \overline{N}\big(r, \frac{1}{f - a_j}\big) - \log r + O(1).$$

3. Proof of Theorem 1

We first consider the following lemma.

Lemma 2. *Let f be a nonconstant meromorphic function on $\mathbf{K}$. Let $a_1, \ldots, a_5$ be distinct small functions with respect to f. We have*

$$2T(r, f) \le \sum_{i=1}^{5} \overline{N}\big(r, \frac{1}{f - a_i}\big) + S(r, f).$$

Proof. By the transformation

$$F = \frac{f-a_2}{f-a_1}\cdot\frac{a_3-a_1}{a_3-a_2},$$

we just need to prove the theorem in the case that $a_1=\infty, a_2=0, a_3=1$, $a_4, a_5 \not\equiv 0, 1, \infty, a_4 \not\equiv a_5$. If one of a_4 and a_5 is constant, then we need to prove nothing according to the second main theorem for constants. Thus, we may assume that both a_4 and a_5 are nonconstant small functions of f. Set

$$H = \begin{vmatrix} ff' & f' & f(f-1) \\ a_4a_4' & a_4' & a_4(a_4-1) \\ a_5a_5' & a_5' & a_5(a_5-1) \end{vmatrix} \tag{3.1}$$

By a simple computation, we get

$$\begin{aligned} H =& f(f-1)a_4(a_4-1)a_5(a_5-1)\Big[\big(\frac{a_4'}{a_4}-\frac{a_5'}{a_5}\big)\big(\frac{f'}{f-1}-\frac{a_5'}{a_5-1}\big) \\ &-\big(\frac{a_4'}{a_4-1}-\frac{a_5'}{a_5-1}\big)\big(\frac{f'}{f}-\frac{a_5'}{a_5}\big)\Big]. \end{aligned} \tag{3.2}$$

We claim that $H\not\equiv 0$. Indeed, on the contrary, assume that $H\equiv 0$. Since f is not constant and $a_4, a_5 \not\equiv 0, 1$, it follows from (3.1) that

$$\Big(\frac{a_4'}{a_4}-\frac{a_5'}{a_5}\Big)\frac{f'}{f-1}-\Big(\frac{a_4'}{a_4-1}-\frac{a_5'}{a_5-1}\Big)\frac{f'}{f} \equiv \Big(\frac{a_4'}{a_4}-\frac{a_5'}{a_5}\Big)\frac{a_5'}{a_5-1}-\Big(\frac{a_4'}{a_4-1}-\frac{a_5'}{a_5-1}\Big)\frac{a_5'}{a_5}. \tag{3.3}$$

We now distinguish four cases

Case 1. $\frac{a_4'}{a_4}\equiv\frac{a_5'}{a_5}$. It follows from (3.3) that $\frac{a_4'}{a_4-1}\equiv\frac{a_5'}{a_5-1}$ or $\frac{f'}{f}\equiv\frac{a_5'}{a_5}$. If $\frac{a_4'}{a_4-1}\equiv\frac{a_5'}{a_5-1}$ then a_4 and a_5 are constants, which contradicts our assumption. This means $\frac{f'}{f}\equiv\frac{a_5'}{a_5}$. Hence, we get $f=ca_5$, where c is a constant. This is a contradiction.

Case 2. $\frac{a_4'}{a_4-1}\equiv\frac{a_5'}{a_5-1}$. By an argument similar to Case 1, we also get a contradiction.

Case 3. $\frac{a_4'}{a_4}-\frac{a_5'}{a_5}\equiv\frac{a_4'}{a_4-1}-\frac{a_5'}{a_5-1}\not\equiv 0$. It follows from (3.3) that

$$\frac{f'}{f-1}-\frac{f'}{f}\equiv\frac{a_5'}{a_5-1}-\frac{a_5'}{a_5},$$

which implies

$$\frac{f-1}{f}\equiv C\frac{a_5-1}{a_5},$$

where C is a constant. Thus, we obtain

$$\frac{1}{f}\equiv 1-C\frac{a_5-1}{a_5}.$$

It follows that

$$T(r,f) = T\big(r,\frac{1}{f}\big) + O(1) = S(r,f).$$

This is a contradiction.

Case 4. $\frac{a_4'}{a_4} \not\equiv \frac{a_5'}{a_5}$, $\frac{a_4'}{a_4-1} \not\equiv \frac{a_5'}{a_5-1}$ and $\frac{a_4'}{a_4} - \frac{a_5'}{a_5} \not\equiv \frac{a_4'}{a_4-1} - \frac{a_5'}{a_5-1}$. Then, it follows from (3.3) that the zeros of $f-1$ can only occur at the zeros or $1-$points or the poles of a_j, $(j=4,5)$, or the zeros of $\frac{a_4'}{a_4} - \frac{a_5'}{a_5}$. Similary, the zeros of f can only occur at the zeros or $1-$points or the poles of a_j, $(j=4,5)$, or the zeros of $\frac{a_4'}{a_4-1} - \frac{a_5'}{a_5-1}$. Furthermore, from (3.3), we can also see that the poles of f can only occur at the zeros or $1-$points or the poles of a_j, $(j=4,5)$, or the zeros of $\frac{a_4'}{a_4} - \frac{a_5'}{a_5} - \frac{a_4'}{a_4-1} + \frac{a_5'}{a_5-1}$. Therefore, we get

$$\overline{N}(r,f) + \overline{N}\big(r,\frac{1}{f}\big) + \overline{N}\big(r,\frac{1}{f-1}\big) = S(r,f). \tag{3.4}$$

By (3.4) and applying the Second Main Theorem for f and $0,1,\infty$, we have

$$T(r,f) \le \overline{N}(r,f) + \overline{N}\big(r,\frac{1}{f}\big) + \overline{N}\big(r,\frac{1}{f-1}\big) - \log r + O(1) = S(r,f).$$

This is a contradiction again.

Thus, we must have $H \not\equiv 0$.

Given a real number $0 < r < \infty$. Let

$$\delta(r) = \min\{1, |a_4|_r, |a_5|_r, |a_4-1|_r, |a_5-1|_r, |a_4-a_5|_r\}.$$

Then, we have

$$\begin{aligned}
\log^+ \frac{1}{\delta(r)} &\le \log^+ \max\{1, \frac{1}{|a_4|_r}, \frac{1}{|a_5|_r}, \frac{1}{|a_4-1|_r}, \frac{1}{|a_5-1|_r}, \frac{1}{|a_4-a_5|_r}\} \\
&\le \log^+ \Big(1 + \frac{1}{|a_4|_r} + \frac{1}{|a_5|_r} + \frac{1}{|a_4-1|_r} + \frac{1}{|a_5-1|_r} + \frac{1}{|a_4-a_5|_r}\Big) \\
&\le \log^+ \frac{1}{|a_4|_r} + \log^+ \frac{1}{|a_5|_r} + \log^+ \frac{1}{|a_4-1|_r} + \log^+ \frac{1}{|a_5-1|_r} \\
&\quad + \log^+ \frac{1}{|a_4-a_5|_r} + \log 6 \\
&= m\big(r,\frac{1}{a_4}\big) + m\big(r,\frac{1}{a_5}\big) + m\big(r,\frac{1}{a_4-1}\big) + m\big(r,\frac{1}{a_5-1}\big) \\
&\quad + m\big(r,\frac{1}{a_4-a_5}\big) + \log 6 \\
&= S(r,f).
\end{aligned}$$

We first consider the case when

$$|f-a_j|_r > \frac{1}{2}\delta(r),$$

for all $2 \le j \le 5$. In this case,

$$m\big(r, \frac{1}{f}\big) + m\big(r, \frac{1}{f-1}\big) + m\big(r, \frac{1}{f-a_4}\big) + m\big(r, \frac{1}{f-a_5}\big) < 5\log^+ \frac{1}{\delta(r)} + O(1) \\ = S(r,f). \tag{3.5}$$

Now let i, $2 \le i \le 5$, be the index among $\{2,3,4,5\}$ such that

$$|f-a_i|_r \le \frac{1}{2}\delta(r).$$

Then for any $j \ne i$, $2 \le j \le 5$, we have

$$\delta(r) \le |a_i - a_j|_r \le |f-a_i|_r + |f-a_j|_r \le \frac{1}{2}\delta(r) + |f-a_j|_r,$$

so

$$|f-a_j|_r \ge \frac{1}{2}\delta(r).$$

Therefore, for $j \ne i$, we have

$$\sum_{\substack{j=2\\ j\ne i}}^{5} m\big(r, \frac{1}{f-a_j}\big) = \sum_{\substack{j=2\\ j\ne i}}^{5} \log^+ \frac{1}{|f-a_j|_r} \le 3\log^+ \frac{1}{\delta(r)}.$$

Combining (3.5) and the above inequality, we get

$$\sum_{\substack{j=2\\ j\ne i}}^{5} m\big(r, \frac{1}{f-a_j}\big) = S(r,f). \tag{3.6}$$

On the other hand, for $2 \le i \le 5$, we can write

$$\begin{aligned} &ff' = (f-a_i)(f'-a_i') + a_i'(f-a_i) + a_i(f'-a_i') + a_i a_i',\\ &f' = (f'-a_i') + a_i',\\ &f(f-1) = f^2 - f = (f-a_i)^2 + (2a_i - 1)(f-a_i) + a_i^2 - a_i. \end{aligned}$$

By substituting the above equalities into (3.1) and using the determinant's properties, we get

$$H = \begin{vmatrix} g_i & f'-a_i' & h_i \\ a_4 a_4' & a_4' & a_4(a_4-1) \\ a_5 a_5' & a_5' & a_5(a_5-1) \end{vmatrix}, \tag{3.7}$$

where

$$g_i = (f-a_i)(f'-a_i') + a_i'(f-a_i) + a_i(f'-a_i'),$$
$$h_i = (f-a_i)^2 + (2a_i-1)(f-a_i)$$

for $2 \le i \le 5$ (note that $a_2 = 0, a_3 = 1$). By the definition of $\delta(r)$, we have $\delta(r) \le 1 + |a_i|_r$. Hence,

$$\log^+ \delta(r) \le \log^+(1 + |a_i|_r) \le \log^+ |a_i|_r + \log 2 = m(r, a_i) + \log 2 = S(r,f).$$

Thus, it follows from (3.7) and the Logarithmic Derivative Lemma that

$$\begin{aligned}\log^+\Big|\frac{H}{f-a_i}\Big|_r &\le \log^+\Big|\frac{f'-a_i'}{f-a_i}\Big|_r + \log^+|f-a_i|_r\\ &\quad + O(\log^+|a_i|_r + \log^+|a_i'|_r + \log^+|a_4|_r + \log^+|a_4'|_r\\ &\quad + \log^+|a_5|_r + \log^+|a_5'|_r)\\ &\le m\big(\frac{f'-a_i'}{f-a_i}\big) + \log^+\delta(r) + S(r,f)\\ &= S(r,f).\end{aligned}$$

Hence, we get

$$\begin{aligned}m\big(r,\frac{1}{f-a_i}\big) = \log^+\frac{1}{|f-a_i|_r} &\le \log^+\Big|\frac{H}{f-a_i}\Big|_r + \log^+\Big|\frac{1}{H}\Big|_r\\ &\le m\big(r,\frac{1}{H}\big) + S(r,f).\end{aligned} \tag{3.8}$$

It follows from (3.5), (3.7) and (3.8) that in any case, we have

$$m\big(r,\frac{1}{f}\big) + m\big(r,\frac{1}{f-1}\big) + m\big(r,\frac{1}{f-a_4}\big) + m\big(r,\frac{1}{f-a_5}\big) \le m\big(r,\frac{1}{H}\big) + S(r,f). \tag{3.9}$$

Hence, by the First Main Theorem, we get

$$\begin{aligned}4T(r,f) \le\ & N\big(r,\frac{1}{f}\big) + N\big(r,\frac{1}{f-1}\big) + N\big(r,\frac{1}{f-a_4}\big) + N\big(r,\frac{1}{f-a_5}\big)\\ & + T(r,H) - N\big(r,\frac{1}{H}\big) + S(r,f).\end{aligned} \tag{3.10}$$

On the other hand, suppose that z_0 be a zero of $f-a_i$, $(2 \le i \le 5)$ of order $s > 1$ which is not a pole of a_4 or a_5. Then, it follows from (3.7) that z_0 is also a zero of H of order at least $s-1$. Hence, from (3.10) and the above observations, we get

$$\begin{aligned}4T(r,f) \le\ & \overline{N}\big(r,\frac{1}{f}\big) + \overline{N}\big(r,\frac{1}{f-1}\big) + \overline{N}\big(r,\frac{1}{f-a_4}\big) + \overline{N}\big(r,\frac{1}{f-a_5}\big)\\ & + T(r,H) + S(r,f).\end{aligned} \tag{3.11}$$

From (3.2), we have

$$\begin{aligned}m(r,H) &\le 2m(r,f) + S(r,f),\\ N(r,H) &\le 2N(r,f) + \overline{N}(r,f) + S(r,f).\end{aligned}$$

Hence, we get

$$T(r,H) \le 2T(r,f) + \overline{N}(r,f) + S(r,f). \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$2T(r,f) \le \overline{N}\big(r,\frac{1}{f}\big) + \overline{N}\big(r,\frac{1}{f-1}\big) + \overline{N}\big(r,\frac{1}{f-a_4}\big) + \overline{N}\big(r,\frac{1}{f-a_5}\big) + S(r,f).$$

This completes the proof of Lemma 2. □

Proof of Theorem 1. By Lemma 2, for every subset $\{i_1, \ldots, i_5\}$ of $\{1, \ldots, q\}$ such that $1 \le i_1 < \cdots < i_5 \le q$, we have

$$2T(r,f) \le \sum_{s=1}^{5} \overline{N}\big(r, \frac{1}{f-a_{i_s}}\big) + S(r,f). \tag{3.13}$$

It is easily seen that the number of such inequalities is C_q^5. Summing up of (3.13) over all subsets $\{i_1, \ldots, i_5\}$ of $\{1, \ldots, q\}$ as above, we get

$$\begin{aligned}2\mathrm{C}_q^5 T(r,f) \le \sum_{\substack{\{i_1,\ldots,i_5\}\subset\{1,\ldots,q\}\\ 1\le i_1<\cdots<i_5\le q}} &\Big(\overline{N}\big(r, \frac{1}{f-a_{i_1}}\big) + \overline{N}\big(r, \frac{1}{f-a_{i_2}}\big) + \overline{N}\big(r, \frac{1}{f-a_{i_3}}\big) \\ &+ \overline{N}\big(r, \frac{1}{f-a_{i_4}}\big) + \overline{N}\big(r, \frac{1}{f-a_{i_5}}\big)\Big) + S(r,f). \end{aligned}\tag{3.14}$$

In (3.14), for each index i_k, the number of terms $\overline{N}\big(r, \frac{1}{f-a_{i_k}}\big)$ is C_{q-1}^4. Hence, from (3.14), we get

$$2\mathrm{C}_q^5 T(r,f) \le \mathrm{C}_{q-1}^4 \sum_{i=1}^{q} \overline{N}\big(r, \frac{1}{f-a_i}\big) + S(r,f).$$

It follows that

$$\frac{2q}{5} T(r,f) \le \sum_{i=1}^{q} \overline{N}\big(r, \frac{1}{f-a_i}\big) + S(r,f).$$

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

To prove Theorem 2, we need to prove the following lemma.

Lemma 3. *Let f and g be nonconstant meromorphic functions on $\mathbf{K}$ and $a_1, \ldots, a_q$ be q distinct small functions with respect to f and g. Let $k_1, \ldots, k_q$ be q positive integers or $+\infty$. Suppose that*

$$\overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \quad (j = 1, \ldots, q).$$

If $f \not\equiv g$, then for every subset $\{i_1, i_2, i_3, i_4\}$ of $\{1, \ldots, q\}$, we have

$$\begin{aligned}\sum_{j\in\{1,\ldots,q\}\setminus\{i_1,\ldots,i_4\}} \overline{N}_{k_j)}\big(r, \frac{1}{f-a_j}\big) \le \sum_{s=1}^{4} &\Big(\overline{N}_{(k_{i_s}+1}\big(r, \frac{1}{f-a_{i_s}}\big) + \overline{N}_{(k_{i_s}+1}\big(r, \frac{1}{g-a_{i_s}}\big)\Big) \\ &+ S(r,f) + S(r,g).\end{aligned}$$

Proof. Without losing generality, we just need to prove that

$$\sum_{i=5}^{q}\overline{N}_{k_i)}\big(r,\frac{1}{f-a_i}\big)\leq\sum_{j=1}^{4}\Big(\overline{N}_{(k_j+1}\big(r,\frac{1}{f-a_j}\big)+\overline{N}_{(k_j+1}\big(r,\frac{1}{g-a_j}\big)\Big)$$
$$+S(r,f)+S(r,g). \tag{4.1}$$

If $\sum_{i=5}^{q}\overline{N}_{k_i)}\big(r,\frac{1}{f-a_i}\big)=S(r,f)+S(r,g)$, then (4.1) obviously holds. Thus, in the following we may assume that

$$\sum_{i=5}^{q}\overline{N}_{k_i)}\big(r,\frac{1}{f-a_i}\big)\neq S(r,f)+S(r,g). \tag{4.2}$$

By using the transformation

$$L(w)=\frac{w-a_1}{w-a_2}\cdot\frac{a_3-a_2}{a_3-a_1}$$

and considering two functions $F=L(f),G=L(g)$ if necessary, we may assume that $a_1=0,a_2=\infty,a_3=1$ and $a_4,\ldots,a_q$ are distinct small functions with respect to f and g, $a_i\not\equiv 0,1,\infty$ for $i=4,\ldots,q$.

Set

$$M:=\frac{f'(a_4'g-a_4g')(f-g)}{f(f-1)g(g-a_4)}-\frac{g'(a_4'f-a_4f')(f-g)}{g(g-1)f(f-a_4)}. \tag{4.3}$$

Then we have

$$M=\frac{(f-g)Q}{f(f-1)(f-a_4)g(g-1)(g-a_4)}, \tag{4.4}$$

where

$$\begin{aligned}Q&=f'(a_4'g-a_4g')(f-a_4)(g-1)-g'(a_4'f-a_4f')(g-a_4)(f-1)\\&=a_4'ff'g^2-a_4'ff'g-a_4(a_4-1)ff'g'-a_4a_4'f'g^2+a_4a_4'f'g-a_4'f^2gg'\\&\quad+a_4'fgg'+a_4(a_4-1)f'gg'+a_4a_4'f^2g'-a_4a_4'fg'.\end{aligned} \tag{4.5}$$

Suppose that $M\equiv 0$. Then from (4.3) we have

$$\frac{f'(a_4'g-a_4g')(f-g)}{f(f-1)g(g-a_4)}\equiv\frac{g'(a_4'f-a_4f')(f-g)}{g(g-1)f(f-a_4)}. \tag{4.6}$$

If a_4 is a constant then $f\equiv g$, which contradicts our assumption. Thus, a_4 is not a constant. It follows from (4.6) that

$$\frac{(f-1)(g-a_4)}{(g-1)(f-a_4)}-1\equiv\frac{f'(a_4'g-a_4g')}{g'(a_4'f-a_4f')}-1,$$

which implies

$$\frac{(f-g)(1-a_4)}{(g-1)(f-a_4)}\equiv\frac{a_4'[(f'-g')g-(f-g)g']}{g'(a_4'f-a_4f')}.$$

This yield that

$$\frac{f'-g'}{f-g} \equiv \frac{(1-a_4)g'(a_4'f-a_4f')}{a_4'g(g-1)(f-a_4)} + \frac{g'}{g}. \tag{4.7}$$

It follows from (4.2) that there exists a point z_0 that is a common zero of $f-a_j$ and $g-a_j$, and it is not neither a zero nor a pole of $a_4, a_4', a_j, a_j-1, a_j-a_4$, for any $5 \le j \le q$. Then, z_0 must be a pole of the left hand side of (4.7), and not be a pole of the right hand side of (4.7). This is a contradiction. Thus $M \not\equiv 0$.

Suppose that z_1 is a common zero of $f-a_j$ and $g-a_j$ and it is not neither a zero nor a pole of a_4, a_j, a_j-1, a_j-a_4 for $5 \le j \le q$. Then, z_1 is a zero of $f-g$ and is not a pole of

$$\frac{Q}{f(f-1)(f-a_4)g(g-1)(g-a_4)},$$

which implies that z_1 is a zero of M. Since $\overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g)$ for any $j = 1, \ldots, q$, we have

$$\begin{aligned}\sum_{i=5}^{q} \overline{N}_{k_i)}\big(r, \frac{1}{g-a_i}\big) &= \sum_{i=5}^{q} \overline{N}_{k_i)}\big(r, \frac{1}{f-a_i}\big) \\ &\le N\big(r, \frac{1}{M}\big) + S(r,f) + S(r,g) \\ &\le m(r,M) + N(r,M) + S(r,f) + S(r,g). \end{aligned} \tag{4.8}$$

We will estimate $m(r,M)$. From (4.3) we get

$$\begin{aligned} M &= \frac{f'}{f-1}\frac{a_4'g-a_4g'}{g(g-a_4)} - \Big(\frac{f'}{f-1} - \frac{f'}{f}\Big)\frac{a_4'g-a_4g'}{g-a_4} \\ &\quad + \frac{g'}{g-1}\frac{a_4'f-a_4f'}{f(f-a_4)} - \Big(\frac{g'}{g-1} - \frac{g'}{g}\Big)\frac{a_4'f-a_4f'}{f-a_4} \\ &= \frac{f'}{f-1}\Big(\frac{g'}{g} - \frac{g'-a_4'}{g-a_4}\Big) - \Big(\frac{f'}{f-1} - \frac{f'}{f}\Big)\Big(a_4' - a_4\frac{g'-a_4'}{g-a_4}\Big) \\ &\quad \frac{g'}{g-1}\Big(\frac{f'}{f} - \frac{f'-a_4'}{f-a_4}\Big) - \Big(\frac{g'}{g-1} - \frac{g'}{g}\Big)\Big(a_4' - a_4\frac{f'-a_4'}{f-a_4}\Big). \end{aligned} \tag{4.9}$$

Combining (4.9) and lemma of the logarithmic derivative, we obtain

$$m(r,M) = S(r,f) + S(r,g). \tag{4.10}$$

Next, we estimate the counting function $N(r,M)$. It follows from (4.3) that the poles of M can only occur at the zeros of $f-a_i$, and $g-a_i$ with $i = 1,2,3,4$ and the poles of a_4', a_4. Remind that $a_1 = 0, a_2 = \infty, a_3 = 1$ and the zeros of $f - \infty$ mean the poles of $\frac{1}{f}$. We consider all the following possibilities.

Case 1: z is a pole of a_4' or a_4. We have $\mathrm{ord}^{\infty}_{a_4'}(z) = \mathrm{ord}^{\infty}_{a_4}(z)+1$, where $\mathrm{ord}^{\infty}_{a}(z)$ denotes the order of a pole of a function a at z. From the formula of M in (4.3), we obtain

$$\mathrm{ord}^{\infty}_{M}(z) \le \mathrm{ord}^{\infty}_{a_4}(z) + 2 \le 3\mathrm{ord}^{\infty}_{a_4}(z).$$

Case 2: For each $i = 1, 3$ or 4, assume that z is a common zero of $f - a_i$ and $g - a_i$, but it is not a pole of a_4. Then z is a zero of $f - g$ of order at least $\min\{\mathrm{ord}^0_{f-a_i}(z), \mathrm{ord}^0_{g-a_i}(z)\}$. From (4.5), z is a zero of Q of order at least $\mathrm{ord}^0_{f-a_i}(z) + \mathrm{ord}^0_{g-a_i}(z) - 1$. From (4.4) we have

$$\begin{aligned}\mathrm{ord}^0_M(z) &\ge \min\{\mathrm{ord}^0_{f-a_i}(z), \mathrm{ord}^0_{g-a_i}(z)\} + \mathrm{ord}^0_{f-a_i}(z) + \mathrm{ord}^0_{g-a_i}(z) - 1\\ &\quad - (\mathrm{ord}^0_{f-a_i}(z) + \mathrm{ord}^0_{g-a_i}(z))\\ &\ge \min\{\mathrm{ord}^0_{f-a_i}(z), \mathrm{ord}^0_{g-a_i}(z)\} - 1\\ &\ge 0.\end{aligned}$$

Hence, z is not a pole of M.

Case 3: z is a common pole of f and g but it is not a pole of a_4. Then, from (4.5), z is a pole of Q of order at most $2\mathrm{ord}^{\infty}_f(z) + 2\mathrm{ord}^{\infty}_g(z) + 1$. Then, z is a pole of $f - g$ of order $\max\{\mathrm{ord}^{\infty}_f(z), \mathrm{ord}^{\infty}_g(z)\}$. Hence, from (4.4) we see that z is a pole of the numerator of M of order at most $2\mathrm{ord}^{\infty}_f(z) + 2\mathrm{ord}^{\infty}_g(z) + 1 + \max\{\mathrm{ord}^{\infty}_f(z), \mathrm{ord}^{\infty}_g(z)\}$ and it is a pole of the denominator of M of order $3\mathrm{ord}^{\infty}_f(z) + 3\mathrm{ord}^{\infty}_g(z)$. Since

$$\begin{aligned}&2\mathrm{ord}^{\infty}_f(z) + 2\mathrm{ord}^{\infty}_g(z) + 1 + \max\{\mathrm{ord}^{\infty}_f(z), \mathrm{ord}^{\infty}_g(z)\} - (3\mathrm{ord}^{\infty}_f(z) + 3\mathrm{ord}^{\infty}_g(z))\\ &= 1 + \max\{\mathrm{ord}^{\infty}_f(z), \mathrm{ord}^{\infty}_g(z)\} - (\mathrm{ord}^{\infty}_f(z) + \mathrm{ord}^{\infty}_g(z))\\ &\le 0,\end{aligned}$$

we see that z is not a pole of M.

Case 4: Assume that z is a zero only of either $f - a_i$ or $g - a_i$, $i = 1, 2, 3$ or 4 and z is not a pole of a_4. By the hypothesis

$$\overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \quad (j = 1, \ldots, q),$$

all zeros order at most k of $f - a_i$ will be zeros of $g - a_i$. Hence, in this case we may assume z to be either a zero of $f - a_i$ or a zero of $g - a_i$ of order at least $k + 1$. The formula (4.9) can be written in terms

$$M = \sum_{i,j=1}^{4} a\Big(\frac{(f - a_i)'}{f - a_i}.\frac{(g - a_j)'}{g - a_j}\Big)$$

when a is either a_4 or a_4'. For each term of the form $f_i := \frac{(f-a_i)'}{f-a_i}$ we have

$$\mathrm{ord}^{\infty}_{f_i}(z) = \min\{1, \mathrm{ord}^0_{f-a_i}(z)\} := \overline{\mathrm{ord}}^0_{f-a_i}(z),$$

which implies

$$\mathrm{ord}^{\infty}_{M}(z) \le \max_{i,j=1,2,3,4} \{\overline{\mathrm{ord}}^{0}_{f-a_i}(z) + \overline{\mathrm{ord}}^{0}_{g-a_j}(z)\},$$

So, from the assumption and the above observations, we get

$$N(r,M) \le \sum_{j=1}^{4} \Big(\overline{N}_{(k_j+1}\big(r,\frac{1}{f-a_j}\big) + \overline{N}_{(k_j+1}\big(r,\frac{1}{g-a_j}\big)\Big) + S(r,f) + S(r,g). \quad (4.11)$$

By combining (4.8), (4.10) and (4.11), we get (4.1). The proof of Lemma 3 is completed. □

Proof of Theorem 2. Suppose that $f \not\equiv g$. By Lemma 3, for every subset $\{i_1,\dots,i_4\}$ of $\{1,\dots,q\}$, we have

$$\begin{aligned}&\sum_{j=1}^{q} \overline{N}_{k_j)}\big(r,\frac{1}{f-a_j}\big) - \sum_{s=1}^{4} \overline{N}_{k_{i_s})}\big(r,\frac{1}{f-a_{i_s}}\big)\\ &\le \sum_{s=1}^{4} \Big(\overline{N}_{(k_{i_s}+1}\big(r,\frac{1}{f-a_{i_s}}\big) + \overline{N}_{(k_{i_s}+1}\big(r,\frac{1}{g-a_{i_s}}\big)\Big) + S(r,f) + S(r,g). \quad (4.12)\end{aligned}$$

Taking summing up of (4.12) over all subsets $\{i_1,\dots,i_4\}$ of $\{1,\dots,q\}$, we get

$$\begin{aligned}&\mathrm{C}^4_q \sum_{j=1}^{q} \overline{N}_{k_j)}\big(r,\frac{1}{f-a_j}\big) - \sum_{\substack{\{i_1,\dots,i_4\}\subset\{1,\dots,q\}\\ 1\le i_1<\dots<i_4\le q}} \sum_{s=1}^{4} \overline{N}_{k_{i_s})}\big(r,\frac{1}{f-a_{i_s}}\big)\\ &\le \sum_{\substack{\{i_1,\dots,i_4\}\subset\{1,\dots,q\}\\ 1\le i_1<\dots<i_4\le q}} \sum_{s=1}^{4} \Big(\overline{N}_{(k_{i_s}+1}\big(r,\frac{1}{f-a_{i_s}}\big) + \overline{N}_{(k_{i_s}+1}\big(r,\frac{1}{g-a_{i_s}}\big)\Big)\\ &\quad + S(r,f) + S(r,g).\end{aligned}$$

In the above inequality, for each index i_s, the number of terms $\overline{N}\big(r,\frac{1}{f-a_{i_s}}\big)$ is C^3_{q-1}. Hence, it follows that

$$\begin{aligned}(q-4)\sum_{j=1}^{q} \overline{N}_{k_j)}\big(r,\frac{1}{f-a_j}\big) \le\, &4\sum_{j=1}^{q} \Big(\overline{N}_{(k_j+1}\big(r,\frac{1}{f-a_j}\big) + \overline{N}_{(k_j+1}\big(r,\frac{1}{g-a_j}\big)\Big)\\ &+ S(r,f) + S(r,g).\end{aligned}$$

By an argument similar, we have

$$\begin{aligned}(q-4)\sum_{j=1}^{q} \overline{N}_{k_j)}\big(r,\frac{1}{g-a_j}\big) \le\, &4\sum_{j=1}^{q} \Big(\overline{N}_{(k_j+1}\big(r,\frac{1}{f-a_j}\big) + \overline{N}_{(k_j+1}\big(r,\frac{1}{g-a_j}\big)\Big)\\ &+ S(r,f) + S(r,g).\end{aligned}$$

Hence, we get

$$(q-4)\sum_{j=1}^{q}\Big(\overline{N}\big(r,\frac{1}{f-a_j}\big)+\overline{N}\big(r,\frac{1}{g-a_j}\big)\Big)$$
$$\leq (q+4)\sum_{j=1}^{q}\Big(\overline{N}_{(k_j+1}\big(r,\frac{1}{f-a_j}\big)+\overline{N}_{(k_j+1}\big(r,\frac{1}{g-a_j}\big)\Big)+S(r,f)+S(r,g). \tag{4.13}$$

By Theorem 1, we have

$$\frac{2q}{5}(T(r,f)+T(r,g))\leq \sum_{j=1}^{q}\Big(\overline{N}\big(r,\frac{1}{f-a_j}\big)+\overline{N}\big(r,\frac{1}{g-a_j}\big)\Big)$$
$$+S(r,f)+S(r,g). \tag{4.14}$$

Combining (4.13) and (4.14), we get

$$\frac{2q(q-4)}{5}(T(r,f)+T(r,g))\leq (q+4)\sum_{j=1}^{q}\Big(\overline{N}_{(k_j+1}\big(r,\frac{1}{f-a_j}\big)+\overline{N}_{(k_j+1}\big(r,\frac{1}{g-a_j}\big)\Big)$$
$$+S(r,f)+S(r,g). \tag{4.15}$$

On the other hand, we have

$$\sum_{j=1}^{q}\Big(\overline{N}_{(k_j+1}\big(r,\frac{1}{f-a_j}\big)+\overline{N}_{(k_j+1}\big(r,\frac{1}{g-a_j}\big)\Big)$$
$$\leq \sum_{j=1}^{q}\frac{1}{k_j+1}\Big(N_{(k_j+1}\big(r,\frac{1}{f-a_j}\big)+N_{(k_j+1}\big(r,\frac{1}{g-a_j}\big)\Big)$$
$$\leq \sum_{j=1}^{q}\frac{1}{k_j+1}(T(r,f)+T(r,g)). \tag{4.16}$$

The inequalities (4.15) and (4.16) imply

$$\Big(\frac{2q(q-4)}{5(q+4)}-\sum_{j=1}^{q}\frac{1}{k_j+1}\Big)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g).$$

Hence, when $\sum_{j=1}^{q}\frac{1}{k_j+1}<\frac{2q(q-4)}{5(q+4)}$, we have a contradiction. Thus, $f\equiv g$. The proof of Theorem 2 is completed. □

5. Proof of Theorem 3

First, we prove the following lemma.

Lemma 4. *Let f be a nonconstant meromorphic function on* $\mathbf{K}$. *Let $a_1, \ldots, a_q$ be q distinct small functions with respect to f. Let k be a positive integer or $+\infty$. Then*

$$\sum_{i=1}^{q} \overline{N}_{(k+1}\big(r, \frac{1}{f-a_i}\big) \le \frac{3q}{5k} T(r,f) + S(r,f).$$

Proof. If $k = +\infty$, then $\overline{N}_{(k+1}\big(r, \frac{1}{f-a_i}\big) = 0$. Therefore, the lemma is always true. From now, we may assume that k is finite.

For each $1 \le i \le q$, we have

$$\begin{aligned} k\overline{N}_{(k+1}\big(r, \frac{1}{f-a_i}\big) + \overline{N}\big(r, \frac{1}{f-a_i}\big) &= (k+1)\overline{N}_{(k+1}\big(r, \frac{1}{f-a_i}\big) + \overline{N}_{k)}\big(r, \frac{1}{f-a_i}\big) \\ &\le N_{(k+1}\big(r, \frac{1}{f-a_i}\big) + N_{k)}\big(r, \frac{1}{f-a_i}\big) \\ &= N\big(r, \frac{1}{f-a_i}\big) \le T(r,f) + S(r,f). \end{aligned}$$

Hence, we get

$$k\overline{N}_{(k+1}\big(r, \frac{1}{f-a_i}\big) \le T(r,f) - \overline{N}\big(r, \frac{1}{f-a_i}\big) + S(r,f).$$

Combining this and Theorem 1, we obtain

$$k\sum_{i=1}^{q} \overline{N}_{(k+1}\big(r, \frac{1}{f-a_i}\big) \le \frac{3q}{5} T(r,f) + S(r,f).$$

This completes the proof of Lemma. □

Proof of Theorem 3. Suppose that $f \not\equiv g$. By arguments similar to the inequality (4.15) in the proof of Theorem 2, we get

$$\begin{aligned} \frac{2q(q-4)}{5}(T(r,f)+T(r,g)) \le &(q+4)\sum_{j=1}^{q}\Big(\overline{N}_{(k+1}\big(r, \frac{1}{f-a_j}\big) + \overline{N}_{(k+1}\big(r, \frac{1}{g-a_j}\big)\Big) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{5.1}$$

By Lemma 4, we have

$$\begin{aligned} \sum_{j=1}^{q}\Big(\overline{N}_{(k+1}\big(r, \frac{1}{f-a_j}\big) + \overline{N}_{(k+1}\big(r, \frac{1}{g-a_j}\big)\Big) \le &\frac{3q}{5k}(T(r,f)+T(r,g)) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we obtain

$$\frac{q}{5}\Big(2(q-4) - \frac{3(q+4)}{k}\Big)(T(r,f)+T(r,g)) \le S(r,f) + S(r,g).$$

Thus, when $k > \frac{3(q+4)}{2(q-4)}$, we have a contradiction. Hence, $f \equiv g$. □

Proof of Corollary 1. Suppose that $f \not\equiv g$. Applying Lemma 3 with $k = \infty$, we show that for every subset $\{i_1, \dots, i_4\}$ of $\{1, \dots, 5\}$, we have

$$\overline{N}\big(r, \frac{1}{f - a_j}\big) = S(r, f) + S(r, g)$$

for $j \in \{1, \dots, 5\} \setminus \{i_1, \dots, i_4\}$. Hence, we obtain

$$\sum_{j=1}^{5} \overline{N}\big(r, \frac{1}{f - a_j}\big) - \sum_{s=1}^{4} \overline{N}\big(r, \frac{1}{f - a_{i_s}}\big) = S(r, f) + S(r, g) \tag{5.3}$$

Summing up of (5.3) over all subsets $\{i_1, \dots, i_4\}$ of $\{1, \dots, 5\}$, we get

$$\mathrm{C}_5^4 \sum_{j=1}^{q} \overline{N}\big(r, \frac{1}{f - a_j}\big) - \sum_{\substack{\{i_1, \dots, i_4\} \subset \{1, \dots, 5\} \\ 1 \le i_1 < \dots < i_4 \le 5}} \sum_{s=1}^{4} \overline{N}\big(r, \frac{1}{f - a_{i_s}}\big) = S(r, f) + S(r, g).$$

In the above equality, for each index i_s, the number of terms $\overline{N}\big(r, \frac{1}{f - a_{i_s}}\big)$ is C_4^3. Hence, it follows that

$$\sum_{j=1}^{5} \overline{N}\big(r, \frac{1}{f - a_j}\big) = S(r, f) + S(r, g).$$

By an argument similar, we have

$$\sum_{j=1}^{5} \overline{N}\big(r, \frac{1}{g - a_j}\big) = S(r, f) + S(r, g).$$

Hence, we get

$$\sum_{j=1}^{5} \big(\overline{N}\big(r, \frac{1}{f - a_j}\big) + \overline{N}\big(r, \frac{1}{g - a_j}\big)\big) = S(r, f) + S(r, g). \tag{5.4}$$

By Theorem 1, we have

$$\begin{aligned} 2(T(r, f) + T(r, g)) \le & \sum_{j=1}^{5} \big(\overline{N}\big(r, \frac{1}{f - a_j}\big) + \overline{N}\big(r, \frac{1}{g - a_j}\big)\big) \\ & + S(r, f) + S(r, g). \end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5), we get

$$2(T(r, f) + T(r, g)) \le S(r, f) + S(r, g).$$

Hence, we have a contradiction. Thus, $f \equiv g$. The proof of Corollary 1 is completed. $\square$

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