Subdiffusive concentration of the graph distance in Bernoulli percolation

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Abstract

Considering supercritical Bernoulli percolation on $\mathbb{Z}^d$. Garet and Marchand [GM09] proved a diffusive concentration for the graph distance. In this paper, we sharpen this result by establishing the subdiffusive concentration inequality in sublinear scale. As consequence, we revisit a recent result by Dembin [Dem22] on the sublinear variance of the distance. The main results also extend the study of the concentration of passage time in first passage percolation by Damron, Hanson, and Sosoe [DHS14] without moment conditions on the edge-weight distribution.

1 Introduction

1.1 Model and main result

Bernoulli percolation is a simple but well-known probabilistic model for porous material introduced by Broadbent and Hammersley [BH57]. Let $d \geq 2$ and $\mathcal{E}(\mathbb{Z}^d)$ be the set of the edges $e = \langle x, y \rangle$ of endpoints $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{Z}_d$ such that $\|x - y\|_1 := \sum_{i=1}^d |x_i - y_i| = 1$. Given the parameter $p \in (0, 1)$, we let each edge $e \in \mathcal{E}(\mathbb{Z}^d)$ be *open* with probability p and *closed* otherwise, independently of the state of other edges. The phase transition of model has been well-known since 1960s. There exists a critical parameter $p_c \in (0, 1)$, such that there is almost surely a unique infinite open cluster $\mathcal{C}_\infty$ if $p > p_c$, whereas all open clusters are finite if $p < p_c$, see [Gri89]. Let $x \in \mathbb{Z}^d$, we denote by x^* the closest point to x in $\mathcal{C}_\infty$ (in $\|.\|_\infty$ distance), called regularized point of x. We define the graph distance as

$$\forall x, y \in \mathbb{Z}^d,\ D^*(x, y) = D(x^*, y^*) = \inf_{\gamma : x^* \to y^*} \#\gamma,$$

where infimum is taken over the set of lattice open paths

$$\gamma = (u_0 = x^*, u_1, \dots, y^* = u_N), \quad \|u_{i+1} - u_i\|_1 = 1.$$

Notice that if these points x, y are not in $\mathcal{C}_\infty$ then $D(x, y)$ might be ∞. Hence, Garet and Marchand [GM09] introduced the definition of graph distance using regularized points, which guarantees that $D^*(x, y) = D(x^*, y^*) < \infty$ almost surely for all x and

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y. Let $\mathbf{e}_1 = (1, 0, \ldots, 0)$ be the first standard basis vector. We aim to study the graph distance from the original 0 to ne_1:

$$D^*_n = D^*(0, n\mathbf{e}_1).$$

First passage percolation. We recall a generalization percolation, known as the first passage percolation. For each edge $e \in \mathcal{E}(\mathbb{Z}^d)$, we assign a random weight t_e taking values in $[0, \infty]$ such that the family $(t_e)_{e \in \mathcal{E}(\mathbb{Z}^d)}$ is independent and identically distributed with distribution ζ. We interpret t_e as the time needed to cross the edge e. Similarly, the quantity we are interested in is the passage time:

$$T(x, y) = \inf_{\gamma: x \to y} \sum_{e \in \gamma} t_e,$$

where infimum is taken over the set of lattice paths. We will assume throughout that

$$\zeta([0, \infty)) > p_c, \quad \zeta(\{0\}) < p_c, \tag{1.1}$$

where p_c is the critical probability for Bernoulli percolation on $\mathbb{Z}^d$. Under the condition (1.1), the supercritical Bernoulli Percolation is indeed a particular case of first passage percolation with the distribution

$$\zeta = \zeta_p = p\delta_1 + (1 - p)\delta_\infty, \quad p > p_c. \tag{1.2}$$

More precisely, each edge e with weight $t_e = 1$ corresponds to an open edge.

Time constant. The first order of growth of D^*_n was described by Cerf and Théret [CT16]: under the assumption (1.1), there exists a constant $\mu(e_1) \in [0, \infty)$ such that,

$$\lim_{n \to \infty} \frac{D^*_n}{n} = \mu(\mathbf{e}_1) \quad \text{a.e and in } L^1.$$

The function μ is the so-called time constant. Moreover, we also obtain lower tail large deviations by Kesten [Kes86]: for any $\varepsilon > 0$ small enough,

$$\lim_{n \to \infty} \frac{\log \mathbb{P}[T(0, n\mathbf{e}_1) \leq (\mu(\mathbf{e}_1) - \varepsilon)n]}{n} = r(\varepsilon) < 0, \tag{1.3}$$

In [BSG21], Basu, Sly and Ganguly have just shown that for any bounded distribution $\zeta \in [0, b]$ with continuity densities and $d = 2$,

$$\forall \varepsilon \in (0, b - \mu(\mathbf{e}_1)), \quad \lim_{n \to \infty} \frac{\log \mathbb{P}[T(0, n\mathbf{e}_1) \geq (\mu(\mathbf{e}_1) + \varepsilon)n]}{n^2} = r(\varepsilon, \zeta) < 0, \tag{1.4}$$

and some further results for unbounded distribution was done by Cosco and Nakajima [CN21] (the speed of large deviation and rate function now depend on the tail assumption of t_e). In the Bernoulli percolation case, Garet and Marchand [GM07] showed that:

$$\forall \varepsilon > 0 \quad \limsup_{n \to \infty} \frac{\log \mathbb{P}\left[\frac{D^*_n}{n\mu(\mathbf{e}_1)} \notin (1 - \varepsilon, 1 + \varepsilon)\right]}{n} < 0,$$

Fluctuation and concentration. It is expected in the physics literature that the variance $T(0, n\mathbf{e}_1)$ should have the order n^α for some $\alpha \in (0, \alpha)$ depending on the dimension d for general distribution ζ. However, these predictions are far from being proved in the first-passage percolation model. In particular, the best known upper bound of variance obtained in [DHS15] by Damron, Hanson and Sosoe for the general distribution ζ: if $\zeta(0) < p_c$ and

$$\mathbb{E}[t_e^2 \log_+ t_e] < \infty, \tag{1.5}$$

then there exists a constant $C > 0$ such that

$$\mathrm{Var}[T(0, n\mathbf{e}_1)] \le C \frac{n}{\log n}. \tag{1.6}$$

Recently, Dembin [Dem22] extend this result to supercritical Bernoulli percolation (note that the moment condition (1.5) is failed). The sublinearity of variance is also called the superconcentration, see e.g. (1.6). Chatterjee [Cha14] discovers the deep connection among properties of superconcentration, chaos and multiplevaleys in, for example, the Gaussian polymer and mixed p-spin model.

Under stronger assumptions on the moments, this phenomenon can be supplemented with concentration results by Damron, Hanson and Sosoe [DHS14]: if $\mathbb{E}[e^{2\alpha t_e}] < \infty$, then there exist $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(|T(0, n\mathbf{e}_1) - \mathbb{E}[T(0, n\mathbf{e}_1)]| \ge \sqrt{\frac{n}{\log n}} \lambda \right) \le c_1 e^{-c_2 \lambda} \text{ for } \lambda \ge 0.$$

either $\mathbb{E}[t_e^2 \log_+ t_e] < \infty$, there exist $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(T(0, n\mathbf{e}_1) - \mathbb{E}[T(0, n\mathbf{e}_1)] \le -\sqrt{\frac{n}{\log n}} \lambda \right) \le c_1 e^{-c_2 \lambda} \text{ for } \lambda \ge 0.$$

To our best knowledge, the moderate deviation of D_n^* (or concentration with diffusive scale) was established in the supercritical Bernoulli percolation by Garet and Marchand [GM09]: for each $c_3 > 0$, there exist some constants c_1, c_2 such that for all $\lambda \in [c_3(1 + \log n), \sqrt{n}]$,

$$\mathbb{P}[|D_n^* - \mathbb{E}[D_n^*]| \ge \sqrt{n}\lambda] \le c_1 e^{-c_2 \lambda}.$$

The main result of our paper is to prove a sub-diffusive concentration of D_n^* for supercritical Bernoulli percolation as follows.

Theorem 1.1. *Let $p > p_c$. There exist some constant $c_1, c_2 > 0$ depending on p and d such that*

$$\mathbb{P}\left(|D_n^* - \mathbb{E}[D_n^*]| \ge \sqrt{\frac{n}{\log n}} \kappa \right) \le c_1 e^{-c_2 \kappa} \textit{ for all } \kappa \ge 0. \tag{1.7}$$

Consequently, we recover the sub-linear bound for the variance:

$$\mathrm{Var}[D_n^*] \le C_0 \frac{n}{\log n}, \tag{1.8}$$

where C_0 is a positive constant depending on p and d.

Remark 1. If $\kappa < 1$ then we can take $c_1 = e^{c_2}$ such that (1.7) holds trivially. From now on, we focus on the case $\kappa \ge 1$ throughout this paper.

1.2 Method of the proof and relevant techniques

In this subsection, we will address the main challenge in extending the previous results of Damron, Hanson and Sosoe [DHS14] to Bernoulli percolation and we will outline our strategy to overcome this issue. In [DHS14], Damron et al. use the ideas of Benaim-Rossignol, to prove the subdiffusive concentration, it suffices to estimate the variance of exponential function of D_n^*. The remaining step can be derived by combining the geometric averaging trick of Benjamini, Kalai, Schramm [BKS11] with the entropy inequalities, following the same sub-linear variance strategy for general distribution [DHS15].

In both [DHS14] and [DHS15], we emphasize the importance of imposing moment conditions on the edge-weight distribution. This is crucial for obtaining good control over the impact of resampling an edge. Specifically, when the distribution is bounded, resampling an edge on the geodesic results in a constant upper bound in the change of the first passage time between two points. However, in the context of the graph distance in Bernoulli percolation, closing an edge on the geodesic can have a significant impact on the graph distance due to the possibility of infinite edge-weight values. To solve this issue, Dembin [Dem22] developed a technique based on the work of Cerf and Dembin [CD22]. Roughly speaking, Dembin prove that, on average, closing an edge on the geodesic modifies the graph distance by at most a constant. The key idea is to construct a detour that bypasses one edge on the geodesic when its status change to closed. Through a complex multiple-scale renormalization process [1], they ensure each an edge e on geodesic can be bypassed by an open path of well-controlled length. In the final step, to prove the sublinear property of variance, Dembin use the concentration inequalities in a similar manner as in [DHS15] with some technical difficulties specific to the graph distance.

In Bernoulli percolation, the general framework presented in [DHS14] does not directly apply to prove subdiffusive concentration for the graph distance due to technical reasons, which will be explained later. To over this issue, we introduced T_n, a modified graph distance, which is derived from a Bernoulli first passage percolation with truncated edge-weights as defined in (3.1). Now our main strategy goes as follows: first, we show that the discrepancy between D_n^* and T_n is small; then, we focus on proving subdiffusive concentration for T_n utilizing entropy inequalities. The key point here lies in controlling the impact of resampling edge e on geodesic of T_n through the construction of the effective radius (refer to Section 4 for more details). In Section 6, we estimate the discrepancy between D_n^* and T_n through a selection process of the suitable family of such detours. Our construction of the effective radius is induced from the good geometrical properties of infinite cluster, which behaves (in a sense made precise) like that of $\mathbb{Z}^d$. This simplifies the approach compared to the work of Cerf and Dembin [CD22]. Lastly, we note that the entropy method requires a large deviation result for lattice animals with dependent weights (Lemma 4.3 (iv)). However, this condition does not hold true in cases where the distributions of edge-weights can take infinite values, such as in Bernoulli percolation.

[1] Renormalization argument was initially developed by [AP96] to understand the basis geometrical structure of cluster. Recently, it has many applications in the study of both supercritical percolation and random conductance models on the infinite cluster for supercritical percolation (refer to [Bar04] [AD18] [BRCD23] for more details)

1.3 Notations and terminologies

- *Metric.* We denote by $\|\cdot\|_1, \|\cdot\|_\infty, \|\cdot\|_2$ correspondence to the l_1, l_∞, l_2 norms.
- *Box and annulus.* Let $x \in \mathbb{Z}^d$ and $R \in \mathbb{N}$, we will denote by $B_x(R) = x + [-R, R]^d$ the box with the center at $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ and radius R. For convenience, we shortly write $B(R)$ for $B_0(R)$. Let $R_2 > R_1 \geq 1$, we introduce an annulus $A_x(R_2, R_1)$ by
$$A_x(R_2, R_1) = B_x(R_2) \setminus B_x(R_1).$$
- *Nearest path and none-nearest path.* A sequence $\gamma = (x_0, x_1, \ldots, x_n) \subseteq \mathbb{Z}^d$ is a *none-nearest path* if $x_i \neq x_j$ for all $0 \leq i \neq j \leq n$. We say that the none-nearest path γ is *nearest path* if for all $1 \leq i \leq n$, $\|x_i - x_{i-1}\|_1 = 1$. Let $L \geq 1$, we define
$$\Xi_L = \{\gamma : \gamma \text{ is non-nearest neighbor path in } B(L);\ \#\gamma \leq L\}.$$
From now on, we shortly write path for nearest path.
- *Crossing path and cluster.* A path $\Gamma \subseteq A_x(R_2, R_1)$ is called a *crossing path* of $A_x(R_2, R_1)$ if $\partial B_x(R_1)$ link to $\partial B_x(R_2)$ by Γ. We say that $\mathcal{C}$ is a *cluster* if for all $x, y \in \mathcal{C}$, x links to y by a path in $\mathbb{Z}^d$, i.e $\mathcal{C}$ is a connected component in graph $(\mathbb{Z}^d, \mathcal{E}(\mathbb{Z}^d))$.
- *Open path, open crossing path and open cluster.* We say that a path or crossing path or cluster is *open path* or *open crossing path* or *open cluster*, respectively if all of its edges is open (1-weight).
- *Diameter.* For $A \subseteq \mathbb{Z}^d$ and $1 \leq i \leq d$, let us define
$$\text{diam}_i(A) = \max_{x,y \in A} |x_i - y_i|,$$
and we thus denote $\text{diam}(A)$ the diameter of A by
$$\text{diam}(A) = \max_{1 \leq i \leq d} \text{diam}_i(A).$$
- *Crossing cluster and open crossing cluster.* Let $M_x(R), M_x(R_2, R_1)$ be the clusters in $B_x(R), A_x(R_2, R_1)$, respectively. We say that $M_x(R), M_x(R_2, R_1)$ cross $B_x(R), A_x(R_2, R_1))$ in the i^{th} direction if $M_x(R), M_x(R_2, R_1)$ contain a path $\gamma = (y^1, \ldots, y^n)$ satisfying $y_i^1 = x_i - R$ and $y_i^n = x_i + R$. In addition, clusters $M_x(R), M_x(R_2, R_1)$ are called crossing clusters of $B_x(R), A_x(R_2, R_1)$ if $M_x(R), M_x(R_2, R_1)$ cross $B_x(R), A_x(R_2, R_1)$ in all directions, respectively. We say that crossing cluster is open if all of its edges is open (1-weight).

1.4 Organization of this paper

In Section 2, we present some standard results of the supercritical percolation and recall the concentration inequalities. Next, we prove Theorem 1.1 by using Theorem 3.2 in Section 3. In Section 4, we prove two key components of the proof. We prove the subdiffusive concentration of the modified graph distance in Section 5. Finally, we remain estimate the discrepancy between the graph distance and its modified version in Section 6.

2 Preliminaries

2.1 Background on Bernoulli Percolation

Let us define for some constants $a, b, c > 0$,

$$\mathrm{Crb}(x,R) = \{\text{there exists a open crossing cluster in } B_x(R)\},$$

$$\mathrm{Cra}(x,R) = \{\text{there exists a open crossing cluster in } A_x(aR,bR)\},$$

$$\mathrm{D}(x,R) = \{\text{there exist two disjoint open clusters in } B_x(R) \text{ having diameter at least } cR\}.$$

The following lemmas show that there exists a unique large crossing cluster in each box with radius R.

Lemma 2.1. *[Gri89] Let $p > p_c$ and $d \geq 2$. There exist positive constants $\beta_1 = \beta_1(p), \beta_2 = \beta_2(p), \beta_3 = \beta_3(p,a,b)$ such that for all $x \in \mathbb{Z}^d$ and $R \in \mathbb{N}$,*

(i)

$$\mathbb{P}[D(x,R)] \leq \beta_1 e^{-\beta_2 R}. \tag{2.1}$$

(ii)

$$\mathbb{P}[Crb(x,R)] \geq 1 - \beta_1 e^{-\beta_2 R}. \tag{2.2}$$

(iii)

$$\mathbb{P}[Cra(x,R)] \geq 1 - \beta_1 e^{-\beta_3 R}. \tag{2.3}$$

The following lemma controls the probability of having a big hole in the infinite cluster.

Lemma 2.2 ([Gri89], Theorem 7)**.** *There exist positive constants β_1 and β_2 depending on p such that for all $x \in \mathbb{Z}^d$ and $R \in \mathbb{N}$,*

$$\mathbb{P}[\mathcal{C}_\infty \cap B_x(R) = \emptyset] \leq \beta_1 \exp(-\beta_2 R).$$

The following lemma show some large deviation estimates for the graph distance that will be used frequently in this paper.

Lemma 2.3 ([GM09], Lemma 2.3)**.** *Let $p > p_c$. There exist positive constants $\rho, \rho_1, \rho_2, \alpha, \beta > 0$, such that for any $x, y \in \mathbb{Z}^d$*

(i)

$$\forall t \geq \rho\|x-y\|_\infty, \quad \mathbb{P}[D^*(x,y) \geq t] \leq \rho_1 e^{-\rho_2 t}. \tag{2.4}$$

(ii)

$$\forall t \geq \rho\|x-y\|_\infty, \quad \mathbb{P}[D(x,y) \geq t, x \text{ links to } y] \leq \rho_1 e^{-\rho_2 t}. \tag{2.5}$$

(iii)

$$\mathbb{E}[e^{\alpha D^*(x,y)}] \leq e^{\beta\|x-y\|_\infty}. \tag{2.6}$$

2.2 Entropy inequalities

Let us enumerate the edges $\mathcal{E}(\mathbb{Z}^d)$ as $e_1, e_2, \ldots$ Let $a, b \in \mathbb{R} \cup \{+\infty\}$. Assume that $(t_{e_i})_{i\geq 1}$ be a family of i.i.d. random variables with the same distribution as

$$\zeta = p\delta_a + (1-p)\delta_b.$$

Let $g : \{a, b\}^{\mathcal{E}(\mathbb{Z}^d)} \to \mathbb{R}$ be a function of $(t_{e_i})_{i\geq 1}$. Fix $\lambda \in \mathbb{R}$, define

$$G = G_\lambda = e^{\lambda g}.$$

We can write

$$G = G(t_{e_i}, t_{e_i^c})$$

to emphasize G is the function of the random variables t_{e_i} and $t_{e_i^c} = (t_{e_j})_{j\neq i}$. We denote a sequence of σ-algebra by

$$\mathcal{F}_0 = \varnothing, \quad \mathcal{F}_i = \sigma(t_{e_1}, \ldots, t_{e_i}),$$

for $i \geq 0$. Now we consider the martingale increments

$$\Delta_i = \mathbb{E}[G \mid \mathcal{F}_i] - \mathbb{E}[G \mid \mathcal{F}_{i-1}] = \mathbb{E}[G(t'_{e_i}, t_{e_i^c}) - G(t_{e_i}, t_{e_i^c}) \mid \mathcal{F}_{i-1}],$$

where t'_{e_i} is an independent copy of t_{e_i} and $G(t'_{e_i}, t_{e_i^c})$ is obtained from $G = G(t_{e_i}, t_{e_i^c})$ by replacing the variable t_{e_i} by t'_{e_i}. It is clear that

$$G - \mathbb{E}[G] = \sum_{i=1}^{\infty} \Delta_i.$$

Combining this with the orthogonality of the $(\Delta_i)_{i=1}^{\infty}$, we have

$$\mathrm{Var}[G] = \sum_{i=1}^{\infty} \mathbb{E}[\Delta_i^2]. \tag{2.7}$$

We will bound the variance of G based on the following entropy inequality by Falik and Samorodnitsky.

Lemma 2.4. *[FS07]*

$$\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] \geq \mathrm{Var}[G] \log \frac{\mathrm{Var}[G]}{\sum_{i=1}^{\infty}(\mathbb{E}[|\Delta_i|])^2}, \tag{2.8}$$

where Ent *denotes the entropy operator:*

$$\mathrm{Ent}[f] = \mathbb{E}\Big[f \log \frac{f}{\mathbb{E}[f]}\Big].$$

We will need the following lemma to control the entropy:

Lemma 2.5. *There exists a constant $C > 0$ depending on p such that*

$$\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] \leq C \sum_{i=1}^{\infty} \mathbb{E}[(G(b, t_{e_i^c}) - G(a, t_{e_i^c}))^2]. \tag{2.9}$$

This lemma is a direct consquence of the two following results.

Lemma 2.6 (Bernoulli log-Sobolev inequalities,[DHS15]). *Assume that $f : \{a, b\} \to \mathbb{R}$ and X be a random variable with the distribution $\zeta = p\delta_a + (1-p)\delta_b$. There exist a constant $C > 0$ depending on p such that*

$$\mathrm{Ent}_\zeta[f^2(X)] \leq C|f(b) - f(a)|^2,$$

where Ent_ζ is the entropy respect to the distribution ζ.

Proposition 2 (Tensorization property, [DHS15]). *Assume that $h : \{a, b\}^{\mathcal{E}(\mathbb{Z}^d)} \to \mathbb{R}$ be a function of $(t_{e_i})_{i\geq 1}$. Then*

$$\mathrm{Ent}[h] \leq \sum_{i=1}^{\infty} \mathbb{E}[\mathrm{Ent}_i[h]], \tag{2.10}$$

where Ent_i is the entropy of h respect to the distribution of t_{e_i}, all other coordinates remain fixed.

3 The modified graph distance and proof of Theorem 1.1

Consider a Bernoulli first passage percolation as follows. Let $(t_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ be i.i.d random weights such that

$$t_e = \begin{cases} 1 & \text{with probability } p, \\ \log^2 n & \text{with probability } 1-p. \end{cases}$$

Now we define a modified graph distance T by

$$T(x, y) = \inf_{\gamma: x\to y} \sum_{e\in\gamma} t_e, \tag{3.1}$$

and set

$$T_n = T(0, n\mathbf{e}_1).$$

Let us couple this first passage percolation with Bernoulli percolation. In this model, we call the open edge by 1-weight edge and closed edge by $\log^2 n$-weight edge. We denote also infinite cluster of 1-weight edges (open) by $\mathcal{C}_\infty$. For each $x \in \mathbb{Z}^d$, we denote x^* for the nearest point of x in $\mathcal{C}_\infty$. Our aim is to show the subdiffusive concentration D_n^* through the modified graph distance T_n. The proof is essentially based on two key components. First, our next theorem, proved in Section 6, controls the discrepancy between D_n^* and $T(0^*, (n\mathbf{e}_1)^*)$.

Theorem 3.1. *There exist positive constants c_1, c_2 such that such that for all $L \geq \log^2 n$,*

$$\mathbb{P}\Big[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \geq L\Big] \leq c_1 \exp(-c_2 \tfrac{L}{\log L}). \tag{3.2}$$

As consequence, we have

$$\mathbb{E}[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)|] \leq \mathcal{O}(\log^2 n), \tag{3.3}$$

Second, we establish the subdiffusive concentration for T_n as follow.

Theorem 3.2. *There exist positive constants c_1, c_2 such that*

$$\mathbb{P}\left(|T_n - \mathbb{E}[T_n]| \geq \sqrt{\tfrac{n}{\log n}}\kappa\right) \leq c_1 e^{-c_2\kappa} \text{ for all } \kappa \geq 0. \tag{3.4}$$

We postpone its proof to Section 5 and give the proof of Theorem 1.1.

Proof of Theorem 1.1. It is straightforward that

$$|D_n^* - \mathbb{E}[D_n^*]| \leq |D_n^* - T_n| + |T_n - \mathbb{E}[T_n]| + |\mathbb{E}[T_n] - \mathbb{E}[D_n^*]|. \tag{3.5}$$

Notice also that

$$|T_n - T(0^*, (n\mathbf{e}_1)^*)| \leq \log^2 n(\|0^*\|_\infty + \|(n\mathbf{e}_1)^* - n\mathbf{e}_1\|_\infty). \tag{3.6}$$

Therefore,

$$\begin{aligned} |D_n^* - T_n| &\leq |D_n^* - T(0^*, (n\mathbf{e}_1)^*)| + |T(0^*, (n\mathbf{e}_1)^*) - T_n| \\ &\leq |D_n^* - T(0^*, (n\mathbf{e}_1)^*)| + \log^2 n(\|0^*\|_\infty + \|(n\mathbf{e}_1)^* - n\mathbf{e}_1\|_\infty) \end{aligned} \tag{3.7}$$

Combining this with triangle inequality, we have

$$\begin{aligned} \mathbb{E}[|D_n^* - T_n|] &\leq \mathbb{E}[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)|] + \log^2 n\mathbb{E}[\|0^*\|_\infty + \|(n\mathbf{e}_1)^* - n\mathbf{e}_1\|_\infty] \\ &= \mathbb{E}[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)|] + 2\log^2 n\mathbb{E}[\|0^*\|_\infty], \end{aligned} \tag{3.8}$$

here for the last line we used the translation invariance. By Lemma 2.2, there exist positive constants β_1, β_2 such that for $t \geq 0$,

$$\mathbb{P}[\|0^*\|_\infty \geq t] \leq \mathbb{P}[\mathcal{C}_\infty \cap [-t, t]^d = \emptyset] \leq \beta_1 \exp(-\beta_2 t), \tag{3.9}$$

and thus

$$\mathbb{E}[\|0^*\|_\infty] = \mathcal{O}(1).$$

Combining this with (3.8), (3.3) and (3.9), we get

$$|\mathbb{E}[D_n^*] - \mathbb{E}[T_n]| \leq \mathcal{O}(\log^2 n).$$

It follows from the above estimate and (3.5) that for all $\kappa \geq 1$ and n large enough,

$$\mathbb{P}\left[|D_n^* - \mathbb{E}[D_n^*]| \geq \sqrt{\tfrac{n}{\log n}}\kappa\right] \leq \mathbb{P}\left[|T_n - \mathbb{E}[T_n]| \geq \tfrac{\kappa}{4}\sqrt{\tfrac{n}{\log n}}\right] + \mathbb{P}\left[|D_n^* - T_n| \geq \tfrac{\kappa}{4}\sqrt{\tfrac{n}{\log n}}\right]. \tag{3.10}$$

By Theorem 3.2,

$$\mathbb{P}\left[|T_n - \mathbb{E}[T_n]| \geq \tfrac{\kappa}{4}\sqrt{\tfrac{n}{\log n}}\right] \leq c_1 \exp(-c_2\kappa/4), \tag{3.11}$$

for some $c_1, c_2 > 0$. It now remains to estimate the second term in (3.10). Again, using (3.7), (3.9) and Theorem 3.1,

$$\begin{aligned} \mathbb{P}\left[|D_n^* - T_n| \geq \tfrac{\kappa}{4}\sqrt{\tfrac{n}{\log n}}\right] &\leq \mathbb{P}\left[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \geq \tfrac{\kappa}{8}\sqrt{\tfrac{n}{\log n}}\right] \\ &\qquad + \mathbb{P}\left[(\|0^*\|_\infty + \|(n\mathbf{e}_1)^* - n\mathbf{e}_1\|_\infty) \geq \tfrac{\kappa}{8}\tfrac{n^{1/2}}{(\log n)^{5/2}}\right] \\ &\leq \mathbb{P}\left[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \geq \tfrac{\kappa}{8}\sqrt{\tfrac{n}{\log n}}\right] + 2\mathbb{P}\left[\|0^*\|_\infty \geq \tfrac{\kappa}{16}\tfrac{n^{1/2}}{(\log n)^{5/2}}\right] \\ &\leq c_1 \exp(-c_2\kappa\tfrac{\sqrt{n}}{(\log n)^{3/2}}) + 2\beta_1 \exp\left(-\beta_2\kappa\tfrac{n^{1/2}}{(\log n)^{5/2}}\right) \\ &\leq c_1' \exp\left(-c_2'\kappa\tfrac{\sqrt{n}}{(\log n)^{5/2}}\right), \end{aligned} \tag{3.12}$$

for some $c_1', c_2' > 0$. Finally, combining (3.10), (3.11) with (3.12), we get Theorem 1.1. □

In the remainder of this paper, we will prove Theorem 3.1 and Theorem 3.2. To do this, we require relevant information about the model, which will be covered in the following section.

4 The effect of resampling

In the first passage percolation models with a general bounded distribution ζ, we define for any $z \in \mathbb{Z}^d$,

$$T_z := T(z, z + n\mathbf{e}_1).$$

We aim to study how the random variable T_z changes when resampling the value of each single edge e. In particular, the change can be estimated as: for $b \geq a$,

$$0 \leq T_z(b, t_{e^c}) - T_z(a, t_{e^c}) \leq (b - a)\mathbb{I}(e \in \gamma_z),$$

where γ_z is the geodesic of $T(a, t_{e^c})$ from z to $z + n\mathbf{e}_1$. However, this bound becomes less effective when b is much larger than a. To address this issue, in our modified model with $\zeta = p\delta_1 + (1-p)\delta_{\log^2 n}$, we will construct a bypass of 1-weight edges avoiding the edge e. The cost of resampling t_e can be bounded by the length of the bypass. Furthermore, we can control this length by a random radius defined in

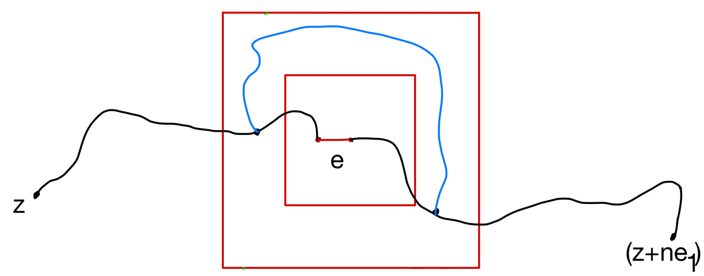

Figure 1: A bypass of 1-weight edges (blue) around one singe edge e (red)

Subsection 4.1. The next question is to estimate the total cost of resampling all the edges in $\mathcal{E}(\mathbb{Z}^d)$. We shall see that this problem leads to an investigation of total weight in a dependent percolation for which we use greedy lattice animal theory to deal with, see more in Subsection 4.2.

4.1 The effective radius

For each $e = (x_e, y_e) \in \mathcal{E}(\mathbb{Z}^d)$, we fix a rule to write an edge e with two endpoints x_e, y_e as $e = (x_e, y_e)$ (for example, $\|x_e\|_\infty < \|y_e\|_\infty$). Now for any $R \in \mathbb{N} \setminus \{0\}$, we define an annulus $A_{x_e}(2R, R)$ by

$$A_{x_e}(2R, R) = B_{x_e}(2R) \setminus B_{x_e}(R). \tag{4.1}$$

For convenience, we shortly write $B_e(R), B_e(2R), A_e(2R,R)$ for $B_{x_e}(R), B_{x_e}(2R)$, $A_{x_e}(2R,R)$, respectively. Let us denote by $\mathscr{C}(A_e)$ the set of all crossing paths of $A_e(2R,R)$. Here we remark that a path is simply a sequence of edges. For any $\Gamma_1, \Gamma_2 \in \mathscr{C}(A_e)$, we define the distance between Γ_1 and Γ_2 by

$$d(\Gamma_1,\Gamma_2) = \inf\{\#\gamma : \gamma \text{ is a path of 1-weight edges that joins } \Gamma_1 \text{ with } \Gamma_2 \text{ in } A_e(2R,R)\}.$$

For all $C > 0$ and $R \geq 1$, let us define some events as

$$\mathcal{V}_e^1(R) = \{\forall\, \Gamma_1, \Gamma_2 \in \mathscr{C}(A_e) : d(\Gamma_1,\Gamma_2) \leq CR\},$$

and

$$\mathcal{V}_e^2(R) = \{\forall x, y \in B_e(2R) : \text{ if } x \text{ links to } y \text{ by 1-weight path in } B_e(2R) \\ \text{then } D(x,y) \leq CR\}.$$

We denote R_e the effective radius of the edge e by

$$R_e = R_e(C) = \inf\{R \geq C : \mathcal{V}_e^1(R) \cap \mathcal{V}_e^2(R) \text{ occurs }\}.$$

We summarize the properties of the effective radius in the following.

Proposition 3. *Let $p > p_c$. Then there exists a constant $C_* > 1$ such that the following holds for all $z \in \mathbb{Z}^d$.*

(i) For all $z \in \mathbb{Z}^d$ and $e \in \mathcal{E}(\mathbb{Z}^d)$,

$$0 \leq T_z(\log^2 n, t_{e^c}) - T_z(1, t_{e^c}) \leq (\log^2 n \mathbb{I}(\mathcal{U}_{z,e}) + \hat{R}_e)\mathbb{I}(e \in \gamma_z), \tag{4.2}$$

where γ_z is the geodesic path of $T_z(1, t_{e^c})$, and

$$\mathcal{U}_{z,e} = \{R_e \geq \tfrac{1}{2}(\|e - z\|_\infty \wedge \|e - (z + n\mathbf{e}_1)\|_\infty\}), \quad \hat{R}_e = C_* R_e \wedge \log^2 n.$$

(ii) There exist positive constants α and β depending on p such that for all $t \in \mathbb{N}$ and $e \in \mathcal{E}(\mathbb{Z}^d)$,

$$\mathbb{P}[R_e \geq t] \leq \alpha \exp(-\beta t). \tag{4.3}$$

(iii) For all $t \in \mathbb{N}$ and $e \in \mathcal{E}(\mathbb{Z}^d)$, the event $\{R_e \leq t\}$ depends only on the status of edges in $B_e(3t)$.

Proof. We first prove the statement (i). Since T_z is increasing, the first inequality in (4.2) is trivial. By the definition of the first passage time, if $e \notin \gamma_z$ then

$$T_z(\log^2 n, t_{e^c}) \leq T_z(1, t_{e^c}).$$

Thus,

$$(T_z(\log^2 n, t_{e^c}) - T_z(1, t_{e^c})) = 0. \tag{4.4}$$

If $e \in \gamma_z$, we can estimate this discrepancy by

$$T_z(\log^2 n, t_{e^c}) - T_z(1, t_{e^c}) \leq \log^2 n \mathbb{I}(e \in \gamma_z). \tag{4.5}$$

In the case where $e \in \gamma_z$ and either z or $z + n\mathbf{e}_1$ is in $B_e(2R_e)$ (i.e. e is near to z or $z + n\mathbf{e}_1$), it is clear that

$$\begin{aligned} T_z((\log n)^2, t_{e^c}) - T_z &\leq (\log^2 n \mathbb{I}(z \in B_e(2R_e) \cup (z + n\mathbf{e}_1) \in B_e(2R_e)))\mathbb{I}(e \in \gamma_z) \\ &= \log^2 n \mathbb{I}(\mathcal{U}_{z,e})\mathbb{I}(e \in \gamma_z). \end{aligned} \tag{4.6}$$

On the other hand, if $e \in \gamma_z$ and both z and $z + n\mathbf{e}_1$ are not in $B_e(2R_e)$, then γ_z crosses the annulus $A_e(2R_e, R_e)$ at least twice. We call the first and last crossing paths (in the order from 0 to $n\mathbf{e}_1$) by Γ_L and Γ_R, respectively. Notice that both on Γ_L and Γ_R are in $\mathscr{C}(A_e)$. Hence, by the definition of R_e, the event $\mathcal{V}_e^1(R_e) \cap \mathcal{V}_e^2(R_e)$ occurs. Thus,

$$T_z(\log^2 n, t_{e^c}) - T_z(1, t_{e^c}) \leq d(\Gamma_L, \Gamma_R)\mathbb{I}(e \in \gamma_z) \leq CR_e \mathbb{I}(e \in \gamma_z).$$

Combining this with (4.5) gives us

$$T_z(\log^2 n, t_{e^c}) - T_z(1, t_{e^c}) \leq (CR_e \wedge \log^2 n)\mathbb{I}(e \in \gamma_z). \tag{4.7}$$

By using (4.6) and (4.7), we claim that

$$T_z(\log^2 n, t_{e^c}) - T_z(1, t_{e^c}) \leq (\log^2 n \mathbb{I}(\mathcal{U}_{z,e}) + (CR_e \wedge \log^2 n))\mathbb{I}(e \in \gamma_z). \tag{4.8}$$

The statement (i) follows from (4.4) and (4.8).

We start the proof of (ii) by noting that for all $t \geq 2$,

$$\begin{aligned} \mathbb{P}[R_e \geq t] \leq \mathbb{P}\left[\bigcap_{R \leq t-1} (\mathcal{V}_e^1(R_e) \cap \mathcal{V}_e^2(R_e))^c\right] &\leq \mathbb{P}\left[(\mathcal{V}_e^1(t-1) \cap \mathcal{V}_e^2(t-1))^c\right] \\ &= 1 - \mathbb{P}[\mathcal{V}_e^1(t-1) \cap \mathcal{V}_e^2(t-1)]. \end{aligned}$$

Thus, we only need to prove that there exists positive constants α, β depending on p satisfy for all $t \geq 1$,

$$\mathbb{P}[\mathcal{V}_e^1(t) \cap \mathcal{V}_e^2(t)] \geq 1 - \alpha \exp(-\beta t). \tag{4.9}$$

Let $\varepsilon > 0$ be a small sufficiently constant. We define

$$A_e(\varepsilon t) := A_e((\tfrac{3}{2} + \varepsilon)t, (\tfrac{3}{2} - \varepsilon)t) = B_e((\tfrac{3}{2} + \varepsilon)t) \setminus B_e((\tfrac{3}{2} - \varepsilon)t).$$

In addition, we divide $A_e(\varepsilon t)$ into a family of consecutive small boxes of radius εt (see Figure 2). Let us denote N for the number of these small boxes. Note that $N \leq (16/\varepsilon)^d = C(\varepsilon, d)$. We enumerate these boxes by $B_0, \ldots, B_N$.

By Lemma 2.1 (iii), there exist positive constants $\beta_1, \beta_3 = \beta_3(p, \varepsilon)$ such that

$$\mathbb{P}[\text{there exists a crossing cluster of 1-weight edges in } A_e(\varepsilon t)] \geq 1 - \beta_1 \exp(-\beta_3 t).$$

On this event, let $Cl(A_e)$ be the open crossing cluster of 1-weight edges in $A_e(\varepsilon t)$. Now we define an event

$$\mathcal{L} = \{\forall \Gamma \in \mathscr{C}(A_e) : \Gamma \text{ links to } Cl(A_e) \text{ by 1-weight paths}\}.$$

It is clear that $\mathcal{L}$ holds for $d = 2$. To estimate the probability of this event for $d \geq 3$, we adapt the argument in [[Gri89], Lemma 7.104] (see more detail in Appendix) to obtain the following result.

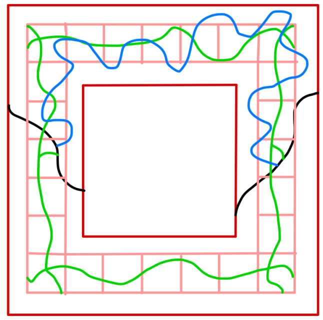

Figure 2: Illustration of the construction of consecutive small boxes control the length of bypass (blue)

Lemma 4.1. *Let $d \geq 3, p > p_c, 0 < c \leq 2$ and $\mathcal{E}$ be the event that there exist two connected sets D_1 and D_2 in $B(R)$ that have diameter at least cR such that D_1 does not link to D_2 by 1-weight paths in $B(R)$. Then there exists $\beta_1 = \beta_1(d), \beta_2 = \beta_2(p) > 0$ such that for all $R \geq 1$,*

$$\mathbb{P}[\mathcal{E}] \leq \beta_1 \exp(-\beta_2 R). \tag{4.10}$$

We note that there exist $q \leq N, c > 0$ such that: (i) $D_1 = \Gamma \cap B_q(\varepsilon t)$ and $diam(D_1) \geq c\varepsilon t$ for all $\Gamma \in \mathscr{C}(A_e)$, and (ii) $D_2 = Cl(A_e) \cap B_q(\varepsilon t)$ and $diam(D_2) \geq c\varepsilon t$. Hence, using Lemma 4.1, it follows that there exist positive constants $\beta_1, \beta_2 = \beta_2(p)$ such that for all $d \geq 2$,

$$\begin{aligned} \mathbb{P}[\mathcal{L}^c] &= \mathbb{P}[\exists \Gamma \in \mathscr{C}(A_e) : \Gamma \text{ does not link to } Cl(A_e) \text{ by 1-weight paths}] \\ &\leq \mathbb{P}[\exists D_1, D_2 : diam(D_1), diam(D_2) \geq c\varepsilon t, D_1 \text{ does not link to } D_2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by 1-weight paths}] \\ &\leq \beta_1 \exp(-\beta_2 \varepsilon t), \end{aligned} \tag{4.11}$$

which implies that

$$\mathbb{P}[\mathcal{L}] = 1 - \mathbb{P}[\mathcal{L}^c] \geq 1 - \beta_1 \exp(-\beta_2 \varepsilon t).$$

We now can take $N' \leq N$ such that

$$\begin{aligned} &x_0 \in \Gamma_L \cap B_0(\varepsilon t) \cap Cl(A_e), \\ &x_{N'} \in \Gamma_R \cap B_{N'}(\varepsilon t) \cap Cl(A_e), \\ &x_i \in Cl(A_e) \cap B_i(\varepsilon t), \quad \forall 1 \leq i \leq N' - 1. \end{aligned}$$

Moreover, on the event $\mathcal{L}$, these points x_i links to x_j by the path of 1-weight edges for all $1 \leq i, j \leq N'$. Then using Lemma 2.3 (ii) and union bound, taking $\varepsilon \leq \frac{1}{16\rho}$ (with ρ as in Lemma 2.3) such that there exist positive constants ρ_1, ρ_2 satisfying

$$\mathbb{P}[\mathcal{V}] \geq 1 - \rho_1 e^{-\rho_2 t},$$

where

$$\mathcal{V} = \Big\{ \max_{1 \le i \le N'} \max_{x \in B_{i-1}(\varepsilon t), y \in B_i(\varepsilon t)} D(x,y) \le t/4 \Big\}. \tag{4.12}$$

On $\mathcal{V}$, we can take $C_* \ge C(\varepsilon, d)$ such that

$$d(\Gamma_1, \Gamma_2) \le \sum_{i=1}^{N'} d(x_i, x_{i+1}) \le C_* t.$$

Therefore,

$$\mathbb{P}[\mathcal{V}_e^1(t)] \ge \mathbb{P}[\mathcal{V}] \ge 1 - \rho_1 e^{-\rho_2 t}. \tag{4.13}$$

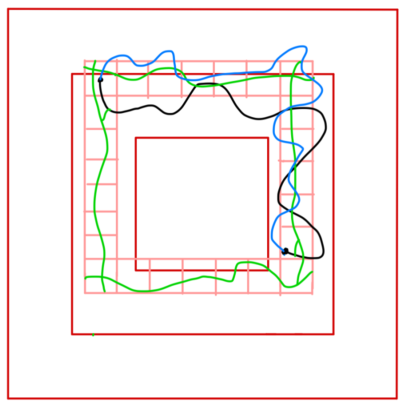

Figure 3: Illustration of the construction of a path of 1-weight edges (blue) that links between two disjoint points

Now by an argument similar to that as in (4.13) (see Figure 3), we also have

$$\mathbb{P}[\mathcal{V}_e^2(t)] \ge 1 - \beta_1 \exp(-\beta_2 t), \tag{4.14}$$

for some positive constants β_1, β_2. Hence, (4.9) follows by combining (4.13) and (4.14). We completed the proof of (ii).

Finally, we observe that the event $\mathcal{V}_e^1(R) \cap \mathcal{V}_e^2(R)$ only depends on the status of edges inside $B_e(3R)$. Thanks to the definition of R_e, it follows that the event $\{R_e \le t\}$ occurs if and only if there exists $R \le t$ such that the event $\mathcal{V}_e^1(R) \cap \mathcal{V}_e^2(R)$ occurs. Therefore, the event $\{R_e \le t\}$ only depends on the status of edges inside $B_e(3t)$. This concludes the proof of (iii).

□

4.2 Lattice animals of dependent weight

We first revisit a result that controls the maximal weight of paths, which is derived from the theory of greedy lattice animals.

Given an integer $M \ge 1$ and positive constants a, A, suppose that $\{I_{e,M}, e \in \mathcal{E}(\mathbb{Z}^d)\}$ is a collection of Bernoulli random variables satisfying

(E1) $\{I_{e,M}, e \in \mathcal{E}(\mathbb{Z}^d)\}$ is $aM-$dependent, i.e. for all $e \in \mathcal{E}(\mathbb{Z}^d)$, the variable $I_{e,M}$ is independent of all variables $\{I_{e',M} : \|e' - e\|_\infty \geq aM\}$.

(E2)

$$q_M = \sup_{e \in \mathcal{E}(\mathbb{Z}^d)} \leq AM^{-d}.$$

For any path γ, we define

$$N(\gamma) = \sum_{e \in \gamma} I_{e,M}, \quad N_{L,M} = \max_{\gamma \in \Xi_L} N(\gamma),$$

where we recall that

$$\Xi_L = \{\gamma : \gamma \text{ is non-nearest neighbor path in } B(L); \#\gamma \leq L\}.$$

Lemma 4.2. *[CN19, Lemma 2.6] Let $M \geq 1, c \geq 1$ and $\{I_{e,M} : e \in \mathcal{E}(\mathbb{Z}^d)\}$ be a collection of random variables satisfying (E1) and (E2). Then there is a positive constant $C = C(a, A, d)$ such that*

(i) For all $L \in \mathbb{N}$

$$\frac{\mathbb{E}[N_{L,M}]}{Lq_M^{1/d}} \leq CM^{d+1}.$$

(ii) if $t \geq CM^d \max\left(1, MLq_M^{1/d}\right)$, then

$$\mathbb{P}[N_{L,M} \geq t] < 2^d \exp(-t/(16M)^d).$$

We aim to extend this result to a family of general distributions. Let a, A be positive constants. Suppose that $\{X_e, e \in \mathcal{E}(\mathbb{Z}^d)\}$ is a collection of random variables satisfying the following for all $M \geq 1$

(P1) there exists a positive constant a such that for all $e \in \mathcal{E}(\mathbb{Z}^d)$, the event $\{X_e \leq M\}$ is independent of status of all edges $\{e' : e' \notin B_e(aM)\}$.

(P2) there exist a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(M) \leq AM^{-d}$ and

$$q_M = \sup_{e \in \mathcal{E}(\mathbb{Z}^d)} \mathbb{P}[X_e \geq M] \leq \phi(M).$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ and consider the following condition:

$$\sum_{M=1}^{\infty} f(M) M^{d+1} (\phi(M))^{1/d} < \infty. \tag{H}$$

Lemma 4.3. *Let $\{X_e, e \in \mathcal{E}(\mathbb{Z}^d)\}$ be a family of random variables such that (P1) and (P2) and $f : \mathbb{R}_+ \to \mathbb{R}_+$. The following holds for all $L \geq 1$ large enough.*

(i) If the function f^2 satisfies (H) then

$$\mathbb{E}\Big[\Big(\max_{\gamma \in \Xi_L} \sum_{e \in \gamma} f(X_e)\Big)^2\Big] = \mathcal{O}(L^2).$$

(ii) Suppose that γ is a random nearest-neighbor path starting from 0 and the functions f^2 and f^4 satisfy (H). If there exists $\varepsilon > 0$ such that $\mathbb{P}(\#\gamma = \ell) = \mathcal{O}(\ell^{-6-\varepsilon})$ then

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2\Big] = \mathcal{O}(L^2).$$

(iii) Suppose that γ is a random path such that $\gamma \subset B(m)$ almost surely for some $m \geq 1$ and the functions f^2 and f^4 satisfy (H). If there exists $\varepsilon > 0$ such that $\mathbb{P}(\#\gamma = \ell) = \mathcal{O}(\ell^{-6-\varepsilon})$ then

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2\Big] = \mathcal{O}((L+m)^2).$$

(iv) Let us define $\hat{X}_e = X_e \wedge \log^2 L$. Suppose that γ is a random nearest-neighbor path starting from 0 and the function f satisfies (H). If $L(\phi(\log^2 L))^{1/d} \geq 1$ then there exist some constants $\alpha, \beta > 0$ such that,

$$\mathbb{P}\Big[\sum_{e\in\gamma} f(\hat{X}_e) \geq \alpha L\Big] \leq \log^2 L \exp(-\beta L(\phi(\log^2 L))^{1/d}) + \mathbb{P}[\#\gamma \geq L].$$

Proof. We first prove (i). By Cauchy-Schwarz inequality,

$$\mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f(X_e)\Big)^2\Big] \leq \mathbb{E}\Big[\max_{\gamma\in\Xi_L}\#\gamma\sum_{e\in\gamma} f^2(X_e)\Big] \leq L\mathbb{E}\Big[\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f^2(X_e)\Big], \tag{4.15}$$

since $\#\gamma \leq L$ for all $\gamma \in \Xi_L$. For any self-avoiding path γ, we define

$$A_M^\gamma = \{e \in \gamma : X_e = M\}.$$

Thus we can express

$$\sum_{e\in\gamma} f^2(X_e) = \sum_{M\geq 1} f^2(M)(\#A_M^\gamma). \tag{4.16}$$

Notice that it follows from the definition of A_M^γ,

$$\#A_M^\gamma = \sum_{e\in\gamma}\mathbb{I}(X_e = M) = \sum_{e\in\gamma} I_{e,M}, \tag{4.17}$$

where

$$I_{e,M} = \mathbb{I}(X_e = M).$$

Plugging this into (4.16), we obtain

$$\begin{aligned}\mathbb{E}\left[\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f^2(X_e)\right] &= \mathbb{E}\left[\sum_{M\geq 1} f^2(M)\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} I_{e,M}\right] \\ &= \sum_{M\geq 1} f^2(M)\mathbb{E}\left[N_{L,M}\right],\end{aligned} \tag{4.18}$$

where

$$N_{L,M} = \max_{\gamma \in \Xi_L} \sum_{e \in \gamma} I_{e,M}.$$

By (P1), $\{I_{e,M}, e \in \mathcal{E}(\mathbb{Z}^d)\}$ is a collection of M-dependent Bernoulli random variables. Moreover, thanks to (P2), there exists $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$q_M = \sup_{e \in \mathcal{E}(\mathbb{Z}^d)} \mathbb{E}[I_{e,M}] = \sup_{e \in \mathcal{E}(\mathbb{Z}^d)} \mathbb{P}[X_e \geq M] \leq \phi(M) = \mathcal{O}(M^{-d}).$$

Therefore, the conditions (E1) and (E2) are satisfied for all $M \geq 1$. Now using Lemma 4.2 (i), we obtain that for all $M \geq 1$,

$$\mathbb{E}[N_{L,M}] \leq CLM^{d+1}\phi(M)^{1/d}. \tag{4.19}$$

By using (4.18) and (4.19), we get that

$$\mathbb{E}\left[\max_{\gamma \in \Xi_L} \sum_{e \in \gamma} f^2(X_e)\right] \leq CL \sum_{M \geq 1} f^2(M) M^{d+1} (\phi(M))^{1/d} = \mathcal{O}(L),$$

here for the last equation holds since f^2 satisfies (H). Finally, combining this with (4.15), we obtain (i). Next, to prove (ii), we decompose

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{e \in \gamma} f(X_e)\Big)^2\Big] &= \mathbb{E}\Big[\Big(\sum_{e \in \gamma} f(X_e)\Big)^2 \mathbb{I}(\#\gamma < L)\Big] + \mathbb{E}\Big[\Big(\sum_{e \in \gamma} f(X_e)\Big)^2 \mathbb{I}(\#\gamma \geq L)\Big] \\
&\leq \mathbb{E}\Big[\Big(\max_{\gamma \in \Xi_L} \sum_{e \in \gamma} f(X_e)\Big)^2 + \sum_{l=L}^{\infty} \mathbb{E}\Big[\#\gamma \sum_{e \in \gamma} f^2(X_e) \mathbb{I}(\#\gamma = l)\Big] \\
&\leq \mathcal{O}(L^2) + \sum_{l=L}^{\infty} l \mathbb{E}\Big[\max_{\gamma \in \Xi_l} \sum_{e \in \gamma} f^2(X_e) \mathbb{I}(\#\gamma = l)\Big].
\end{aligned} \tag{4.20}$$

Here for the last line we used (i). Moreover, since f^4 satisfies (H),

$$\begin{aligned}
\mathbb{E}\Big[\max_{\gamma \in \Xi_l} \sum_{e \in \gamma} f^2(X_e) \mathbb{I}(\#\gamma = l)\Big] &\leq \mathbb{E}\left[\left(\max_{\gamma \in \Xi_l} \sum_{e \in \gamma} f^2(X_e)\right)^2\right]^{1/2} \mathbb{E}\left[\mathbb{I}(\#\gamma = l)\right]^{1/2} \\
&\leq \mathcal{O}(l) (\mathbb{P}[(\#\gamma = l)])^{1/2}.
\end{aligned}$$

Combining this with (4.20), we obtain

$$\mathbb{E}\Big[\Big(\sum_{e \in \gamma} f(X_e)\Big)^2\Big] \leq \mathcal{O}(L^2) + \mathcal{O}(1) \sum_{l=L}^{\infty} l^2 (\mathbb{P}[(\#\gamma = l)])^{1/2}. \tag{4.21}$$

Moreover, if $\mathbb{P}(\#\gamma = \ell) = \mathcal{O}(\ell^{-6-\varepsilon})$ for some $\varepsilon > 0$ then the second term of right-hand side is bounded by $\mathcal{O}(1)$. As consequence, we have

$$\mathbb{E}\Big[\Big(\sum_{e \in \gamma} f(X_e)\Big)^2\Big] = \mathcal{O}(L^2). \tag{4.22}$$

We prove the statement (iii) by the same manner. First, we have

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2\Big] = \mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2 \mathbb{I}(\#\gamma \le L)\Big] + \mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2 \mathbb{I}(\#\gamma > L)\Big]. \tag{4.23}$$

According to the hypothesis that γ is a random path such that $\gamma \subset B(m)$ almost surely, it follows that if $\#\gamma \le t$ then $\gamma \in \Xi_{t+m}$ for all $t \ge 1$. Hence, applying (i), we have

$$\begin{aligned}\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(\hat{X}_e)\Big)^2 \mathbb{I}(\#\gamma \le L)\Big] \le \mathbb{E}\Big[\max_{\gamma\in\Xi_{L+m}} \Big(\sum_{e\in\gamma} f(X_e)\Big)^2\Big] &= \mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_{L+m}} \sum_{e\in\gamma} f(X_e)\Big)^2\Big] \\ &= \mathcal{O}((L+m)^2).\end{aligned}$$

To estimate the second term of (4.23), we note that

$$\begin{aligned}\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2 \mathbb{I}(\#\gamma > L)\Big] &= \sum_{l=L}^{\infty} \mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2 \mathbb{I}(\#\gamma = l)\Big] \\ &\le \sum_{l=L}^{\infty} \mathbb{E}\Big[\#\gamma \sum_{e\in\gamma} f^2(X_e)\mathbb{I}(\#\gamma = l)\Big] \\ &\le \sum_{l=L}^{\infty} l\mathbb{E}\Big[\max_{\gamma\in\Xi_{l+m}} \sum_{e\in\gamma} f^2(X_e)\mathbb{I}(\#\gamma = l)\Big] \\ &\le \sum_{l=L}^{\infty} l\Big(\mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_{l+m}} \sum_{e\in\gamma} f^2(X_e)\Big)^2\Big]\Big)^{1/2} (\mathbb{P}[(\#\gamma = l)])^{1/2} \\ &\le \mathcal{O}(1) \sum_{l=L}^{\infty} l(l+m)(\mathbb{P}[(\#\gamma = l)])^{1/2}.\end{aligned} \tag{4.24}$$

Here the last line thanks to (i). Combining this with (4.23), it follows that

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2\Big] \le \mathcal{O}((L+m)^2) + \mathcal{O}(1) \sum_{l=L}^{\infty} l(l+m)(\mathbb{P}[(\#\gamma = l)])^{1/2}.$$

Moreover, if $\mathbb{P}(\#\gamma = \ell) = \mathcal{O}(\ell^{-6-\varepsilon})$ for some $\varepsilon > 0$ then the second term of right-hand side is bounded by $\mathcal{O}((L+m)^2)$. As consequence, we have

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(X_e)\Big)^2\Big] = \mathcal{O}((L+m)^2). \tag{4.25}$$

Finally, we show (iv). For any $\alpha > 0$,

$$\mathbb{P}\Big[\sum_{e\in\gamma} f(\hat{X}_e) \ge \alpha L\Big] \le \mathbb{P}\Big[\sum_{e\in\gamma} f(\hat{X}_e) \ge \alpha L, \#\gamma \le L\Big] + \mathbb{P}\Big[\#\gamma \ge L\Big]. \tag{4.26}$$

Notice that $\hat{X}_e$ is bounded by $\log^2 L$. Using the representation as in (4.16), we have

$$\mathbb{P}\Big[\sum_{e\in\gamma} f(\hat{X}_e) \ge \alpha L, \#\gamma \le L\Big] \le \mathbb{P}\Big[\sum_{M=1}^{\log^2 L} f(M) N_{L,M} \ge \alpha L\Big],$$

where we recall that

$$N_{L,M} = \max_{\gamma \in \Xi_L} \sum_{e \in \gamma} I_{e,M}, \quad I_{e,M} = \mathbb{I}(X_e = M).$$

Furthermore, the conditions (E1) and (E2) are satisfied for all $M \geq 1$. Since f satisfies (H), we can choose α such that $\alpha > \sum_{M=1}^{\infty} f(M)(\phi(M))^{1/d} M^{d+1}$. By Lemma 4.2 (ii),

$$\begin{aligned}
\mathbb{P}\Big[\sum_{M=1}^{\log^2 L} f(M) N_{L,M} \geq \alpha L\Big] &\leq \mathbb{P}\Big[\sum_{M=1}^{\log^2 L} f(M) N_{L,M} \geq \sum_{M=1}^{\log^2 L} f(M)(\phi(M))^{1/d} L M^{d+1}\Big] \\
&\leq \sum_{M=1}^{\log^2 L} \mathbb{P}\Big[N_{L,M} \geq (\phi(M))^{1/d} L M^{d+1}\Big] \\
&\leq (\log L)^2 \exp(-L(\phi((\log L)^2))^{1/d}/16^d).
\end{aligned}$$

Combining this with (4.26), it follows that there exists a constant $\beta > 0$ such that

$$\mathbb{P}\Big[\sum_{e \in \gamma} f(\hat{X}_e) \geq \alpha L\Big] \leq \log^2 L \exp(-\beta L(\phi(\log^2 L))^{1/d}) + \mathbb{P}[\#\gamma \geq L],$$

as stated in (iv). □

5 Subdiffusive concentration of T_n

5.1 Proof of Theorem 3.2

In order to show the subdiffusive concentration of T_n, we will use the strategy of Benjamini, Kalai and Schramm [BKS11] (called BKS trick). First of all, we define a spatial average version of T_n,

$$F_m = \frac{1}{\#B(m)} \sum_{z \in B(m)} T(z, z+x) = \frac{1}{\#B(m)} \sum_{z \in B(m)} T_z, \tag{5.1}$$

where $T(z, z+x) = T_z$ and

$$B(m) = \{x : \|x\|_\infty \leq m\}, \quad m = n^{1/4}.$$

The main point behind this technique is to help us get around the difficulty: proving that each individual edge has a small probability of having any effect on the first passage time, known as small influence. This phenomenon does not hold true for edges near the origin or $n\mathbf{e}_1$ on the first passage time T_n, but it is effective when considering spatial averaging F_m.

To prove Theorem 3.2, it now suffices to show the following variance bound:

Theorem 5.1. *There exists a constant $c > 0$ such that*

$$\mathrm{Var}[e^{\lambda F_m/2}] \leq K\lambda^2 \mathbb{E}[e^{\lambda F_m}] < \infty \text{ for } |\lambda| < \frac{1}{2\sqrt{K}}, \tag{5.2}$$

where $K = \dfrac{cn}{\log n}$.

The following result is a direct consequence of Theorem 5.1. We refer the reader to [BR08] for a proof.

Corollary 5.2. *There exist positive constants c'_1, c'_2 such that*

$$\mathbb{P}\Big[|F_m - \mathbb{E}[F_m]| \geq \sqrt{\frac{n}{\log n}}\kappa\Big] \leq c'_1 e^{-c'_2\kappa}, \quad \forall \kappa \geq 0. \tag{5.3}$$

Since $\mathbb{E}[F_m] = \mathbb{E}[T_n]$,

$$\begin{aligned} |T_n - \mathbb{E}[T_n]| = |F_m - \mathbb{E}[T_n] + T_n - F_m| &= |F_m - \mathbb{E}[F_m] + T_n - F_m| \\ &\leq |F_m - \mathbb{E}[F_m]| + |T_n - F_m|. \end{aligned} \tag{5.4}$$

Thus, for all $M \geq 1$, using the union bound, we have

$$\mathbb{P}[|T_n - \mathbb{E}[T_n]| \geq 4M] \leq \mathbb{P}[|F_m - \mathbb{E}[F_m]| \geq 2M] + \mathbb{P}[|T_n - F_m| \geq 2M]. \tag{5.5}$$

By subadditivity property,

$$\begin{aligned} |T_n - F_m| = \Big|T_n - \frac{1}{\#B(m)} \sum_{z \in B(m)} T_z\Big| &\leq \frac{1}{\#B(m)} \sum_{z \in B(m)} |T(0, n\mathbf{e}_1) - T(z, z + n\mathbf{e}_1)| \\ &\leq \frac{1}{\#B(m)} \sum_{z \in B(m)} (T(0, z) + T(n\mathbf{e}_1, n\mathbf{e}_1 + z)). \end{aligned} \tag{5.6}$$

Observe that if the event $\Big\{\frac{1}{\#B(m)} \sum_{z \in B(m)} (T(0,z) + T(n\mathbf{e}_1, n\mathbf{e}_1 + z)) \geq 2M\Big\}$ occurs,

$$\max_{z \in B(m)} T(0, z) \geq M \text{ or } \max_{z \in B(m)} T(n\mathbf{e}_1, n\mathbf{e}_1 + z) \geq M. \tag{5.7}$$

Combining this with union bound, it yields that

$$\begin{aligned} &\mathbb{P}\Big[\frac{1}{\#B(m)} \sum_{z \in B(m)} (T(0, z) + T(n\mathbf{e}_1, n\mathbf{e}_1 + z)) \geq 2M\Big] \\ &\quad \leq \mathbb{P}[\max_{z \in B(m)} T(0, z) \geq M] + \mathbb{P}[\max_{z \in B(m)} T(n\mathbf{e}_1, n\mathbf{e}_1 + z) \geq M] \\ &\quad = 2\mathbb{P}[\max_{z \in B(m)} T(0, z) \geq M] \\ &\quad \leq 2(\#B(m)) \max_{z \in B(m)} \mathbb{P}[T(0, z) \geq M], \end{aligned} \tag{5.8}$$

where for the equation we used the translation invariant.

To estimate the probability in (5.8), we need the following lemma.

Lemma 5.3. *Let $x, y \in \mathbb{Z}^d$ and $\gamma_{x,y}$ be a geodesic of $T(x, y)$. Then there exist positive constants ρ, ρ_1, ρ_2 such that for all $t \geq \rho\|x - y\|_\infty$,*

$$\mathbb{P}[\#\gamma_{x,y} \geq t] \leq \mathbb{P}[T(x, y) \geq t] \leq \rho_1 \exp(-\rho_2 t / \log n). \tag{5.9}$$

Proof. By triangle inequality,

$$\begin{aligned}\mathbb{P}[\#\gamma_{x,y} \geq t] &\leq \mathbb{P}[T(x,y) \geq t] \leq \mathbb{P}[T(x,x^*) + T(y,y^*) + T(x^*,y^*) \geq t] \\ &\leq \mathbb{P}[D^*(x,y) \geq t/2] + \mathbb{P}[\|x - x^*\|_\infty \geq \tfrac{t}{4\log n}] + \mathbb{P}[\|y - y^*\|_\infty \geq \tfrac{t}{4\log n}].\end{aligned}$$

By using similar argument as in (3.9), the two last terms are bounded by $\beta_1 \exp(\frac{-\beta_2 t}{\log n})$, for some positive constants β_1, β_2. Hence, there exist constants $\rho, \rho_1, \rho_2 > 0$ such that for all $t \geq \rho \|x - y\|_\infty$,

$$\begin{aligned}\mathbb{P}[\#\gamma_{x,y} \geq t] \leq \mathbb{P}[T_{x,y} \geq t] &\leq \mathbb{P}[T(x,x^*) + T(y,y^*) + T(x^*,y^*) \geq t] \\ &\leq \mathbb{P}[D^*(x,y) \geq t/2] + 2\beta_1 \exp(-\beta_2 t/\log n) \\ &\leq \rho_1 \exp(-\rho_2 t/\log n),\end{aligned}$$

where for the last line we used Lemma 2.3 (i). □

Taking $M = \frac{1}{4}\sqrt{\frac{n}{\log n}}\kappa$. Since $m = o(M)$, using Lemma 5.3, there exist some positive constants ρ_1, ρ_2 such that for any $z\mathbb{Z}^d$,

$$\mathbb{P}[T(0,z) \geq M] \leq e^{-\rho_2 M/\log^2 n}. \tag{5.10}$$

Using this estimate and (5.6), (5.8) gives

$$\mathbb{P}\Big[|T_n - F_m| \geq \frac{\kappa}{2}\sqrt{\frac{n}{\log n}}\Big] \leq \mathcal{O}(n^d) \exp\Big(-\rho_2 \frac{\sqrt{n}}{4\sqrt{\log^5 n}}\kappa\Big).$$

Combining this with (5.3) and (5.5), it follows that there exist some constants $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(|T_n - \mathbb{E}[T_n]| \geq \sqrt{\frac{n}{\log n}}\kappa\right) \leq c_1 e^{-c_2\kappa}, \qquad \forall \kappa \geq 0, \tag{5.11}$$

as desired. □

In the rest of Section 5, we will estimate $\mathrm{Var}[e^{\lambda F_m}]$ by utilizing Falik-Samorodnitsky inequality (Lemma 2.4) with $G = G_\lambda = e^{\lambda F_m}$. This approach involves controlling two crucial ingredients: the influence and entropy on the right and left sides of this inequality, respectively.

5.2 Bound on influences

Proposition 4. *Let $d \geq 2$. There exists a constant $C_1 > 0$,*

$$\sum_{i=1}^{\infty} (\mathbb{E}[|\Delta_i|])^2 \leq C_1 \lambda^2 \mathbb{E}\big[e^{2\lambda F_m}\big] n^{(9-d)/8}, \quad \forall \lambda \in \mathbb{R}.$$

The above proposition is a direct consequence of the following propositions with the notice that $m = n^{1/4}$.

Proposition 5. *Let $d \geq 2$. There exists a constant $C_2 > 0$ such that*

$$\sup_{i \geq 1} \mathbb{E}[|\Delta_i|] \leq C_2 |\lambda| m^{(1-d)/2} \big(\mathbb{E}\big[e^{2\lambda F_m}\big]\big)^{1/2}, \quad \forall \lambda \in \mathbb{R}. \tag{5.12}$$

Proposition 6. *Let $d \geq 2$. There exists a constant $C_3 > 0$ such that*

$$\sum_{i=1}^{\infty} \mathbb{E}[|\Delta_i|] \leq C_3 |\lambda| n \big(\mathbb{E}\big[e^{2\lambda F_m}\big]\big)^{1/2}, \quad \forall \lambda \in \mathbb{R}. \tag{5.13}$$

5.2.1 Proof of Proposition 5

First, we note that for all $i \geq 1$,

$$\Delta_i = \mathbb{E}[G|\mathcal{F}_i] - \mathbb{E}[G|\mathcal{F}_{i-1}] = \mathbb{E}[G(t'_{e_i}, t_{e_i^c}) - G(t_{e_i}, t_{e_i^c})|\mathcal{F}_{i-1}]. \tag{5.14}$$

Thus,

$$\mathbb{E}[|\Delta_i|] \leq \mathbb{E}[|G(t'_{e_i}, t_{e_i^c}) - G(t_{e_i}, t_{e_i^c})|] = 2\mathbb{E}\big[\big(e^{\lambda F_m(t'_{e_i}, t_{e_i^c})} - e^{\lambda F_m(t_{e_i}, t_{e_i^c})}\big)_+\big],$$

where t'_{e_i} is the independent copy of t_{e_i}. Furthermore, using the inequality that $(e^{\lambda a} - e^{\lambda b})_+ \leq |\lambda|(e^{\lambda a} + e^{\lambda b})(a-b)_+$, we get

$$\begin{aligned}\mathbb{E}[|\Delta_i|] &\leq 2|\lambda|\mathbb{E}\big[\big(e^{\lambda F_m(t'_{e_i}, t_{e_i^c})} + e^{\lambda F_m(t_{e_i}, t_{e_i^c})}\big)(F_m(t'_{e_i}, t_{e_i^c}) - F_m(t_{e_i}, t_{e_i^c}))_+\big] \\ &= 4|\lambda|\mathbb{E}\big[e^{\lambda F_m(t_{e_i}, t_{e_i^c})}(F_m(t'_{e_i}, t_{e_i^c}) - F_m(t_{e_i}, t_{e_i^c}))_+\big]. \end{aligned} \tag{5.15}$$

Recall that γ_z is the geodesic of $T_z(1, t_{e_i^c})$ for each $z \in \mathbb{Z}^d$. By Proposition 3 (i), there exists a positive constant C_* and random variable R_{e_i} such that

$$\begin{aligned}T_z(t'_{e_i}, t_{e_i^c}) - T_z(t_{e_i}, t_{e_i^c}) \leq \big(T_z(t'_{e_i}, t_{e_i^c}) - T_z(t_{e_i}, t_{e_i^c})\big)_+ &= T_z(\log^2 n, t_{e_i^c}) - T_z(1, t_{e_i^c}) \\ &\leq (\log^2 n\mathbb{I}(\mathcal{U}_{z,e_i}) + \hat{R}_{e_i})\mathbb{I}(e_i \in \gamma_z),\end{aligned}$$

where

$$\mathcal{U}_{z,e_i} = \{R_{e_i} \geq r_{z,e_i}\}, \quad r_{z,e_i} = \tfrac{1}{2}\|e_i - z\|_\infty \wedge \|e_i - (z + n\mathbf{e}_1)\|_\infty,$$

and

$$\hat{R}_{e_i} = C_* R_{e_i} \wedge \log^2 n.$$

Therefore,

$$\begin{aligned}F_m(t'_{e_i}, t_{e_i^c}) - F_m(t_{e_i}, t_{e_i^c}) \leq \big(F_m(t'_{e_i}, t_{e_i^c}) - F_m(t_{e_i}, t_{e_i^c})\big)_+ &= F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c}) \\ &= \tfrac{1}{\#B(m)} \sum_{z \in B(m)} (T_z(\log^2 n, t_{e_i^c}) - T_z(1, t_{e_i^c})) \leq S_i, \end{aligned}\tag{5.16}$$

where

$$S_i = \tfrac{1}{\#B(m)} \sum_{z \in B(m)} \big(\log^2 n\mathbb{I}(\mathcal{U}_{z,e_i}) + \hat{R}_{e_i}\big)\mathbb{I}(e_i \in \gamma_z).$$

We decompose S_i as

$$\begin{aligned}S_i = \tfrac{1}{\#B(m)} \sum_{z \in B(m)} &\big(\log^2 n\mathbb{I}(\mathcal{U}_{z,e_i})\mathbb{I}(r_{z,e_i} \geq \tfrac{\log^2 n}{C_*}) + \hat{R}_{e_i}\big)\mathbb{I}(e_i \in \gamma_z) \\ &+ \tfrac{1}{\#B(m)} \sum_{z \in B(m)} \log^2 n\mathbb{I}(\mathcal{U}_{z,e_i})\mathbb{I}(r_{z,e_i} < \tfrac{\log^2 n}{C_*})\mathbb{I}(e_i \in \gamma_z).\end{aligned}$$

If the event $\mathcal{U}_{z,e_i} \cap \{r_{z,e_i} \geq \log^2 n/C_*\}$ occurs, then $C_* R_{e_i} \geq \log^2 n$. Therefore, thanks to the definition of $\hat{R}_{e_i}$ and (5.16), we can assert that

$$\big(F_m(t'_{e_i}, t_{e_i^c}) - F_m(t_{e_i}, t_{e_i^c})\big)_+ \leq S_i \leq A_i,$$

where

$$A_i = \tfrac{2}{\#B(m)} \sum_{z\in B(m)} \hat{R}_{e_i}\mathbb{I}(e_i \in \gamma_z) \\ + \tfrac{1}{\#B(m)} \sum_{z\in B(m)} \log^2 n \mathbb{I}(\{r_{z,e_i} \le \tfrac{\log^2 n}{C_*}\} \cap \{e_i \in \gamma_z\}). \tag{5.17}$$

Combining this with (5.15) and Cauchy-Schwarz inequality yields

$$\mathbb{E}[|\Delta_i|] \le 4|\lambda|\mathbb{E}\big[e^{\lambda F_m} A_i\big] \tag{5.18}$$

$$\le 4|\lambda|\mathbb{E}\big[e^{2\lambda F_m}\big]^{1/2}\mathbb{E}\big[A_i^2\big]^{1/2}. \tag{5.19}$$

Here for the first line, we remark that $F_m(t_{e_i}, t_{e_i^c}) = F_m$.

Next we will estimate for $\mathbb{E}[A_i^2]$. Note that

$$\sum_{z\in\mathbb{Z}^d} \mathbb{I}(r_{z,e_i} \le \tfrac{\log^2 n}{C_*}) = \mathcal{O}(\log^{3d} n). \tag{5.20}$$

Thus, by Cauchy-Schwarz inequality,

$$A_i^2 \le \tfrac{8}{\#B(m)} \sum_{z\in B(m)} \hat{R}_{e_i}^2 \mathbb{I}(e_i \in \gamma_z) + \tfrac{\mathcal{O}((\log n)^{3d+4})}{\#B(m)}. \tag{5.21}$$

By the translation invariant, we have

$$\mathbb{E}[A_i^2] \le \tfrac{8}{\#B(m)}\mathbb{E}\Big[\sum_{z\in B(m)} \hat{R}_{e_i-z}^2 \mathbb{I}(e_i - z \in \gamma_0)\Big] + \tfrac{\mathcal{O}((\log n)^{3d+4})}{\#B(m)}$$

$$= \tfrac{8}{\#B(m)}\mathbb{E}\Big[\sum_{e\in\gamma} \hat{R}_e^2\Big] + \tfrac{\mathcal{O}((\log n)^{3d+4})}{\#B(m)}, \tag{5.22}$$

where

$$\gamma = \gamma_0 \cap \{e_i - B(m)\},\ \{e_i - B(m)\} = \{(x_{e_i} - z, y_{e_i} - z) : z \in B(m)\}.$$

Let us now estimate the first term in (5.22). For any pair $x, y \in \mathbb{Z}^d$, we call $\gamma_{x,y}$ a geodesic from x to y. By the union bound,

$$\mathbb{P}[\#\gamma \ge l] \le \mathbb{P}\left[\exists x, y \in e_i - B(m) : \#\gamma_{x,y} \ge l\right] \le (2m+1)^{2d} \max_{x,y\in V} \mathbb{P}[\#\gamma_{x,y} \ge l]. \tag{5.23}$$

Notice that $\|x - y\|_\infty \le 2m$ for all $x, y \in e_i - B(m)$. Then by using Lemma 5.3 and (5.23), we have for all $l \ge 2\rho m$,

$$\mathbb{P}[\#\gamma \ge l] \le \rho_1(2m+1)^{2d}\exp(-\rho_2 l/\log^2 n) \le \mathcal{O}(1)\exp(-\tfrac{\rho_2 l}{2\log^2 n}). \tag{5.24}$$

It can be seen that the functions f^2, f^4 satisfy (H) with $f(x) = x^2$ and $X_e = R_e$ satisfies the conditions (P1) and (P2) in Proposition 3. Therefore, thanks to (5.24), applying Lemma 4.3 (iii) to $f(x) = x^2$, $X_e = R_e$, $\gamma = \gamma_0 \cap \{e_i - B(m)\}$, $L = 2\rho m$, we obtain that

$$\mathbb{E}\Big[\sum_{e\in\gamma} \hat{R}_e^2\Big] \le C_*^2\mathbb{E}\Big[\sum_{e\in\gamma} R_e^2\Big] \le C_*^2\Big(\mathbb{E}\Big[\Big(\sum_{e\in\gamma} R_e^2\Big)^2\Big]\Big)^{1/2} \overset{\text{Lem 4.3 (iii)}}{=} \mathcal{O}(m). \tag{5.25}$$

Combining (5.22) and (5.25), it yields that for all $i \geq 1$,

$$\mathbb{E}[A_i^2] = \mathcal{O}(m^{1-d}). \tag{5.26}$$

Finally, we conclude from (5.19) and (5.26) that there exists a positive constant C_2 such that

$$\sup_{i\geq 1} \mathbb{E}[|\Delta_i|] \leq C_2|\lambda|\big(\mathbb{E}\big[e^{2\lambda F_m}\big]\big)^{1/2} m^{(1-d)/2}, \tag{5.27}$$

and the result follows. □

5.2.2 Proof of Proposition 6

Using (5.18) and Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}\sum_{i=1}^{\infty} \mathbb{E}[|\Delta_i|] &\leq 4|\lambda|\mathbb{E}\Big[e^{\lambda F_m}\sum_{i=1}^{\infty} A_i\Big] \\ &\leq 4|\lambda|\big(\mathbb{E}\big[e^{2\lambda F_m}\big]\big)^{1/2}\Big(\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty} A_i\Big)^2\Big]\Big)^{1/2},\end{aligned} \tag{5.28}$$

where A_i is defined as in (5.17). By Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty} A_i\Big)^2\Big] &\leq \tfrac{8}{\#B(m)} \sum_{z\in B(m)} \mathbb{E}\Big[\Big(\sum_{i=1}^{\infty} \hat{R}_{e_i}\mathbb{I}(e_i \in \gamma_z)\Big)^2\Big] \\ &\quad + \tfrac{2\log^4 n}{\#B(m)} \sum_{z\in B(m)} \mathbb{E}\Big[\Big(\sum_{i=1}^{\infty} \mathbb{I}(\{r_{z,e_i} \leq \tfrac{\log^2 n}{C_*}\} \cap \{e_i \in \gamma_z\})\Big)^2\Big] \\ &= \tfrac{8}{\#B(m)} \sum_{z\in B(m)} \mathbb{E}\Big[\Big(\sum_{e\in\gamma_z} \hat{R}_e\Big)^2\Big] \\ &\quad + \tfrac{2\log^4 n}{\#B(m)} \sum_{z\in B(m)} \mathbb{E}\Big[\Big(\sum_{e\in\gamma_z} \mathbb{I}(r_{z,e} \leq \tfrac{\log^2 n}{C_*})\Big)^2\Big].\end{aligned} \tag{5.29}$$

The second term of (5.29) can be bounded by

$$\tfrac{2\log^4 n}{\#B(m)} \sum_{z\in B(m)} \Big(\sum_{e\in\mathcal{E}(\mathbb{Z}^d)} \mathbb{I}\big(r_{z,e} \leq \tfrac{\log^2 n}{C_*}\big)\Big)^2 \leq \mathcal{O}((\log n)^{6d+4}). \tag{5.30}$$

We next estimate the first term. It follows from Lemma 5.3 that

$$\mathbb{P}[\#\gamma_z \geq \rho n] \leq \rho_1 \exp(-\rho_2 n/\log^2 n).$$

Then applying Lemma 4.3 (ii) to $f(x) = x, X_e = R_e, \gamma = \gamma_z, L = \rho n$, we have

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma_z} \hat{R}_e\Big)^2\Big] \leq C_*^2\mathbb{E}\Big[\Big(\sum_{e\in\gamma_z} R_e\Big)^2\Big] \overset{\text{Lem 4.3 (ii)}}{=} \mathcal{O}(n^2). \tag{5.31}$$

Combining (5.29) and (5.30) with (5.31) yields

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty} A_i\Big)^2\Big] = \mathcal{O}(n^2). \tag{5.32}$$

Plugging this into (5.28), it follows that

$$\sum_{i=1}^{\infty} \mathbb{E}[|\Delta_i|] \leq \mathcal{O}(1)|\lambda| n \big(\mathbb{E}\big[e^{2\lambda F_m}\big]\big)^{1/2},$$

as desired. □

5.3 Entropy bound

The goal of this subsection is to prove the following upper bound for entropy.

Proposition 7. *Let $d \geq 2$. There exist constants $C_4 > 0$ such that*

$$\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] \leq C_4 \lambda^2 n \mathbb{E}\big[e^{2\lambda F_m}\big], \quad \forall \lambda \leq \tfrac{1}{\log^{2d+11} n}. \tag{5.33}$$

Proof. First, we get the upper bound of the entropy by using Lemma 2.5.

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] &\leq C \sum_{i=1}^{\infty} \mathbb{E}\Big[\big(G(\log^2 n, t_{e_i^c}) - G(1, t_{e_i^c})\big)^2\Big] \\
&= C \sum_{i=1}^{\infty} \mathbb{E}\Big[\Big(e^{\lambda F_m(\log^2 n, t_{e_i^c})} - e^{\lambda F_m(1, t_{e_i^c})}\Big)^2\Big] \\
&\leq 2C|\lambda|^2 \sum_{i=1}^{\infty} \mathbb{E}\Big[\Big(e^{2\lambda F_m(\log^2 n, t_{e_i^c})} + e^{2\lambda F_m(1, t_{e_i^c})}\Big) \\
&\qquad\qquad \times \big(F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c})\big)^2\Big].
\end{aligned} \tag{5.34}$$

Here for the last line we used the inequality that $|e^{\lambda a} - e^{\lambda b}| \leq |\lambda|(e^{\lambda a} + e^{\lambda b})(a - b)$ for all $a \geq b$. Notice that

$$\begin{aligned}
&\mathbb{E}\Big[e^{2\lambda F_m(\log^2 n, t_{e_i^c})}\big(F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c})\big)^2\Big] \\
&\qquad = \tfrac{1}{1-p} \mathbb{E}\Big[e^{2\lambda F_m(\log^2 n, t_{e_i^c})}\big(F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c})\big)^2 \mathbb{I}(t_e = \log^2 n)\Big] \\
&\qquad \leq \tfrac{1}{1-p} \mathbb{E}\Big[e^{2\lambda F_m}\big(F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c})\big)^2\Big].
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[e^{2\lambda F_m(1, t_{e_i^c})}\big(F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c})\big)^2\Big] \\
&\qquad = \tfrac{1}{p} \mathbb{E}\Big[e^{2\lambda F_m(1, t_{e_i^c})}\big(F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c})\big)^2 \mathbb{I}(t_e = 1)\Big] \\
&\qquad \leq \tfrac{1}{p} \mathbb{E}\Big[e^{2\lambda F_m}\big(F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c})\big)^2\Big].
\end{aligned}$$

Combining these inequalities with (5.34), we get

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathrm{Ent}[\Delta_i^2] &\leq \mathcal{O}(1)\lambda^2 \sum_{i=1}^{\infty} \mathbb{E}\Big[e^{2\lambda F_m}\big(F_m(\log^2 n, t_{e_i^c}) - F_m(1, t_{e_i^c})\big)^2\Big] \\
&\leq \mathcal{O}(1)\lambda^2 \sum_{i=1}^{\infty} \mathbb{E}\big[e^{2\lambda F_m} A_i^2\big],
\end{aligned} \tag{5.35}$$

where for the last line we used (5.16) and (5.17). Now we recall that

$$A_i = \tfrac{1}{\#B(m)} \sum_{z\in B(m)} \big(2\hat{R}_{e_i} + \log^2 n \mathbb{I}(r_{z,e_i} \le \tfrac{\log^2 n}{C_*})\big)\mathbb{I}(e_i \in \gamma_z).$$

Thanks to Cauchy-Schwarz inequality,

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathbb{E}\Big[e^{2\lambda F_m} A_i^2\Big] &\le \tfrac{1}{\#B(m)} \sum_{z\in B(m)} \mathbb{E}\Big[e^{2\lambda F_m} \sum_{i=1}^{\infty} \big(2\hat{R}_{e_i} + \log^2 n \mathbb{I}(r_{z,e_i} \le \tfrac{\log^2 n}{C_*})\big)^2 \mathbb{I}(e_i \in \gamma_z)\Big] \\
&\le \tfrac{1}{\#B(m)} \sum_{z\in B(m)} \mathbb{E}\Big[e^{2\lambda F_m} \sum_{e\in\gamma_z} \big(2\hat{R}_e + \log^2 n \mathbb{I}(r_{z,e} \le \tfrac{\log^2 n}{C_*})\big)^2\Big] \\
&\le \tfrac{2}{\#B(m)} \sum_{z\in B(m)} \mathbb{E}\Big[e^{2\lambda F_m} \sum_{e\in\gamma_z} \big(4\hat{R}_e^2 + \log^4 n \mathbb{I}(r_{z,e} \le \tfrac{\log^2 n}{C_*})\big)\Big] \\
&= \tfrac{2}{\#B(m)} \sum_{z\in B(m)} \mathbb{E}\Big[e^{2\lambda F_m} Y_{\gamma_z}\Big]. \qquad (5.36)
\end{aligned}$$

where

$$Y_{\gamma_z} = \sum_{e\in\gamma_z} \big(4\hat{R}_e^2 + \log^4 n \mathbb{I}(r_{z,e} \le \tfrac{\log^2 n}{C_*})\big)\Big].$$

Next, we consider large deviation estimates for Y_{γ_z}. Using (5.30) gives us

$$Y_{\gamma_z} \le 4 \sum_{e\in\gamma_z} \hat{R}_e^2 + \mathcal{O}((\log n)^{3d+4}). \qquad (5.37)$$

It is easy to check that $f(x) = x^2, \phi(x) = 1/x^{d^2+5d}$ satisfy (H). Additionally, there exists a positive constant L_0 depending on d such that $L(\phi(\log^2 L))^{1/d} \ge 1$ for all $L \ge L_0$. Then, applying Lemma 4.3 (iv) to $f(x) = x^2$, $X_e = C_* R_e$, $\hat{X}_e = X_e \wedge \log^2 L \ge \hat{R}_e$, $\phi(x) = 1/x^{d^2+5d}$, $\gamma = \gamma_z$, it yields that there exist positive constants α, β satisfying for all $L \ge L_0$,

$$\mathbb{P}\Big[\sum_{e\in\gamma_z} \hat{R}_e^2 \ge \alpha L\Big] \le \mathbb{P}\Big[\sum_{e\in\gamma_z} \hat{X}_e^2 \ge \alpha L\Big] \le \log^2 L \exp\big(-\beta \tfrac{L}{\log^{2d+10} L}\big) + \mathbb{P}[\#\gamma_z \ge L].$$

Furthermore, thanks to Lemma 5.3, taking $L = \rho n$,

$$\mathbb{P}[\#\gamma_z \ge \rho n] \le \rho_1 \exp(-\rho_2 \rho n / \log^2 n).$$

The above inequalities gives

$$\mathbb{P}\Big[\sum_{e\in\gamma_z} \hat{R}_e^2 \ge \alpha\rho n\Big] \le \mathcal{O}(1) \log^2 n \exp\big(-\beta \tfrac{n}{\log^{2d+10} n}\big), \qquad (5.38)$$

which together with (5.37) implies

$$\mathbb{P}[Y_{\gamma_z} \ge 8\alpha\rho n] \le \mathcal{O}(1) \log^2 n \exp\big(-\beta \tfrac{n}{\log^{2d+10} n}\big). \qquad (5.39)$$

On the other hand,

$$\mathbb{E}\Big[e^{2\lambda F_m} Y_{\gamma_z}\Big] \le 8\alpha\rho n \mathbb{E}\Big[e^{2\lambda F_m}\Big] + \mathbb{E}\Big[e^{2\lambda F_m} Y_{\gamma_z} \mathbb{I}(Y_{\gamma_z} \ge 8\alpha\rho n)\Big]. \qquad (5.40)$$

It follows from the Cauchy-Schwarz inequality that

$$\mathbb{E}\Big[e^{2\lambda F_m} Y_{\gamma_z}\mathbb{I}(Y_{\gamma_z} \geq 8\alpha\rho n)\Big] \leq (\mathbb{E}[e^{8\lambda F_m}])^{1/4}(\mathbb{P}[Y_{\gamma_z} \geq 8\alpha\rho n])^{1/4}(\mathbb{E}[Y_{\gamma_z}^2])^{1/2}. \tag{5.41}$$

By using transition invariance and Lemma 5.3, there exist some positive constants ρ, ρ_1, ρ_2 such that for all $\lambda \leq \frac{1}{\log^{2d+11} n}$,

$$\begin{aligned}\mathbb{E}[e^{8\lambda F_m}] &\leq \frac{1}{\#B(m)} \sum_{z \in B(m)} \mathbb{E}[e^{8\lambda T_z}] = \mathbb{E}[e^{8\lambda T_n}] \leq e^{8\rho\lambda n} + \sum_{t=\rho n}^{\infty} \mathbb{P}[T_n \geq t] e^{8\lambda t} \\ &\leq e^{8\rho\lambda n} + \sum_{t=\rho n}^{\infty} \rho_1 e^{-\rho_2 t/\log^2 n} e^{8\lambda t} \leq \mathcal{O}(1) \exp\left(\tfrac{8\rho n}{\log^{2d+11} n}\right).\end{aligned} \tag{5.42}$$

Moreover, it follows from (5.31) and (5.37) that

$$\mathbb{E}[Y_{\gamma_z}^2] \leq 32\mathbb{E}\Big[\sum_{e \in \gamma_z} \hat{R}_e^2\Big] + \mathcal{O}((\log n)^{6d+8}) = \mathcal{O}(n^2). \tag{5.43}$$

Combining this with (5.39), (5.41) and (5.42) gives

$$\mathbb{E}\big[e^{2\lambda F_m} Y_{\gamma_z}\mathbb{I}(Y_{\gamma_z} \geq 8\alpha\rho n)\big] = \mathcal{O}(1).$$

Plugging this into (5.40) yields

$$\mathbb{E}\big[e^{2\lambda F_m} Y_{\gamma_z}\big] \leq \mathcal{O}(1) n \mathbb{E}\big[e^{2\lambda F_m}\big].$$

Using this estimate, (5.35) and (5.36), we conclude that

$$\sum_{i=1}^{\infty} \mathrm{Ent}_\zeta[\Delta_i^2] \leq \mathcal{O}(1)\lambda^2 n \mathbb{E}\big[e^{2\lambda F_m}\big].$$

□

5.4 Proof of Theorem 5.1

By Lemma 2.4, Theorem 4, Proposition 7, we have

$$\mathrm{Var}\left[e^{\lambda F_m}\right] \leq C_4\Big(\log \frac{\mathrm{Var}[e^{\lambda F_m}]}{C_1|\lambda|^2 n^{(1-d)/8}\mathbb{E}\big[e^{2\lambda F_m}\big]}\Big)^{-1} |\lambda|^2 n \mathbb{E}\big[e^{2\lambda F_m}\big]. \tag{5.44}$$

We can assume that

$$\mathrm{Var}[e^{\lambda F_m}] \geq C_1|\lambda|^2 n^{15/16}\mathbb{E}\left[e^{2\lambda F_m}\right], \tag{5.45}$$

since otherwise there is nothing left to prove. By (5.44) and (5.45), there exist constants $c, C > 0$ such that for any $\lambda \leq \frac{C}{\log^{13} n}$,

$$\mathrm{Var}\left[e^{\lambda F_m}\right] \leq c|\lambda|^2 \frac{n}{\log n}\mathbb{E}\big[e^{2\lambda F_m}\big].$$

This concludes the proof of Theorem 5.1 by substituting $\lambda/2$ for λ. □

6 Discrepancy between the graph distance and first passage time

The difference between D_n^* and $T(0^*, (n\mathbf{e}_1)^*)$ is due to the change of $T(0^*, (n\mathbf{e}_1)^*)$ in weight distribution of edges from $\zeta = p\delta_1 + (1-p)\delta_{\log^2 n}$ to $\zeta = p\delta_1 + (1-p)\delta_\infty$. We can avoid one $\log^2 n$-weight edge e on the geodesic of $T(0^*, (n\mathbf{e}_1)^*$ using the bypass as in Subsection 4.1. In fact, this bypass can avoid all edges of the geodesic of $T(0^*, (n\mathbf{e}_1)^*)$ inside $B_e(R_e)$. Therefore, we can construct a new path consisting of 1-weight edges using a simple procedure to control this discrepancy.

The following lemma is key result to prove Theorem 3.1.

Lemma 6.1. *Let γ_n be a geodesic of $T(0^*, (n\mathbf{e}_1)^*)$. Suppose that $R_e \geq n/2$ for all $e \in \gamma_n$. There exists a deterministic rule to find a subset $\gamma_n' \subset \gamma_n$ such that the following holds.*

(i) If $e \in \gamma_n'$ then $t_e = \log^2 n$,

(ii) for all $e, f \in \gamma_n'$,

$$\|e - f\|_\infty \geq \max\{R_e, R_f\},$$

(iii)

$$|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \leq C_* \sum_{e \in \gamma_n'} R_e,$$

where $(R_e)_{e \in \mathcal{E}(\mathbb{Z}^d)}$ are the effective radii defined in Subsection 4.1, and C_* is the constant as in Proposition 3.

Proof. For each $e \in \gamma$ and $t_e = \log^2 n$, let us recall that $A_e(R_e) = B_e(2R_e) \setminus B_e(R_e)$ the annulus with the center e and radius R_e. Notice that $A_e(R_e)$ satisfies the good connectivity property, i.e. for any two crossing paths, there exists a link of 1-weight edges with length at most $C_* R_e$ between them, and the graph distance between them two connected points is always smaller than $C_* R_e$.

We consider a procedure of detecting bad boxes as follows.

Input:

- γ a path from 0^* to $(n\mathbf{e}_1)^*$,
- $\gamma' \subset \mathcal{E}(\mathbb{Z}^d)$.

Output: an edge e, and $\tilde{\gamma}$ a path from $0^* \to (n\mathbf{e}_1)^*$ such that

- $clo(\tilde{\gamma}) \subset clo(\gamma) \setminus \{e\}$ where $clo(\gamma) = \{e \in \gamma : t_e = \log^2 n\}$,
- $\tilde{\gamma} \cap B_e(R_e) = \emptyset$; $\tilde{\gamma}' = \gamma' \cup \{e\}$.

Procedure: We always define

$$e = \arg\max\{R_f : f \in \gamma;\, t_f = \log^2 n\};$$
$$\tilde{\gamma}' = \gamma' \cup \{e\}.$$

In the case that there several maximize edges, we choose one of them in a deterministic rule. Next, we construct $\tilde{\gamma}$ as follows.

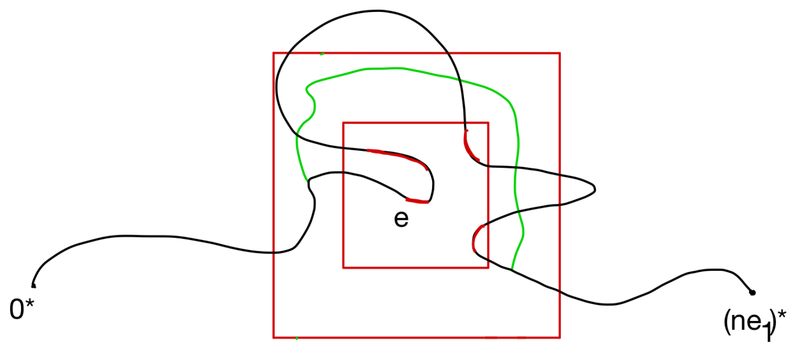

Figure 4: Illustration of the case where e is not close to both 0^* and $(n\mathbf{e}_1)^*$)

Case 1: Both 0^* and $(n\mathbf{e}_1)^*$ are not in $B_e(2R_e)$.

It follows that γ crosses the annulus $A_e(2R_e, R_e)$ at least twice. On the event $\mathcal{V}_e^1(R_e) \cap \mathcal{V}_e^2(R_e)$, there exists a bypass of e in $A_e(2R_e, R_e)$ denoted by γ_e, which links between these crossing paths. We assume that γ_e intersects with the two crossing paths at H_e and Q_e. We define a new path,

$$\tilde{\gamma} := (\gamma \setminus \gamma_{H_e Q_e}) \cup \gamma_e, \tag{6.1}$$

where $\gamma_{H_e Q_e}$ is the subpath of γ starting and ending at H_e and Q_e, respectively. Thanks to the definition of R_e in Subsection 4.1,

$$\#\gamma_e \leq C_* R_e.$$

Case 2: At least 0^* or $(n\mathbf{e}_1)^*$ is in $B_e(2R_e)$.

By assumption that $R_e \leq n/2$, there is only one 0^* or $(n\mathbf{e}_1)^*$ is in $B_e(2R_e)$. Suppose that $0^* \in B_e(2R_e)$. The remaining case is similar and omitted. The path γ crosses the annulus $A_e(2R_e, R_e)$ at least once. We call the last crossing path by Γ. Furthermore, by the definition of 0^*, there is a path of 1-weight edges connecting 0^* to ∞ denoted by γ_∞. Obviously, γ_∞ crosses the annulus $A_e(R_e)$ and we call the first crossing path by Γ'. By the definition of R_e, there exists a path of 1-weight edges γ_e linking γ and γ'. We call the intersection of γ_e and these crossing path by H_e and Q_e, respectively. Now we define

$$\tilde{\gamma} := \gamma_{0^*, Q_e} \cup \gamma_e \cup (\gamma \setminus \gamma_{0^*, H_e}), \tag{6.2}$$

where γ_{0^*, H_e} is the subpath of γ starting and ending at 0^* and H_e, respectively and γ_{0^*, Q_e} is the subpath of γ_∞ starting and ending at 0^* and Q_e, respectively. Furthermore, thanks to the definition of R_e again,

$$\max\{\gamma_{0^* Q_e}, \#\gamma_e\} \leq C_* R_e.$$

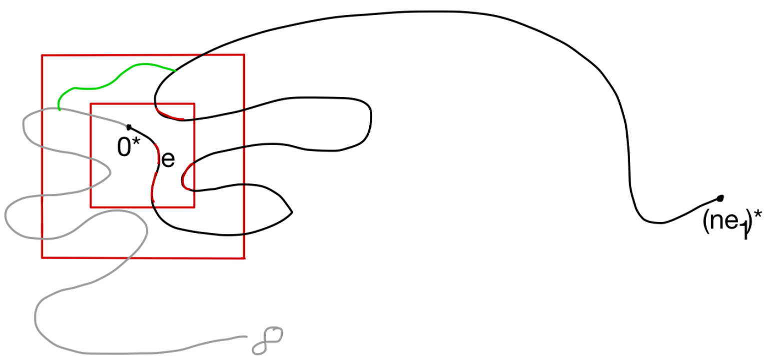

Figure 5: Illustration of the case where e is close to 0^*

We iteratively apply this procedure to γ_n until there is no $\log^2 n$-weight edges. We call the final set of maximizer edges by this procedure by γ_n' and the final path by $\tilde{\gamma}_n$. It follows that $\tilde{\gamma}_n$ has no edges with $\log^2 n$-weight.

Additionally,

$$0 \le D_n^* - T(0^*, (ne_1)^*) \le \#\tilde{\gamma}_n - T(0^*, (ne_1)^*) \le 2C_* \sum_{e \in \gamma_n'} R_e. \tag{6.3}$$

We now enumerate the path γ_n' as $\{e_1, \ldots, e_{l'}\}$. By construction of γ_n', we note that

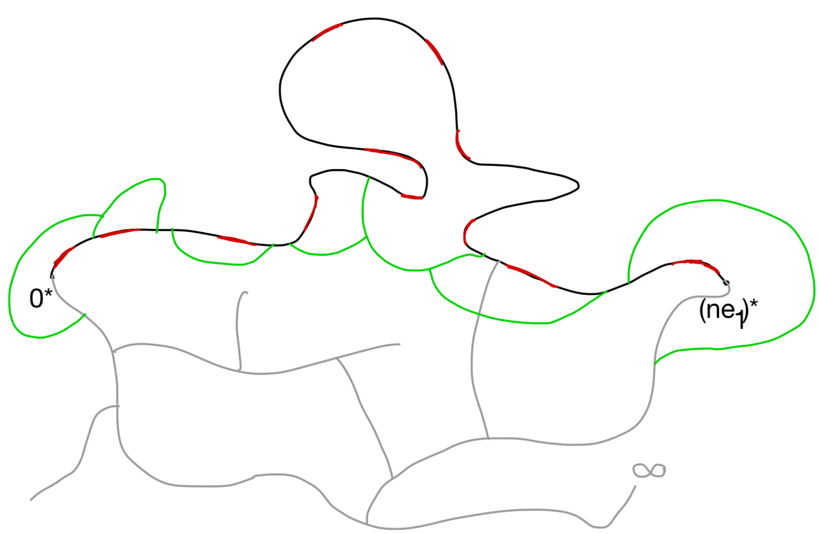

Figure 6: Illustration of a path of 1-weight edges $\tilde{\gamma}_n$ avoiding all $\log^2 n$-weight edges (red) from 0^* to $(n\mathbf{e}_1)^*$

$(R_{e_i})_{1\le i\le l'}$ is non-increasing sequence of radii. Moreover,

$$B_{e_i}(R_{e_i}) \cap B_{e_j}(R_{e_j}) = \emptyset, \quad \forall 1 \le i < j \le l'.$$

Hence, we have for all $1 \le j < k \le l'$,

$$\|e_j - e_k\|_\infty \ge R_{e_j} = \max(R_{e_j}, R_{e_k}),$$

and thus the statement (ii) follows. □

Proof of Theorem 3.1. For the convenience, we recall the desired statement: there exist $c_1, c_2 > 0$ such that for all $L \ge \log^2 n$,

$$\mathbb{P}\Big[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \ge L\Big] \le c_1 \exp(-c_2 L/\log L).$$

Case 1: $L \ge \rho n$ where ρ is the positive constant in Lemma 2.3.

Thank to Lemma 2.3 (i),

$$\mathbb{P}\Big[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \ge L\Big] \le \mathbb{P}\Big[D_n^* \ge L\Big] \le \rho_1 \exp(-\rho_2 L). \tag{6.4}$$

Case 2: $L \le \rho n$.

By using Lemma 6.1, we have for n large enough,

$$\begin{aligned}
&\mathbb{P}\Big[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \ge L\Big] \\
&\quad \le \mathbb{P}[\exists e \in \gamma_n : R_e > \tfrac{L}{2\rho}] + \mathbb{P}\Big[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \ge L; R_e \le \tfrac{L}{2\rho}\, \forall e \in \gamma_n\Big] \\
&\quad \le \mathbb{P}[\exists e \in \gamma_n : R_e > \tfrac{L}{2\rho}] + \mathbb{P}\Big[\sum_{j=1}^{l'} 2C_* R_{e_{i_j}} \mathbb{I}(R_{e_{i_j}} \le \tfrac{L}{2\rho}) \ge L\Big],
\end{aligned} \tag{6.5}$$

where $\gamma_n' = \{e_{i_1}, e_{i_2}, \ldots, e_{i_{l'}}\}$ is the subset of γ_n obtained in Lemma 6.1.

For the first term of (6.5), using Propositions 3 and 2.3 (i), we can estimate

$$\begin{aligned}
&\mathbb{P}[\exists e \in \gamma_n : R_e > \tfrac{L}{2\rho}] \\
&\le \mathbb{P}[\exists e \in \gamma_n : R_e > \tfrac{L}{2\rho}, \#\gamma_n \le \rho n] + \mathbb{P}[\#\gamma_n \ge \rho n] \\
&\le \alpha_1 (2\rho n)^d \exp(-\tfrac{\alpha_2 L}{2\rho}) + \rho_1 \exp(-\rho_2 n) \le \mathcal{O}(1) \exp(-\tfrac{\alpha_2 L}{4\rho}).
\end{aligned} \tag{6.6}$$

To compute the second term of (6.5), we first set $M_0 = \log^2 n$, $M_q = M_0 2^q$ and $M_{a_n} = L$ with $a_n = \lceil \log_2 L - 2\log_2 \log n \rceil$. It is clear that

$$\sum_{i=1}^{l'} R_{e_i} \mathbb{I}(\mathbb{R}_{e_i} \le \tfrac{L}{2\rho}) \le \sum_{q=0}^{a_n} 2M_q N_q,$$

where

$$N_q = \#\{1 \le i \le l' : R_{e_i} \in [M_q, 2M_q]\}.$$

Therefore, by using union bound we obtain

$$\mathbb{P}\Big[\sum_{i=1}^{l'} 2C_* R_{e_i} \mathbb{I}(\mathbb{R}_{e_i} \le \tfrac{L}{2\rho_1}) \ge L\Big] \le \sum_{q=0}^{[a_n]} \mathbb{P}\Big[N_q \ge b_q\Big], \tag{6.7}$$

where

$$b_q := \Big\lfloor \frac{L}{4C_* M_q \log_2 L} \Big\rfloor.$$

Note that if R_{e_j} and R_{e_k} are in $[M_q, 2M_q]$ for some $1 \leq j, k \leq l'$ then

$$\|e_j - e_k\|_\infty \geq \max(R_{e_j}, R_{e_k}) \geq M_q. \tag{6.8}$$

Moreover, since $t_e \in \{1, \log^2 n\}$, we get a brutal bound for the length of geodesic γ_n of $T(0^*, (n\mathbf{e}_1)^*)$

$$\#\gamma_n \leq T(0^*, (n\mathbf{e}_1)^*) \leq (\log n)^2 n, \tag{6.9}$$

which implies that for all $1 \leq i \leq l'$,

$$\|e_i\|_\infty \leq (\log n)^2 n. \tag{6.10}$$

Using (6.8), (6.10) and the definition of N_q, we have

$$\mathbb{P}\Big[N_q \geq b_q\Big] \leq S_q, \tag{6.11}$$

where

$$\begin{aligned} S_q :=& \mathbb{P}\big[\text{there exist } b_q \text{ edges } f_1, \ldots, f_{b_q} \subset [-(\log n)^2 n, (\log n)^2 n]^d \text{ such that} \\ & R_{f_j} \in [M_q, 2M_q]\, \forall 1 \leq j \leq b_q \text{ and } \|f_j - f_k\|_\infty \geq M_q \,\forall 1 \leq j \neq k \leq [b_q]\big]. \end{aligned} \tag{6.12}$$

The following claim is straightforward.

Claim 8. *There exists a constant $c = c(d) > 0$ such that the following holds: for any $M \in \mathbb{N}$ and $\Lambda \subset \mathbb{Z}^d$ satisfying*

$$\|u - v\|_\infty \geq M, \quad \forall u, v \in \Lambda,$$

we can find $\Lambda' \subset \Lambda$ such that $|\Lambda'| \geq c|\Lambda|$ and $\|u - v\|_\infty \geq 14M$ for all $u, v \in \Lambda'$.

By this claim, there exists a positive constant c depending on d such that for any $1 \leq q \leq a_n$ if the event in (6.12) occurs then we can find $\Lambda' \subset \{f_1, \ldots, f_{b_q}\}$ such that $|\Lambda'| \geq \lfloor cb_q \rfloor$ and $\|e - f\|_\infty \geq 14M_q$ for all $e, f \in \Lambda'$. As a result, we have

$$\begin{aligned} S_q \leq& \mathbb{P}\big[\text{there exist } \lfloor cb_q \rfloor \text{ edges } f'_1, \ldots, f'_{\lfloor cb_q \rfloor} \text{ such that} \\ & \|f'_j - f'_k\|_\infty \geq 14M_q \forall 1 \leq j \neq k \leq \lfloor cb_q \rfloor \text{ and } R_{f'_j} \in [M_q, 2M_q] \quad \forall 1 \leq j \leq \lfloor cb_q \rfloor\big] \\ \leq& \sum_{(f'_1, \ldots, f'_{\lfloor cb_q \rfloor}) \in \mathcal{T}} \mathbb{P}[R_{f'_j} \in [M_q, 2M_q] \quad \forall 1 \leq j \leq \lfloor cb_q \rfloor], \end{aligned} \tag{6.13}$$

where

$$\mathcal{T} = \{(f'_1, \ldots, f'_{\lfloor cb_q \rfloor}) \in [-(\log n)^2 n, (\log n)^2 n]^d : \|f'_j - f'_k\|_\infty \geq 14M_q \forall 1 \leq j, k \leq \lfloor cb_q \rfloor\}.$$

It is clear that $|\mathcal{T}| \leq (2(\log n)^2 n)^{cdb_q}$. We remark that the event $R_e \in [M_q, 2M_q]$ only depends on the status of edges in $B_e(6M_q)$. Therefore, given $(f'_1, \ldots, f'_{\lfloor cb_q \rfloor}) \in \mathcal{T}$, the family of the events $(\{R_{f'_j})_{1 \leq j \leq \lfloor cb_q \rfloor}$ are independent. Hence,

$$\mathbb{P}[R_{f'_j} \in [M_q, 2M_q] \quad \forall 1 \leq j \leq \lfloor cb_q \rfloor] = \prod_{j=1}^{\lfloor cb_q \rfloor} \mathbb{P}[R_{f'_j} \in [M_q, 2M_q]] \leq \exp(-\alpha_2 c M_q b_q), \tag{6.14}$$

where for the inequality we used Lemma 3 (ii) with $\alpha_2 > 0$ a positive constant. Combining this estimate and (6.13), we obtain that

$$\begin{aligned} S_q \leq |\mathcal{T}| \exp(-\alpha_2 c M_q b_q) &\leq (C(\log n)^2 n)^{cdb_q} \exp(-\alpha_2 c M_q b_q) \\ &\leq \exp(-\alpha_2 c M_q b_q/2) \\ &\leq \exp(-\alpha_2 L/\log L). \end{aligned}$$

Using this with (6.11), (6.7),(6.6) and (6.5) yields

$$\begin{aligned} \mathbb{P}\big[|D_n^* - T(0^*, (n\mathbf{e}_1)^*)| \geq L\big] &\leq \mathcal{O}(1)\exp(-\tfrac{\alpha_2 L}{4\rho}) + \mathcal{O}(1)\log L \exp\big(-\alpha_2 \tfrac{L}{\log L}\big) \\ &\leq c_1 \exp(-c_2 \tfrac{L}{\log L}), \end{aligned}$$

for some constants $c_1, c_2 > 0$. $\square$

Appendix

Lemma 6.2. *Let $d \geq 3, p > p_c, 0 < c \leq 2$ and $\mathcal{E}$ be the event that there exist two connected sets D_1 and D_2 in $B(R)$ that have diameter at least cR such that D_1 does not link to D_2 by 1-weight paths in $B(R)$. Then there exists $\beta_1 = \beta_1(d), \beta_2 = \beta_2(p) > 0$ such that for all $R \geq 1$,*

$$\mathbb{P}[\mathcal{E}] \leq \beta_1 \exp(-\beta_2 R). \tag{6.15}$$

Proof. Let us define

$$\mathrm{Crb}(R) = \{\text{there exists a crossing cluster in } B(R)\}.$$

By Lemma 2.1 (ii), there exist positive constants $\beta_1, \beta_2 = \beta_2(p)$ such that

$$\mathbb{P}[\mathrm{Crb}(R)] \geq 1 - \beta_1 \exp(-\beta_2 R).$$

On the above event, we suppose that $Cl(B)$ is the crossing cluster in $B(R)$. We note that if the event $\mathcal{E}$ occurs, then either D_1 or D_2 does not link to $Cl(B)$ by 1-weight paths. Therefore,

$$\mathbb{P}[\mathcal{E}] \leq \mathbb{P}[\mathcal{E}^*], \tag{6.16}$$

where

$$\begin{aligned} \mathcal{E}^* = &\{\text{there exist a connected set } D \subseteq B(R) : Diam(D) \geq cR; \\ &\qquad\qquad D \text{ does not link to } Cl(B) \text{ by 1-weight paths}\}. \end{aligned}$$

Now we only need prove that $\mathbb{P}[\mathcal{E}^*]$ decreasing exponentially by R. For $-R \leq q \leq R$, let

$$H_q = \{x = (x_1, \ldots, x_d) \in B(R) : x_1 = q\} = \{q\} \times [-R, R]^{d-1}.$$

and for $-R \leq i < j \leq R$, define

$$K_{i,j} = [i, j) \times [-R, R]^{d-1}.$$

Now for two distinct points x, y of $B(R)$ with the first coordinate $x_1 = y_1 = a$, we define

$$Q_k(x,y) = \{y \text{ link to } H_{a+k} \text{ by 1-weight paths in } B(R)\} \\ \cap \{x \text{ does not link to } y \text{ in } K_{a,a+k} \text{ by 1-weight paths}\}.$$

Without loss of generalization, we assume that the diameter of D is achieved in the first coordinate direction, i.e.

$$\text{there exist } u, v \in D \text{ with } v_1 - u_1 = \text{diam}(D).$$

It can be seen that if the event $\mathcal{E}^*$ occurs, there exists an integer a satisfying $-R \leq a \leq (1-c)R$ and $x, y \in H_a$ such that the event $Q_{cR}(x,y)$ occurs. Therefore, by union bound,

$$\begin{aligned} \mathbb{P}[\mathcal{E}^*] &\leq d \sum_{-R \leq a \leq (1-c)R} \sum_{x,y \in H_a} \mathbb{P}[Q_{cR}(x,y)] \\ &\leq d(2R)^{2d} \sup_{x,y \in H_{-R}} \mathbb{P}[Q_{cR}(x,y)]. \end{aligned} \tag{6.17}$$

Let L be a positive integer that we choose later. We can write

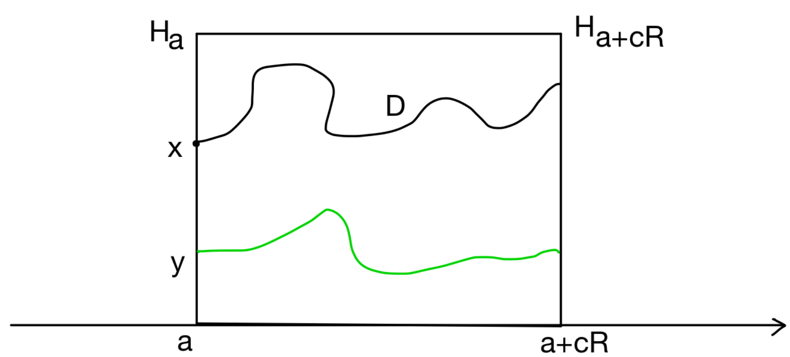

Figure 7: Illustration of the relation between $\mathcal{E}^*$ and $Q_{cR}(x,y)$

$$cR = KL + r, \quad 0 \leq r < L, \tag{6.18}$$

where K is a non-negative integer. It is clear that

$$Q_{cR}(x,y) \subseteq Q_{KL}(x,y) \subseteq Q_{(K-1)L}(x,y) \subseteq \cdots \subseteq Q_L(x,y),$$

which implies that

$$\begin{aligned}\mathbb{P}[Q_{cR}(x,y)] &\le \mathbb{P}\Big[Q_{KL}(x,y)|\bigcap_{i=1}^{K-1} Q_{iL}(x,y)\Big]\mathbb{P}\Big[\bigcap_{i=1}^{K-1} Q_{iL}(x,y)\Big]\\ &= \mathbb{P}\Big[Q_{KL}(x,y)|Q_{(K-1)L}(x,y)\Big]\mathbb{P}\Big[\bigcap_{i=1}^{K-1} Q_{iL}(x,y)\Big]\\ &= \prod_{i=1}^{K}\mathbb{P}[Q_{iL}(x,y)|Q_{(i-1)L}(x,y)]. \end{aligned} \tag{6.19}$$

For $u \in H_{-R}$ and $i \ge 0$, let $O_i(u)$ denote the set of all vertices $z \in H_{-R+iL}$ which

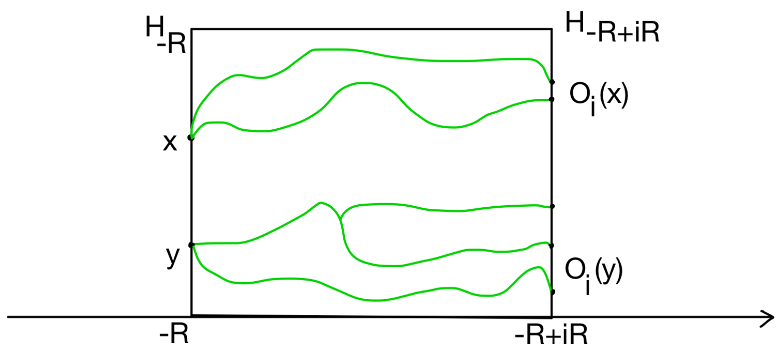

Figure 8: Illustration of $O_i(x)$ and $O_i(y)$

link to u by a 1-weight path. Note that for all $u \in H_{-R}$, the set $O_i(u)$ only depend the status of edges having at least one endvertex to the left of H_{-R+iL}. Moreover, on the event $Q_{iL}(x,y)$, the sets $O_i(x)$ and $O_i(y)$ are non-empty and disjoint. Given the event $Q_{(i-1)L}(x,y)$, if the event $Q_{iL}(x,y)$ occurs then u does not link to v by 1-weight paths in $K_{-R+(i-1)L,-R+iL}$ for all $u \in O_{i-1}(x)$ and $v \in O_{i-1}(y)$. By [[Gri89], Lemma 7.78], there exist L and $\delta = \delta(p,L) > 0$ such that

$$\begin{aligned}\mathbb{P}[Q_{iL}(x,y)|Q_{(i-1)L}(x,y)] &\le \sup_{u,v\in H_{-R+(i-1)L}} \mathbb{P}[u \text{ does not link to } v \text{ in } K_{-R+(i-1)L,-R+iL}]\\ &= \sup_{u,v\in H_0} \mathbb{P}[u \text{ does not link to } v \text{ in } K_{0,L}]\\ &\le 1-\delta.\end{aligned}$$

Combining this estimates, (6.18) and (6.19), it yields that

$$\mathbb{P}[Q_{cR}(x,y)] \le (1-\delta)^K \le (1-\delta)^{cR/L}, \quad \forall x,y \in H_{-R}. \tag{6.20}$$

The claim of lemma follows from combining (6.16), (6.17) and (6.20). $\square$

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