A SHARP BOUND FOR THE RESURGENCE OF SUMS OF IDEALS

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Abstract. We prove a sharp upper bound for the resurgence of sums of ideals involving distinct sets of variables, strengthening work of Bisui–Hà–Jayanthan–Thomas. Complete solutions are delivered for two conjectures proposed by these authors. We employ partially work of DiPasquale et al. on the interpretation of the asymptotic resurgence in terms of integral closure and Rees valuations.

1. Introduction

Let $\mathbb{k}$ be a field, $R = \mathbb{k}[x_1, \dots, x_n]$ be a polynomial ring with variables of degree 1. Let I be an ideal of R. Denote by $I^{(n)}$ the n-th symbolic power of I (defined in terms of associated primes):

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(I)} I^n R_{\mathfrak{p}} \cap R. \tag{1.1}$$

A classical problem in commutative algebra is the comparison between ordinary and symbolic powers. One of the celebrated results in this area is

Theorem 1.1 (Hochster–Huneke, Ein–Lazarsfeld–Smith [8, 14])**.** *Let I be a homogeneous ideal in a polynomial ring $R = \mathbb{k}[x_1, \dots, x_n]$. Assume that $n = \dim(R) \geq 2$. Denote by h the big height of I, namely the maximal height of an associated prime of I. Then for all $s \geq 1$, there is a containment*

$$I^{(sh)} \subseteq I^s.$$

In particular, we always have $I^{((n-1)s)} \subseteq I^s$ for all $s \geq 1$.

Inspired by this result, the resurgence and asymptotic resurgence of I, denoted by $\rho(I)$ and $\rho_a(I)$, were introduced in [2, 10] as follows:

$$\rho(I) = \sup\left\{\frac{m}{r} : I^{(m)} \not\subseteq I^r, m \geq 1, r \geq 1\right\},$$
$$\rho_a(I) = \sup\left\{\frac{m}{r} : I^{(mt)} \not\subseteq I^{rt} \text{ for } t \gg 0\right\}.$$

This is a measure of the difference between the ordinary and symbolic powers of I. Recently, many authors have paid attention to the resurgence and asymptotic resurgence of homogeneous ideals; see, for example [1, 2, 3, 6, 7, 9, 10, 15]. We refer to [4] for a survey on various topics concerning symbolic powers.

It is clear that $1 \leq \rho_a(I) \leq \rho(I)$. Some additional bounds for $\rho(I)$ and $\rho_a(I)$ are given in [2, 6, 10, 13, 15], for instance if I is a squarefree monomial ideal, then $\rho(I) \leq d(I)$, the maximal generating degree of I by [13, Corollary 3.6]. As far as we know, there is still no algorithm to compute $\rho(I)$ and $\rho_a(I)$, even when I is a monomial ideal. DiPasquale et al. [6, Theorem 2.23, Corollary 2.24] have obtained an interesting recursive formula for the asymptotic resurgence of monomial ideals. Apparently no analogous formula was established for the resurgence.

Let A, B be standard graded polynomial rings over $\Bbbk$. Let $I \subseteq A$, $J \subseteq B$ be non-zero proper homogeneous ideals. In [1], the resurgence and asymptotic resurgence of the ideal of $R = A \otimes_{\Bbbk} B$ generated by I and J, simply written as $I + J$, was studied. The construction of the ideal $I + J$ is a classical construction and it is relevant in commutative algebra and algebraic geometry since it corresponds to the notions of tensor products of $\Bbbk$-algebras and fiber products of schemes over $\mathrm{Spec}(\Bbbk)$. For various classical invariants, the value at $I + J$ (or $R/(I + J)$) are determined by the values at I and J (or the corresponding quotient rings). This is the case, for example, for the Krull dimension, depth, graded Betti numbers, and hence the Castelnuovo–Mumford regularity. It turns out that we can also determine the asymptotic resurgence of $I + J$ from those of I and J: It is proved in [1, Theorem 2.6] proved that there is an equality

$$\rho_a(I + J) = \max\{\rho_a(I), \rho_a(J)\}. \tag{1.2}$$

On the other hand, for the resurgence of $I+J$, the best known information is given by the following inequalities [1, Theorem 2.7]:

$$\max\{\rho(I), \rho(J)\} \le \rho(I + J) \le \rho(I) + \rho(J). \tag{1.3}$$

It is also noted in [1, Remark 2.9] that the authors are not aware of any case where the upper bound is attained. Some partial improvements of (1.3) was provided in recent work by Jayanthan, Kumar, and Mukundan [15, Theorems 3.6, 3.9]. As our first main result, we prove the following improved upper bound for the resurgence of sums of ideals. Our result also confirms that the upper bound (1.3) is really strict: As $\rho(I), \rho(J) \ge 1$, the result below implies that $\rho(I+J) \le \rho(I)+\rho(J)-2/3$ always holds.

Theorem 1.2 (=Theorem 2.3)**.** *Let A, B be standard graded polynomial rings over $\Bbbk$. Let $I \subseteq A$, $J \subseteq B$ be non-zero proper homogeneous ideals. Then there are inequalities*

$$\max\{\rho(I), \rho(J)\} \le \rho(I + J) \le \max\left\{\rho(I), \rho(J), \frac{2(\rho(I) + \rho(J))}{3}\right\},$$

and the upper bound is sharp. In particular, if the inequality $\max\{\rho(I), \rho(J)\} \ge 2\min\{\rho(I), \rho(J)\}$ *holds then there is an equality*

$$\rho(I + J) = \max\{\rho(I), \rho(J)\}.$$

The last assertion of this result seems to be rather unexpected. The proof of Theorem 2.3 is somewhat similar to, but differs in a crucial way, from the proof method of [1, Theorem 2.7], namely we employ more efficiently the binomial expansion formula for "associated" symbolic powers of $I + J$. This formula was proved first for "minimal" symbolic powers (defined in terms of minimal primes) in [12, Theorem 3.4], and later for "associated" symbolic powers as well in [11, Theorem 4.1]. We stress that, in this paper, we focus solely on the "associated" symbolic powers defined by Formula (1.1). For related results on the behaviour of (asymptotic) resurgence under taking sum, product, and intersection, we refer to [15].

The next two main results answer completely two conjectures proposed by Bisui–Hà–Jayanthan–Thomas in [1, Conjectures 3.8 and 3.9], one negatively and the other positively. In [1, Conjecture 3.8], the following conjecture was raised.

Conjecture 1.3 (Bisui et al.)**.** *Let I be a nonzero proper homogeneous ideal in $R = \Bbbk[x_1, \dots, x_n]$. Let $X = \operatorname{Proj}(R/I)$ and for each $m \geq 1$, let $I^{[m]}$ be the defining ideal of the fiber product $\underbrace{X \times_{\Bbbk} \cdots \times_{\Bbbk} X}_{m \text{ times}}$, as a closed subscheme of $\underbrace{\mathbb{P}^{n-1}_{\Bbbk} \times_{\Bbbk} \cdots \times_{\Bbbk} \mathbb{P}^{n-1}_{\Bbbk}}_{m \text{ times}}$. Then there exists a choice of I such that $\limsup_{m\to\infty} \rho(I^{[m]}) = \infty$.*

Using the improved upper bound established in Theorem 2.3, we give a negative answer to Conjecture 1.3. Interestingly, the strict inequality $\rho(I^{[m]}) < 2\rho(I)$ always holds; see Corollary 2.8.

There are equalities $0 \leq \rho(I) - \rho_a(I) \leq \dim(A) - 1$ thanks to Theorem 1.1. But so far it is not clear whether by varying the number of variables, $\rho(I) - \rho_a(I)$ may become arbitrarily large. This issue was raised in [1, Conjecture 3.9]. Using a formula for asymptotic resurgence [6, Theorem 2.23], we give a positive answer to this conjecture. We show that for certain sequence (P_m) of squarefree monomial ideals, each of them generated in a single degree, the difference $\rho(P_m) - \rho_a(P_m)$ is unbounded.

Theorem 1.4 (Cf. Theorem 3.4)**.** *Let $m \geq 2$ be an integer, $R = \Bbbk[x_{1,i}, x_{2,i}, x_{3,i} : 1 \leq i \leq 2m-1]$ and for each $1 \leq j \leq 3$, let I_j be the ideal generated by all the products of m distinct variables among the $(2m-1)$ variables $x_{j,1}, \dots, x_{j,2m-1}$:*

$$I_j = (x_{j,i_1} x_{j,i_2} \cdots x_{j,i_m} : 1 \leq i_1 < i_2 < \cdots < i_m \leq 2m-1).$$

Denote $P_m = I_1 + I_2 + I_3$. Then $\rho(P_m) \geq \dfrac{3m}{4}$ and $\rho_a(P_m) = \dfrac{m^2}{2m-1}$. In particular, $\dfrac{2m^2 - 3m}{4(2m-1)} \leq \rho(P_m) - \rho_a(P_m)$, so $\lim\limits_{m\to\infty} (\rho(P_m) - \rho_a(P_m)) = \infty$.

Organization. The new upper bound for the resurgence of sums of ideals is given in Section 2. We construct for any given integer $d \geq 1$, a monomial ideal I with three generators in three variables, such that the equality between ordinary and symbolic powers holds for the first d powers, but fails for the $(d+1)$ ones (Lemma 2.6). This seemingly new construction is useful for showing that the upper bound of Theorem 1.4 is sharp. As an application, we prove for every $m \geq 1$ the strict inequality $\rho(I^{[m]}) < 2\rho(I)$, where $I^{[m]}$ is as in Conjecture 1.3. That the difference between the resurgence and asymptotic resurgence can be arbitrarily large is proved in Section 3, employing work of DiPasquale, Francisco, Mermin, and Schweig.

2. Sharp upper bound for the resurgence of sums of ideals

Let A be a noetherian ring and I an ideal of A. In this paper, we only work with the following notion of the n-th symbolic power of I:

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(I)} (I^n A_{\mathfrak{p}} \cap A).$$

Thus symbolic powers in our sense are defined in terms of *associated primes*, not minimal primes. It is well-known that $I^{(n)} \subseteq I^{(1)} = I$ always holds for every $n \geq 1$.

The following result is from recent work of Hà et al. [11, Theorem 4.1].

Theorem 2.1. *Let A, B be standard graded polynomial rings over a field $\Bbbk$ and $I \subseteq A, J \subseteq B$ be nonzero proper homogeneous ideals. Then, for any $s \in \mathbb{N}$, we have*

$$(I+J)^{(s)} = \sum_{i=0}^{s} I^{(i)} J^{(s-i)}. \tag{2.1}$$

Lemma 2.2. *Let $A_1, \dots, A_p$ be standard graded polynomial rings over $\Bbbk$, and $I_i \subseteq A_i$ be a proper homogeneous ideal. Assume that for $1 \le i \le p$, $m_i, r_i \ge 1$ be integers such that $I_i^{(m_i)} \not\subseteq I_i^{r_i}$. Denote $P = I_1 + \cdots + I_p \subseteq A_1 \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_p$. Then*

$$P^{(m_1+\cdots+m_p)} \not\subseteq P^{r_1+\cdots+r_p-p+1}.$$

In particular, if $p = v+1$, $m_1 = \cdots = m_{v+1} = m$, and $r_1 = \cdots = r_{v+1} = v$ for some $v \ge 1$, then $P^{(m(v+1))} \not\subseteq P^{v^2}$.

Proof. For the first assertion, it suffices to consider the case $p = 2$, as the general case follows by induction on p. In this case, denote $A = A_1, I = I_1, B = A_2, J = I_2$ for simplicity, so $P = I + J \subseteq R = A \otimes_{\Bbbk} B$.

We have to show that $P^{(m_1+m_2)} \not\subseteq P^{r_1+r_2-1}$ given that $I^{(m_1)} \not\subseteq I^{r_1}$ and $J^{(m_2)} \not\subseteq J^{r_2}$. From [1, Lemma 3.2], as soon as $f \in A, g \in B$, $f \in I^{(m_1)} \setminus I^{r_1}$, and $g \in J^{(m_2)} \setminus J^{r_2}$,

$$fg \notin P^{r_1+r_2-1}.$$

Clearly $fg \in I^{(m_1)} J^{(m_2)} \subseteq P^{(m_1+m_2)}$, so $P^{(m_1+m_2)} \not\subseteq P^{r_1+r_2-1}$. The first assertion follows. The remaining assertion is a simple accounting. ☐

The result [1, Theorem 2.7] has shown that the resurgence of $I+J$ is bounded above by the sum of the resurgences of I and J. The following proposition give a sharp upper bound for the resurgence of $I+J$.

Theorem 2.3. *Let A, B be standard graded polynomial rings over $\Bbbk$. Let $I \subseteq A$, $J \subseteq B$ be non-zero proper homogeneous ideals. Then there are inequalities*

$$\max\{\rho(I), \rho(J)\} \le \rho(I+J) \le \sup_{\substack{m,n \in \mathbb{Z} \\ m,n \ge 2}} \left\{ \rho(I), \rho(J), \frac{m\rho(I) + n\rho(J)}{m+n-1} \right\}.$$

More precisely, this is equivalent to

$$\max\{\rho(I), \rho(J)\} \le \rho(I+J) \le \max\left\{ \rho(I), \rho(J), \frac{2(\rho(I)+\rho(J))}{3} \right\},$$

and the upper bound is sharp. Furthermore, if $\max\{\rho(I), \rho(J)\} \ge 2\min\{\rho(I), \rho(J)\}$ then $\rho(I+J) = \max\{\rho(I), \rho(J)\}$.

The inequality $\max\{\rho(I), \rho(J)\} \le \rho(I+J)$ was proved in [1, Theorem 2.7]. We remark that the upper bound in Theorem 2.3 is better than that one in [1, Theorem 2.7]. In fact, as $\min\{\rho(I), \rho(J)\} \ge 1$,

$$\begin{aligned} \max\left\{ \rho(I), \rho(J), \frac{2}{3}(\rho(I)+\rho(J)) \right\} &\le \rho(I) + \rho(J) - \frac{2}{3}\min\{\rho(I), \rho(J)\} \\ &\le \rho(I) + \rho(J) - 2/3. \end{aligned}$$

It therefore gives an explanation to [1, Remark 2.9].

The proof of Theorem 2.3 employs the following lemma.

Lemma 2.4. *Let a, b be non-negative real numbers. Then*
$$\sup_{\substack{m,n\in\mathbb{Z}\\ m,n\geq 2}} \left\{a, b, \frac{ma+nb}{m+n-1}\right\} = \max\left\{a, b, \frac{2(a+b)}{3}\right\}.$$

Proof. Choosing $m = n = 2$, we see that the left-hand side is not smaller than the right-hand side. Hence it remains to prove the reverse inequality.

It is harmless to assume that $a \geq b$. Note that
$$\max\left\{a, b, \frac{2(a+b)}{3}\right\} = \begin{cases} a, & \text{if } a \geq 2b,\\ 2(a+b)/3, & \text{if } b \leq a < 2b.\end{cases}$$

For integers $m, n \geq 2$, since $a - b, b \geq 0$,
$$\begin{aligned}\frac{ma+nb}{m+n-1} &= \frac{ma-(m-1)b}{m+n-1} + b = \frac{m(a-b)+b}{m+n-1} + b\\ &\leq \frac{m(a-b)+b}{m+1} + b = a + \frac{2b-a}{m+1}.\end{aligned}$$

If $a \geq 2b$, then $a + \dfrac{2b-a}{m+1} \leq a$. If $a < 2b$, using $m \geq 2$,
$$a + \frac{2b-a}{m+1} \leq a + \frac{2b-a}{3} = \frac{2(a+b)}{3}.$$

This finishes the proof. □

Proof of Theorem 2.3. Take $h, r \geq 1$ such that
$$h/r > D = \sup_{\substack{m,n\in\mathbb{Z}\\ m,n\geq 2}} \left\{\rho(I), \rho(J), \frac{m\rho(I)+n\rho(J)}{m+n-1}\right\}.$$

We show that $(I+J)^{(h)} \subseteq (I+J)^r$. Note that $(I+J)^{(h)} \subseteq (I+J)^{(1)} = I+J$, so it suffices to consider the case $r \geq 2$. Without loss of generality, we assume $\rho(I) \geq \rho(J)$.

By Theorem 2.1, there is an equality $(I+J)^{(h)} = \sum_{i=0}^{h} I^{(i)}J^{(h-i)}$. Using this, we want to show that $I^{(i)}J^{(h-i)} \subseteq (I+J)^r$ for all $0 \leq i \leq h$. If $i = 0$, $h/r > \rho(I)$, so $I^{(h)} \subseteq I^r$. Similarly, if $i = h$ then $J^{(h)} \subseteq J^r$. Hence it remains to consider the case $1 \leq i \leq h-1$.

There exists a unique integer $m \geq 0$ such that $m\rho(I) < i \leq (m+1)\rho(I)$. Note that $I^{(i)} \subseteq I^m$. If $m \geq r-1$, then using $h - i \geq 1$,
$$I^{(i)}J^{(h-i)} \subseteq I^m J \subseteq I^{r-1}J \subseteq (I+J)^r,$$
so we are done. Assume that $m \leq r-2$.

If $m = 0$, then $i \leq \rho(I)$. Since $h/r > \rho(I)$, we get $h - i \geq h - \rho(I) > (r-1)\rho(I) \geq (r-1)\rho(J)$. Therefore $I^{(i)}J^{(h-i)} \subseteq IJ^{r-1} \subseteq (I+J)^r$.

Assume that $1 \leq m \leq r-2$. Denote $n = r-m$, then $n \geq 2$ and $r = (m+1)+n-1$. Since

$$h/r > D \geq \frac{(m+1)\rho(I)+n\rho(J)}{m+n} = \frac{(m+1)\rho(I)+n\rho(J)}{r},$$

it follows that $h > (m+1)\rho(I)+n\rho(J)$. Thus $h-i \geq h-(m+1)\rho(I) > n\rho(J)$. This yields $J^{(h-i)} \subseteq J^n$, consequently $I^{(i)}J^{(h-i)} \subseteq I^mJ^n \subseteq (I+J)^r$. Hence $\rho(I+J) \leq D$.

The second assertion holds since by Lemma 2.4,

$$D = \max\left\{\rho(I), \rho(J), \frac{2(\rho(I)+\rho(J))}{3}\right\}.$$

That the upper bound is sharp follows from part (2) of Lemma 2.5 below, where we give an example with $\rho(I) = \rho(J) = 1$ and $\rho(I+J) = 4/3$.

When $\max\{\rho(I), \rho(J)\} \geq 2\min\{\rho(I), \rho(J)\}$, we have

$$\max\left\{\rho(I), \rho(J), \frac{2(\rho(I)+\rho(J))}{3}\right\} = \max\{\rho(I), \rho(J)\}.$$

Hence $\rho(I+J) = \max\{\rho(I), \rho(J)\}$ in this case. The proof is completed. $\square$

Lemma 2.5. *Let $A = \Bbbk[x,y,z]$, $I = (x^3, xy^2, y^3) \cap (x,z) = (x^3, xy^2, y^3z)$.*

(1) *For all $n \geq 2$, there is a chain $I^{(n)} = I^n + x^3y^3I^{n-2} \subseteq I^{n-1}$. In particular, $\rho(I) = 1$. Moreover $x^3y^3 \in I^{(2)} \setminus I^2$.*
(2) *Let $B = \Bbbk[u,v,w]$, and $J = (u^3, uv^2, v^3w) \subseteq B$. Then $\rho(J) = \rho(I) = 1$ while $\rho(I+J) = 4/3$.*

Proof. (1): This follows from the more general Lemma 2.6 below.

(2): By part (1), $x^3y^3 \in I^{(2)} \setminus I^2, u^3v^3 \in J^{(2)} \setminus J^2$. This implies

$$x^3y^3u^3v^3 \in I^{(2)}J^{(2)} \setminus (I^2+J^2) \subseteq (I+J)^{(4)} \setminus (I+J)^3.$$

Hence $\rho(I+J) \geq 4/3$ and by Theorem 2.3, $\rho(I+J) = 4/3$. $\square$

Lemma 2.6. *Let $d \geq 1$ be an integer, $I = (x^{2d+1}, x^{2d-1}y^2, y^{2d+1}z) \subseteq A = \Bbbk[x,y,z]$. Then the following statements hold.*

(1) *For each $1 \leq n \leq d$, there is an equality $I^{(n)} = I^n$.*
(2) *For each $n \geq d+1$, there is a chain*

$$I^{(n)} = I^n + (x^dy)^{2d+1}I^{n-d-1} \subseteq I^{n-1}.$$

In particular, $\rho(I) = 1$. Moreover $(x^dy)^{2d+1} \in I^{(d+1)} \setminus I^{d+1}$.

Proof. The irredundant primary decomposition

$$I = (x^{2d+1}, x^{2d-1}y^2, y^{2d+1}) \cap (x^{2d-1}, z)$$

implies that for all $n \geq 1$,

$$I^{(n)} = (x^{2d+1}, x^{2d-1}y^2, y^{2d+1})^n \cap (x^{2d-1}, z)^n. \tag{2.2}$$

STEP 1: Let $f = x^a y^b z^c$ be a monomial in $I^{(n)}$. Note that belonging to the ideal $(x^{2d+1}, x^{2d-1}y^2, y^{2d+1})^n$, $x^a y^b$ has a divisor $(x^{2d+1})^g (x^{2d-1}y^2)^h (y^{2d+1})^i = x^{(2d+1)g+(2d-1)h} y^{2h+(2d+1)i}$, where

$$g, h, i \geq 0, g + h + i = n. \tag{2.3}$$

We deduce

$$a \geq (2d+1)g + (2d-1)h, \tag{2.4}$$
$$b \geq 2h + (2d+1)i, \tag{2.5}$$
$$\left\lfloor \frac{a}{2d-1} \right\rfloor + c \geq n. \tag{2.6}$$

The last inequality holds since $f \in (x^{2d-1}, z)^n$.

Adding (2.4) and (2.5), then using (2.3), it follows that

$$a \geq (2d+1)n - b. \tag{2.7}$$

Note that $x^a y^b z^c \in I^n$ if and only if it has a divisor $(x^{2d+1})^p (x^{2d-1}y^2)^q (y^{2d+1}z)^r$ where $p, q, r \geq 0, p + q + r = n$. Equivalently, $f \in I^n$ if and only if the following system has an integral solution

$$\begin{cases} p, q, r & \geq 0, p + q + r = n, \\ a & \geq (2d+1)p + (2d-1)q, \\ b & \geq 2q + (2d+1)r, \\ c & \geq r. \end{cases}$$

Since $(2d+1)p + (2d-1)q = (2d+1)n - (2q + (2d+1)r)$ and $r = n - (p+q)$, the last system has an integral solution (p, q, r) if and only if the following system has an integral solution (p, q):

$$p, q \geq 0, \tag{2.8}$$
$$n - c \leq p + q \leq n, \tag{2.9}$$
$$(2d+1)n - b \leq (2d+1)p + (2d-1)q \leq a. \tag{2.10}$$

STEP 2: We have a crucial observation.

Claim 1: For any $n \geq 1$, if $f \notin I^n$ then the following conditions are simultaneously satisfied

$$i \geq c + 1,\ b \leq 2n + (2d-1)c - 1 \text{ and } b + c \text{ is odd.} \tag{2.11}$$

Proof of Claim 1: If $i \leq c$, then $g + h = n - i \geq n - c$. Thus the system (2.8)–(2.10) has a solution $(p, q) = (g, h)$, thanks to the hypotheses (2.3), (2.4), (2.5). It remains to consider the case $i \geq c + 1$.

Note that this yields $c \leq i - 1 \leq n - 1$ and (2.5) implies that

$$b \geq 2h + (2d+1)i \geq 2h + (2d+1)(c+1) \geq 2d + 1. \tag{2.12}$$

Case 1: $b \geq 2n + (2d-1)c$. The system (2.8)–(2.10) has a solution $(p, q) = (0, n - c)$, as per (2.6), $(2d-1)(n-c) \leq a$.

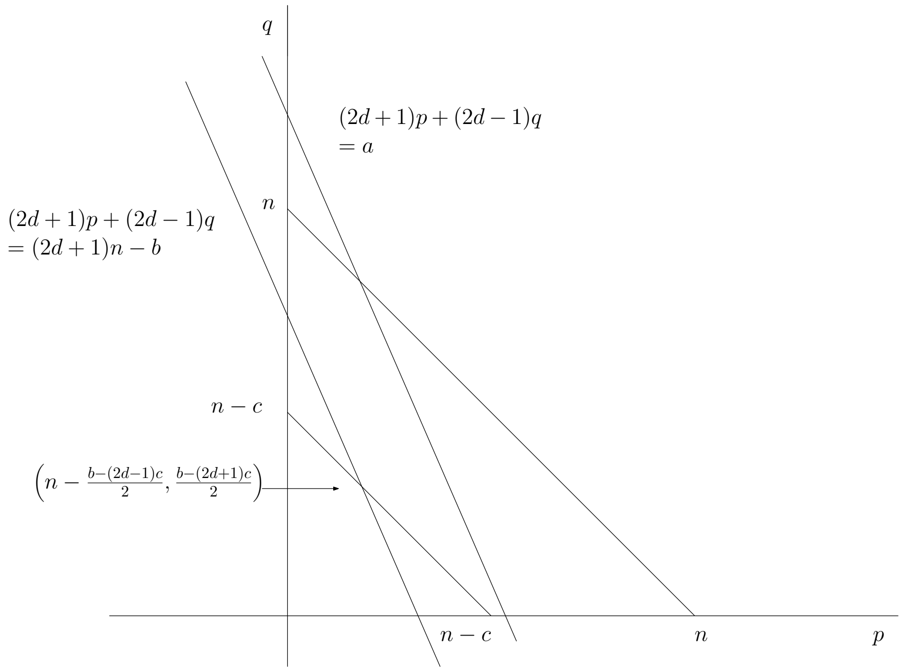

FIGURE 1. The representation of the system (2.8)–(2.10) in the coordinate plane.

Case 2: $b \le 2n + (2d-1)c - 1$ and $b + c$ is even. Solving for

$$\begin{cases} p+q & = n-c, \\ (2d+1)p+(2d-1)q & = (2d+1)n-b, \end{cases}$$

we get

$$(p,q) = \left(n - \frac{b-(2d-1)c}{2}, \frac{b-(2d+1)c}{2}\right).$$

Since $b+c$ is even, $p, q \in \mathbb{Z}$. Since $b \le 2n + (2d-1)c - 1$, p is non-negative, and thanks to (2.12), it follows that $q \ge 0$. Hence (2.8)–(2.10) admits an integral solution.

Now we are left with the case $i \ge c+1$, $b \le 2n + (2d-1)c - 1$ and $b + c$ is odd, namely Claim 1 is true.

STEP 3: Assume that $I^{(n)} \ne I^n$ for some $n \ge 1$. There exists a monomial $f = x^a y^b z^c \in I^{(n)} \setminus I^n$. Choose the integers g, h, i as in Step 1. By Claim 1, we have $i \ge c+1$ and $b \le 2n + (2d-1)c - 1$.

Using (2.5),

$$2n + (2d-1)c - 1 \ge b \ge 2h + (2d+1)i \ge 2h + (2d+1)(c+1).$$

Simplifying, this yields,

$$n \ge h + c + d + 1 \ge d + 1. \tag{2.13}$$

In particular, this shows that $I^{(n)} = I^n$ for all $1 \le n \le d$.

STEP 4: We prove for each $n \geq d+1$ that

$$I^{(n)} = I^n + (x^d y)^{2d+1} I^{n-d-1}.$$

The containment $I^n + (x^d y)^{2d+1} I^{n-d-1} \subseteq I^{(n)}$ is elementary. Indeed, it suffices to show that $f_0 = (x^d y)^{2d+1} = x^{2d^2+d} y^{2d+1} \in I^{(d+1)}$. In fact $f_0 = (x^{2d+1})^d y^{2d+1} \in (x^{2d+1}, x^{2d-1}y^2, y^{2d+1})^{d+1}$ and

$$f_0 \in (x^{2d^2+d-1}) \subseteq (x^{2d-1}, z)^{d+1}.$$

Hence $f_0 \in I^{(d+1)}$.

For the reverse containment, take any monomial $f = x^a y^b z^c \in I^{(n)}$. Define the numbers g, h, i as in Step 1. Assuming $f \notin I^n$, we claim that $f \in (x^d y)^{2d+1} I^{n-d-1}$. By Claim 1, $i \geq c+1$, $b \leq 2n + (2d-1)c - 1$ and $b+c$ is odd.

Claim 2: We have inequalities $b \geq 2d+1, a \geq 2d^2 + d$.

That $b \geq 2d+1$ follows from (2.12). Assume that $a \leq 2d^2 + d - 1 = (2d-1)(d+1)$, then $a/(2d-1) \leq d+1$. As in (2.13), we get $n - c \geq h + d + 1$. Hence together with (2.5),

$$h + d + 1 \leq n - c \leq \frac{a}{2d-1} \leq d + 1.$$

This implies that $h = 0$ and $d + 1 = n - c = \dfrac{a}{2d-1}$. Again by (2.5),

$$(2d+1)i \leq b \leq 2n + (2d-1)c - 1 = 2n + (2d-1)(n-d-1) - 1 = (2d+1)(n-d),$$

so $i \leq n - d$. But then $g = n - h - i \geq d$, and (2.4) yields

$$a \geq (2d+1)g \geq 2d^2 + d,$$

a contradiction. Thus the above assumption is wrong, and $a \geq 2d^2 + d$.

Now $f = x^a y^b z^c = x^{2d^2+d} y^{2d+1} x^{a'} y^{b'} z^c$, where $a' = a - (2d^2+d), b' = b - (2d+1)$. Denote $n' = n - d - 1$, then (2.13) implies that $n' \geq c \geq 0$. We wish to show that $x^{a'} y^{b'} z^c \in I^{n'}$. As in Step 1, this means the following system has an integral solution

$$\begin{cases} p', q' & \geq 0, \\ n' - c & \leq p' + q' \leq n', \\ (2d+1)n' - b' & \leq (2d+1)p' + (2d-1)q' \leq a'. \end{cases}$$

Solving for

$$\begin{cases} p' + q' & = n' - c, \\ (2d+1)p' + (2d-1)q' & = (2d+1)n' - b', \end{cases}$$

we get that

$$q' = \frac{b' - (2d+1)c}{2} = \frac{b - (2d+1)(c+1)}{2}, \quad p' = n' - c - q'.$$

Since $b + c$ is odd, $p', q' \in \mathbb{Z}$. It remains to check that $0 \leq q' \leq n' - c$. The first inequality holds thanks to (2.12). The second inequality can be rewritten as

$$b - (2d+1)(c+1) \leq 2(n - d - 1 - c),$$

equivalently $b \le 2n + (2d-1)c - 1$, which is valid by (2.11). Hence the desired containment $f \in (x^d y)^{2d+1} I^{n-d-1}$ holds, and we finish the proof that for every $n \ge d+1$,

$$I^{(n)} = I^n + (x^d y)^{2d+1} I^{n-d-1}.$$

STEP 5: To prove (2), it remains to show that for each $n \ge d+1$,

$$I^n + (x^d y)^{2d+1} I^{n-d-1} \subseteq I^{n-1}.$$

This is clear since $(x^d y)^{2d+1} \in ((x^{2d+1})^d) \subseteq I^d$. Now $\rho(I) = 1$ follows since $I^{(n)} \subseteq I^{n-1}$ for all $n \ge 1$. Finally, we check that $(x^d y)^{2d+1} \in I^{(d+1)} \setminus I^{d+1}$. The containment was established in Step 4. Assume that $(x^d y)^{2d+1} \in I^{d+1} = (x^{2d+1}, x^{2d-1}y^2, y^{2d+1}z)^{d+1}$. Inspecting supports, we deduce

$$(x^d y)^{2d+1} \in (x^{2d+1}, x^{2d-1}y^2)^{d+1},$$

and after simplifying common factors,

$$xy^{2d+1} \in (x^2, y^2)^{d+1}.$$

This is clearly a contradiction. The proof is completed. $\square$

Remark 2.7. The two inequalities in Theorem 2.3 can be both strict. For example, let $I = (x^3, xy^2, y^3 z) \subseteq A = \Bbbk[x,y,z]$ and

$$J = (t^5, t^3u^2, u^5 v) \subseteq B = \Bbbk[t,u,v].$$

By Lemma 2.6 for $d = 2$, $\rho(J) = 1$ and $t^{10}u^5 \in J^{(3)} \setminus J^3$.

We have two strict inequalities in the chain

$$\max\{\rho(I), \rho(J)\} = 1 < \rho(I+J) = \frac{5}{4} < \max\left\{\rho(I), \rho(J), \frac{2(\rho(I)+\rho(J))}{3}\right\} = \frac{4}{3}.$$

Indeed, this can be seen using [15, Theorem 3.9], or by direct arguments as follows. Denote $P = I + J$. We have $x^3y^3 \in I^{(2)} \setminus I^2$, $t^{10}u^5 \in J^{(3)} \setminus J^3$, hence

$$x^3y^3t^{10}u^5 \in I^{(2)}J^{(3)} \setminus (I^2 + J^3) \subseteq P^{(5)} \setminus P^4.$$

Hence $\rho(P) \ge 5/4$. Using the fact that $I^{(n+1)} \subseteq I^n, J^{(n+1)} \subseteq J^n$ for all $n \ge 1$, we get that $P^{(n)} \subseteq P^{n-2}$ for all $n \ge 2$. Moreover by direct inspection $P^{(n)} \subseteq P^{n-1}$ for $1 \le n \le 4$. Hence

$$\rho(P) \le \sup_{\substack{2 \le m \le 4 \\ n \ge 5}} \left\{\frac{m}{m}, \frac{n}{n-1}\right\} = \frac{5}{4}.$$

The desired conclusion follows.

Now we give a complete answer to [1, Conjecture 3.8] about the resurgence numbers of iterated sums of an ideal. For an ideal I in a polynomial ring A, the *d-th iterated sum* $I^{[d]}$ of I is the defining ideal of the tensor product

$$\underbrace{(A/I) \otimes_{\Bbbk} (A/I) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} (A/I)}_{d \text{ times}}.$$

as a quotient ring of $A^{\otimes d}$. For example, for $I = (x^2, xy) \subseteq \Bbbk[x,y]$, the first three iterated sums of I are given by:

- $I^{[1]} = (x_1^2, x_1y_1) \subseteq \Bbbk[x_1, y_1]$,
- $I^{[2]} = (x_1^2, x_1y_1, x_2^2, x_2y_2) \subseteq \Bbbk[x_1, y_1, x_2, y_2]$,

- $I^{[3]} = (x_1^2, x_1y_1, x_2^2, x_2y_2, x_3^2, x_3y_3) \subseteq \Bbbk[x_1, y_1, x_2, y_2, x_3, y_3]$.

From Theorem 2.3, [1, Conjecture 3.8] admits a negative answer.

Corollary 2.8. *Let I be any nonzero proper homogeneous ideal in a polynomial ring A. Then for all $d \in \mathbb{N}$ we have a strict inequality*

$$\rho(I^{[d]}) < 2\rho(I).$$

Proof. We will prove the above inequality by induction. By definition of iterated sums, we have

$$\rho(I^{[d+1]}) = \rho(I + I^{[d]}) \leq \max\left\{\rho(I), \rho(I^{[d]}), \frac{2}{3}(\rho(I) + \rho(I^{[d]}))\right\}$$

for all $d \in \mathbb{N}, d \geq 1$. For $d = 1$,

$$\rho(I^{[2]}) \leq \max\{\rho(I^{[1]}), \rho(I^{[1]}), \frac{4}{3}\rho(I^{[1]})\} = \frac{4}{3}\rho(I^{[1]}) < 2\rho(I).$$

Assume that the conclusion is true up to d, then

$$\rho(I^{[d+1]}) \leq \max\left\{\rho(I), \rho(I^{[d]}), \frac{2}{3}(\rho(I) + \rho(I^{[d]}))\right\}.$$

On the other hand

$$\rho(I^{[d]} < 2\rho(I), \quad \frac{2}{3}(\rho(I) + \rho(I^{[d]})) < \frac{2}{3}(\rho(I) + 2\rho(I)) = 2\rho(I).$$

Hence $\rho(I^{[d+1]}) < 2\rho(I)$. The proof is concluded. □

3. Large difference between the resurgence and asymptotic resurgence

In the polynomial ring $A = \Bbbk[x_1, \dots, x_d]$, let I be a homogeneous ideal. Recall that the *Waldschmidt constant* of I is defined to be

$$\widehat{\alpha}(I) = \lim_{n\to\infty} \frac{\alpha(I^{(n)})}{n},$$

where $\alpha(I^{(n)})$ is the smallest degree of a nonzero element in $I^{(n)}$.

Assume furthermore that I is a squarefree monomial ideal of A with the irredundant primary decomposition

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_s.$$

The *symbolic polyhedron* $\mathrm{SP}(I)$ of I is the subset of $\mathbb{R}^d$ defined by

$$\mathrm{SP}(I) = \left\{(a_1, \dots, a_d) \in \mathbb{R}^d : \sum_{x_i \in Q_j} a_i \geq 1 \quad \text{for each } 1 \leq j \leq s\right\}.$$

By [6, Lemma 2.14(3)-(4)] for the functional $v(a_1, \dots, a_d) = a_1 + \cdots + a_d$, there is an equality

$$\widehat{\alpha}(I) = \min\{a_1 + \cdots + a_d : (a_1, \dots, a_d) \in \mathrm{SP}(I)\}. \tag{3.1}$$

For integers m, d such that $1 \leq m \leq d$, let $I_{m,d}$ be the ideal generated by products of $d - m + 1$ different variables in the polynomial ring $A = \Bbbk[x_1, \dots, x_d]$. The first result of this section is

Proposition 3.1. *For all integers m, d such that $1 \le m \le d$, we have a primary decomposition*
$$I_{m,d} = \bigcap_{1 \le i_1 < \cdots < i_m \le d} (x_{i_1}, \dots, x_{i_m}).$$
Moreover, there is an equality
$$\rho_a(I_{m,d}) = \frac{m(d-m+1)}{d}.$$

The proof of Proposition 3.1 requires the following recursive formula.

Lemma 3.2 (DiPasquale–Francisco–Mermin–Schweig [6, Corollary 2.24])**.** *Let $I \subseteq A = \Bbbk[x_1, \dots, x_d]$ be a squarefree monomial ideal generated in a single degree $\alpha(I)$. For each non-empty subset U of $[d]$, let I_U be the monomial ideal obtained from I by setting $x_i = 1$ for every $i \in U$. Let $\widehat{\alpha}(I)$ be the Waldschmidt constant of I. Let L be the affine span of the exponent vectors of the minimal generators of I. Then there is an inequality*
$$\rho_a(I) \le \max_{U \subseteq [d], U \ne \emptyset} \left\{ \rho_a(I_U), \frac{\alpha(I)}{\widehat{\alpha}(I)} \right\}.$$
The equality happens if $\dim L = d - 1$.

Proof. This is immediate from Corollary 2.24 in [6], and the fact that if I is generated in a single degree, then the degree valuation $v(a_1, \dots, a_d) = a_1 + \cdots + a_d$ is constant on L. □

Proof of Proposition 3.1. The primary decomposition is the algebraic translation of the fact that: a subset U of $[d]$ meets all the m-element subsets of $[d]$ if and only if $|U| \ge d - m + 1$.

If $m = d$, then $I_{d,d} = (x_1, \dots, x_d)$, which is a complete intersection. Hence its asymptotic resurgence is 1. Similarly, if $m = 1$ then $I_{1,d} = (x_1 \cdots x_d)$ is also a complete intersection. Thus we can assume $2 \le m \le d - 1$.

Denote $I = I_{m,d}$.

STEP 1: We note that $I^{(m(d-m+1)n)} \not\subseteq I^{dn+1}$ for all $n \ge 1$.

In fact, we show that $f = (x_1 \cdots x_d)^{(d-m+1)n} \in I^{(m(d-m+1)n)} \setminus I^{dn+1}$. Note that I is generated in degree $d - m + 1$, so any generator of I^{dn+1} has degree at least $(dn + 1)(d - m + 1) > d(d - m + 1)n = \deg f$. From the primary decomposition
$$I^{(m(d-m+1)n)} = \bigcap_{1 \le i_1 < \cdots < i_m \le d} (x_{i_1}, \dots, x_{i_m})^{m(d-m+1)n}$$
we have $f \in I^{(m(d-m+1)n)}$.

Therefore $I^{(m(d-m+1)n)} \not\subseteq I^{dn+1}$ for every $n \ge 1$. This also implies that for each $s \ge 1$,
$$I^{(m(d-m+1)sn)} \not\subseteq I^{(ds+1)n} \quad \text{for all } n \ge 1,$$
as $I^{(ds+1)n} \subseteq I^{dsn+1}$, which does not contain $I^{(m(d-m+1)sn)}$. By the definition of asymptotic resurgence, for each $s \ge 1$,
$$\rho_a(I) \ge \frac{m(d-m+1)s}{ds+1}.$$

In particular, letting $s \to \infty$,

$$\rho_a(I) \geq \frac{m(d-m+1)}{d}.$$

Step 2: We prove by induction on $d \geq 1$ that

$$\rho_a(I_{m,d}) \leq \frac{m(d-m+1)}{d}.$$

This is true if $d = 1$, since $d = m = 1$ in that case. Assume that $d \geq 2$. As noted above, there is nothing to do if $m = d$, so we can assume $m \leq d-1$.

By Lemma 3.2 and the fact that I is generated in degree $d-m+1$,

$$\rho_a(I) \leq \max_{U \subseteq [d], U \neq \emptyset} \left\{ \rho_a(I_U), \frac{d-m+1}{\widehat{\alpha}(I)} \right\}.$$

For $U \subseteq [d], U \neq \emptyset$, let $r = |U|$. Then I_U is generated by products of $d-r-m+1$ distinct variables among $d-r$ variables. Hence $I_U = (1)$ if $r \geq d-m+1$, in which case $\rho_a(I_U) = 1$.

Assume that $r \leq d-m$. Then $I_U \cong I_{m,d-r}$, so by the induction hypothesis on d,

$$\rho_a(I_U) = \rho_a(I_{m,d-r}) \leq \frac{m(d-r-m+1)}{d-r}.$$

The symbolic polyhedron of I is given by

$$\mathrm{SP}(I) = \left\{ (a_1, \ldots, a_d) \in \mathbb{R}^d : a_{i_1} + \cdots + a_{i_m} \geq 1 \quad \text{for all } 1 \leq i_1 < \cdots < i_m \leq d \right\}.$$

By (3.1),

$$\widehat{\alpha}(I) = \min\{a_1 + \cdots + a_d : (a_1, \ldots, a_d) \in \mathrm{SP}(I)\}.$$

Claim: The minimum value is attained by $(\underbrace{1/m, \ldots, 1/m}_{d \text{ times}})$, giving $\widehat{\alpha}(I) = d/m$.

Proof of the claim: Consider the $d \times m$ matrix where for $1 \leq i \leq d, 1 \leq j \leq m$, at the (i,j)-position we write a_{i+j}, with $(i+j)$ taken modulo d. Consider the sum s of all the entries in the matrix. In each column, we have a permutation of $\{1, 2, \ldots, d\}$, so $s = m(a_1 + \cdots + a_d)$. In each row, for fixed $1 \leq i \leq d$, as $m \leq d$, the numbers $i+1, \ldots, i+m$ are pairwise distinct modulo d. Hence by the assumption the sum in each row is at least 1, and so $m(a_1 + \cdots + a_d) = s \geq d$. Hence $a_1 + \cdots + a_d \geq d/m$, as claimed.

From the claim,

$$\frac{d-m+1}{\widehat{\alpha}(I)} = \frac{m(d-m+1)}{d}.$$

Putting everything together,

$$\rho_a(I_{m,d}) \leq \max_{1 \leq r \leq d-m} \left\{ 1, \frac{m(d-r-m+1)}{d-r}, \frac{m(d-m+1)}{d} \right\} = \frac{m(d-m+1)}{d}.$$

The induction step is completed.

From Steps 1 and 2, we finish the proof that $\rho_a(I) = (m(d-m+1))/d$. $\square$

Corollary 3.3. *Let $m \geq 2$ be an integer, $A = \Bbbk[x_1, \ldots, x_{2m-1}]$ and I be the ideal generated by all the products of m distinct variables:*

$$I = (x_{i_1} x_{i_2} \cdots x_{i_m} : 1 \leq i_1 < i_2 < \cdots < i_m \leq 2m-1).$$

Then $x_1 \cdots x_{2m-1} \in I^{(m)} \setminus I^2$ and $\rho_a(I) = \dfrac{m^2}{2m-1}$.

Proof. It is clear that $x_1 \cdots x_{2m-1} \notin I^2$. One has

$$I^{(m)} = \bigcap_{1 \leq i_1 < i_2 < \cdots < i_m \leq 2m-1} (x_{i_1}, x_{i_2}, \ldots, x_{i_m})^m.$$

Because $x_1 \cdots x_{2m-1} \in (x_{i_1}, x_{i_2}, \ldots, x_{i_m})^m$ for all $1 \leq i_1 < i_2 < \cdots < i_m \leq 2m-1$, we obtain the first assertion.

That $\rho_a(I) = m^2/(2m-1)$ follows from Proposition 3.1 with $d = 2m-1$. ☐

The following result answers in the positive [1, Conjecture 3.9].

Theorem 3.4. *There exists a sequence of polynomial rings R_n and squarefree monomial ideal $P_n \subseteq R_n$ generated by forms of the same degree, such that $\rho(P_n) - \rho_a(P_n) \to \infty$ when $n \to \infty$.*

Proof. For each integer $m \geq 2$, choose I as in Corollary 3.3. Using Lemma 2.2 for $r = 2, m_1 = m_2 = m_3 = m$, we get a squarefree monomial ideal P_m with

$$\rho(P_m) \geq \frac{3m}{4}, \rho_a(P_m) = \frac{m^2}{2m-1}.$$

The equality follows from Equation (1.2). Hence $\frac{2m^2-3m}{4(2m-1)} \leq \rho(P_m) - \rho_a(P_m) \to \infty$ when $m \to \infty$. Since I is generated by forms of the same degree m, so is P_m. ☐

The rationality of resurgence and asymptotic resurgence is an interesting topic considered in [6, 5, 15]. We do not know whether if $\rho(I)$ and $\rho(J)$ are rational numbers, then so is $\rho(I+J)$. This has a positive answer if $\rho(I) = \rho(J) = 1$ [15, Theorem 3.9] or if I and J are both monomial ideals [5, Theorem 3.7], as the symbolic Rees algebras of monomial ideals are known to be noetherian. By Theorem 2.3, we also have a positive answer in the case $\max\{\rho(I), \rho(J)\} \geq 2\min\{\rho(I), \rho(J)\}$.

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