Regularity and large-time behavior of solutions for fractional semilinear mobile-immobile equations

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Abstract. This paper addresses the regularity and large-time behavior of solutions for the fractional semilinear mobile-immobile equations where the nonlinearity term admits various kind of growth conditions. Concerning the associated linear Cauchy problem, a variation of parameters formula of mild solution via the relaxation functions and the eigenfunction expansions is established and the C^1-regularity in time of this solution is proved. In addition, based on the theory of completely positive functions, local estimates and fixed point arguments, some results on existence, regularity and stability of solutions to above mentioned semilinear problem are shown. Furthermore, we prove a result on convergence to equilibrium of solutions with polynomial rate in the case when the nonlinearity function is globally Lispschitzian.

Keywords. Fractional mobile–immobile equation; regularity; mild solution; asymptotic stability.

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1. Introduction

Nonlocal partial differential equation models arise directly and naturally from applications. In recent years, researchers have shown that time fractional partial differential equations are effective tools to describe many different processes in mathematical physics such as anomalous diffusion in porous media [10, 13, 19, 25], behavior of non-Newtonian flows in a viscous and elastic media [2], homogenization of a single phase flow in a porous medium containing a thin layer [1],..., etc. Details of the history and recent development of fractional partial differential equations can be found in the monographs [7, 14, 15, 17] and the documents cited therein.

A fractal mobile/immobile model for solute transport assumes power law waiting times in the immobile zone, leading to a fractional time derivative in the model equations. The equations are equivalent to previous models of mobile/immobile transport with memory functions under in term of a law power and are the limiting equations that govern continuous time random walks with heavy tailed random waiting times.

As dissolved solutes move through an aquifer or stream, they may sorb to solids or diffuse into regions where the advective flux is negligible. To sufficiently describe the mobile solute concentrations and masses, ones have to formulate some functional relationship between the concentrations in the relatively mobile and immobile regions (phases). Typically, this is done with the first-order kinetic mass transfer, commonly called the mobile/immobile (MIM) model. This method has been successfully applied to a large number of tracer tests. It predicts exponential

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decline of the late portion of a breakthrough curve. However, several recent field tests that have resolved very low concentrations show breakthrough curves with heavier, power law, tails.

In 2003, R. Schumer and his co-authors have proposed a fractal mobile/immobile model for solute transport assumes power law waiting times in the immobile zone, leading to a fractional time derivative in the model equations (FrMIM). The equations are equivalent to previous models of mobile/immobile transport with memory functions under in term of a law power. After [23], there have been many studies discussing numerical solutions for FrMIMs [11, 16, 21, 26, 27, 28]. Concerning a special form of nonlinear FrMIMs, in [22], under some suitable conditions on nonlinear terms and coefficients, Sánchez and V. Vergara have constructed a suitable Lyapunov energy functional and used the Lojasiewicz-Simon inequality to prove results on convergence to the equilibrium point of non-trivial solutions.

Let $\Omega \subset \mathbb{R}^d, d \geq 1$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we consider the following problem

$$\partial_t u + \nu \partial_{0+}^{\alpha} u - \beta \Delta u = f(t, u) \text{ in } \Omega, t > 0, \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega, t \geq 0, \tag{1.2}$$

$$u(0, \cdot) = \xi, \text{ in } \Omega, \tag{1.3}$$

where $\nu, \beta > 0$ and $\partial_{0+}^{\alpha}, \alpha \in (0, 1)$, stands for the Caputo derivative of order α defined by

$$\partial_{0+}^{\alpha} u(t, x) := \frac{d}{dt}\big(g_{1-\alpha} * [u(s, x) - u(0, x)]\big)(t), x \in \Omega, t > 0,$$

where $g_{1-\alpha}(t) = t^{-\alpha}/\Gamma(1-\alpha), t > 0, \alpha \in (0, 1)$, $\Gamma(\cdot)$ is the Gamma function and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \xi : \Omega \to \mathbb{R}$ are given functions. On the characteristics of functions having fractional derivatives (in the Caputo or Riemann-Liouville sense), we introduce the reader to the interesting paper by Vainikko [24]. Our main aim is to find sufficient conditions on the nonlinear term to achieve the following results

(1) Global solvability;
(2) Regularity of mild solutions;
(3) Large-time behavior of solutions.

The article is organized as follows. Based on the theory of integral equations with completely positive kernel and relaxation functions, we describe in detail the asymptotic behavior of its solutions in some special cases. Then, by using the eigenfunction expansion, we present the formula of a mild solution to linear FrMIMs. Some important properties of operators containing the formula of the mild solution are obtained. This section is ended with a new Gronwall type inequality which plays an important role in our analysis in the next steps. Section 3 is devoted to studying the global solvability of the problem (1.1)-(1.3), where the nonlinear term allows both sublinear and superlinear growth conditions. The regularity (including the regularity Hölder and C^1-regularity) of solutions to the linear and nonlinear problems is proved in Section 4. In the last part, we propose the results on asymptotic stability and dissipativity for solutions. In addition, when the external force function f is independent of time, say $f = f(u)$, we prove a result of convergence to an equilibrium point of nontrivial solutions with the polynomial rate.

2. Preliminaries

In this section, we aim to present formula of a mild solution to linear problems. Let $\{e_n\}_{n=1}^{\infty}$ be the orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Laplace $-\Delta$ subject to homogeneous Dirichlet boundary condition, that is

$$-\Delta e_n = \lambda_n e_n \text{ in } \Omega, e_n = 0 \text{ on } \partial\Omega,$$

where we can assume that $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence, $\lambda_n > 0$ and $\lambda_n \to \infty$ as $n \to \infty$, see for example [6, Sect. 6.5, p. 354]. For $\beta \in \mathbb{R}$, the fractional power operator $(-\Delta)^{\beta}$ is defined as follows

$$(-\Delta)^{\beta} v = \sum_{n=1}^{\infty} \lambda_n^{\beta}(v, e_n)e_n,$$

$$D((-\Delta)^{\beta}) = \{v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\beta}(v, e_n)^2 < \infty\},$$

here the notation $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$. Let $\mathbb{V}_{\beta} = D((-\Delta)^{\beta})$. It should be noted that $\mathbb{V}_{\beta}$ is a Hilbert space equipped with the norm

$$\|z\|_{\mathbb{V}_{\beta}} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\beta} (z, e_n)^2 \right)^{\frac{1}{2}}, \ z \in D((-\Delta)^{\beta}).$$

Furthermore, for each $\beta > 0$, we can identify $\mathbb{V}_{-\beta} = D((-\Delta)^{-\beta})$ with $\mathbb{V}_{\beta}^*$, the dual space of $\mathbb{V}_{\beta}$.

Let $E_{\alpha,\beta}$ with $\alpha, \beta > 0$ be the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, z \in \mathbb{C}.$$

Consider the following integral equation

$$\ell + \nu g_{1-\alpha} * \ell = 1, \text{on } [0, \infty), \tag{2.1}$$

here and in the sequel the notation '*' is used to indicate the convolution with respect to the time t of locally integrable two functions m, v defined on $\mathbb{R}^+$, i.e.,

$$(m * v)(t) = \int_0^t m(t-s)v(s)ds.$$

Using the Laplace transform, we find from (2.1) that

$$\ell(t) = E_{1-\alpha,1}(-\nu t^{1-\alpha}), t \geq 0. \tag{2.2}$$

To motivate for a definition of a mild solution to the equation (1.1) (which is defined in Section 3), we will give a representation for a solution of the following linear equation

$$\partial_t u + \nu \partial_{0+}^{\alpha} u - \beta \Delta u = F \text{ in } \Omega, t > 0, \tag{2.3}$$

$$u = 0 \text{ on } \partial\Omega, t \geq 0, \tag{2.4}$$

$$u(0, \cdot) = \xi \text{ in } \Omega. \tag{2.5}$$

where $F \in L^1_{\mathrm{loc}}\big(\mathbb{R}^+; L^2(\Omega)\big)$. Assume that

$$u(t, \cdot) = \sum_{n=1}^{\infty} u_n(t)e_n, F(t) = \sum_{n=1}^{\infty} F_n(t)e_n.$$

Substituting into (2.3)-(2.5) one has

$$u_n'(t) + \nu \frac{d}{dt}\big(g_{1-\alpha} * [u_n(\cdot) - u_n(0)]\big)(t) + \beta\lambda_n u_n(t) = F_n(t), t > 0 \tag{2.6}$$

$$u_n(0) = \xi_n := (\xi, e_n). \tag{2.7}$$

In order to find a representation of u_n, we consider the following scalar Volterra integral equations

$$s(t) + \mu(\ell * s)(t) = 1, \ t \geq 0, \tag{2.8}$$

$$r(t) + \mu(\ell * r)(t) = \ell(t), \ t \geq 0, \tag{2.9}$$

where $\mu > 0$ and $\ell(t) = E_{1-\alpha,1}(-\nu t^{1-\alpha})$ is given by (2.2). Let us remark that the function ℓ is completely monotonic, that is,

$$(-1)^n \ell^{(n)}(t) \geq 0, \ \text{for all } n = 0, 1, 2, \ldots, t > 0,$$

thanks to [9, Proposition 3.23, p. 47] (see also [20]). Some useful other properties of ℓ are listed in the following proposition.

Proposition 2.1. *Let ℓ be given by* (2.2)*. Then the following estimates hold.*

(i) $\dfrac{1}{1+\nu\Gamma(\alpha)t^{1-\alpha}} \leq \ell(t) \leq \dfrac{1}{1+\frac{\nu}{\Gamma(2-\alpha)}t^{1-\alpha}}$, *for all $t \geq 0$.*

(ii) $0 < -\ell'(t) < \nu t^{-\alpha}$, *for almost all $t > 0$.*

Proof. In order to prove the statement (i), we use the following bounds of the Mittag-Leffler $E_{\alpha,1}(\cdot)$ (see, e.g., [25, Inequality (39), p. 227]):

$$\frac{1}{1+\Gamma(1-\alpha)t} \leq E_{\alpha,1}(-t) \leq \frac{1}{1+\frac{t}{\Gamma(1+\alpha)}}, \ \text{for all } t \geq 0. \tag{2.10}$$

Thus the statement (i) is followed by exploiting the inequality (2.10) above with α and t replaced by $1-\alpha$ and $\mu t^{1-\alpha}$, respectively. Since $E_{1-\alpha,1}(z)$ is entire in z, we can directly differentiate $E_{1-\alpha,1}(-\nu t^{1-\alpha}) = \sum_{k=0}^{\infty} \frac{(-\nu t^{1-\alpha})^k}{\Gamma\left((1-\alpha)k+1\right)}$ term by term to get

$$\begin{aligned}\frac{d}{dt}E_{1-\alpha,1}(-\nu t^{1-\alpha}) &= \sum_{k=1}^{\infty} \frac{(-\nu)^k t^{(1-\alpha)(k-1)-\alpha}}{\Gamma\big((1-\alpha)k\big)} \\ &= -\nu t^{-\alpha} \sum_{k=1}^{\infty} \frac{(-\nu)^{k-1} t^{(1-\alpha)(k-1)}}{\Gamma\big((1-\alpha)(k-1)+1-\alpha\big)} \\ &= -\nu t^{-\alpha} E_{1-\alpha,1-\alpha}(-\nu t^{1-\alpha}), t > 0. \end{aligned} \tag{2.11}$$

By [9, Lemma 4.25, p. 86], we know that $E_{1-\alpha,1-\alpha}(\cdot)$ is a completely monotonic function on $(0,\infty)$. Thus

$$0 < E_{1-\alpha,1-\alpha}(-\nu t^{1-\alpha}) < 1, \ \text{for all } t > 0. \tag{2.12}$$

The bounds in the statement (ii) then follows by combining (2.11) and (2.12). This completes the proof. □

From Proposition 2.1, we know (see, e.g. , [8, Theorem 2.3.5]) that the equations (2.8) and (2.9) are uniquely solved. Denote by $s_\alpha(\cdot,\mu)$ and $r_\alpha(\cdot,\mu)$ the solutions of (2.8) and (2.9), respectively. It should be noted that, for each fixed $\mu > 0$, $s_\alpha(\cdot,\mu)$ and $r_\alpha(\cdot,\mu)$ are continuous functions on $[0,\infty)$, thanks to [8, Theorem 2.3.5] again.

In addition, some important other properties of $s_\alpha(\cdot,\mu)$ and $r_\alpha(\cdot,\mu)$ are provided in the next proposition.

Proposition 2.2. *Let ℓ be given by* (2.2). *The following assertions are true.*

(i) *For every $\mu>0$, the function $r_\alpha(\cdot,\mu)$ is nonnegative and the following two equalities hold*

$$s_\alpha(t,\mu)=1-\mu\int_0^t r_\alpha(\tau,\mu)d\tau=r_\alpha(t,\mu)+\nu\big(g_{1-\alpha}*r_\alpha(\cdot,\mu)\big)(t),\ t\geq 0. \tag{2.13}$$

(ii) *For every $\mu>0$, the function $s_\alpha(\cdot,\mu)$ is nonnegative and nonincreasing. Moreover,*

$$s_\alpha(t,\mu)\left[1+\mu\int_0^t\ell(\tau)d\tau\right]\leq 1,\ \forall t\geq 0. \tag{2.14}$$

(iii) *For every $\mu>0$, the following estimates hold*

$$\mu r_\alpha(t,\mu)\leq\frac{1}{t},\ \textit{for all } t>0 \textit{ and } r_\alpha(t,\mu)\leq\ell(t),\ \textit{for all } t\geq 0. \tag{2.15}$$

(iv) *For each $t>0$, the functions $\mu\mapsto s_\alpha(t,\mu)$ and $\mu\mapsto r_\alpha(t,\mu)$ are nonincreasing.*

Proof. We first prove part (i). The nonnegativeness of $r_\alpha(\cdot,\mu)$ is shown by applying [18, Theorem 2, p. 322]. It remains to show two equalities in (2.13). For $\mu>0$, convolving the equation (2.9) with $g_{1-\alpha}$ and using (2.1), we find that

$$\begin{aligned} g_{1-\alpha}*r_\alpha(\cdot,\mu)+\mu g_{1-\alpha}*\ell*r_\alpha(\cdot,\mu)&=g_{1-\alpha}*\ell,\\ g_{1-\alpha}*r_\alpha(\cdot,\mu)+\mu\ell*(g_{1-\alpha}*r_\alpha(\cdot,\mu))&=g_{1-\alpha}*\ell,\\ g_{1-\alpha}*r_\alpha(\cdot,\mu)+\mu\ell*(g_{1-\alpha}*r_\alpha(\cdot,\mu))&=\nu^{-1}(1-\ell). \end{aligned} \tag{2.16}$$

Then, by combining (2.8) and (2.16), one has

$$\big(s_\alpha(\cdot,\mu)-\nu g_{1-\alpha}*r_\alpha(\cdot,\mu)\big)+\mu\ell*(s_\alpha(\cdot,\mu)-\nu g_{1-\alpha}*r_\alpha(\cdot,\mu))=\ell. \tag{2.17}$$

Because $r_\alpha(\cdot,\mu)$ is a unique solution of (2.9), it follows from (2.17) that

$$r_\alpha(\cdot,\mu)=s_\alpha(\cdot,\mu)-\nu g_{1-\alpha}*r_\alpha(\cdot,\mu),$$

which is equivalent to

$$s_\alpha(\cdot,\mu)=r_\alpha(\cdot,\mu)+\nu g_{1-\alpha}*r_\alpha(\cdot,\mu). \tag{2.18}$$

Thus the second equality in (2.13) is testified. Regarding the remainder equality, by virtue of (2.1), (2.8), (2.9) and (2.18), we see that

$$\begin{aligned} \mu\big(1*r_\alpha(\cdot,\mu)\big)&=\mu\big(\ell*r_\alpha(\cdot,\mu)\big)+\mu\big(\nu g_{1-\alpha}*\ell*r_\alpha(\cdot,\mu)\big)\\ &=\mu\big(\ell*r_\alpha(\cdot,\mu)\big)+\mu\big(\nu g_{1-\alpha}*r_\alpha(\cdot,\mu)*\ell\big)\\ &=\mu\big(\ell*r_\alpha(\cdot,\mu)\big)+\mu\big([s_\alpha(\cdot,\mu)-r_\alpha(\cdot,\mu)]*\ell\big)\\ &=\mu(s_\alpha(\cdot,\mu)*\ell)=1-s_\alpha(\cdot,\mu). \end{aligned}$$

Before proving part (ii), we first note that the nonnegativity of $s_\alpha(\cdot,\mu)$ is followed by applying [18, Theorem 1, p. 321]. On the other hand, by Proposition 2.1, one can take the derivative the first equality in (2.13) for almost every t and obtain

$$s'_\alpha(t,\mu)=-\mu r_\alpha(t,\mu). \tag{2.19}$$

The later equality and the nonnegativity of $r_\alpha(\cdot,\mu)$ imply that $s(\cdot,\mu)$ is a nonincreasing function on $(0,\infty)$. Using this fact and the equation (2.8), it is easy to get the inequality (2.14).

To prove part (iii), notice that the convolution term in (2.9) is nonnegative, we conclude that $r_\alpha(t,\mu) \le \ell(t)$, for all $t \ge 0$. On the other hand, since ℓ is a nonincreasing function, it implies from (2.18) that

$$r_\alpha(t,\mu) + \mu\ell(t)\int_0^t r_\alpha(\tau,\mu)d\tau \le \ell(t).$$

Moreover, using the first equality in (2.13), one has

$$\begin{aligned}\int_0^t r_\alpha(\tau,\mu)d\tau = \mu^{-1}(1-s_\alpha(t,\mu))) &\ge \mu^{-1}\Big(1-\frac{1}{1+\mu(1*\ell)(t)}\Big)\\ &= \frac{(1*\ell)(t)}{1+\mu(1*\ell)(t)},\end{aligned}$$

thanks to the statement (ii). Therefore

$$\begin{aligned} r_\alpha(t,\mu) &\le \ell(t)\Big(1-\mu\int_0^t r_\alpha(\tau,\mu)d\tau\Big)\\ &\le \ell(t)\Big(1-\frac{\mu(1*\ell)(t)}{1+\mu(1*\ell)(t)}\Big)\\ &= \frac{\ell(t)}{1+\mu(1*\ell)(t)}\\ &\le \frac{\ell(t)}{1+\mu t\ell(t)} \text{ for all } t\ge 0, \end{aligned} \tag{2.20}$$

thanks to $(1*\ell)(t) \ge t\ell(t), \forall t \ge 0$. From the inequality (2.20), it implies

$$\mu r_\alpha(t,\mu) \le \frac{1}{t}, \text{ for all } t>0.$$

We finally prove part (iv). By the above arguments, the Laplace transform of the functions $s_\alpha(\cdot,\mu), r_\alpha(\cdot,\mu)$ exist and given by

$$\widehat{s_\alpha(\cdot,\mu)}(\lambda) = \frac{1}{\lambda(1+\mu\widehat{\ell})}, \widehat{r_\alpha(\cdot,\mu)}(\lambda) = \frac{\widehat{\ell}}{1+\mu\widehat{\ell}}, \Re(\lambda)>0.$$

Differentiating with respect to μ we see that

$$\begin{aligned}\frac{\partial}{\partial\mu}\widehat{s_\alpha(\cdot,\mu)}(\lambda) &= -\frac{\widehat{\ell}}{\lambda\big(1+\mu\widehat{\ell}\big)^2} = -\widehat{s_\alpha(\cdot,\mu)}(\lambda)\widehat{r_\alpha(\cdot,\mu)}(\lambda),\\ \frac{\partial}{\partial\mu}\widehat{r_\alpha(\cdot,\mu)}(\lambda) &= -\frac{\widehat{\ell}}{\big(1+\mu\widehat{\ell}\big)^2} = -\widehat{r_\alpha(\cdot,\mu)}(\lambda)\widehat{r_\alpha(\cdot,\mu)}(\lambda).\end{aligned}$$

Applying the formula for the inverse Laplace transform and using the convolution rule, we conclude that

$$\begin{aligned}\frac{\partial}{\partial\mu}s_\alpha(t,\mu) &= -s_\alpha(\cdot,\mu)*r_\alpha(\cdot,\mu)(t)\le 0,\\ \frac{\partial}{\partial\mu}r_\alpha(t,\mu) &= -r_\alpha(\cdot,\mu)*r_\alpha(\cdot,\mu)(t)\le 0, t>0.\end{aligned}$$

The proof is complete. $\square$

Remark 2.1. (i) *According to the estimates* (2.14), (2.15), *for each fixed* $\mu > 0$, *one has*

$$s_\alpha(t,\mu) = O(t^{-\alpha}) \text{ and } r_\alpha(t,\mu) = O(t^{\alpha-1}) \text{ as } t \to \infty.$$

(ii) *In view of the representation* (2.19) *and the inequality* (2.15), *for each fixed* $\mu > 0$, *we have*

$$0 \le -s'_\alpha(t,\mu) = \mu r_\alpha(t,\mu) \le \frac{1}{t}, \text{ for all } t > 0.$$

Let us now consider the following initial value problem

$$v'(t) + \nu\frac{d}{dt}\big(g_{1-\alpha} * [v - v_0]\big)(t) + \beta\mu v(t) = \omega(t), t > 0, \tag{2.21}$$

$$v(0) = v_0. \tag{2.22}$$

where $\mu > 0$ and $\omega \in L^1_{loc}(\mathbb{R}^+)$. The following proposition gives a representation for the solution of (2.21)-(2.22).

Proposition 2.3. *The function*

$$v(t) = s_\alpha(t,\beta\mu)v_0 + \big(r_\alpha(\cdot,\beta\mu) * \omega\big)(t), t \ge 0, \tag{2.23}$$

be the unique solution of (2.21)-(2.22).

Proof. Assume that v is given by the formula (2.23). We will show that v is a solution to the problem (2.21)-(2.22). Indeed, by the formulation of v, we have $v(0) = s_\alpha(0,\beta\mu)v_0 = v_0$, thanks to the fact that $s_\alpha(0,\beta\mu) = 1$. Furthermore,

$$\begin{aligned} v + \nu g_{1-\alpha} * [v - v_0] &= s_\alpha(\cdot,\beta\mu)v_0 + r_\alpha(\cdot,\beta\mu) * \omega + \nu g_{1-\alpha} * [s_\alpha(\cdot,\beta\mu) - 1]v_0 \\ &\quad + \nu g_{1-\alpha} * r_\alpha(\cdot,\beta\mu) * \omega \\ &= s_\alpha(\cdot,\beta\mu)v_0 + r_\alpha(\cdot,\beta\mu) * \omega + \nu g_{1-\alpha} * [s_\alpha(\cdot,\beta\mu) - 1]v_0 \\ &\quad + \big(s_\alpha(\cdot,\beta\mu) - r_\alpha(\cdot,\beta\mu)\big)*\omega \\ &= s_\alpha(\cdot,\beta\mu)v_0 + \nu g_{1-\alpha} * [s_\alpha(\cdot,\beta\mu) - 1]v_0 + s_\alpha(\cdot,\beta\mu) * \omega, \end{aligned} \tag{2.24}$$

thanks to (2.18). Straightforward differentiation of (2.24) and using (2.19), Proposition 2.2(i), Remark 2.1(ii) leads to

$$\begin{aligned} v' + \nu\frac{d}{dt}\big(g_{1-\alpha} * [v - v_0]\big) &= s'_\alpha(\cdot,\beta\mu)v_0 + \nu g_{1-\alpha} * s'_\alpha(\cdot,\beta\mu)v_0 + s_\alpha(0,\beta\mu)\omega \\ &\quad + s'_\alpha(\cdot,\beta\mu) * \omega \\ &= -\beta\mu r_\alpha(\cdot,\beta\mu)v_0 - \beta\mu\nu g_{1-\alpha} * r_\alpha(\cdot,\beta\mu)v_0 + \omega \\ &\quad - \beta\mu r_\alpha(\cdot,\beta\mu) * \omega \\ &= -\beta\mu r_\alpha(\cdot,\beta\mu)v_0 - \beta\mu\big(s_\alpha(\cdot,\beta\mu) - r_\alpha(\cdot,\beta\mu)\big)v_0 + \omega \\ &\quad - \beta\mu r_\alpha(\cdot,\beta\mu) * \omega \\ &= -\beta\mu s_\alpha(\cdot,\beta\mu)v_0 - \beta\mu r_\alpha(\cdot,\beta\mu) * \omega + \omega \\ &= -\beta\mu v + \omega, \end{aligned}$$

which implies the equality (2.21) as asserted.

Conversely, let v is a solution of (2.21)-(2.22). Noting that, as shown in the proof of part (iv) in Proposition 2.2, the Laplace transform of the functions $s_\alpha(\cdot,\beta\mu), r_\alpha(\cdot,\beta\mu)$ admit the following representations

$$\begin{aligned}\widehat{s_\alpha(\cdot,\beta\mu)}(\lambda) &= \frac{1}{\lambda(1+\beta\mu\widehat{\ell})}\\ &= \frac{1+\nu\lambda^{\alpha-1}}{\lambda+\nu\lambda^{\alpha-1}+\beta\mu}, \Re(\lambda)>0,\end{aligned}\tag{2.25}$$

and

$$\begin{aligned}\widehat{r_\alpha(\cdot,\beta\mu)}(\lambda) &= \frac{\widehat{\ell}}{1+\beta\mu\widehat{\ell}}\\ &= \frac{1}{\lambda+\nu\lambda^{\alpha-1}+\beta\mu}, \Re(\lambda)>0.\end{aligned}\tag{2.26}$$

Now taking the Laplace transform of both sides of the equation (2.21), then we get

$$\lambda\widehat{v}(\lambda)-v_0+\nu(\lambda^\alpha\widehat{v}(\lambda)-\lambda^{\alpha-1}v_0)+\beta\mu\widehat{v}(\lambda)=\widehat{\omega}(\lambda),$$

or equivalently

$$\widehat{v}(\lambda)=\frac{1+\nu\lambda^{\alpha-1}}{\lambda+\nu\lambda^\alpha+\beta\mu}v_0+\frac{1}{\lambda+\nu\lambda^\alpha+\beta\lambda_n}\widehat{\omega}(\lambda).\tag{2.27}$$

From (2.25), (2.26) and (2.27), we have

$$\widehat{v}(\lambda)=\widehat{s_\alpha(\cdot,\beta\mu)}(\lambda)v_0+\widehat{r_\alpha(\cdot,\beta\mu)}(\lambda)\widehat{\omega}(\lambda).$$

Taking the inverse Laplace transform yields $v(t)=s_\alpha(t,\beta\mu)v_0+\big(r_\alpha(\cdot,\beta\mu)*\omega\big)(t)$, which is (2.23). The proof is complete. $\square$

According to Proposition 2.23, the solution of problem (2.6)-(2.7) is given by

$$u_n(t)=s_\alpha(t,\beta\lambda_n)\xi_n+\int_0^t r_\alpha(t-\tau,\beta\lambda_n)F_n(\tau)d\tau,\ t\geq 0.$$

Therefore

$$u(t,\cdot)=\mathcal{S}_\alpha(t)\xi+\int_0^t\mathcal{R}_\alpha(t-\tau)F(\tau)d\tau,\ t\geq 0,\tag{2.28}$$

where

$$\mathcal{S}_\alpha(t)v=\sum_{n=1}^\infty s_\alpha(t,\beta\lambda_n)v_ne_n,\tag{2.29}$$

$$\mathcal{R}_\alpha(t)v=\sum_{n=1}^\infty r_\alpha(t,\beta\lambda_n)v_ne_n.\tag{2.30}$$

It is easily seen that $\mathcal{S}_\alpha(t)$ and $\mathcal{R}_\alpha(t)$ are linear. We collect some basic properties of these operators in the following lemma.

Lemma 2.4. *Let $\{\mathcal{S}_\alpha(t)\}_{t\geq 0}$ and $\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ be the families of linear operators defined by* (2.29) *and* (2.30)*, respectively. Then*

(a) *For each* $v \in L^2(\Omega)$ *and* $T > 0$, $\mathcal{S}_\alpha(\cdot)v \in C([0,T]; L^2(\Omega))$ *and* $\Delta\mathcal{S}_\alpha(\cdot)v \in C((0,T]; L^2(\Omega))$. *Moreover,*

$$\|\mathcal{S}_\alpha(t)v\| \le s_\alpha(t, \beta\lambda_1)\|v\|,\ t \in [0,T], \tag{2.31}$$

$$\|\mathcal{S}_\alpha(t)v\|_{\mathbb{V}_1} \le \frac{\|v\|}{(1*\ell)(t)},\ t \in (0,T]. \tag{2.32}$$

In addition, $\mathcal{S}_\alpha(\cdot)$ *is differentiable on* $(0,\infty)$ *and the following estimate holds*

$$\|\mathcal{S}'_\alpha(t)v\| \le \frac{\|v\|}{t}, \forall v \in L^2(\Omega), \forall t > 0. \tag{2.33}$$

(b) *Let* $v \in L^2(\Omega), T > 0$ *and* $g \in C([0,T]; L^2(\Omega))$. *Then* $\mathcal{R}_\alpha(\cdot)v \in C([0,T]; L^2(\Omega))$ *and* $\mathcal{R}_\alpha * g \in C([0,T]; \mathbb{V}_{1/2})$. *Furthermore,*

$$\|\mathcal{R}_\alpha(t)v\| \le r_\alpha(t, \beta\lambda_1)\|v\|,\ t \in [0,T], \tag{2.34}$$

$$\|(\mathcal{R}_\alpha * g)(t)\| \le \int_0^t r_\alpha(t-\tau, \beta\lambda_1)\|g(\tau)\| d\tau,\ t \in [0,T], \tag{2.35}$$

$$\|(\mathcal{R}_\alpha * g)(t)\|_{\mathbb{V}_{1/2}} \le \left(\int_0^t r_\alpha(t-\tau, \beta\lambda_1)\|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}},\ t \in [0,T]. \tag{2.36}$$

Moreover, $\mathcal{R}_\alpha(\cdot)$ *is differentiable on* $(0,\infty)$ *and the following estimate holds*

$$\|\mathcal{R}'_\alpha(t)v\| \le \left(t^{-1} + \nu t^{-\alpha}\right)\|v\|, \forall v \in L^2(\Omega), \forall t > 0. \tag{2.37}$$

Proof. The proof of the first parts of assertions (a), (b) are similar to the ones proposed in [12, Lemma 2.3] and thus we omit them. We now derive the proof for the estimates (2.33), (2.37). Consider the series

$$\sum_{n=1}^\infty s'_\alpha(t, \beta\lambda_n)v_n e_n,\ t > 0, v_n = (v, e_n),\ v \in L^2(\Omega), \tag{2.38}$$

one sees that $s'(\cdot, \beta\lambda_n)$ is continuous on $(0,\infty)$ and

$$|s'_\alpha(t, \beta\lambda_n)| \le \frac{1}{t},\ \text{for all } n = 1, 2, \dots,$$

thanks to Remark 2.1(ii). By this and the Weierstrass's criterion, the series (2.38) uniformly converges on $[\epsilon, T]$ for every $\epsilon \in (0,T)$ and it holds that

$$\mathcal{S}'_\alpha(t)v = \sum_{n=1}^\infty s'_\alpha(t, \beta\lambda_n)v_n e_n,\ \|\mathcal{S}'_\alpha(t)v\| \le t^{-1}\|v\|,\ \forall t > 0.$$

Regarding the quality $\mathcal{S}'_\alpha(t)v$, we first study the series

$$\sum_{n=1}^\infty r'_\alpha(t, \beta\lambda_n)v_n e_n,\ t > 0, v_n = (v, e_n),\ v \in L^2(\Omega).$$

Observing that, for each $\mu > 0$, then

$$\begin{aligned} |\big(\ell' * r_\alpha(\cdot, \mu)\big)(t)| &\le \int_0^t \nu(t-\tau)^{-\alpha}\ell(\tau)d\tau \\ &\le \nu(1-\alpha)^{-1}t^{1-\alpha} < \infty,\ \text{for each } t > 0, \end{aligned}$$

thanks to Proposition 2.1(ii) and Proposition 2.2(iii). Therefore, the convolution term in equation (2.9) is differentiable almost everywhere on $(0,\infty)$. From this observation and the differentiability of ℓ, one concludes that $r_\alpha(\cdot,\mu)$ is also differentiable almost everywhere on $(0,\infty)$. Therefore, by differentiating on both sides of (2.9) with respect to t, one obtains

$$r'_\alpha(t,\mu)+\mu[\big(\ell'*r_\alpha(\cdot,\mu)\big)(t)+\ell(0)r_\alpha(t,\mu)]=\ell'(t), t>0,$$

thanks to [8, Corollary 2.7.4(ii), p. 101]. Then

$$r'_\alpha(t,\mu)+\mu r_\alpha(t,\mu)-\ell'(t)\geq 0, \tag{2.39}$$

thanks to the facts that $\ell(0)=1$ and $\big(\ell'*r_\alpha(\cdot,\mu)\big)(t)\leq 0$, for all $t>0$. Using this fact, one has

$$\begin{aligned}-r'_\alpha(t,\mu)&\leq \mu r_\alpha(t,\mu)-\ell'(t)\\ &\leq t^{-1}+\nu t^{-\alpha},\end{aligned}$$

thanks to Proposition 2.1(ii) and Proposition 2.2(iii). Moreover, using the differentiability of $r_\alpha(\cdot,\mu)$, we also have

$$r'_\alpha(t,\mu)+\mu[\big(\ell*r'_\alpha(\cdot,\mu)\big)(t)+\ell(t)r_\alpha(0,\mu)]=\ell'(t), t>0.$$

By this equality and the nonincreasing of ℓ, it implies that

$$r'_\alpha(t,\mu)+\mu\ell(t)(1*r'_\alpha(\cdot,\mu))(t)+\mu\ell(t)\leq \ell'(t),$$

or equivalently

$$r'_\alpha(t,\mu)+\mu\ell(t)\big(r(t,\mu)-1\big)+\mu\ell(t)\leq \ell'(t).$$

The later inequality shows that

$$r'_\alpha(t,\mu)\leq \ell'(t)-\mu\ell(t)r_\alpha(t,\mu), \text{ for all } t>0.$$

By combining the above estimates, we have

$$0\leq -r'_\alpha(t,\mu)\leq t^{-1}+\nu t^{-\alpha}, \text{ for all } t>0.$$

The remainder of the proof is similar to those given as above. We have finished the proof of Lemma 2.4. □

Remark 2.2. *The first statement of Lemma 2.4 guarantees that the operator $\mathcal{S}_\alpha(t): L^2(\Omega)\to L^2(\Omega)$ is compact for any $t>0$, due to the compactness of the embedding $\mathbb{V}_1\hookrightarrow L^2(\Omega)$.*

In the following, for sake of simplicity we make use of the notation $u(t)$ for $u(t,\cdot)$ and consider u as a function defined on $[0,T]$, taking values in $L^2(\Omega)$. The notation $\|\cdot\|$ will be understood as the standard norm in $L^2(\Omega)$ and $\|\cdot\|_{op}$ stands for the operator norm of bounded linear operators on $L^2(\Omega)$. Exploiting the properties of $\mathcal{R}_\alpha$, we prove the compactness of the Cauchy operator defined by

$$\begin{aligned}&\mathcal{Q}_\alpha: C([0,T];L^2(\Omega))\to C([0,T];L^2(\Omega)),\\ &\mathcal{Q}_\alpha(g)(t)=\big(\mathcal{R}_\alpha*g\big)(t),\end{aligned} \tag{2.40}$$

in the next lemma.

Lemma 2.5. *The operator $\mathcal{Q}_\alpha$ defined by (2.40) is compact.*

Proof. To prove this lemma, we use the Arzelà-Ascoli theorem. Let $D \subset C([0,T];L^2(\Omega))$ be a bounded set and denote $\|g\|_\infty = \sup_{t\in[0,T]} \|g(t)\|$ for $g \in C([0,T];L^2(\Omega))$. We first testify that $(-\Delta)^{1/2}\mathcal{Q}_\alpha(D)(t)$ is bounded in $L^2(\Omega)$ for each $t \geq 0$. Indeed, by using Lemma 2.4(b), we get

$$\|(-\Delta)^{1/2}\mathcal{Q}_\alpha(g)(t)\|^2 \leq \int_0^t r_\alpha(t-\tau,\beta\lambda_1)\|g(\tau)\|^2 d\tau,\ \forall t \geq 0, g \in D,$$

which ensures the boundedness of $(-\Delta)^{1/2}\mathcal{Q}_\alpha(D)(t)$ in $L^2(\Omega)$ for all $t \geq 0$. Since the embedding $\mathbb{V}_{1/2} \hookrightarrow L^2(\Omega)$ is compact (see, e.g., [3, Theorem 1.1]), we obtain the relative compactness of $\mathcal{Q}_\alpha(D)(t)$ for each $t \geq 0$.

Now we show that $\mathcal{Q}_\alpha(D)$ is equicontinuous. Let $g \in D$, $t \in (0,T)$, and $h \in (0, T-t]$, then one sees that

$$\begin{aligned}\|\mathcal{Q}_\alpha(g)(t+h) - \mathcal{Q}_\alpha(g)(t)\| &\leq \int_0^t \|[\mathcal{R}_\alpha(t+h-\tau) - \mathcal{R}_\alpha(t-\tau)]g(\tau)\|d\tau \\ &\quad + \int_t^{t+h} \|\mathcal{R}_\alpha(t+h-\tau)g(\tau)\|d\tau \\ &= I_1(t) + I_2(t).\end{aligned}$$

It is easy to see that $I_2(t) \to 0$ as $h \to 0$ uniformly in $g \in D$. Regarding $I_1(t)$, we observe that

$$I_1(t) \leq \|g\|_\infty \int_0^t \|\mathcal{R}_\alpha(t+h-\tau) - \mathcal{R}_\alpha(t-\tau)\|_{op} d\tau.$$

Put $I_h(t) = \|\mathcal{R}_\alpha(t+h-\tau) - \mathcal{R}_\alpha(t-\tau)\|_{op}$. Then, for every $\tau \in (0,t)$, $\lim_{h\to 0} I_h(\tau) = 0$. Furthermore, in accordance to Proposition 2.1(i) and Lemma 2.4(b), we have

$$\begin{aligned}I_h(t) &\leq r_\alpha(t+h-\tau,\beta\lambda_1) + r_\alpha(t-\tau,\beta\lambda_1) \\ &\leq \ell(t+h-\tau) + \ell(t-\tau) \\ &\leq \frac{1}{1 + \frac{\nu}{\Gamma(2-\alpha)}(t+h-\tau)^{1-\alpha}} + \frac{1}{1 + \frac{\nu}{\Gamma(2-\alpha)}(t-\tau)^{1-\alpha}} \\ &\leq \frac{2\Gamma(2-\alpha)}{\nu(t-\tau)^{1-\alpha}} = I(t).\end{aligned}$$

Since $I \in L^1(0,t)$, it follows from the Lebesgue dominated convergence theorem that

$$I_1(t) \leq \|g\|_\infty \int_0^t I_h(\tau) d\tau \to 0 \text{ as } h \to 0 \text{ uniformly in } g \in D.$$

Finally, for $h \in (0,T)$, we have

$$\begin{aligned}\|\mathcal{Q}_\alpha(g)(h) - \mathcal{Q}_\alpha(g)(0)\| &\leq \int_0^h \|\mathcal{R}_\alpha(h-\tau)g(\tau)\|d\tau \\ &\leq \|g\|_\infty \int_0^h r_\alpha(h-\tau,\beta\lambda_1)d\tau \\ &= \frac{\|g\|_\infty}{\beta\lambda_1}\big(1 - s_\alpha(h,\beta\lambda_1)\big) \to 0 \text{ as } h \to 0 \text{ uniformly in } g \in D.\end{aligned}$$

Therefore, $\mathcal{Q}_\alpha(D)$ is equicontinuous. Thus the proof of Lemma 2.5 is completed. □

We are in a position to prove a Gronwall type inequality which plays an important role in our analysis in the next sections.

Lemma 2.6. *Let v be a nonnegative function satisfying*

$$v(t) \le s_\alpha(t,\beta\mu)v_0 + \int_0^t r_\alpha(t-\tau,\beta\mu)[a(\tau)+bv(\tau)]d\tau,\ t \ge 0, \tag{2.41}$$

for $b \in (0,\beta\mu), v_0 \ge 0$ and $a \in L^1_{loc}(\mathbb{R}^+)$. Then

$$v(t) \le s_\alpha(t,\beta\mu-b)v_0 + \int_0^t r_\alpha(t-\tau,\beta\mu-b)a(\tau)d\tau,\ t \ge 0.$$

Especially, if a is constant then

$$v(t) \le s_\alpha(t,\beta\mu-b)v_0 + \frac{a}{\beta\mu-b}\big(1-s_\alpha(t,\beta\mu-b)\big),\ t \ge 0.$$

Proof. Let $y(t)$ be the right hand side of (2.41). Then $v(t) \le y(t)$ and y satisfies the equation

$$y'(t) + \nu\frac{d}{dt}\big(g_{1-\alpha} * [y(\cdot)-y(0)]\big)(t) + \beta\mu y(t) = a(t) + bv(t),\ t>0, y(0)=v_0,$$

as stated by Proposition 2.3. It follows that

$$y'(t) + \nu\frac{d}{dt}\big(g_{1-\alpha} * [y(\cdot)-y(0)]\big)(t) + (\beta\mu-b)y(t) = a(t) + b[v(t)-y(t)],\ t>0, y(0)=v_0,$$

and then y admits the representation

$$\begin{aligned} y(t) &= s_\alpha(t,\beta\mu-b)v_0 + \int_0^t r_\alpha(t-\tau,\beta\mu-b)\big(a(\tau)+b[v(\tau)-y(\tau)]\big)d\tau \\ &\le s_\alpha(t,\beta\mu-b)v_0 + \int_0^t r_\alpha(t-\tau,\beta\mu-b)a(\tau)d\tau, \end{aligned}$$

thanks to the positivity of $r_\alpha(\cdot,\beta\mu-b)$ and the fact that $v(\tau)-y(\tau) \le 0$ for $\tau \ge 0$. In addition, if a is constant then

$$\begin{aligned} y(t) &\le s_\alpha(t,\beta\mu-b)v_0 + \int_0^t r_\alpha(t-\tau,\beta\mu-b)a(\tau)d\tau, \\ &= s_\alpha(t,\beta\mu-b)v_0 + \frac{a}{\beta\mu-b}\big(1-s_\alpha(t,\beta\mu-b)\big), \end{aligned}$$

thanks to Proposition 2.2(i). So we get the conclusion as stated. □

3. Solvability

This section is devoted to study the existence of mild global solutions of semilinear evolution problems (1.1)-(1.3) on the bounded time intervals $[0,T]$, where T is arbitrary but fixed. Based on the representation (2.28), we give the following definition of mild solution for (1.1)-(1.3).

Definition 3.1. *A function $u \in C([0,T];L^2(\Omega))$ is said to be a mild solution to the problem* (1.1)-(1.3) *on $[0,T]$ iff*

$$u(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)f(\tau,u(\tau))d\tau \text{ for any } t \in [0,T].$$

Let $\Phi : C([0,T]; L^2(\Omega)) \to C([0,T]; L^2(\Omega))$ be the operator defined by

$$\Phi(u)(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau) f\big(\tau, u(\tau)\big) d\tau.$$

This operator is continuous if f is a continuous map. Obviously, u is a fixed point of Φ iff u is a mild solution of (1.1)-(1.3). So we call Φ *the solution operator*.

In the first result on the existence, the nonlinear function f is assumed to be locally Lipschitzian.

Theorem 3.1. *Assume that the nonlinearity function $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ satisfies*

(**F**1) $f(\cdot, 0) = 0$ *and is locally Lipschitz, i.e., for each $r > 0$, there exists a nonnegative constant $\kappa(r)$ such that*

$$\|f(t, v_1) - f(t, v_2)\| \le \kappa(r)\|v_1 - v_2\|, \forall v_1, v_2 \in B_r, t \in [0,T], \tag{3.1}$$

where B_r is the closed ball in $L^2(\Omega)$ with radius r and centered at origin and κ is a nonnegative function such that $\limsup_{r\to 0} \kappa(r) = l \in [0, \beta\lambda_1)$.

Then there exists $\delta > 0$ such that the problem (1.1)-(1.3) *has a unique mild solution on $[0,T]$, provided $\|\xi\| \le \delta$.*

Proof. To prove this theorem, we use the contraction mapping principle. Since $l \in [0, \beta\lambda_1)$, we can choose $\epsilon > 0$ such that $l + \epsilon < \beta\lambda_1$. On the other hand, the definition of lim sup implies that there exists $R > 0$ satisfying

$$\kappa(R) \le l + \epsilon.$$

Let us denote by B_R the closed ball in $C([0,T]; L^2(\Omega))$ centered at the origin with radius R.

Set $\delta = \dfrac{(l+\epsilon)R}{\beta\lambda_1}$, we will show that $\Phi(\mathsf{B}_R) \subset \mathsf{B}_R$, provided $\|\xi\| \le \delta$. Taking $u \in \mathsf{B}_R$, then $\|u(\tau)\| \le R$ for all $\tau \in [0,T]$. Therefore

$$\begin{aligned}
\|\Phi(u)(t)\| &\le \|\mathcal{S}_\alpha(t)\xi\| + \Big\|\int_0^t \mathcal{R}_\alpha(t-\tau) f(\tau, u(\tau)) d\tau\Big\| \\
&\le s_\alpha(t, \beta\lambda_1)\|\xi\| + \int_0^t r_\alpha(t-\tau, \beta\lambda_1)\kappa(R)\|u(\tau)\| d\tau \\
&\le s_\alpha(t, \beta\lambda_1)\|\xi\| + (l+\epsilon)R \int_0^t r_\alpha(t-\tau, \beta\lambda_1) d\tau \\
&\le s_\alpha(t, \beta\lambda_1)\delta + \frac{(l+\epsilon)R}{\beta\lambda_1}\left(1 - s_\alpha(t, \beta\lambda_1)\right) \\
&= s_\alpha(t, \beta\lambda_1)\left[\delta - \frac{(l+\epsilon)R}{\beta\lambda_1}\right] + \frac{(l+\epsilon)R}{\beta\lambda_1} \\
&\le R, \ \text{ for all } t \in [0,T],
\end{aligned}$$

thanks to Lemma 2.4 and the choice for δ. We now consider $\Phi : \mathsf{B}_R \to \mathsf{B}_R$. For $u_1, u_2 \in \mathsf{B}_R$ one gets

$$\begin{aligned}\|\Phi(u_1)(t) - \Phi(u_2)(t)\| &\leq \int_0^t r_\alpha(t-\tau, \beta\lambda_1)\kappa(R)\|u_1(\tau) - u_2(\tau)\|d\tau \\ &\leq (l+\epsilon)\int_0^t r_\alpha(t-\tau, \beta\lambda_1)\sup_{\theta\in[0,\tau]}\|u_1(\theta) - u_2(\theta)\|d\tau \\ &\leq (l+\epsilon)\|u_1 - u_2\|_\infty \int_0^t r_\alpha(\tau, \beta\lambda_1)d\tau \\ &\leq \frac{l+\epsilon}{\beta\lambda_1}\|u_1 - u_2\|_\infty, \forall t \in [0,T],\end{aligned}$$

thanks to Proposition 2.2. Hence, the solution operator Φ is a contraction operator on B_R. We can conclude that there is a mild solution to (1.1)-(1.3).

We then prove the uniqueness of solutions. Observing that, if $u, v \in C([0,T]; L^2(\Omega))$ are two solutions of (1.1)-(1.3), then one can assume that $u, v \in \mathsf{B}_{\bar{R}}$ for some $\bar{R} > 0$. We have the following estimates

$$\begin{aligned}\|u(t) - v(t)\| &\leq \int_0^t r_\alpha(t-\tau, \beta\lambda_1)\kappa(\bar{R})\|u(\tau) - v(\tau)\|d\tau \\ &\leq \kappa(\bar{R})\int_0^t \sup_{\theta\in[0,\tau]}\|u(\theta) - v(\theta)\|d\tau,\ \forall t \in [0,T],\end{aligned}$$

thanks to the fact that $u(0) = v(0) = \xi$ and $r_\alpha(t, \beta\lambda_1) \leq 1$ for $t \geq 0$. Since the last inequality is nondecreasing in t, we get

$$\sup_{\tau\in[0,t]}\|u(\tau) - v(\tau)\| \leq \kappa(\bar{R})\int_0^t \sup_{\theta\in[0,\tau]}\|u(\theta) - v(\theta)\|d\tau,\ \forall t \in [0,T].$$

Employing the classical Gronwall inequality, we get

$$\sup_{\tau\in[0,t]}\|u(\tau) - v(\tau)\| = 0,$$

for all $t \in [0,T]$, which implies that $u = v$. The proof is complete. □

In the next result, we relax condition imposed on initial datum. However, the function f should be global Lipschitzian.

Theorem 3.2. *Suppose that the nonlinearity function $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ satisfies*

($\mathbf{F}2$) $\|f(t, v_1) - f(t, v_2)\| \leq q(t)\|v_1 - v_2\|, \forall t \in [0,T], v_1, v_2 \in L^2(\Omega)$,

where $q \in L^1([0,T])$ is a nonnegative function. Then the problem (1.1)-(1.3) *has a unique mild solution.*

Proof. Since $\lim\limits_{\eta\to\infty}\sup\limits_{[0,T]}\int_0^t e^{-\eta(t-\tau)}q(\tau)d\tau = 0$ (see [5, Lemma 2.7]), we can choose a fixed number $\eta > 0$ such that

$$\sup_{[0,T]}\int_0^t e^{-\eta(t-\tau)}q(\tau)d\tau < 1.$$

We denote $\|v\|_\eta = \sup\limits_{[0,T]} e^{-\eta t}\|v(t)\|$ for each $v \in C([0,T]; L^2(\Omega))$. Then $\|\cdot\|_\eta$ is equivalent to the sup norm in $C([0,T]; L^2(\Omega))$.

For arbitrary $u_1, u_2 \in C([0,T]; L^2(\Omega))$, one has

$$\begin{aligned}\|\Phi(u_1)(t) - \Phi(u_2)(t)\| &\le \int_0^t r_\alpha(t-\tau, \beta\lambda_1) q(\tau) \|u_1(\tau) - u_2(\tau)\| d\tau \\ &\le \int_0^t r_\alpha(t-\tau, \beta\lambda_1) q(\tau) \sup_{\theta\in[0,\tau]} \|u_1(\theta) - u_2(\theta)\| d\tau.\end{aligned}$$

Hence

$$\begin{aligned}\|\Phi(u_1) - \Phi(u_2)\|_\eta &= \sup_{t\in[0,T]} e^{-\eta t} \|\Phi(u_1)(t) - \Phi(u_2)(t)\| \\ &\le \|u_1 - u_2\|_\eta \sup_{[0,T]} \int_0^t e^{-\eta(t-\tau)} r_\alpha(t-\tau, \beta\lambda_1) q(\tau) d\tau, \\ &\le \|u_1 - u_2\|_\eta \sup_{[0,T]} \int_0^t e^{-\eta(t-\tau)} q(\tau) d\tau,\end{aligned}$$

due to the fact that $r_\alpha(t-\tau, \beta\lambda_1) \le 1$, for $t \ge \tau$. The last relation implies that Φ is a contraction operator. The proof is complete. □

In the following theorem, we employ the Schauder fixed point Theorem to obtain an existence result where the function f may have a superlinear growth.

Theorem 3.3. *Assume that the nonlinearity function $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ satisfies*

(**F**3) *f is continuous such that*

$$\|f(t,v)\| \le q(t)\psi_f(\|v\|),\ \forall t \in [0,T], v \in L^2(\Omega),$$

where $q \in L^1([0,T])$ is a nonnegative function and $\psi_f \in C(\mathbb{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{r\to 0} \frac{\psi_f(r)}{r} \cdot \sup_{t\in[0,T]} \int_0^t r_\alpha(t-\tau, \beta\lambda_1) q(\tau) d\tau < 1. \tag{3.2}$$

Then there exists $\delta > 0$ such that the problem (1.1)-(1.3) *has at least one mild solution on* $[0,T]$ *provided* $\|\xi\| \le \delta$.

Proof. The idea for our proof is to utilize the Schauder fixed point theorem. We first find a number $\varrho > 0$ such that $\Phi(\mathsf{B}_\varrho) \subset \mathsf{B}_\varrho$, where B_ϱ be the closed ball in $C([0,T]; L^2(\Omega))$ centered at origin with radius ϱ. Let

$$\tilde\beta = \limsup_{r\to 0} \frac{\psi_f(r)}{r} \text{ and } M = \sup_{t\in[0,T]} \int_0^t r_\alpha(t-\tau, \beta\lambda_1) q(\tau) d\tau.$$

From (3.2), we can choose $\epsilon > 0$ small enough such that

$$(\tilde\beta + \epsilon) M < 1.$$

In addition, one can find $\varrho > 0$ such that

$$\frac{\psi_f(\varrho)}{\varrho} \le \tilde\beta + \epsilon.$$

Choosing $\delta = \varrho[1 - (\tilde{\beta} + \epsilon)M]$, it is obvious that $\delta \leq \varrho$ and $\delta > 0$. Now, we take $\xi \in L^2(\Omega)$ such that $\|\xi\| \leq \delta$ and $u \in \mathsf{B}_\varrho$. Then one has

$$\begin{aligned}
\|\Phi(u)(t)\| &\leq s_\alpha(t, \beta\lambda_1)\|\xi\| + \int_0^t r_\alpha(t-\tau, \beta\lambda_1)\|f(\tau, u(\tau)\|d\tau \\
&\leq \|\xi\| + \int_0^t r_\alpha(t-\tau, \beta\lambda_1)q(\tau)\psi_f(\|u(\tau)\|)d\tau \\
&\leq \|\xi\| + \int_0^t r_\alpha(t-\tau, \beta\lambda_1)q(\tau)\psi_f(\varrho)d\tau \\
&\leq \delta + \varrho(\tilde{\beta} + \epsilon)\int_0^t r_\alpha(t-\tau, \beta\lambda_1)q(\tau)d\tau \\
&\leq \delta + \varrho(\tilde{\beta} + \epsilon)M \\
&\leq \varrho[1 - (\tilde{\beta} + \epsilon)M] + \varrho(\tilde{\beta} + \epsilon)M \\
&\leq \varrho, \forall t \in [0, T]. \qquad (3.3)
\end{aligned}$$

The inequality (3.3) implies that $\Phi(\mathsf{B}_\varrho) \subset \mathsf{B}_\varrho$, provided $\|\xi\| \leq \delta$. We are allowed to consider $\Phi : \mathsf{B}_\varrho \to \mathsf{B}_\varrho$. It is easily seen that Φ is continuous because of the continuity of f. By the representation

$$\begin{aligned}
&\Phi(u) = \mathcal{S}_\alpha(\cdot)\xi + \mathcal{Q}_\alpha \circ N_f(u), \\
&N_f(u)(t) = f\big(t, u(t)\big),
\end{aligned}$$

we realize that Φ is a compact operator, thanks to the compactness of $\mathcal{Q}_\alpha$ which is proved in Lemma 2.5. Hence, by the Schauder fixed point theorem, we get the desired conclusion. The proof is complete. □

If the nonlinearity function has a sublinear growth then the smalless of given data is omitted. We get the global solvability in the next theorem.

Theorem 3.4. *Assume that f satisfies the condition*

(**F**4) *f is continuous such that $\|f(t, v)\| \leq q(t)\|v\| + a(t)$, for all $t \in [0, T]$, $v \in L^2(\Omega)$, where $q, a \in L^1([0, T])$ are nonnegative functions.*

Then the problem (1.1)-(1.3) *has at least one mild solution on* $[0, T]$.

Proof. Denote

$$\mathcal{D} = \{u \in C([0, T]; L^2(\Omega)) : \sup_{\tau \in [0,t]} \|u(\tau)\| \leq \vartheta(t),\ \forall t \in [0, T]\},$$

where ϑ is the unique solution of the integral equation

$$\vartheta(t) = \|\xi\| + \sup_{t \in [0,T]} \big(a * r_\alpha(\cdot, \beta\lambda_1)\big)(t) + \int_0^t r_\alpha(t-\tau, \beta\lambda_1)q(\tau)\vartheta(\tau)d\tau, t \in [0, T].$$

Then $\mathcal{D}$ is a closed, bounded and convex set in $C([0,T];L^2(\Omega))$. Considering the solution operator Φ on $\mathcal{D}$, we see that

$$\begin{aligned}\|\Phi(u)(t)\| &\le \|\mathcal{S}_\alpha(t)\xi\| + \int_0^t \|\mathcal{R}_\alpha(t-\tau)\|_{op}\|f(\tau,u(\tau))\|d\tau\\ &\le s_\alpha(t,\beta\lambda_1)\|\xi\| + \int_0^t r_\alpha(t-\tau,\beta\lambda_1)\big(q(\tau)\|u(\tau)\|+a(\tau)\big)d\tau\\ &\le \|\xi\| + \sup_{t\in[0,T]}\big(a*r_\alpha(\cdot,\beta\lambda_1)\big)(t)\\ &\qquad + \int_0^t r_\alpha(t-\tau,\beta\lambda_1)q(\tau)\sup_{\theta\in[0,\tau]}\|u(\theta)\|d\tau, \end{aligned}\tag{3.4}$$

for any $u\in\mathcal{D}$, thanks to (**F**4). Since the function $t\mapsto \sup_{\tau\in[0,t]}\|u(\tau)\|$ is nondecreasing, the integral term in the right hand side of (3.4) is nondecreasing in t as well. Thus

$$\begin{aligned}\sup_{\tau\in[0,t]}\|\Phi(u)(\tau)\| &\le \|\xi\| + \sup_{t\in[0,T]}\big(a*r_\alpha(\cdot,\beta\lambda_1)\big)(t)\\ &\qquad + \int_0^t r_\alpha(t-\tau,\beta\lambda_1)q(\tau)\sup_{\theta\in[0,\tau]}\|u(\theta)\|d\tau\\ &\le \|\xi\| + \sup_{t\in[0,T]}\big(a*r_\alpha(\cdot,\beta\lambda_1)\big)(t)\\ &\qquad + \int_0^t r_\alpha(t-\tau,\beta\lambda_1)q(\tau)\vartheta(\tau)d\tau \text{ for all } t\in[0,T]. \end{aligned}\tag{3.5}$$

The inequality (3.5) leads to $\Phi(\mathcal{D})\subset\mathcal{D}$. Then, by the same arguments in the proof of Theorem 3.3, we get the conclusion. The proof is complete. □

4. Regularity of mild solutions

This section is devoted to analyze the regularity of solutions to the semilinear problem (1.1)-(1.3). Our first result is the following.

Theorem 4.1. *Suppose that the assumptions of Theorem 3.3 hold with $q\in L^\infty(0,T;\mathbb{R}^+)$. Then every mild solutions of the problem (1.1)-(1.3) belong to $C^\gamma([\rho,T];L^2(\Omega))$, for any $\rho\in(0,T)$, here $\gamma\in(\gamma_0,1)$, $\gamma_0 := \max\{\alpha,1-\alpha\}$.*

Proof. Fixing δ,ϱ as in the proof of Theorem 3.3, where we know that for each $\xi\in L^2(\Omega)$ with $\|\xi\|\le\delta$ the problem (1.1)-(1.3) has a mild solution u which belongs to B_ϱ. Recalling that

$$u(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)f(\tau,u(\tau))d\tau, t\in[0,T]. \tag{4.1}$$

We next show that u given by (4.1) belongs to $C^\gamma([\rho,T];L^2(\Omega))$ for any $\rho\in(0,T)$. For $t\in[\rho,T]$ and $h\in(0,T-t]$, we find that

$$\begin{aligned}\|u(t+h)-u(t)\| &\le \|[\mathcal{S}_\alpha(t+h)-\mathcal{S}_\alpha(t)]\xi\|\\ &\qquad + \int_0^t \|[\mathcal{R}_\alpha(t+h-\tau)-\mathcal{R}_\alpha(t-\tau)]f(\tau,u(\tau))\|d\tau\\ &\qquad + \int_t^{t+h}\|\mathcal{R}_\alpha(t+h-\tau)f(\tau,u(\tau))\|d\tau. \end{aligned}\tag{4.2}$$

We now estimate the right hand side of (4.2) term by term as follows. For the first summand, we have

$$\begin{aligned}
\|[\mathcal{S}_\alpha(t+h)-\mathcal{S}_\alpha(t)]\xi\| &= h\|\int_0^1 \mathcal{S}'_\alpha(t+\theta h)\xi d\theta\| \\
&\leq h\|\xi\|\int_0^1 \frac{d\theta}{t+\theta h} \\
&= \|\xi\|\ln\big(1+\frac{h}{t}\big) \\
&\leq \gamma^{-1}\|\xi\|\Big(\frac{h}{t}\Big)^\gamma = \gamma^{-1}\|\xi\|t^{-\gamma}h^\gamma \\
&\leq \gamma^{-1}\|\xi\|\rho^{-\gamma}h^\gamma,
\end{aligned}$$

thanks to Lemma 2.4(a).

For the second summand, we first get

$$\begin{aligned}
&\|[\mathcal{R}_\alpha(t+h-\tau)-\mathcal{R}_\alpha(t-\tau)]f(\tau,u(\tau))\| \\
\leq& h\int_0^1 \|\mathcal{R}'_\alpha(t+\theta h-\tau)f(\tau,u(\tau))\|d\theta \\
\leq& h\|a\|_{L^\infty}\psi_f(\|u(\tau)\|)\int_0^1\big((t+\theta h-\tau)^{-1}+\nu\Gamma(1-\alpha)^{-1}(t+\theta h-\tau)^{-\alpha}\big)d\theta \\
\leq& \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho \\
&\quad\times\Big[\ln\big(1+\frac{h}{t-\tau}\big)+\nu\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}\big((t+h-\tau)^{1-\alpha}-(t-\tau)^{1-\alpha})\big)\Big] \\
\leq& \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho\Big[\gamma^{-1}(t-\tau)^{-\gamma}h^\gamma+\nu\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}h^{1-\alpha}\Big] \\
\leq& \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho\Big[\gamma^{-1}(t-\tau)^{-\gamma}+\nu\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}\Big]h^\gamma,
\end{aligned}$$

thanks to Lemma 2.4(b), here we have used the basic inequality $|t_1^\gamma - t_2^\gamma| \leq |t_1 - t_2|^\gamma$ for all $t_1, t_2 \geq 0$. From which we infer that

$$\begin{aligned}
&\int_0^t \|[\mathcal{R}_\alpha(t+h-\tau)-\mathcal{R}_\alpha(t-\tau)]f(\tau,u(\tau))\|d\tau \\
\leq& \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho h^\gamma\int_0^t\Big[\gamma^{-1}(t-\tau)^{-\gamma}+\nu\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}\Big]d\tau \\
=& \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho[\gamma^{-1}(1-\gamma)^{-1}t^{1-\gamma}+\nu\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}t]h^\gamma \\
\leq& \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho[\gamma^{-1}(1-\gamma)^{-1}T^{1-\gamma}+\nu\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}T]h^\gamma.
\end{aligned}$$

For the third summand, by using Proposition 2.1 and Lemma 2.4, we obtain

$$\begin{aligned}
&\int_t^{t+h} \|\mathcal{R}_\alpha(t+h-\tau)f(\tau,u(\tau))\|d\tau \\
&\le \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho\int_t^{t+h} r_\alpha(t-\tau,\beta\lambda_1)d\tau \\
&\le \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho\int_t^{t+h} \ell(t-\tau)d\tau \\
&\le \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho\int_t^{t+h} \frac{1}{1+\nu\Gamma(2-\alpha)^{-1}(t-\tau)^{1-\alpha}}d\tau \\
&\le \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho\nu^{-1}\Gamma(2-\alpha)\int_0^h \tau^{\alpha-1}d\tau \\
&= \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho\nu^{-1}\Gamma(2-\alpha)\alpha^{-1}h^\alpha \\
&\le \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho\nu^{-1}\Gamma(2-\alpha)\alpha^{-1}h^\gamma.
\end{aligned}$$

By combining all these previous estimates, it leads to

$$\|u(t+h)-u(t)\| \le \big(\gamma^{-1}\|\xi\|\rho^{-\gamma} + \|a\|_{L^\infty}(\alpha_f+\epsilon)\varrho C_0\big)h^\gamma, t>0, \tag{4.3}$$

where

$$C_0 = \gamma^{-1}(1-\gamma)^{-1}T^{1-\gamma} + \nu\Gamma(1-\alpha)^{-1}(1-\alpha)^{-1}T + \nu^{-1}\Gamma(2-\alpha)\alpha^{-1}.$$

The proof is complete. □

Remark 4.1. *In the case f possesses the sublinear growth condition* (**F**4) *with $a,b,c \in L^\infty(0,T;\mathbb{R}^+)$, then the Hölder regularity of mild solutions for the problem* (1.1)-(1.3) *can be proved without assuming the smallness of initial condition. In this case, the existence of mild solution is received by Theorem 3.4 and the Hölder regularity is proved the same lines as the proof of Theorem 4.1.*

We next deal with C^1-regularity of mild solutions for the problem (1.1)-(1.3). To do this, we first establish a result on C^1-regularity of mild solutions for the linear problem (2.3)-(2.5).

Theorem 4.2. *Suppose that $F \in C^\gamma([0,T];L^2(\Omega))$, $\xi \in L^2(\Omega)$. Let u be the coresponding mild solution of problem* (2.3)-(2.5)*, then the following statements are true:*

(i) *u is Hölder continuous on $(0,T]$ with exponent γ;*
(ii) *$u \in C^1((0,T];L^2(\Omega))$;*
(iii) *$\Delta u \in C((0,T];L^2(\Omega))$;*
(iv) *$\partial_{0+}^\alpha u \in C((0,T];L^2(\Omega))$.*

Consequently, u is a strong solution in time variable.

Proof. Let H_F be the Hölder constant of F. Recalling that

$$u(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)F(\tau)d\tau, t\in[0,T].$$

We now split the proof of this theorem into four claims.

Claim 1. u is Hölder continuous on $(0,T]$ with exponent γ. Indeed, for $t \in (0,T]$ and $h \in (0, T-t]$, by using the similar arguments as in the proof of Theorem 4.1, we have that

$$\|u(t+h) - u(t)\| \le \big(\gamma^{-1}\|\xi\| t^{-\gamma} + \|F\|_\infty C_0\big) h^\gamma, t > 0,$$

where C_0 is given in Theorem 4.1.
Claim 2. We now check that $u \in C^1((0,T]; L^2(\Omega))$. Noting that

$$u(t) = \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)F(\tau)d\tau = u_1(t) + u_2(t),$$

it is evidently clear from the formula above that $u_1 = \mathcal{S}_\alpha(\cdot)\xi \in C^1((0,T], L^2(\Omega))$, according to the statement of Lemma 2.4(a). It remains to testify that u_2 is a continuously differentiable function on $(0,T]$.

Observe that

$$\frac{du_2}{dt}(t) = F(t) + \int_0^t \mathcal{R}'_\alpha(t-\tau)F(\tau)d\tau.$$

The later term has meaning because

$$\begin{aligned}
\|\int_0^t \mathcal{R}'_\alpha(t-\tau)F(\tau)d\tau\| &\le \|\int_0^t \mathcal{R}'_\alpha(t-\tau)[F(\tau)-F(t)]d\tau\| + \|\int_0^t \mathcal{R}'_\alpha(t-\tau)F(t)d\tau\| \\
&\le \int_0^t \|\mathcal{R}'_\alpha(t-\tau)\|_{op}\|F(\tau)-F(t)\|d\tau + \|\big(I-\mathcal{R}_\alpha(t)\big)F(t)\| \\
&\le H_F \int_0^t [(t-\tau)^{-1} + \nu(t-\tau)^{-\alpha}](t-\tau)^\gamma d\tau \\
&\quad + \|\big(I-\mathcal{R}_\alpha(t)\big)F(t)\| \\
&= H_F\big(\gamma^{-1}t^\gamma + \nu(\gamma-\alpha+1)^{-1}t^{\gamma-\alpha+1}\big) \\
&\quad + \|\big(I-\mathcal{R}_\alpha(t)\big)F(t)\| < \infty, \forall t > 0,
\end{aligned}$$

thanks to Lemma 2.4(b). We next show that the mapping $t \mapsto \Phi(t) := \int_0^t \mathcal{R}'_\alpha(t-\tau)F(\tau)d\tau$ is continuous on $(0,T]$. For $t \in (0,T]$ and $h \in (0, T-t)$, the following estimate holds

$$\begin{aligned}
\|\Phi(t+h) - \Phi(t)\| &\le \|\int_0^t [\mathcal{R}'_\alpha(t+h-\tau) - \mathcal{R}'_\alpha(t-\tau)][F(\tau)-F(t)]d\tau\| \\
&\quad + \|\int_0^t [\mathcal{R}'_\alpha(t+h-\tau) - \mathcal{R}'_\alpha(t-\tau)]F(t)d\tau\| \\
&\quad + \|\int_t^{t+h} \mathcal{R}'_\alpha(t+h-\tau)F(\tau)d\tau\| \\
&= \Phi_1(t) + \Phi_2(t) + \Phi_3(t).
\end{aligned} \tag{4.4}$$

Since, for each $t \in (0,T)$, the mapping $\tau \mapsto \mathcal{R}'_\alpha(t+h-\tau)F(\tau)$ is integrable on $(0,t)$ and hence $\Phi_3(t) \to 0$ as $h \to 0$. Regarding $\Phi_1(t)$, we get

$$\Phi_1(t) \le H_F \int_0^t \|\mathcal{R}'_\alpha(t+h-\tau) - \mathcal{R}'_\alpha(t-\tau)\|_{op}(t-\tau)^\gamma d\tau.$$

Let $G_h(t) = \|\mathcal{R}'_\alpha(t+h-\tau) - \mathcal{R}'_\alpha(t-\tau)\|_{op}(t-\tau)^\gamma$. Clearly, for any $\tau \in (0,t)$, $\lim_{h\to 0} G_h(t) = 0$. In addition, with the formula of G_h, one has

$$\begin{aligned}G_h(t) &\le \big(\|\mathcal{R}'_\alpha(t+h-\tau)\|_{op} + \|\mathcal{R}'_\alpha(t-\tau)\|_{op}\big)(t-\tau)^\gamma\\ &\le \big((t+h-\tau)^{-1} + \nu(t+h-\tau)^{-\alpha} + (t-\tau)^{-1}\\ &\quad + \nu(t-\tau)^{-\alpha}\big)(t-\tau)^\gamma\\ &\le 2\big((t-\tau)^{\gamma-1} + \nu(t-\tau)^{\gamma-\alpha}\big) = G(t).\end{aligned}$$

Since $G \in L^1(0,t)$, it follows from the Lebesgue dominated convergence theorem that

$$\Phi_1(t) \le H_F \int_0^t G_h(\tau)d\tau \to 0 \text{ as } h \to 0.$$

With respect to Φ_2, we have

$$\Phi_2(t) = \|F(t)[\mathcal{R}_\alpha(t+h) - \mathcal{R}_\alpha(t) + I - \mathcal{R}_\alpha(h)]\| \to 0 \text{ as } h \to 0.$$

Therefore Φ is continuous on $(0,T]$ and hence $u \in C^1((0,T], L^2(\Omega))$.

Claim 3. $\Delta u \in C((0,T]; L^2(\Omega))$. Noting that $\Delta u = \Delta u_1 + \Delta u_2$ and $\Delta u_1(\cdot) = \Delta\mathcal{S}_\alpha(\cdot)\xi \in C((0,T]; L^2(\Omega))$, thanks to Lemma 2.4(a), it remains to show $\Delta u_2 \in C((0,T]; L^2(\Omega))$. To do this, we first observe that $\mathcal{S}'_\alpha(t)v = \beta\Delta\mathcal{R}_\alpha(t)v$ for all $v \in L^2(\Omega)$. And then

$$\begin{aligned}\|\beta\Delta u_2(t)\| &= \|\int_0^t \beta\Delta\mathcal{R}_\alpha(t-\tau)F(\tau)d\tau\|\\ &\le \|\int_0^t \mathcal{S}'_\alpha(t-\tau)[F(\tau)-F(t)]d\tau\| + \|\int_0^t \mathcal{S}'_\alpha(t-\tau)F(t)d\tau\|\\ &\le H_F\int_0^t (t-s)^{-1}(t-s)^\gamma d\tau + \|\int_0^t \mathcal{S}'_\alpha(\tau)F(t)d\tau\|\\ &\le \gamma^{-1}H_F t^\gamma + \|F(t)(I - \mathcal{S}_\alpha(t))\| < \infty, \text{ for all } t > 0,\end{aligned}$$

thanks to Lemma 2.4(a) again and then Δu_2 makes sense. We now prove that the mapping $t \mapsto \Delta u_2(t)$ is continuous on $(0,T]$. Let $t > 0$ and $h \in (0, T-t]$, one has

$$\begin{aligned}&\|\Delta u_2(t+h) - \Delta u_2(t)\|\\ \le&\|\int_t^{t+h} \mathcal{S}'_\alpha(t+h-\tau)F(\tau)d\tau\|\\ &+ \|\int_0^t [\mathcal{S}'_\alpha(t+h-\tau) - \mathcal{S}'_\alpha(t-\tau)]F(t)d\tau\|\\ &+ \|\int_0^t [\mathcal{S}'_\alpha(t+h-\tau) - \mathcal{S}'_\alpha(t-\tau)][F(\tau)-F(t)]d\tau\|\\ =&\Phi_4(t) + \Phi_5(t) + \Phi_6(t).\end{aligned}$$

According to Lemma 2.4(a), the function $t \mapsto \Delta u_2(t)$ is integrable, it implies that $\Phi_4(t) \to 0$ as $h \to 0$. For same reason, one obtains the following estimates for Φ_5:

$$\begin{aligned}
\Phi_5(t) &= \| \int_h^{t+h} \mathcal{S}'_\alpha(\tau)F(t)d\tau - \int_0^t \mathcal{S}'_\alpha(\tau)F(t)d\tau \| \\
&\leq \| \int_t^{t+h} \mathcal{S}'_\alpha(\tau)F(t)d\tau \| + \| \int_0^h \mathcal{S}'_\alpha(\tau)F(t)d\tau \| \\
&\to 0 \text{ as } h \to 0.
\end{aligned}$$

Concerning $\Phi_6(t)$, for all $\tau \in (0,t)$, we see

$$I_h(t) := \| [\mathcal{S}'_\alpha(t+h-\tau) - \mathcal{S}'_\alpha(t-\tau)][F(\tau) - F(t)] \| \to 0 \text{ as } h \to 0,$$

thanks to the fact that $\mathcal{S}'_\alpha(\cdot)z \in C((0,T];L^2(\Omega))$ for any $z \in L^2(\Omega)$. Furthermore, due to Lemma 2.4(a) and the Hölder continuity of F, we have that

$$\begin{aligned}
I_h(t) &\leq [(t+h-\tau)^{-1} + (t-\tau)^{-1}]H_F(t-\tau)^\gamma \\
&\leq 2H_F\Gamma(2-\alpha)(t-\tau)^{\gamma-1} =: I^*(\tau).
\end{aligned}$$

Obviously, $I^* \in L^1(0,t)$, so that

$$\Phi_5(t) \leq \int_0^t I_h(\tau)d\tau \to \text{ as } h \to 0,$$

thanks to the Lebesgue dominated convergence theorem.

Claim 4. Finally we show $\partial^\alpha_{0+}u \in C((0,T];L^2(\Omega))$ and u satisfies the equation (2.3) on $(0,T]$.

Recalling that

$$\begin{aligned}
u(t) &= \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)F(\tau)d\tau \\
&= \sum_{n=1}^\infty [s_\alpha(t,\beta\lambda_n)\xi_n + \big(r_\alpha(\cdot,\beta\lambda_n) * F_n\big)(t)]e_n,
\end{aligned}$$

where $F_n(t) = \big(F(t), e_n\big)$. Then

$$\begin{aligned}
\big(g_{1-\alpha} * [u(\cdot) - \xi]\big)(t) = \sum_{n=1}^\infty \Big[& \big(g_{1-\alpha} * [s_\alpha(\cdot,\beta\lambda_n) - 1]\big)(t)\xi_n \\
& + \big(g_{1-\alpha} * r_\alpha(\cdot,\beta\lambda_n) * F_n\big)(t)\Big]e_n.
\end{aligned}$$

Consider the series

$$\sum_{n=1}^\infty [g_{1-\alpha} * s_\alpha(\cdot,\beta\lambda_n)(t)\xi_n + \big(g_{1-\alpha} * r_\alpha(\cdot,\beta\lambda_n) * F_n\big)(t)]'(t)e_n. \tag{4.5}$$

Note that

$$g_{1-\alpha} * [s_\alpha(\cdot,\beta\lambda_n) - 1] + \beta\lambda_n\big(g_{1-\alpha} * \ell * s_\alpha(\cdot,\beta\lambda_n)\big) = 0,$$

thus

$$\begin{aligned}\nu\frac{d}{dt}g_{1-\alpha}*[s_\alpha(\cdot,\beta\lambda_n)-1] &= -\beta\lambda_n\frac{d}{dt}\big(\nu g_{1-\alpha}*\ell*s_\alpha(\cdot,\beta\lambda_n)\big)\\ &= -\beta\lambda_n\frac{d}{dt}[(1-\ell)*s_\alpha(\cdot,\beta\lambda_n)]\\ &= -\beta\lambda_n s_\alpha(\cdot,\beta\lambda_n)+\beta\lambda_n\frac{d}{dt}\ell*s_\alpha(\cdot,\beta\lambda_n)\\ &= -\beta\lambda_n s_\alpha(\cdot,\beta\lambda_n)-s'_\alpha(\cdot,\beta\lambda_n),\end{aligned}$$

thanks to the differentiability of $s_\alpha(\cdot,\beta\lambda_n)$ and the equation (2.8). Furthermore, by using Proposition 2.2(iii), we obtain

$$\nu\frac{d}{dt}g_{1-\alpha}*r_\alpha(\cdot,\beta\lambda_n)=s'_\alpha(\cdot,\beta\lambda_n)-r'_\alpha(\cdot,\beta\lambda_n).$$

Therefore

$$\nu\frac{d}{dt}\big(g_{1-\alpha}*r_\alpha(\cdot,\beta\lambda_n)*F_n\big)=s'_\alpha(\cdot,\beta\lambda_n)*F_n-r'_\alpha(\cdot,\beta\lambda_n)*F_n.$$

From the above calculations, one can write

$$\begin{aligned}&\sum_{n=1}^\infty \nu[g_{1-\alpha}*s_\alpha(\cdot,\beta\lambda_n)(t)\xi_n+\big(g_{1-\alpha}*r_\alpha(\cdot,\beta\lambda_n)*F_n\big)(t)]'(t)e_n\\ &=-\sum_{n=1}^\infty\beta\lambda_n s_\alpha(t,\beta\lambda_n)\xi_n e_n-\sum_{n=1}^\infty s'_\alpha(\cdot,\beta\lambda_n)\xi_n e_n\\ &\quad+\sum_{n=1}^\infty s'_\alpha(\cdot,\beta\lambda_n)*F_n(t)e_n-\sum_{n=1}^\infty r'_\alpha(\cdot,\beta\lambda_n)*F_n(t)e_n. \qquad (4.6)\end{aligned}$$

According to the representation (4.6), for any $\epsilon\in(0,T)$ we have that the first and second series are uniformly convergent on $[\epsilon,T]$ to $-\beta\Delta\mathcal{S}_\alpha(t)\xi$, $\mathcal{S}'_\alpha(t)\xi$, respectively, thanks to Lemma 2.4. We now show that the third and fourth series are also uniformly convergent on $[\epsilon,T]$. Indeed, put

$$\mathcal{J}_1=\sum_{n=1}^\infty s'_\alpha(\cdot,\beta\lambda_n)*F_n(\cdot)e_n, \mathcal{J}_2=\sum_{n=1}^\infty r'_\alpha(\cdot,\beta\lambda_n)*F_n(\cdot)e_n.$$

It is clear that $\mathcal{J}_1$ converges pointwise in $L^2(\Omega)$ to $\mathcal{S}'_\alpha*F(\cdot)$ on $[\epsilon,T]$. Hence, it only needs to prove that $\mathcal{J}_1$ is also uniformly convergent on $[\epsilon,T]$. Since $F\in C^\gamma([0,T];L^2(\Omega))$, it follows that

$$\|F(t)-F(\tau)\|^2=\sum_{n=1}^\infty|F_n(t)-F_n(\tau)|^2\le H_F^2|t-\tau|^{2\gamma},\forall t,\tau\in[0,T].$$

Denote

$$\Upsilon_n(t,\tau)=\frac{|F_n(t)-F_n(\tau)|}{|t-\tau|^\gamma},t,\tau\in[0,T],t\ne\tau,n=1,2,\dots$$

Then

$$
\begin{aligned}
\mathcal{J}^2_{1,n}(t) &:= \big(s'_\alpha(\cdot,\beta\lambda_n) * F_n(t)\big)^2 \\
&\leq \bigg(\int_0^t \beta\lambda_n r_\alpha(t-\tau,\beta\lambda_n)|F_n(t)-F_n(\tau)|d\tau \\
&\qquad + \int_0^t \beta\lambda_n r_\alpha(t-\tau,\beta\lambda_n)|F_n(t)|d\tau\bigg)^2 \\
&\leq 2\bigg(\int_0^t \beta\lambda_n r_\alpha(t-\tau,\beta\lambda_n)|F_n(t)-F_n(\tau)|d\tau\bigg)^2 \\
&\qquad + 2\bigg(\int_0^t \beta\lambda_n r_\alpha(t-\tau,\beta\lambda_n)|F_n(t)|d\tau\bigg)^2 \\
&= 2\bigg(\int_0^t \beta\lambda_n r_\alpha(t-\tau,\beta\lambda_n)(t-\tau)^\gamma \Upsilon_n(t,\tau)d\tau\bigg)^2 \\
&\qquad + 2|F_n(t)|^2\bigg(\int_0^t \beta\lambda_n r_\alpha(\tau,\beta\lambda_n)d\tau\bigg)^2 \\
&\leq 2\bigg(\int_0^t (t-\tau)^{\gamma-1}\Upsilon_n(t,\tau)d\tau\bigg)^2 + 2|F_n(t)|^2\bigg(\int_0^t \beta\lambda_n r_\alpha(\tau,\beta\lambda_n)d\tau\bigg)^2 \\
&\leq 2\bigg(\int_0^t (t-\tau)^{\gamma-1}\Upsilon_n(t,\tau)d\tau\bigg)^2 + 2|F_n(t)|^2, \qquad (4.7)
\end{aligned}
$$

here we have utilized the inequalities $\int_0^t \beta\lambda_n r(\tau,\beta\lambda_n)d\tau \leq 1, \forall t \geq 0$ and $\beta\lambda_n r_\alpha(t-\tau,\beta\lambda_n) < (t-\tau)^{-1}, \forall t > \tau$. In the light of the Hölder inequality, the first term in (4.7) can be estimated as the following

$$
\begin{aligned}
\bigg(\int_0^t (t-\tau)^{\gamma-1}\Upsilon_n(t,\tau)d\tau\bigg)^2 &\leq \int_0^t (t-\tau)^{\gamma-1}\Upsilon_n(t,\tau)^2 d\tau \int_0^t (t-\tau)^{\gamma-1}d\tau \\
&\leq \gamma^{-1}T^\gamma \int_0^t (t-\tau)^{\gamma-1}\Upsilon_n(t,\tau)^2 d\tau. \qquad (4.8)
\end{aligned}
$$

It is to be noticed that, after defining values at diagonal, the function

$$
\Upsilon^*(t,\tau) = \frac{\|F(t)-F(\tau)\|}{|t-\tau|^\gamma}, t,\tau \in [0,T],
$$

is continuous on $[0,T]\times[0,T]$. Furthermore, the series $\sum_{n=1}^\infty \Upsilon_n^2(t,\tau)$ converges uniformly to $\Upsilon^*(t,\tau)$ on $[0,T]\times[0,T]$. Therefore, for $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$
\sum_{n=N}^{N+p} \Upsilon_n^2(t,\tau) < \epsilon, \text{ for all } p \in \mathbb{N}, t,\tau \in [0,T]. \qquad (4.9)
$$

Similarly, due to the uniform convergence of the series $\sum_{n=1}^\infty |F_n(t)|^2, t \in [0,T]$, we also have

$$
\sum_{n=N}^{N+p} |F_n(t)|^2 < \epsilon, \text{ for all } p \in \mathbb{N}, t \in [0,T]. \qquad (4.10)
$$

From the estimates (4.7)-(4.10), it implies

$$\sum_{n=N}^{N+p} \mathcal{J}_{2,n}^2 \leq 2\gamma^{-1}T^{\gamma}\epsilon \int_0^t (t-\tau)^{\varsigma} d\tau + 2\epsilon$$
$$\leq 2\big(\gamma^{-2}T^{2\gamma} + 1\big)\epsilon,$$

where $\mathcal{J}_{1,n}^2(t) := \big(r'_\alpha(\cdot, \beta\lambda_n) * F_n(t)\big)^2$. We have the conclusion as required. By the same arguments, we can prove that $\mathcal{J}_2$ is uniformly convergent on $[\epsilon, T]$ to $\big(\mathcal{R}'_\alpha * F\big)(\cdot)$. From the previous considerations, it turns out that $\partial_{0+}^\alpha u \in C((0,T], L^2(\Omega))$. Thus we get the desired claim.

Finally, we check that u satisfies equation (2.3). By the proof of Claim 2, one has

$$\partial_t u(t,\cdot) = \mathcal{S}'_\alpha(t)\xi + F(t,\cdot) + \int_0^t \mathcal{R}'_\alpha(t-\tau)F(\tau,\cdot)d\tau. \tag{4.11}$$

Furthermore, by (4.6), we get

$$\begin{aligned}\nu\partial_{0+}^\alpha u &= \beta\Delta\mathcal{S}_\alpha(t)\xi - \mathcal{S}'_\alpha(t)\xi + \mathcal{S}'_\alpha * F(t) - \big(\mathcal{R}'_\alpha * F\big)(t)\\ &= \beta\Delta\big(\mathcal{S}_\alpha(t)\xi + \big(\mathcal{R}_\alpha * F\big)(t)\big) - \mathcal{S}'_\alpha(t)\xi - \big(\mathcal{R}'_\alpha * F\big)(t)\\ &= \beta\Delta u - \mathcal{S}'_\alpha(t)\xi - \big(\mathcal{R}'_\alpha * F\big)(t). \end{aligned}\tag{4.12}$$

Combining (4.11) and (4.12) yields

$$\partial_t u + \nu\partial_{0+}^\alpha u - \beta\Delta u = F, t \in (0,T],$$

which completes the proof of the theorem. □

Now we are in a position to formulate the result on C^1-regularity of mild solutions to the semilinear problem (1.1)-(1.3).

Theorem 4.3. *Assume that the nonlinearity function* $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ *satisfies* $f(\cdot, 0) = 0$ *and a locally Lipschitz-Hölder condition*

$$\|f(t_1, v_1) - f(t_2, v_2)\| \leq \kappa(r)\big(|t_1 - t_2|^\gamma + \|v_1 - v_2\|\big), \tag{4.13}$$

for all $t_i \in [0,T]$, $\|v_i\| \leq r, i \in \{1,2\}$, *where* κ *is a nonnegative function such that* $\limsup_{r\to 0} \kappa(r) = l \in \big[0, \beta\lambda_1\big)$. *Then there exists* $\delta > 0$ *such that the problem* (1.1)-(1.3) *has a unique mild solution* u *on* $[0,T]$ *obeying* $u \in C([0,T]; L^2(\Omega)) \cap C^1((0,T]; L^2(\Omega))$, *provided that* $\|\xi\| \leq \delta$.

Proof. Note that, the assumption (4.13) implies (3.1). Thus, due to Theorem 3.1, for each ξ with $\|\xi\| \leq \delta$, the problem (1.1)-(1.3) has a unique mild $u \in \mathsf{B}_R$, where δ, R are taken as in the proof of Theorem 3.1. Let $F(t) = f(t, u(t)), t \in [0,T]$. Using the same arguments as in the proof of Theorem 4.1 and of Claim 1 in Theorem 4.2, the mild solution u given by (4.1) is Hölder continuous on $[0,T]$ with exponent γ. According to this consideration and assumption (4.13), F is also Hölder continuous on $[0,T]$ with exponent γ. Therefore the conclusion of this theorem follows by applying Theorem 4.2. □

Remark 4.2. *If* f *enjoys global Lipschitz condition, i.e.*

$$\|f(t, v_1) - f(t, v_2)\| \leq L_f\|v_1 - v_2\|, \tag{4.14}$$

for all $t \in [0,T]$, $v_i \in L^2(\Omega), i \in \{1,2\}$, *then the* C^1-*regularity of mild solution for the problem* (1.1)-(1.3) *can be proved without any assumptions* $L_f \in \big[0, \beta\lambda_1\big)$ *or the* *smallness of initial condition. In this case, the existence and uniqueness of mild solution is received by Theorem 3.2 and the C^1-regularity is proved the same lines as the proof of Theorem 4.3.*

5. Large-time behavior of solutions

Our main object in this section is to analyze the asymptotic behavior of mild solutions of problem (1.1)-(1.3) on $[0,\infty)$. Using the Gronwall type inequality stated in Lemma 2.6, we establish several results on asymptotic stability and dissipativity of solutions. Furthermore, in the special case when the nonlinearity function f does not depend on the time variable, say $f(t,u)=f(u)$, we prove a result on the convergence rate of nontrivial solutions to the equilibrium.

5.1. **Stability and dissipativity.**

Theorem 5.1. *Consider the system* (1.1)-(1.3)*. Suppose that the function f satisfies the condition*

(**F**4)' *f is continuous such that $\|f(t,v)\| \le q(t)\|v\| + a(t)$, for all $t \in \mathbb{R}^+$, $v \in L^2(\Omega)$, where q,a are nonnegative functions with $q \in L^\infty(\mathbb{R}^+)$, $a \in L^1_{loc}(\mathbb{R}^+)$, $\|q\|_{L^\infty} < \beta\lambda_1$ and $r_\alpha * a$ is a bounded function.*

Then, there exists an absorbing set for its solutions. Moreover, if $a=0$, then the trivial solution is asymptotically stable.

Proof. Let u be a solution of (1.1)-(1.3). Then, by using Lemma 2.4 and the estimate of f, we obtain that

$$\begin{aligned}\|u(t)\| &\le s_\alpha(t,\beta\lambda_1)\|\xi\| + \int_0^t r_\alpha(t-\tau,\beta\lambda_1)\big(a(\tau)+q(\tau)\|u(\tau)\|\big)d\tau\\ &\le s_\alpha(t,\beta\lambda_1)\|\xi\| + \int_0^t r_\alpha(t-\tau,\beta\lambda_1)[a(\tau)+\|q\|_{L^\infty}\|u(\tau)\|]d\tau.\end{aligned}$$

Applying the Gronwall type inequality given in Lemma 2.6, we get

$$\|u(t)\| \le s_\alpha(t,\beta\lambda_1-\|q\|_{L^\infty})\|\xi\| + \int_0^t r_\alpha(t-\tau,\beta\lambda_1-\|q\|_{L^\infty})a(\tau)d\tau.$$

Put

$$R = 1+\sup_{t\ge 0}\int_0^t r_\alpha(t-\tau,\beta\lambda_1-\|q\|_{L^\infty})a(\tau)d\tau,$$

then the ball B_R becomes an absorbing set for solutions of (1.1)-(1.3), by virtue of the fact that

$$s_\alpha(t,\beta\lambda_1-\|q\|_{L^\infty}) \to 0 \text{ as } t\to\infty.$$

Finally, if $a=0$ then (1.1) admits the zero solution and it holds that

$$\|u(t)\| \le s_\alpha(t,\beta\lambda_1-\|q\|_{L^\infty})\|\xi\|,\ \forall t\ge 0,$$

which means that the zero solution is asymptotically stable. The proof is complete. □

Theorem 5.2. *Suppose that the hypotheses of Theorem 3.1 hold for any $T>0$. Then the trivial solution of* (1.1) *is asymptotically stable.*

Proof. Take ρ, δ, and ϵ as in the proof of Theorem 3.1. Then for every $\|\xi\| \leq \delta$, there exists a unique mild solution to (1.1)-(1.3) such that $\|u(t)\| \leq \rho$ for all $t > 0$ and it holds that

$$\begin{aligned}\|u(t)\| &\leq \|\mathcal{S}_\alpha(t)\xi\| + \int_0^t \|\mathcal{R}_\alpha(t-\tau)\|_{op}\|f(\tau, u(\tau)) - f(\tau, 0)\|d\tau \\ &\leq s_\alpha(t, \beta\lambda_1)\|\xi\| + \int_0^t r_\alpha(t-\tau, \beta\lambda_1)\kappa(\rho)\|u(\tau)\|d\tau \\ &\leq s_\alpha(t, \beta\lambda_1)\|\xi\| + \int_0^t r_\alpha(t-\tau, \beta\lambda_1)(l+\epsilon)\|u(\tau)\|d\tau, \ \forall t \geq 0.\end{aligned}$$

Using the Gronwall type inequality in Lemma 2.6 again, we obtain

$$\|u(t)\| \leq s_\alpha(t, \beta\lambda_1 - l - \epsilon)\|\xi\|, \ \forall t \geq 0. \tag{5.1}$$

Since $\beta\lambda_1 - l - \epsilon > 0$, we have

$$s_\alpha(t, \beta\lambda_1 - l - \epsilon) \to 0 \text{ as } t \to \infty.$$

Hence inequality (5.1) ensures the stability and attractivity of the zero solution. The proof is complete. □

Considering the case when f is globally Lipschitzian, we obtain a stronger result.

Theorem 5.3. *If f satisfies the Lipschitz condition* (4.14) *with $L_f \in [0, \beta\lambda_1)$, then every mild solution of* (1.1)-(1.2) *is asymptotically stable.*

Proof. Let u and v be solutions of (1.1)-(1.2). Then

$$\begin{aligned}\|u(t) - v(t)\| &\leq \|\mathcal{S}_\alpha(t)[u(0) - v(0)]\| \\ &\quad + \int_0^t \|\mathcal{R}_\alpha(t-\tau)\|_{op}\|f(\tau, u(\tau)) - f(\tau, v(\tau))\|d\tau \\ &\leq s_\alpha(t, \beta\lambda_1)\|u(0) - v(0)\| + \int_0^t r_\alpha(t-\tau, \beta\lambda_1)L_f\|u(\tau) - v(\tau)\|d\tau.\end{aligned}$$

Applying Lemma 2.6 yields

$$\|u(t) - v(t)\| \leq s_\alpha(t, \beta\lambda_1 - L_f)\|u(0) - v(0)\|, \ \forall t \geq 0,$$

which implies that every solution of (1.1)-(1.2) is asymptotically stable. □

5.2. **Convergence to equilibrium.** In the special case the nonlinearity function f being not dependent on the variable time t, i.e., $f(t, u) = f(u)$, we obtain a result on the convergence to equilibrium point with algebraic rate of solutions to problem (1.1)-(1.3) in the following theorem.

Theorem 5.4. *Let* (**F**1) *holds and u be a global mild solution of* (1.1)-(1.3). *Assume that $\partial\Omega \in C^2$. If there exists the limit $\lim_{t\to\infty} u(t) = u^*$ in $L^2(\Omega)$, then u^* is a strong solution of the elliptic problem*

$$-\beta\Delta v = f(v) \text{ in } \Omega \tag{5.2}$$

$$v = 0 \text{ on } \partial\Omega. \tag{5.3}$$

In addition, if f satisfies the Lipschitz condition (4.14) *with $L_f \in [0, \beta\lambda_1)$ then the problem* (5.2)-(5.3) *has a unique strong solution u^*, and for each $\xi \in L^2(\Omega)$, the mild solution of* (1.1)-(1.3) *converges to u^* with algebraic rate as $t \to \infty$.*

Proof. We prove by the following two steps. By the continuity of f we obtain $\lim_{t\to\infty} f(u(t)) = f(u^*)$. That means for $\epsilon > 0$ there exists $T^* > 0$ such that

$$\|f(u(t)) - f(u^*)\| \leq \epsilon, \text{ for all } t > T^*.$$

On the other hand, by the representation formula of mild solution, we can write

$$\begin{aligned} u(t) &= \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau) f(u(\tau)) d\tau \\ &= \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau) f(u^*) d\tau + \int_0^t \mathcal{R}_\alpha(t-\tau)[f(u(\tau)) - f(u^*)] d\tau \\ &= \mathcal{N}_1(t) + \mathcal{N}_2(t) + \mathcal{N}_3(t). \end{aligned}$$

Regarding $\mathcal{N}_1$, we have that

$$\|\mathcal{N}_1(t)\| \leq s_\alpha(t, \beta\lambda_1)\|\xi\| \to 0 \text{ as } t \to \infty.$$

As for $\mathcal{N}_3$, we find that

$$\begin{aligned} \|\mathcal{N}_3(t)\| &= \left(\int_0^{T^*} + \int_{T^*}^t\right) \mathcal{R}_\alpha(t-\tau)[f(u(\tau)) - f(u^*)] d\tau \\ &\leq \int_0^{T^*} r_\alpha(t-\tau, \beta\lambda_1)\kappa(R)\|u(\tau) - u^*\| d\tau + \epsilon \int_{T^*}^t r_\alpha(t-\tau, \beta\lambda_1) d\tau \\ &\leq 2R\kappa(R) \int_0^{T^*} r_\alpha(t-\tau, \beta\lambda_1) d\tau + \epsilon \int_{T^*}^t r_\alpha(\tau, \beta\lambda_1) d\tau \\ &\leq 2R\kappa(R) \int_0^{T^*} r_\alpha(t-\tau, \beta\lambda_1) d\tau + \epsilon\beta^{-1}\lambda_1^{-1} \\ &\leq 2R\kappa(R) \int_{t-T^*}^t r_\alpha(\tau, \beta\lambda_1) d\tau + \epsilon\beta^{-1}\lambda_1^{-1}, \text{ for } t > T^*, \end{aligned}$$

thanks to Proposition 2.2, where $R = \|u\|_\infty + \|u^*\|$. Since $r_\alpha(\cdot, \beta\lambda_1) \in L^1(\mathbb{R}^+)$, there exists $T_1 > 0$ such that

$$\int_{T_1}^t r_\alpha(\tau, \beta\lambda_1) d\tau \leq \epsilon, \text{ for all } t > T_1.$$

Therefore for $t > T^* + T_1$ we have

$$\|\mathcal{N}_3(t)\| \leq [2R\kappa(R) + \beta^{-1}\lambda_1^{-1}]\epsilon,$$

which guarantees that $\mathcal{N}_3(t) \to 0$ as $t \to \infty$. Since $\lim_{t\to\infty} u(t) = u^*$, it implies that

$$u^* = \lim_{t\to\infty} \mathcal{N}_2(t) = \int_0^\infty \mathcal{R}_\alpha(\tau) f(u^*) d\tau = \widehat{\mathcal{R}_\alpha}(0) f(u^*).$$

Noting that $\mathcal{S}_\alpha(\cdot)$ be the solution operator of the homogeneous problem

$$\partial_t u + \nu\partial_{0+}^\alpha u - \beta\Delta u = 0, \text{ in } \Omega, t > 0, \tag{5.4}$$

$$u = 0 \text{ on } \partial\Omega, t \geq 0, \tag{5.5}$$

$$u(0, \cdot) = \xi \text{ in } \Omega. \tag{5.6}$$

that is, $u(t) = \mathcal{S}_\alpha(t)\xi$, $t \geq 0$. Furthermore, by integration the equation (5.4) in t and using the relation (2.1), this problem can be rewritten in $L^2(\Omega)$ under the form

$$u(t) + \big(\ell * (-\beta\Delta)u\big)(t) = \xi, t \geq 0.$$

With the aid of Laplace transform, we find that

$$\widehat{u}(\lambda) = \xi\lambda^{-1}\big(I + \widehat{\ell}(\lambda)(-\beta\Delta)\big)^{-1},$$

thus

$$\widehat{\mathcal{S}_\alpha}(\lambda) = \lambda^{-1}\big(I + \widehat{\ell}(\lambda)(-\beta\Delta)\big)^{-1}.$$

Since $\mathcal{S}_\alpha'(t) = \beta\Delta\mathcal{R}_\alpha(t)$, it follows that

$$\lambda\widehat{\mathcal{S}_\alpha}(\lambda) - \mathcal{S}_\alpha(0) = \beta\Delta\widehat{\mathcal{R}_\alpha}(\lambda),$$

or equivalently

$$\beta\Delta\widehat{\mathcal{R}_\alpha}(\lambda) = \big(I + \widehat{\ell}(\lambda)(-\beta\Delta)\big)^{-1} - I.$$

By letting the limit as $\lambda \to 0$ in the last relation, we find that

$$\beta\Delta\widehat{\mathcal{R}_\alpha}(0) = \lim_{\lambda\to 0}\big[\big(I + \widehat{\ell}(\lambda)(-\beta\Delta)\big)^{-1} - I\big] = -I,$$

thanks to the fact that $\lim\limits_{\lambda\to 0}\widehat{\ell}(\lambda) = \lim\limits_{\lambda\to 0}\frac{1}{\lambda+\lambda^\alpha} = \infty$. Therefore,

$$\widehat{\mathcal{R}_\alpha}(0) = (-\beta\Delta)^{-1},$$

and then

$$u^* = (-\beta\Delta)^{-1}f(u^*).$$

According to $\partial\Omega \in C^2$, we obtain $u^* \in H^2(\Omega)$ and $-\beta\Delta u^* = f(u^*)$, thanks to [6, Sect. 6.3.2].

We now consider the elliptic problem (5.2)-(5.3) where the nonlinear function f satisfies the global Lipschitz hypothesis (4.14) with $L_f \in [0, \beta\lambda_1)$. Let C_{e} be the constant of embedding $H_0^1(\Omega) \subset L^2(\Omega)$. By the smoothness of $\partial\Omega$, we find that

$$C_{\mathsf{e}}^{-2} = \inf_{v\in H_0^1(\Omega)\setminus\{0\}}\frac{\|\nabla v\|^2}{\|v\|^2} = \lambda_1.$$

Since the Lipschitz constant L_f fulfills $L_f < \beta\lambda_1$, then the problem (5.2)-(5.3) admits a unique weak solution $v \in H_0^1(\Omega)$, due to [4, Theorem 7.4.1]. Observing that if u^* be a weak solution of (5.2)-(5.3) then $f(u^*) \in L^2(\Omega)$. Using this fact and the regularity result in [6, Sect. 6.3.2] again, it may be checked that $u^* \in H^2(\Omega)$ and then u^* be a unique strong solution of (5.2)-(5.3).

On the one hand, by the formula of mild solution, we have

$$\begin{aligned} u(t) - u^* &= \mathcal{S}_\alpha(t)\xi - (-\beta\Delta)^{-1}f(u^*) + \int_0^t \mathcal{R}_\alpha(t-\tau)f(u(\tau))d\tau \\ &= \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(\tau)f(u^*)d\tau - (-\beta\Delta)^{-1}f(u^*) \\ &\quad + \int_0^t \mathcal{R}_\alpha(t-\tau)[f(u(\tau)) - f(u^*)]d\tau. \end{aligned} \tag{5.7}$$

On the other hand, by a direct computation, we obtain

$$\int_0^t \mathcal{R}_\alpha(\tau)f(u^*)d\tau - (-\beta\Delta)^{-1}f(u^*) = -(-\beta\Delta)^{-1}\mathcal{S}_\alpha(t)f(u^*). \tag{5.8}$$

The above relation (5.8) together with (5.7) yields

$$\begin{aligned}\|u(t)-u^*\| &\leq \|\mathcal{S}_\alpha(t)\xi-(-\beta\Delta)^{-1}\mathcal{S}_\alpha(t)f(u^*)\|\\ &\quad+\|\int_0^t \mathcal{R}_\alpha(t-\tau)[f(u(\tau))-f(u^*)]d\tau\|\\ &\leq s_\alpha(t,\beta\lambda_1)\big(\|\xi\|+\|(-\beta\Delta)^{-1}\|_{op}\|f(u^*)\|\big)\\ &\quad+L_f\int_0^t r_\alpha(t-\tau,\beta\lambda_1)\|u(\tau)-u^*\|d\tau.\end{aligned}$$

Using the Gronwall type inequality, one sees that

$$\|u(t)-u^*\|\leq s_\alpha(t,\beta\lambda_1-L_f)\big(\|\xi\|+\|(-\beta\Delta)^{-1}\|_{op}\|f(u^*)\|\big).$$

From the last inequality and Remark 2.1(i), we obtain

$$\|u(t)-u^*\|=O(t^{-\alpha}) \text{ as } t\to\infty.$$

The proof is complete. □

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Conflicts of interest

There are no conflicts of interest to this work.

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