PROPORTIONAL LOCAL ASSIGNABILITY OF THE DICHOTOMY SPECTRUM OF ONE-SIDED DISCRETE TIME-VARYING LINEAR SYSTEMS*

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Abstract. We consider a problem of assignability of dichotomy spectrum for one-sided discrete time-varying linear systems. Our purpose is to prove that uniform complete controllability is a sufficient condition for proportional local assignability of the dichotomy spectrum.

Key words. dichotomy spectrum; discrete time-varying linear systems; pole placement theorem; uniform complete controllability

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1. Introduction.

In 1967 W. M. Wonham in the famous paper [38] proved that in a time-invariant linear continuous-time control system, the poles of a closed system can be arbitrarily assignable through a linear feedback if and only if the system is controllable. This result is a theoretical base for of the one of the fundamental methods of designing control which is called pole placement method or pole-shifting or the spectrum assignment method (see [36]). The idea of this method is to construct a feedback in such a way that the eigenvalues of the closed loop system have a priory given location.

When one wants to generalize this methodology to time-varying systems many difficulties arise. One of them is that there are no direct equivalents of poles for time-varying systems. The role of their real parts for continuous - time systems or modules for discrete-time systems play Lyapunov, Bohl, Perron exponents or dichotomy spectra. Another difficulty is that for time-varying systems there are many non-equivalent definitions of controllability (see [20]).

The problem of generalization of the pole placement theorem to time-varying systems has been so far studied mainly for Lyapunov exponents as a counterpart of poles and the results are summarized for continuous-time systems in [23] and for discrete-time systems in [8, 5] and [4]. Very recently, several results on the assignability of dichotomy spectrum for time-varying control systems have been established. Concretely, it was shown in [15], [6] and [7] that both for discrete time and continuous time systems with bounded time-varying coefficients, considered both on the half-line and on the whole line, the dichotomy spectrum is assignable if and only if the system is uniformly completely controllable.

The concept of the dichotomy spectrum comes in a natural way from the concept of exponential dichotomy which plays an important role in many aspect of linear systems, see [24, 25] and [13]. Based on the notion of exponential dichotomy, in [31] the authors developed a spectral theory for linear differential equations over a

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compact base. Later, this spectrum, which is now called dichotomy spectrum, has been extended to a linear systems over non-compact base, see [34], [22] and [27].

Now dichotomy spectrum is an important tool in the qualitative theory of time-varying dynamical systems which has steadily growing interest in the last thirty years due to its applicability in modelling many real world phenomena in biology, economy, climate change,..., see [21, 22]. The more reasons for importance and usefulness of dichotomy spectrum are as follows. This spectrum describes uniform exponential stability as follows: if the dichotomy spectrum lies left of zero, then the uniform exponential stability of nonlinearly perturbed systems is guaranteed [10]. This concept may be also used to discuss the existence and the smoothness of invariant manifolds for time-varying differential and difference equations [2, 29], to obtain a version of the Hartman–Grobman theorem for non-autonomous systems [26, 12], to characterize the existence of center manifolds [32] and in the theory of Lyapunov regularity [11]. The dichotomy spectrum together with the spectral manifolds completely describe the dynamical skeleton of a linear system. Using the resonance of the dichotomy spectrum to study the normal forms of non-autonomous system, in [35] a finite order normal form were obtained, and in [39] analytic normal forms of a class of analytic non-autonomous differential systems were presented. Finally, information on the fine structure of the dichotomy spectrum allows to classify various types of non-autonomous bifurcations [30, 27].

In this paper, we consider a local version of the assignability of dichotomy spectrum for linear discrete time-varying systems, whereas in [15], [6] and [7] a global version was investigated. Our aim is to obtain sufficient conditions to place the dichotomy spectrum of the closed-loop system in an arbitrary position within some neighborhood of the dichotomy spectrum of the free system using some non-stationary linear feedback. Moreover, we assume that the norm of the feedback matrix should be bounded from above by the Hausdorff distance between these two spectra, with some constant coefficients. We say that the dichotomy spectrum is proportionally locally assignable if all these assumptions are valid. The main result (Theorem 2.8) states that uniform complete controllability is a sufficient condition for this type of assignability.

The paper is organized as follows. In the first part of the next section (Subsection 2.1), we collect some basic definitions and theorems connecting to the concept of exponential dichotomy and dichotomy spectrum. The problem and the main result of this paper are formulated and stated in Subsection 2.2. The result of assignability of the dichotomy spectrum by multiplicative perturbation is stated and proved in Section 3. We continue this consideration in Section 4 and show the result of proportional local assignability of dichotomy spectrum by multiplicative perturbation under the assumption that this spectrum consist of only one interval. Section 5 is devoted to the proofs of the main results. We provide an example in Section 6 to illustrate the obtained theoretical results. The last section contains conclusions.

The following notations will be used throughout this paper: Let $\mathcal{K}$ denote the set of all compact subsets of $\mathbb{R}$. For $U, V \in \mathcal{K}$, the Hausdorff distance d_H is defined as

$$d_H(U,V) := \max\left\{\max_{x\in U}\min_{y\in V}|x-y|, \max_{y\in V}\min_{x\in U}|x-y|\right\}.$$

For matrices $M_1 \in \mathbb{R}^{d_1\times d_1}, \dots, M_k \in \mathbb{R}^{d_k\times d_k}$, let $\operatorname{diag}(M_1,\dots,M_k)$ denote the square matrix of dimension $d_1 + \cdots + d_k$ of the form

$$\operatorname{diag}(M_1, \dots, M_k) = \begin{pmatrix} M_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_k \end{pmatrix}.$$

Let $\mathbb{R}^d$ be endowed with the standard Euclidean norm. For $s, d \in \mathbb{N}$, let $\mathcal{L}^\infty(\mathbb{N}, \mathbb{R}^{s\times d})$ be the set of all sequences $M : \mathbb{N} \to \mathbb{R}^{s\times d}$ such that

$$\|M\|_\infty := \sup_{n\in\mathbb{N}} \|M(n)\| < \infty.$$

Denote by $\mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ the set of invertible matrices $M : \mathbb{N} \to \mathbb{R}^{d\times d}$ satisfying that

$$\|M\|_{\mathrm{Lya}} := \max\left(\sup_{n\in\mathbb{N}} \|M(n)\|, \sup_{n\in\mathbb{N}} \|M(n)^{-1}\|\right) < \infty.$$

On the set $\mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$, the sets of elements near identity are particularly important in our further study. For any $\delta > 0$ let $\mathcal{I}_\delta(d)$ denote the set of all sequences $R = \big(R(n)\big)_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ satisfying that $\|R - I\|_\infty \le \delta$.

2. Preliminaries and the statement of the main results.

2.1. Dichotomy spectra and reducibility for linear discrete time - varying systems. Consider a one-sided discrete time-varying linear system

$$x\,(n+1) = M(n)x(n) \qquad \text{for } n \in \mathbb{N}, \tag{2.1}$$

where $M := (M(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$. Let $\Phi_M(\cdot,\cdot) : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^{d\times d}$ denote the *evolution operator* generated by (2.1), i.e.

$$\Phi_M(m, n) := \begin{cases} M(m-1)\dots M(n), & \text{if } m > n, \\ I, & \text{if } m = n, \\ M^{-1}(m)\dots M^{-1}(n-1), & \text{if } m < n. \end{cases}$$

Next, we introduce the notion of dichotomy spectrum of (2.1). This notion is defined in terms of exponential dichotomy. Recall that system (2.1) is said to admit an *exponential dichotomy* (ED) if there exist $K, \alpha > 0$ and a family of invariant projections $P(\cdot) : \mathbb{N} \to \mathbb{R}^{d\times d}$, i.e. $P(n+1)M(n) = M(n)P(n)$ for all $n \in \mathbb{N}$, such that for all $m, n \in \mathbb{N}$) we have

$$\begin{aligned} \|\Phi_M(m, n)P(n)\| &\le Ke^{-\alpha(m-n)} && \text{for } m \ge n, \\ \|\Phi_M(m, n)(I - P(n))\| &\le Ke^{\alpha(m-n)} && \text{for } m \le n, \end{aligned}$$

see [28]. The *dichotomy spectrum* of (2.1) is defined by

$$\Sigma_{\mathrm{ED}}(M) := \big\{\gamma \in \mathbb{R} : x(n+1) = e^{-\gamma}M(n)x(n) \text{ has no ED}\big\}.$$

Our aim is now to state the result on the structure of the dichotomy spectrum of (2.1) and the result on reducing (2.1) to a system of block diagonal form corresponding to the dichotomy spectral intervals. For this purpose, we first recall the notion of dynamical equivalence between two discrete time-varying linear systems. System (2.1) is said to be *dynamically equivalent*[1] to

$$y(n+1) = N(n)y(n) \qquad \text{for } n \in \mathbb{N},$$

where $(N(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ if there exists $(L(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ satisfying that

$$L(n+1)M(n) = N(n)L(n) \qquad \text{for all } n \in \mathbb{N}.$$

THEOREM 2.1. *The dichotomy spectrum $\Sigma_{\mathrm{ED}}(M)$ consists of k disjoint closed intervals, $k \le d$. Furthermore, let $\Sigma_{\mathrm{ED}}(M) = \bigcup_{i=1}^{k}[\alpha_i, \beta_i]$, where $\alpha_1 \le \beta_1 < \alpha_2 \le \beta_2 < \cdots < \alpha_k \le \beta_k$. Then, there exist $(M_i(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d_i\times d_i})$, $i = 1,\ldots,k$, such that (2.1) is dynamically equivalent to the following system*

$$y(n+1) = diag(M_1(n),\ldots,M_k(n))y(n) \quad \text{for } n \in \mathbb{N}, \tag{2.2}$$

and for $i = 1,\ldots,k$ the dichotomy spectrum $\Sigma_{\mathrm{ED}}(M_i)$ of the subsystem

$$z(n+1) = M_i(n)z(n)$$

is $\Sigma_{\mathrm{ED}}(M_i) = [\alpha_i, \beta_i]$. The dimensions $d_1,\ldots,d_k$ are independent on the choice of reducing system (2.2).

Proof. See Appendix. ∎

When we also want to emphasize the information of dimension of subspaces corresponding to the dichotomy spectral intervals, we arrive at the following definition of the repeated dichotomy spectrum. We refer the readers to [14] for a similar manner in defining repeated Lyapunov spectrum.

DEFINITION 2.2. *The repeated dichotomy spectrum $\Sigma^r_{\mathrm{ED}}(A)$ of (2.1) is defined by*

$$\Sigma^r_{\mathrm{ED}}(M) = \Big\{ \underbrace{[\alpha_1,\beta_1],\ldots,[\alpha_1,\beta_1]}_{d_1\, times},\ldots,\underbrace{[\alpha_k,\beta_k],\ldots,[\alpha_k,\beta_k]}_{d_k\, times} \Big\}, \tag{2.3}$$

where $d_1,\ldots,d_k$ are the dimension of subsystems corresponding to the spectral intervals $[\alpha_1,\beta_1],\ldots,[\alpha_k,\beta_k]$, respectively.

Remark 2.3. From Definition 2.2, two spectral intervals of a repeated dichotomy spectrum are either disjoint or the same. Then, a collection of d closed intervals $[\alpha_1,\beta_1],\ldots,[\alpha_d,\beta_d]$ is said to be *admissible for repeated dichotomy spectrum of a linear discrete time-varying system on $\mathbb{R}^d$* (for short admissible closed intervals) if

$$[\alpha_i,\beta_i] = [\alpha_j,\beta_j] \quad \text{or} \quad [\alpha_i,\beta_i]\cap[\alpha_j,\beta_j] = \emptyset \qquad \text{for } i \ne j.$$

2.2. Problem formulation and the statement of the main results. Consider a discrete time-varying linear control system

$$x(n+1) = A(n)x(n) + B(n)u(n) \quad \text{for } n \in \mathbb{N}, \tag{2.4}$$

where $A = (A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$, $B = (B(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\infty}(\mathbb{N}, \mathbb{R}^{d\times s})$ and $(u(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\infty}(\mathbb{N}, \mathbb{R}^{s})$. Let $x(\cdot, k_0, \xi, u)$ denote the solution of (2.4) satisfying that $x(k_0) = \xi$. We will also consider the free system associated with (2.4) of the form

$$x(n+1) = A(n)x(n) \quad \text{for } n \in \mathbb{N}. \tag{2.5}$$

[1]This notion is known as Lyapunov equivalence, kinematical equivalence.

Now, we recall the notion of uniform complete controllability of (2.4), see e.g. [18] and also [37].

Definition 2.4 (Uniform complete controllability). *System (2.4) is called* uniformly completely controllable *if there exist $\alpha > 0$ and $K \in \mathbb{N}$ such that for all $\xi \in \mathbb{R}^d$ and $k_0 \in \mathbb{N}$ there exists a control sequence $u(n)$, $n = k_0,\ k_0+1,\ \ldots,\ k_0+K-1$ such that $x(k_0+K, k_0, 0, u) = \xi$ and*

$$\|u(n)\| \leq \alpha\|\xi\| \qquad \text{for all } n = k_0,\ k_0+1,\ \ \ldots,\ \ k_0+K-1.$$

Remark 2.5. To verify condition of uniform complete controllability of discrete time-varying systems we can use Kalman controllability matrix

$$W(k,n) = \sum_{j=k}^{n} \Phi_A(k,j+1)B(j)B^T(j)\Phi_A^T(k,j+1)$$

where $n > k, n, k \in \mathbb{N}$ (see [4, 18, 37]). Under the assumption that A is a Lyapunov sequence and B is bounded, the condition uniformly completely controllable of (2.4) is equivalent to the condition of existing a positive γ and a natural K such that

$$W(k_0, k_0+K) \geq \gamma I.$$

For systems with a large number of parameters, numerical calculations using the Matlab environment can be used [16].

For a sequence $U = (U(n))_{n\in\mathbb{N}} \in \mathcal{L}^\infty(\mathbb{N}, \mathbb{R}^{s\times d})$ we consider a linear feedback control for system (2.4)

$$u(n) = U(n)x(n) \qquad \text{for } n \in \mathbb{N}.$$

Definition 2.6. *A bounded sequence $U \in \mathcal{L}^\infty(\mathbb{N}, \mathbb{R}^{s\times d})$ is said to be an admissible feedback control for system (2.4) if $(A\,(n) + B\,(n)\,U\,(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$.*

Let $U = (U(n))_{n\in\mathbb{N}}$ be any admissible feedback control for system (2.4). Then for a closed-loop system

$$x(n+1) = (A(n) + B(n)U(n))\,x(n) \qquad \text{for } n \in \mathbb{N}, \tag{2.6}$$

its dichotomy spectrum and its repeated dichotomy spectrum are denoted by $\Sigma_{\mathrm{ED}}(A+BU)$ and $\Sigma_{\mathrm{ED}}^r(A+BU)$, respectively. Now, we introduce the notion of proportional local assignability of repeated dichotomy spectrum.

Definition 2.7. *Denote the repeated dichotomy spectrum of the system (2.5) by $\Sigma_{\mathrm{ED}}^r(A) = \Big\{[a_1, b_1], \ldots, [a_d, b_d]\Big\}$, where $[a_1, b_1], \ldots, [a_d, b_d]$ are admissible closed intervals. The repeated dichotomy spectrum of (2.6) is called proportionally locally assignable if there exist $\delta,\ \ell > 0$ such that for arbitrary admissible closed intervals $[\widehat{a}_1, \widehat{b}_1], \ldots, [\widehat{a}_d, \widehat{b}_d]$ with $\max_{1\leq i\leq d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i])) \leq \delta$ there exists an admissible feedback control $U = (U(n))_{n\in\mathbb{N}}$ satisfying that*

$$\|U\|_\infty \leq \ell \max_{1\leq i\leq d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i])), \ \ \Sigma_{\mathrm{ED}}^r(A+BU) = \Big\{[\widehat{a}_i, \widehat{b}_i]\Big\}_{i=1}^d.$$

We now state the main result of this paper about the fact that uniform complete controllability implies proportional local assignability of repeated dichotomy spectrum.

THEOREM 2.8 (Proportional local assignability of repeated dichotomy spectrum). *Suppose that system* (2.4) *is uniformly completely controllable. Then, the repeated dichotomy spectrum of* (2.6) *is proportionally locally assignable.*

Remark 2.9. The proceeding result leads to several natural questions about analog results for non-invertible linear control systems and continuous time-varying linear control systems. Concerning the first class of systems, we can use the result in [3] to formulate properly the problem of assigning dichotomy spectrum. However, an immediate difficulty arising is that due to non-invertibility, there has been no result in transforming the systems to block-diagonal form based on the structure of the dichotomy spectrum (cf. Theorem 2.1). Concerning the generalization to continuous-time varying linear control systems, the authors believe that it is doable. The technical issue one might need to consider is the regularity of the linear control term to ensure that an analogous result of multiplicative perturbation (cf. Theorem 3.1) holds and we refer the reader to [23, 7] and the references therein for more details. We leave these open questions for future research.

3. Assignability of dichotomy spectrum by multiplicative perturbation. In this section, we discuss the problem of assignability of dichotomy spectrum by multiplicative perturbation. The main motivation for studying this problem comes from the fact that given an uniformly completely controllable system there exists a set of admissible multiplicative perturbation such that the perturbed system is dynamically equivalent to the closed-loop linear system (Theorem 3.1). The main result of this section is about a relation between proportional local assignability of repeated dichotomy spectrum by multiplicative perturbation and proportional local assignability of the control systems (Theorem 3.4).

3.1. Multiplicative perturbation. Together with system (2.5), we will consider the perturbed system

$$z(n+1) = A(n)R(n)z(n) \quad \text{for } n \in \mathbb{N}. \tag{3.1}$$

The perturbation $\big(R(n)\big)_{n\in\mathbb{N}}$ will be called a multiplicative perturbation of the system (2.5). Let $\Phi_{AR}(n,k)$ be the evolution operator of system (3.1). The following theorem will play an important role in our further consideration.

THEOREM 3.1. *If system (2.4) is uniformly completely controllable, then there exist $\delta, \ell > 0$ such that for each $R = \big(R(n)\big)_{n\in\mathbb{N}} \in \mathcal{I}_\delta(d)$ there exists an admissible feedback control $U = \big(U(n)\big)_{n\in\mathbb{N}}$ for system (2.6) such that $\|U\|_\infty \le \ell\|R - I\|_\infty$ and system (2.6) is dynamically equivalent to system (3.1).*

Proof. See [4]. ◻

3.2. Relation between assigning of dichotomy spectrum by multiplicative perturbation and proportional local assignability. Thanks to Theorem 3.1, it is natural to study the assigning of dichotomy spectrum by multiplicative perturbation. Let us introduce a concept of assignability of dichotomy spectrum of a linear system by multiplicative perturbation. To do this, we study an arbitrary linear discrete time-varying system (2.1). Let

$$\Sigma^r_{\mathrm{ED}}(M) = \Big\{[\alpha_1, \beta_1], \dots, [\alpha_d, \beta_d]\Big\},$$

where $[\alpha_1, \beta_1], \dots [\alpha_d, \beta_d]$ are admissible closed intervals.

Definition 3.2 (Proportionally locally assignable spectrum by multiplicative perturbation). *The repeated dichotomy spectrum of (2.1) is called* proportionally locally assignable by multiplicative perturbation *if there exist* $\delta, \ell > 0$ *such that for arbitrary admissible closed intervals* $[\widehat{\alpha}_1, \widehat{\beta}_1], \dots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ *satisfying that* $\max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i])) \le \delta$ *there exists* $R = (R(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ *such that*

$$\|R - I\|_\infty \le \ell \max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i])), \ \Sigma^r_{\mathrm{ED}}(MR) = \left\{[\widehat{\alpha}_i, \widehat{\beta}_i]\right\}_{i=1}^d. \tag{3.2}$$

In the following lemma, we show the persistence of proportional local assignability of repeated dichotomy spectrum by multiplicative perturbation via dynamical equivalence.

Lemma 3.3. *Proportional local assignability of repeated dichotomy spectrum by multiplicative perturbation persists via dynamical equivalence.*

Proof. Consider a system

$$y(n+1) = N(n)y(n) \qquad \text{for } n \in \mathbb{N}$$

which is dynamically equivalent to (2.1) via the transformation $L = (L(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$, i.e. $N(n) = L(n+1)M(n)L^{-1}(n)$. Suppose that the repeated dichotomy spectrum of (2.1) is proportionally locally assignable by multiplicative perturbation with respect to δ, ℓ as in Definition 3.2. Let $[\widehat{\alpha}_1, \widehat{\beta}_1], \dots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ be arbitrary admissible closed intervals satisfying $\max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i])) \le \delta$. Then, by Definition 3.2 there exists $R = (R(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ satisfying

$$\|R - I\|_\infty \le \ell \max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i])), \ \Sigma^r_{\mathrm{ED}}(MR) = \left\{[\widehat{\alpha}_i, \widehat{\beta}_i]\right\}_{i=1}^d. \tag{3.3}$$

Let

$$\widehat{R} = \left(\widehat{R}(n)\right)_{n\in\mathbb{N}} = \left(L(n)R(n)L^{-1}(n)\right)_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d}).$$

Then, we have the following claims

$$\|\widehat{R} - I\|_\infty \le \ell \|L\|^2_{\mathrm{Lya}} \max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i])), \ \Sigma^r_{\mathrm{ED}}(N\widehat{R}) = \left\{[\widehat{\alpha}_i, \widehat{\beta}_i]\right\}_{i=1}^d.$$

The first claim follows from the inequality

$$\begin{aligned}
\|\widehat{R}(n) - I\| &= \|L(n)R(n)L^{-1}(n) - I\| \\
&= \|L(n)\left(R(n) - I\right)L^{-1}(n)\| \\
&\le \|L(n)\| \, \|R(n) - I\| \, \left\|L^{-1}(n)\right\| \\
&\le \ell \|L\|^2_{\mathrm{Lya}} \max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i])).
\end{aligned}$$

The second one is deduced from (3.3) and the fact that $(M(n)R(n))_{n\in\mathbb{N}}$ and $\left(N(n)\widehat{R}(n)\right)_{n\in\mathbb{N}}$ are dynamically equivalent, since

$$L^{-1}(n+1)N(n)\widehat{R}(n)L(n) = M(n)R(n) \qquad \text{for } n \in \mathbb{N}.$$

∎

We now state and prove the main result of this section in which we describe a relation between proportional local assignability of the dichotomy spectrum of (3.1) by multiplicative perturbation and proportional local assignability of (2.6).

THEOREM 3.4. *Suppose that system (2.4) is uniformly completely controllable. If the repeated dichotomy spectrum of the associated free system (2.5) is proportionally locally assignable by multiplicative perturbation, then the dichotomy spectrum of (2.6) is proportionally locally assignable.*

Proof. From the proportional local assignability of the dichotomy spectrum of (2.5) by multiplicative perturbation it follows that there exist $\delta_1 > 0$ and $\ell_1 > 0$ such that for any admissible closed intervals $[\widehat{a}_1, \widehat{b}_1], \dots, [\widehat{a}_d, \widehat{b}_d]$ with $\max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i])) \le \delta_1$ there exists a sequence $R = \big(R(n)\big)_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ satisfying the estimate

$$\|R - I\|_\infty \le \ell_1 \max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]))$$

and providing the validity of relation

$$\Sigma^r_{\mathrm{ED}}(AR) = \left\{[\widehat{a}_i, \widehat{b}_i]\right\}_{i=1}^{d}. \tag{3.4}$$

According to Theorem 3.1, there exist $\delta_2 > 0$ and $\ell_2 > 0$ such that for each system (3.1) with $R \in \mathcal{I}_{\delta_2}(d)$ there exists an admissible feedback control $U = (U(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$, such that $\|U\|_\infty \le \ell_2\|R - I\|_\infty$ and the corresponding closed-loop system (2.6) is dynamically equivalent to system (3.1). Let

$$\delta := \min\left\{\frac{\delta_2}{\ell_1}, \delta_1\right\}, \qquad \ell := \ell_1\ell_2. \tag{3.5}$$

To conclude the proof, choose and fix an arbitrary admissible closed intervals $[\widehat{a}_1, \widehat{b}_1], \dots, [\widehat{a}_d, \widehat{b}_d]$ such that

$$\max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i])) \le \delta.$$

By definition of δ and δ_1, there exists a sequence $R \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ such that

$$\|R - I\|_\infty \le \ell_1 \max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i])) \le \ell_1\delta \le \delta_2$$

and (3.4) is satisfied. For this sequence R and by definition of δ_2 there exists an admissible feedback control U for system (2.6) such that

$$\begin{aligned}\|U\|_\infty \le \ell_2\|R - I\|_\infty &\le \ell_2\ell_1 \max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]))\\ &= \ell \max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i])),\end{aligned}$$

and such that systems (3.1) and (2.6) are dynamically equivalent. Since equivalent systems have the same dichotomy spectrum the proof is completed. ∎

4. Proportional local assignability of one dichotomy spectral interval by multiplicative perturbation. In this section, we pay attention to discrete time-varying linear systems of the form (2.1) with the property that the dichotomy spectrum consists of only one interval, i.e.

$$\Sigma_{\mathrm{ED}}(M) = [\alpha, \beta]. \tag{4.1}$$

Our main result is to show that for these systems the dichotomy spectra are proportionally locally assignable by multiplicative perturbation:

- Theorem 4.8 (Subsection 4.2) for the case that the dichotomy spectrum is singleton, i.e. $\alpha = \beta$. The main ingredient in the proof of this result is the existence of two sequences of time realizing the end points of the dichotomy spectrum (Lemma 4.3 and Lemma 4.7).
- Theorem 4.9 (Subsection 4.3) for the case that the dichotomy spectrum is a non-trivial interval, i.e. $\alpha < \beta$. The main ingredient in the proof is an explicit formula of dichotomy spectra for a special class of upper-triangular systems (Lemma 4.10).

In addition to Subsection 4.2 and Subsection 4.3, we prove several preparatory results for the properties of dichotomy spectra in Subsection 4.1.

4.1. Preparatory results. In the following lemma, we establish a presentation of α and β. A proof of this formula for the scalar system can be found in [28, Proposition 2.4] and [19, Proposition 3.3.14].

LEMMA 4.1. *Consider system (2.1) and suppose that* $\Sigma_{\mathrm{ED}}(M) = [\alpha, \beta]$. *Then,*

$$\beta = \lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left\|\prod_{i=k}^{k+j-1} M(i)\right\|\right)\right),$$

$$\alpha = -\lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left\|\left(\prod_{i=k}^{k+j-1} M(i)\right)^{-1}\right\|\right)\right).$$

Proof. At first observe that by the Fekete Lemma (see [17]) the limits in the statement of the theorem exist. By $\Sigma_{\mathrm{ED}}(M) = [\alpha, \beta]$, the resolvent set $\rho(M) = (-\infty, \alpha) \cup (\beta, \infty)$ consists of only two open intervals. Note that the projection associated with (β, ∞) and $(-\infty, \alpha)$ is, respectively, identity and zero, cf. [22, Lemma 5.4]. Thus, that for any $\varepsilon > 0$ there exists $K > 1$ such that

$$\|\Phi_M(m, n)\| \le K e^{(m-n)(\beta+\varepsilon)} \qquad \text{for all } m \ge n \tag{4.2}$$

and

$$\|\Phi_M(m, n)\| \le K e^{(m-n)(\alpha-\varepsilon)} \qquad \text{for all } m \le n. \tag{4.3}$$

By (4.2) we have that for any $k \in \mathbb{N}$ and $j \in \mathbb{N}$ we have

$$\ln\left(\left\|\prod_{i=k}^{k+j-1} M(i)\right\|\right) \le \ln K + j\,(\beta + \varepsilon)$$

and therefore

$$\lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left\|\prod_{i=k}^{k+j-1} M(i)\right\|\right)\right) \le \beta. \tag{4.4}$$

Similarly from (4.3) we get

$$\lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left\|\left(\prod_{i=k}^{k+j-1} M(i)\right)^{-1}\right\|\right)\right) \le -\alpha. \tag{4.5}$$

Suppose that the inequality (4.4) is strict, i.e.

$$\tau := \lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left\|\prod_{i=k}^{k+j-1} M(i)\right\|\right)\right) < \beta. \tag{4.6}$$

Since $\Sigma_{\mathrm{ED}}(M) = [\alpha, \beta]$, the scaled equation

$$x(n+1) = e^{-\beta} M(n)x(n) \quad \text{for } n \in \mathbb{N}, \tag{4.7}$$

does not have a ED. Denote by $\Phi_{M,\beta}$ the evolution operator generated by (4.7). Consider $\varepsilon > 0$ such that $\tau + \varepsilon < \beta$. The definition of τ implies that there exists $K > 0$ such that

$$\left\|\prod_{i=k}^{k+j-1} M(i)\right\| \le K e^{(\tau+\varepsilon)j} \quad \text{for all } k \in \mathbb{N}, j \in \mathbb{N}.$$

Thus, for all $k \in \mathbb{N}$, $j \in \mathbb{N}$ we have

$$\|\Phi_{M,\beta}(k+j,k)\| = e^{-j\beta}\|\Phi_M(k+j,k)\| \le K e^{j(\tau+\varepsilon-\beta)}.$$

Since $\tau + \varepsilon - \beta < 0$, then the last inequality means that (4.7) poses an ED. This contradictions shows that $\tau = \beta$. In the same way we may show that the inequality (4.5) may not be strict. ∎

Remark 4.2. When (2.1) is scalar then the dichotomy spectrum consists of one interval $\Sigma_{\mathrm{ED}}(M) = [\alpha, \beta]$, then Lemma 4.1 becomes

$$\beta = \lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{i=k}^{k+j-1} M(i)\right|\right)\right),$$

$$\alpha = \lim_{j\to\infty} \frac{1}{j}\left(\inf_{k\in\mathbb{N}} \ln\left(\left|\prod_{i=k}^{k+j-1} M(i)\right|\right)\right).$$

Next we show results about the sequence realizing the upper end points of the dichotomy spectrum of a discrete-time linear system.

Lemma 4.3. *Consider system (2.1) and suppose that $\Sigma_{\mathrm{ED}}(M) = [\alpha, \beta]$. Then, there exists a pair of sequences of natural numbers $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ satisfying that $m_k < n_k < m_{k+1}$ for $k \in \mathbb{N}$, $\lim_{k\to\infty} m_k = \infty$ and $\lim_{k\to\infty}(n_k - m_k) = \infty$ and*

$$\beta = \lim_{k\to\infty} \frac{1}{n_k - m_k} \ln\left(\left\|\prod_{i=m_k}^{n_k-1} M(i)\right\|\right).$$

Proof. Each sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ of integer such that $\lim_{k\to\infty}(n_k - m_k) = \infty$ and

$$\beta = \lim_{k\to\infty} \frac{1}{n_k - m_k} \ln\left(\left\|\prod_{i=m_k}^{n_k-1} M(i)\right\|\right)$$

will be called sequences realizing the maximal point of the dichotomy spectrum of system (2.1). We first show that there exist realizing sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$

satisfying additionally that $\lim_{k\to\infty} m_k = \infty$. By Lemma 4.1, there are two realizing sequences $(\widetilde{n}_k)_{k\in\mathbb{N}}$, $(\widetilde{m}_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\widetilde{n}_k - \widetilde{m}_k) = \infty$ and

$$\beta = \lim_{k\to\infty} \frac{1}{\widetilde{n}_k - \widetilde{m}_k} \ln\left(\left\|\prod_{i=\widetilde{m}_k}^{\widetilde{n}_k - 1} M(i)\right\|\right). \tag{4.8}$$

If the sequence $(\widetilde{m}_k)_{k\in\mathbb{N}}$ is unbounded, then it contains a subsequence $(\widetilde{m}_{k_l})_{l\in\mathbb{N}}$ divergent to ∞. In this case the sequences $(\widetilde{n}_{k_l})_{l\in\mathbb{N}}$, $(\widetilde{m}_{k_l})_{l\in\mathbb{N}}$ are the desired ones. If $(\widetilde{m}_k)_{k\in\mathbb{N}}$ is bounded, i.e. $\widetilde{m}_k < c$, then $(\widetilde{n}_k)_{k\in\mathbb{N}}$ contains a subsequence diverging to ∞. Taking a subsequence of $(\widetilde{n}_k)_{k\in\mathbb{N}}$, if necessary, we may assume that $\widetilde{n}_k > 0$ and $\widetilde{n}_{k+1} > \widetilde{n}_k$, $k \in \mathbb{N}$. Denote $m_k = \left\lceil\sqrt{\widetilde{n}_k}\right\rceil$, where $\lceil x\rceil$ means the smallest integer not less than x. It is clear that $m_k \to \infty$ when $k \to \infty$. Now we will show that $(\widetilde{n}_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ are realizing sequences. By the formula of β as in Lemma 4.1,

$$\beta \geq \limsup_{k\to\infty} \frac{1}{\widetilde{n}_k - m_k} \ln\left(\left\|\prod_{i=m_k}^{\widetilde{n}_k - 1} M(i)\right\|\right).$$

Then, it is sufficient to show that

$$\widetilde{\beta} := \liminf_{k\to\infty} \frac{1}{\widetilde{n}_k - m_k} \ln\left(\left\|\prod_{i=m_k}^{\widetilde{n}_k - 1} M(i)\right\|\right) \geq \beta.$$

Using the fact that $\left\|\prod_{i=m_k}^{\widetilde{n}_k - 1} M(i)\right\| \geq \frac{\left\|\prod_{i=\widetilde{m}_k}^{\widetilde{n}_k - 1} M(i)\right\|}{\left\|\prod_{i=\widetilde{m}_k}^{m_k - 1} M(i)\right\|}$, we obtain that

$$\ln\left(\left\|\prod_{i=m_k}^{\widetilde{n}_k - 1} M(i)\right\|\right) \geq \ln\left(\left\|\prod_{i=\widetilde{m}_k}^{\widetilde{n}_k - 1} M(i)\right\|\right) - \ln\left(\left\|\prod_{i=\widetilde{m}_k}^{m_k - 1} M(i)\right\|\right).$$

This together with the definition of $\widetilde{\beta}$ implies that

$$\widetilde{\beta} \geq \liminf_{k\to\infty} \frac{\ln\left(\left\|\prod_{i=\widetilde{m}_k}^{\widetilde{n}_k - 1} M(i)\right\|\right)}{\widetilde{n}_k - m_k} - \limsup_{k\to\infty} \frac{\ln\left(\left\|\prod_{i=\widetilde{m}_k}^{m_k - 1} M(i)\right\|\right)}{\widetilde{n}_k - m_k}. \tag{4.9}$$

Since $\widetilde{m}_k \leq c$ for all k it follows that

$$\lim_{k\to\infty} \frac{\widetilde{n}_k - m_k}{\widetilde{n}_k - \widetilde{m}_k} = \lim_{k\to\infty} \frac{\widetilde{n}_k - \left\lceil\sqrt{\widetilde{n}_k}\right\rceil}{\widetilde{n}_k} \frac{\widetilde{n}_k}{\widetilde{n}_k - \widetilde{m}_k} = 1.$$

Consequently, by (4.8) we arrive at

$$\liminf_{k\to\infty} \frac{\ln\left(\left\|\prod_{i=\widetilde{m}_k}^{\widetilde{n}_k - 1} M(i)\right\|\right)}{\widetilde{n}_k - m_k} = \liminf_{k\to\infty} \frac{\ln\left(\left\|\prod_{i=\widetilde{m}_k}^{\widetilde{n}_k - 1} M(i)\right\|\right)}{\widetilde{n}_k - \widetilde{m}_k} = \beta. \tag{4.10}$$

From $(M(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$, there exists $M \geq 1$ such that $\|M(n)\| \leq M$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned}\limsup_{k\to\infty} \frac{\ln\left(\left\|\prod_{i=\widetilde{m}_k}^{m_k-1} M(i)\right\|\right)}{\widetilde{n}_k - m_k} &\leq \limsup_{k\to\infty} \frac{(m_k - \widetilde{m}_k)}{\widetilde{n}_k - m_k} \ln M \\ &\leq \limsup_{k\to\infty} \frac{\lceil\sqrt{\widetilde{n}_k}\rceil}{\widetilde{n}_k - \lceil\sqrt{\widetilde{n}_k}\rceil} \ln M = 0.\end{aligned}$$

This together with (4.9) and (4.10) implies that $\widetilde{\beta} \geq \beta$ and therefore $\widetilde{\beta} = \beta$. Finally, we start with realizing sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} m_k = \infty$. Taking the subsequences of $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$, if necessary, we can assume that $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ are strictly increasing and satisfy

$$m_k < n_k < m_{k+1} < n_{k+1}.$$

The proof is complete. □

Being similar to the preceding lemma, we have the following result about the realizing sequences for the lower end point of the dichotomy spectrum of (2.1).

LEMMA 4.4. *Consider system (2.1) and suppose that* $\Sigma_{\mathrm{ED}}(M) = [\alpha, \beta]$. *Then, there exists a pair of sequences of natural numbers* $(q_k)_{k\in\mathbb{N}}$, $(p_k)_{k\in\mathbb{N}}$ *satisfying that* $p_k < q_k < p_{k+1}$ *for* $k \in \mathbb{N}$, $\lim_{k\to\infty} p_k = \infty$ *and* $\lim_{k\to\infty}(q_k - p_k) = \infty$ *and*

$$\alpha = -\lim_{k\to\infty} \frac{1}{q_k - p_k} \ln\left(\left\|\left(\prod_{i=p_k}^{q_k-1} M(i)\right)^{-1}\right\|\right).$$

Now, we recall a result from [28] about a relation between dichotomy spectra of block upper-triangular systems and the dichotomy spectra of subsystems corresponding to entries on the diagonal.

PROPOSITION 4.5. *Consider a system*

$$x(n+1) = N(n)x(n) = \begin{pmatrix} P(n) & S(n) \\ & \\ 0 & Q(n) \end{pmatrix} x(n), \tag{4.11}$$

where $(P(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{p\times p}), (Q(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{q\times q})$ *and* $(S(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\infty}(\mathbb{N}, \mathbb{R}^{p\times q})$. *Denote the dichotomy spectra of subsystems*

$$y(n+1) = P(n)y(n), \qquad z(n+1) = Q(n)z(n) \quad \text{for } n \in \mathbb{N},$$

by $\Sigma_{\mathrm{ED}}(P)$ *and* $\Sigma_{\mathrm{ED}}(Q)$, *respectively. Then, the dichotomy spectrum* $\Sigma_{\mathrm{ED}}(N)$ *of (4.11) is given by*

$$\Sigma_{\mathrm{ED}}(N) = \Sigma_{\mathrm{ED}}(P) \cup \Sigma_{\mathrm{ED}}(Q).$$

Proof. See [28, Theorem 4.8, Corollary 4.4]. □

4.2. Proof of proportional local assignability by multiplicative perturbation for systems with one trivial dichotomy spectra. Throughout this subsection, we consider a linear discrete time-varying system (2.1) and suppose additionally that the dichotomy spectrum $[\alpha, \beta]$ of (2.1) is singleton, i.e.

$$\Sigma_{\mathrm{ED}}(M) = \{\alpha\}. \tag{4.12}$$

First, we show that under the preceding assumption system (2.1) is dynamically equivalent to an upper-triangular linear system.

Lemma 4.6. *There exists a sequence of upper-triangular matrices $(N(n))_{n\in\mathbb{N}} = (N_{ij}(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$, i.e. $N_{ij}(n) = 0$ for $i > j$ such that (2.1) is dynamically equivalent to*

$$y(n+1) = N(n)y(n) \qquad \text{for } n \in \mathbb{N} \tag{4.13}$$

and for $i = 1, \dots, d$ the dichotomy spectrum $\Sigma_{\mathrm{ED}}(N_{ii})$ of subsystem

$$z(n+1) = N_{ii}(n)z(n) \qquad \text{for } n \in \mathbb{N}, \tag{4.14}$$

satisfies $\Sigma_{\mathrm{ED}}(N_{ii}) = \{\alpha\}$.

Proof. It is well known that there exists an upper-triangular system (4.13) which is dynamically equivalent to (2.1) (see e.g. Algorithm 5.1 in [28]). Since the spectrum of ED is preserved under kinematic similarity, then $\Sigma_{\mathrm{ED}}(N) = \{\alpha\}$. We will prove by induction with respect to the d that for each upper-triangular system (4.13) such that $\Sigma_{\mathrm{ED}}(N_{ii}) = \{\alpha\}$ for the diagonal elements N_{ii} the equality (4.14) holds. For $d = 1$ it is nothing to prove. Suppose that the statement holds for certain $d \in \mathbb{N}$ and consider a $d + 1$ upper-triangular system (4.13). Let

$$P(n) = (N_{ij}(n))_{1\le i,j\le d} \;\text{ and }\; Q(n) = N_{d+1,d+1}(n) \;\text{ for } n \in \mathbb{N}.$$

By Proposition 4.5 and the fact that $\Sigma_{\mathrm{ED}}(N) = \{\alpha\}$, we have

$$\Sigma_{\mathrm{ED}}(P) \cup \Sigma_{\mathrm{ED}}(Q) = \{\alpha\}.$$

Thus, $\Sigma_{\mathrm{ED}}(N_{d+1,d+1}) = \Sigma_{\mathrm{ED}}(P) = \{\alpha\}$. This together with the induction assumption completes the proof. ∎

Next, we show that the dichotomy spectrum of each subsystem (4.14) is proportionally locally assignable by multiplicative perturbation.

Lemma 4.7. *For each $i = 1, \dots, d$, the dichotomy spectrum of each scalar subsystem (4.14) is proportionally locally assignable by multiplicative perturbation. More precisely, for an arbitrary interval $[a, b]$ with $d_H([a, b], \{\alpha\}) < 1$ there exists $(r(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{1\times 1})$ satisfying that*

$$\sup_{n\in\mathbb{Z}} |r(n) - 1| \le e\, d_H([a, b], \{\alpha\}) \;\text{ and }\; \Sigma_{\mathrm{ED}}(N_{ii}r) = [a, b]. \tag{4.15}$$

Proof. Choose and fix an arbitrary interval $[a, b]$ satisfying $d_H([a, b], \{\alpha\}) < 1$. By Lemma 4.3, there exist sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ satisfying that $m_k < n_k < m_{k+1}$, $\lim_{k\to\infty} m_k = \infty$, $\lim_{k\to\infty}(n_k - m_k) = \infty$ and

$$\alpha = \lim_{k\to\infty} \frac{1}{n_k - m_k} \ln\left(\left|\prod_{j=m_k}^{n_k-1} N_{ii}(j)\right|\right). \tag{4.16}$$

Define the sequence $r = (r(n))_{n\in\mathbb{N}}$ as follows

$$r(n) = \begin{cases} e^{b-\alpha} & \text{for } n \in [n_{2k}, m_{2k}), k \in \mathbb{N}, \\ e^{a-\alpha} & \text{for } n \in [n_{2k+1}, m_{2k+1}), k \in \mathbb{N}, \\ e^{\frac{a+b}{2}-\alpha} & \text{otherwise} . \end{cases}$$

Obviously, $r \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{1\times 1})$. Define $[\widehat{a}, \widehat{b}] := \Sigma_{\mathrm{ED}}(N_{ii}r)$ and we will verify that $[\widehat{a}, \widehat{b}] = [a, b]$. In fact, we only show that $b = \widehat{b}$ and use analogous arguments to obtain $a = \widehat{a}$. Firstly, by Remark 4.2 we have

$$\begin{aligned} \widehat{b} &= \lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{n=k}^{k+j-1} r(n)N_{ii}(n)\right|\right)\right) \\ &\geq \limsup_{k\to\infty} \frac{\ln\left(\prod_{j=m_{2k}}^{n_{2k}-1} |r(j)N_{ii}(j)|\right)}{n_{2k}-m_{2k}} \\ &= b - \alpha + \lim_{k\to\infty} \frac{\ln\left(\prod_{j=m_{2k}}^{n_{2k}-1} |N_{ii}(j)|\right)}{n_{2k}-m_{2k}} = b, \end{aligned}$$

where we use (4.16) to obtain the last equality. Finally, by definition of r we have $|N_{ii}(n)r(n)| \leq e^{b-\alpha}|N_{ii}(n)|$. Thus,

$$\begin{aligned} \widehat{b} &= \lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{n=k}^{k+j-1} r(n)N_{ii}(n)\right|\right)\right) \\ &\leq \lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{n=k}^{k+j-1} e^{b-\alpha}N_{ii}(n)\right|\right)\right) \\ &= b-\alpha+\lim_{j\to\infty} \frac{1}{j}\left(\sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{n=k}^{k+j-1} N_{ii}(n)\right|\right)\right) = b, \end{aligned}$$

which implies that $b = \widehat{b}$. To conclude the proof, it remains to estimate $\sup_{n\in\mathbb{N}} |r(n)-1|$. By definition of r and inequality $|e^t - 1| \leq e^{|t|} - 1$ for all $t \in \mathbb{R}$ we have that

$$|r(n) - 1| \leq \max\left\{e^{|a-\alpha|} - 1, e^{\left|\frac{a+b}{2}-\alpha\right|} - 1, e^{|b-\alpha|} - 1\right\}.$$

By Mean Value Theorem, we obtain that $e^{|a-\alpha|} - 1 \leq e\,|a-\alpha|, e^{\left|\frac{a+b}{2}-\alpha\right|} - 1 \leq e\left|\frac{a+b}{2}-\alpha\right|, e^{|b-\alpha|} - 1 \leq e\,|b-\alpha|$. Thus, $|r(n)-1| \leq e\max\{|a-\alpha|, |b-\alpha|\}$ and the proof is complete. ∎

THEOREM 4.8. *Consider a linear discrete time-varying system (2.1). Suppose additionally that the dichotomy spectrum $[\alpha, \beta]$ of (2.1) is singleton, i.e. $\alpha = \beta$. Then, the repeated dichotomy spectrum of (2.1) is proportionally locally assignable by multiplicative perturbation.*

Proof. By Lemma 4.6, system (2.1) is dynamically equivalent to

$$x(n+1) = N(n)x(n) \qquad \text{for } n \in \mathbb{N}, \tag{4.17}$$

where $(N(n))_{n\in\mathbb{N}} = (N_{ij}(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ satisfying $N_{ij}(n) = 0$ for $i > j$ and for $i = 1, \dots, d$ the dichotomy spectrum of the subsystem

$$z(n+1) = N_{ii}(n)z(n) \qquad \text{for } n \in \mathbb{N}, \tag{4.18}$$

is $\Sigma_{\mathrm{ED}}(N_{ii}) = \{\alpha\}$. By virtue of Lemma 3.3, it is sufficient to show that the repeated dichotomy spectrum of (4.17) is proportionally locally assignable by multiplicative perturbation. For this purpose, let $[\widehat{\alpha}_1, \widehat{\beta}_1], \dots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ be an arbitrary admissible closed intervals of the form

$$\Big\{ \underbrace{[a_1, b_1], \dots, [a_1, b_1]}_{d_1 \text{times}}, \dots, \underbrace{[a_k, b_k], \dots, [a_k, b_k]}_{d_k \text{times}} \Big\}$$

satisfying that $\max_{1\le j\le k} d_H([a_j, b_j], \{\alpha\})) \le 1$ and $a_1 > b_2, \dots, a_{k-1} > b_k$. Let $j \in \{1, \dots, k\}$ be arbitrary. In view of Lemma 4.7, for arbitrary $i \in \{d_1 + \dots + d_{j-1} + 1, d_1 + \dots + d_j\}$ there exists $(r_i(n))_{n\in\mathbb{Z}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{1\times 1})$ satisfying that

$$\sup_{n\in\mathbb{Z}} |r_i(n) - 1| \le e\, d_H([a_j, b_j], \{\alpha\}) \text{ and } \Sigma_{\mathrm{ED}}(N_{ii} r_i) = [a_j, b_j]. \tag{4.19}$$

Let $R(n) = \mathrm{diag}(r_1(n), \dots, r_d(n))$. Then, on one hand by (4.19) we have

$$\sup_{n\in\mathbb{Z}} \|R(n) - I\| \le e \max_{1\le j\le k} d_H([a_j, b_j], \{\alpha\}).$$

On the other hand, by Proposition 4.5

$$\Sigma^r_{\mathrm{ED}}(NR) = \Big\{ \underbrace{[a_1, b_1], \dots, [a_1, b_1]}_{d_1 \text{times}}, \dots, \underbrace{[a_k, b_k], \dots, [a_k, b_k]}_{d_k \text{times}} \Big\}.$$

The proof is complete. ∎

4.3. Proof of proportional local assignability by multiplicative perturbation for systems with one non-trivial dichotomy spectral interval. We now state the main result of this subsection about proportional local assignability by multiplicative perturbation for systems with only one non-trivial dichotomy spectral interval.

THEOREM 4.9. *Suppose that the dichotomy spectrum $[\alpha, \beta]$ of (2.1) is not singleton, i.e. $\alpha < \beta$. Then, the repeated dichotomy spectrum of (2.1) is proportionally locally assignable by multiplicative perturbation.*

Before proving the preceding theorem, we need the following preparatory result.

PROPOSITION 4.10. *Consider a upper-triangular time-varying linear system*

$$x(n+1) = N(n)x(n), \qquad N(n) = \begin{pmatrix} N_{11}(n) & N_{12}(n) & \cdots & N_{1d}(n) \\ 0 & N_{22}(n) & \cdots & N_{2d}(n) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{dd}(n) \end{pmatrix}. \tag{4.20}$$

Suppose that $(N(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$ and the dichotomy spectrum $\Sigma_{\mathrm{ED}}(N)$ of (4.20) consists of only one interval denoted by $[\alpha, \beta]$. Let $\zeta, \eta \in \mathbb{R}$ be arbitrary. Define

$N^{\zeta,\eta}(n) = (N_{ij}^{\zeta,\eta}(n))_{1\le i,j\le d}$ *by*

$$N_{ij}^{\zeta,\eta}(n) := \begin{cases} N_{ij}(n), & \text{if } i \neq j, \\ e^{-\zeta}|N_{ij}(n)|^{\eta} N_{ij}(n), & \text{if } i = j. \end{cases}$$

Then, the dichotomy spectrum of the system

$$x(n+1) = N^{\zeta,\eta}(n)x(n) \qquad \text{for } n \in \mathbb{N}, \tag{4.21}$$

is given by $\Sigma_{\mathrm{ED}}(N^{\zeta,\eta}) = [(1+\eta)\alpha - \zeta, (1+\eta)\beta - \zeta]$.

Proof. Let $\ell \in \{1,\dots,d\}$ be arbitrary. Let $[\alpha_\ell, \beta_\ell]$ be the dichotomy spectrum of the scalar system

$$z(n+1) = N_{\ell\ell}(n)z(n) \qquad \text{for } n \in \mathbb{N}.$$

We now verify that the dichotomy spectrum of the following system

$$z(n+1) = N_{\ell\ell}^{\zeta,\eta}(n)z(n) \qquad \text{for } n \in \mathbb{N}.$$

is given by $[(1+\eta)\alpha_\ell - \zeta, (1+\eta)\beta_\ell - \zeta]$. For this purpose, using the fact that

$$\begin{aligned} \sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{i=k}^{k+j-1} N_{\ell\ell}^{\zeta,\eta}(i)\right|\right) &= \sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{i=k}^{k+j-1} e^{-\zeta}|N_{\ell\ell}(i)|^{\eta} N_{\ell\ell}(i)\right|\right) \\ &= -\zeta j + (1+\eta)\sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{i=k}^{k+j-1} N_{\ell\ell}(i)\right|\right), \end{aligned}$$

we obtain that

$$\lim_{j\to\infty} \frac{1}{j} \sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{i=k}^{k+j-1} N_{\ell\ell}^{\zeta,\eta}(i)\right|\right) = -\zeta + (1+\eta) \lim_{j\to\infty} \frac{1}{j} \sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{i=k}^{k+j-1} N_{\ell\ell}(i)\right|\right).$$

Then, by Lemma 4.1 we have

$$\lim_{j\to\infty} \frac{1}{j} \sup_{k\in\mathbb{N}} \ln\left(\left|\prod_{i=k}^{k+j-1} N_{\ell\ell}^{\zeta,\eta}(i)\right|\right) = -\zeta + (1+\eta)\beta_\ell.$$

Similarly, we have

$$-\lim_{j\to\infty} \frac{1}{j} \sup_{k\in\mathbb{N}} \ln\left(\left|\left(\prod_{i=k}^{k+j-1} N_{\ell\ell}^{\zeta,\eta}(i)\right)^{-1}\right|\right) = -\zeta + (1+\eta)\alpha_\ell.$$

Consequently, using Lemma 4.1, we obtain $\Sigma_{\mathrm{ED}}(N_{\ell\ell}^{\zeta,\eta}) = [(1+\eta)\alpha_\ell - \zeta, (1+\eta)\beta_\ell - \zeta]$. Thus, by virtue of Proposition 4.5 we arrive at

$$\Sigma_{\mathrm{ED}}(N^{\zeta,\eta}) = \bigcup_{\ell=1}^{d} \Sigma_{\mathrm{ED}}(N_{\ell\ell}^{\zeta,\eta}) = \bigcup_{\ell=1}^{d} [(1+\eta)\alpha_\ell - \zeta, (1+\eta)\beta_\ell - \zeta]. \tag{4.22}$$

On the other hand, since $\Sigma_{\mathrm{ED}}(N) = [\alpha, \beta]$ it follows that

$$[\alpha, \beta] = \bigcup_{\ell=1}^{d} \Sigma_{\mathrm{ED}}(N_{\ell\ell}) = \bigcup_{\ell=1}^{d} [\alpha_\ell, \beta_\ell]$$

which together with (4.22) implies that $\Sigma_{\mathrm{ED}}(N^{\zeta,\eta}) = [(1+\eta)\alpha - \zeta, (1+\eta)\beta - \zeta]$. The proof is complete. $\square$

We are now in the position to prove the main result of this subsection.

Proof of Theorem 4.9. Due to the fact that there exists an upper-triangular system (4.13) which is dynamically equivalent to (2.1) (see e.g. Algorithm 5.1 in [28]) and proportional local assignability of repeated dichotomy spectrum by multiplicative perturbation persists via dynamical equivalence (Lemma 3.3), it is sufficient to prove this theorem under the assumption that $M(n)$ is an upper-triangular matrix for all $n \in \mathbb{N}$. Let

$$\delta := \frac{b-a}{3+|a|+|b|} \quad \text{and} \quad \ell := \frac{e\left(1+\|M\|^2_{\mathrm{Lya}}\right)(2+|a|+|b|)}{b-a}. \tag{4.23}$$

Now, let $[\widehat{\alpha}_1, \widehat{\beta}_1], \dots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ be arbitrary admissible closed intervals satisfying that $\max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [a,b])) \le \delta$. By (4.23) and $d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [a,b])) \le \delta$, we have $[\frac{2a+b}{3}, \frac{a+2b}{3}] \subset [\widehat{\alpha}_i, \widehat{\beta}_i]$ for all $i = 1, \dots, d$. Thus, by virtue of Remark 2.3 all intervals $[\widehat{\alpha}_1, \widehat{\beta}_1], \dots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ coincide and let $\widehat{\alpha} := \widehat{\alpha}_i$ and $\widehat{\beta} := \widehat{\beta}_i$. Let

$$\eta := \frac{(\widehat{\beta}-b)-(\widehat{\alpha}-a)}{b-a}, \qquad \zeta := \frac{a\widehat{\beta}-\widehat{\alpha}b}{b-a}. \tag{4.24}$$

Then, a direct computation yields that

$$(1+\eta)a - \zeta = \widehat{\alpha}, \qquad (1+\eta)b - \zeta = \widehat{\beta} \tag{4.25}$$

and

$$|\eta| \le \frac{2}{b-a}\max\{|\widehat{\alpha}-a|, |\widehat{\beta}-b|\},\ |\zeta| \le \frac{|a|+|b|}{b-a}\max\{|\widehat{\alpha}-a|, |\widehat{\beta}-b|\}.$$

Thus,

$$\max(|\eta|, |\zeta|) \le \frac{\max(2, |a|+|b|)}{b-a} d_H([a,b], [\widehat{\alpha}, \widehat{\beta}]) \le 1. \tag{4.26}$$

By Definition 3.2, to complete the proof it is sufficient to find $R = (R(n))_{n\in\mathbb{N}}$ satisfying

$$\|R - I\|_\infty \le \ell \max\{|\widehat{\alpha}-a|, |\widehat{\beta}-b|\},\ \Sigma_{\mathrm{ED}}(MR) = [\widehat{\alpha}, \widehat{\beta}]. \tag{4.27}$$

For this purpose, let

$$R(n) := e^{-\zeta}\mathrm{diag}(|M_{11}(n)|^\eta, \dots, |M_{dd}(n)|^\eta) \qquad \text{for all } n \in \mathbb{N},$$

where ζ and η are defined as in (4.24). Thus, by virtue of Proposition 4.10 and (4.25) we have

$$\Sigma_{\mathrm{ED}}(MR) = [(1+\eta)a - \zeta, (1+\eta)b - \zeta] = [\widehat{\alpha}, \widehat{\beta}].$$

Furthermore, by definition of $R(n)$ we have

$$\|R(n) - I\| = \max_{1\le \ell\le d}\left|e^{-\zeta}|M_{\ell\ell}(n)|^\eta - 1\right|. \tag{4.28}$$

For each $\ell \in \{1,\dots,d\}$ by the Mean Value Theorem[2], there exists $\widehat{\zeta}$ and $\widehat{\eta}$ with $|\widehat{\zeta}| \le |\zeta|, |\widehat{\eta}| \le |\eta|$ and

$$\left|e^{-\zeta}|M_{\ell\ell}(n)|^{\eta}-1\right| = \left\langle \left(-e^{-\widehat{\zeta}}|M_{\ell\ell}(n)|^{\widehat{\eta}}, \widehat{\eta}e^{-\widehat{\zeta}}|M_{\ell\ell}(n)|^{\widehat{\eta}-1}\right), (-\zeta,\eta)\right\rangle,$$

which together with (4.26) implies that

$$\begin{aligned}\left|e^{-\zeta}|M_{\ell\ell}(n)|^{\eta}-1\right| &\le \max\left(e^{-\widehat{\zeta}}|M_{\ell\ell}(n)|^{\widehat{\eta}}, |\widehat{\eta}|e^{-\widehat{\zeta}}|M_{\ell\ell}(n)|^{\widehat{\eta}-1}\right)\max(|\eta|,|\zeta|)\\ &\le e\max(|M_{\ell\ell}(n)|^{\widehat{\eta}}, |M_{\ell\ell}(n)|^{\widehat{\eta}-1})\max(|\eta|,|\zeta|)\\ &\le e(1+\|M\|^2_{\mathrm{Lya}})\max(|\eta|,|\zeta|)\\ &\le \frac{e\left(1+\|M\|^2_{\mathrm{Lya}}\right)(2+|a|+|b|)}{b-a} d_H([a,b],[\widehat{\alpha},\widehat{\beta}]).\end{aligned}$$

Thus, by (4.23) and (4.28) R satisfies all properties in (4.27). The proof is complete.□

5. Proof of the main results.

Proof of Theorem 2.8. Thanks to Theorem 3.4, to show the proportional local assignability of the dichotomy spectrum of (2.6) it is sufficient to verify the proportional local assignability of the dichotomy spectrum by multiplicative perturbation of system (2.5). Let the repeated dichotomy spectrum $\Sigma^r_{\mathrm{ED}}(A)$ be of the following form

$$\Sigma^r_{\mathrm{ED}}(A) = \Big\{[a_1,b_1],\dots,[a_d,b_d]\Big\} = \Big\{\underbrace{[a_1^*,b_1^*],\dots,[a_1^*,b_1^*]}_{d_1\text{times}},\dots,\underbrace{[a_k^*,b_k^*],\dots,[a_k^*,b_k^*]}_{d_k\text{times}}\Big\},$$

where

$$a_d \le b_d \le a_{d-1} \le b_{d-1} \le \dots \le a_1 \le b_1$$

and

$$b_k^* \le a_k^* < a_{k-1}^* \le b_{k-1}^* \dots < a_1^* \le b_1^*.$$

Then, we have for all $i = 1,\dots,k$

$$[a_i^*, b_i^*] = [a_j, b_j] \quad \text{for } d_1+\dots+d_{i-1}+1 \le j \le d_1+\dots+d_i. \tag{5.1}$$

In light of Theorem 2.1, system (2.5) is dynamically equivalent to a block-diagonal system

$$y(n+1) = \mathrm{diag}(M_1(n),\dots,M_k(n))y(n) \qquad \text{for } n\in\mathbb{N}, \tag{5.2}$$

where $(M_i(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{d_i\times d_i})$ for $i=1,\dots,k$ satisfies that

$$\Sigma_{\mathrm{ED}}(M_i) = [a_i^*, b_i^*] \qquad \text{for } i = 1,\dots,k.$$

By Remark 3.3, to conclude the proof we verify proportional local assignability of the dichotomy spectrum by multiplicative perturbation of (5.2). Note that by virtue of

[2]Let $f:\mathbb{R}^2\to\mathbb{R}$ be a C^2 function. Then, for any ζ,η there exist $\widehat{\zeta},\widehat{\eta}$ with $|\widehat{\zeta}| \le |\zeta|, |\widehat{\eta}| \le |\eta|$ and $f(\zeta,\eta) = f(0,0) + \langle\nabla f(\widehat{\zeta},\widehat{\eta}),(\zeta,\eta)\rangle$.

Theorem 4.8 and Theorem 4.9, for $i = 1, \dots, k$ the repeated dichotomy spectrum of each subsystem

$$y_i(n+1) = M_i(n) y_i(n) \quad \text{for } n \in \mathbb{N}, \tag{5.3}$$

is proportionally locally assignable by multiplicative perturbation. This implies that for each $i = 1, \dots, k$ there exist δ_i and ℓ_i such that for each admissible intervals $[a_1^i, b_1^i], \dots, [a_{d_i}^i, b_{d_i}^i]$ satisfying $\sup_{1\le j\le d_i} d_H([a_j^i, b_j^i], [a_i^*, b_i^*]) \le \delta_i$ there exists $R_i = (R_i(n))_{n\in\mathbb{N}}$ such that

$$\|R_i - I\|_\infty \le \ell_i \max_{1\le j\le d_i} d_H([a_j^i, b_j^i], [a_i^*, b_i^*])), \quad \Sigma_{\mathrm{ED}}^r(M_i R_i) = \left\{[a_j^i, b_j^i]\right\}_{j=1}^{d_i}.$$

Define

$$\delta := \min\left\{\min_{1\le i\le k} \delta_i, \min_{1\le i\le k-1} \frac{(a_i^* - b_{i+1}^*)}{3}\right\}, \ \ell := \max_{1\le i\le k} \ell_i. \tag{5.4}$$

To complete the proof, let $[\widehat{a}_1, \widehat{b}_1], \dots, [\widehat{a}_d, \widehat{b}_d]$ be arbitrary admissible closed intervals satisfying that $\max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i])) \le \delta$. Using the fact that $\delta \le \min_{1\le i\le k-1} \frac{(a_i^* - b_{i+1}^*)}{3}$, for $i = 1, \dots, k$ there exist exact d_i intervals $[\widehat{a}_1, \widehat{b}_1], \dots, [\widehat{a}_d, \widehat{b}_d]$ whose Hausdorff distance to $[a_i^*, b_i^*]$ is smaller than δ. More precisely, for $i = 1 \dots, k$ we have

$$d_H([\widehat{a}_j, \widehat{b}_j]_{j=d_1+\dots d_{i-1}+1}^{d_1+\dots d_i}, [a_i^*, b_i^*]) \le \delta.$$

Since $\delta \le \delta_i$ it follows that there exists $R_i(n)$ such that

$$\|R_i - I\|_\infty \le \ell_i \max_{d_1+d_{i-1}+1\le j\le d_1+\dots+d_i} d_H([\widehat{a}_j, \widehat{b}_j], [a_i^*, b_i^*])) \tag{5.5}$$

and

$$\Sigma_{\mathrm{ED}}^r(M_i R_i) = \left\{[\widehat{a}_j, \widehat{b}_j]\right\}_{j=d_1+\dots+d_{i-1}+1}^{d_1+\dots+d_i}. \tag{5.6}$$

Let $R(n) = \mathrm{diag}(R_1(n), \dots, R_k(n))$. Then, by (5.5) and (5.1) we have

$$\begin{aligned}\|R - I\|_\infty &\le \max_{1\le i\le k} \sup_{n\in\mathbb{N}} \ell_i \max_{d_1+d_{i-1}+1\le j\le d_1+\dots+d_i} d_H([\widehat{a}_j, \widehat{b}_j], [a_i^*, b_i^*])) \\ &\le \ell \max_{1\le j\le d} d_H([\widehat{a}_j, \widehat{b}_j], [a_j, b_j]).\end{aligned}$$

Furthermore, by (5.6) we have $\Sigma_{\mathrm{ED}}^r(MR) = \bigcup_{i=1}^k \Sigma_{\mathrm{ED}}^r(M_i R_i) = \left\{[\widehat{a}_j, \widehat{b}_j]\right\}_{j=1}^d$. The proof is complete. ∎

6. Example. Consider the following discrete time-varying linear control system

$$x(n+1) = A(n)x(n) + B(n)u(n) \quad \text{for } n \in \mathbb{N}, \tag{6.1}$$

where $A(n) = \begin{pmatrix} 1 & 1 \\ 0 & e^{\alpha_{n+1}-\alpha_n} \end{pmatrix}$ and $B(n) = I$ with $\alpha_n := (n+1)\sin(\ln(n+1))$ for $n \in \mathbb{N}$. We will use the theoretical result in the preceding sections to construct explicit linear state feedback for the proportional local assignability of the dichotomy spectrum of the free system. For this purpose, we first establish several properties of the sequence (α_n).

LEMMA 6.1. *We have*

$$\lim_{n\to\infty} |(\alpha_{n+1}-\alpha_n)-(\sin(\ln(n+2))+\cos(\ln(n+2)))| = 0. \tag{6.2}$$

Consequently,

$$\limsup_{n\to\infty}(\alpha_{n+1}-\alpha_n)=\sqrt{2}, \quad \liminf_{n\to\infty}(\alpha_{n+1}-\alpha_n)=-\sqrt{2} \tag{6.3}$$

and

$$\lim_{n\to\infty}((\alpha_{n+2}-\alpha_{n+1})-(\alpha_{n+1}-\alpha_n))=0. \tag{6.4}$$

Proof. By the Mean Value Theorem, for each $n\in\mathbb{N}$ there exists $\xi_n\in(\ln(n+1),\ln(n+2))$ such that

$$\begin{aligned}\alpha_{n+1}-\alpha_n &= \sin(\ln(n+2))+(n+1)(\sin(\ln(n+2))-\sin(\ln(n+1)))\\ &= \sin(\ln(n+2))+(n+1)(\ln(n+2)-\ln(n+1))\cos(\xi_n). \end{aligned}\tag{6.5}$$

Note that $\lim_{n\to\infty}(n+1)\ln(\frac{n+2}{n+1})=1$. Applying again the Mean Value Theorem, we obtain that

$$|\cos(\xi_n)-\cos(\ln(n+2))| = |\xi_n-\ln(n+2)||\sin(\eta_n)| \le \ln(n+2)-\ln(n+1),$$

where $\eta_n\in(\xi_n,\ln(n+2))$. Consequently,

$$\lim_{n\to\infty}(n+1)(\ln(n+2)-\ln(n+1))(\cos(\xi_n)-\cos(\ln(n+2)))=0,$$

which together with (6.5) proves (6.2). Note that

$$\limsup_{n\to\infty}\sin(\ln(n+2))+\cos(\ln(n+2)) = \limsup_{n\to\infty}\sqrt{2}\sin\big(\ln(n+2)+\frac{\pi}{4}\big)=\sqrt{2}.$$

Thus, $\limsup_{n\to\infty}(\alpha_{n+1}-\alpha_n)=\sqrt{2}$ and analogously $\liminf_{n\to\infty}(\alpha_{n+1}-\alpha_n)=-\sqrt{2}$ and (6.3) is verified. Finally, by using (6.2) we have

$$\begin{aligned}&\limsup_{n\to\infty}|(\alpha_{n+2}-\alpha_{n+1})-(\alpha_{n+1}-\alpha_n)|\\ =&\limsup_{n\to\infty}|(\sin(\ln(n+3))-\sin(\ln(n+2)))+(\cos(\ln(n+3))-\cos(\ln(n+2)))|.\end{aligned}$$

Thus, by using the Mean Value Theorem we have

$$\limsup_{n\to\infty}|(\alpha_{n+2}-\alpha_{n+1})-(\alpha_{n+1}-\alpha_n)| \le 2\limsup_{n\to\infty}(\ln(n+3)-\ln(n+2))=0,$$

which proves (6.4). The proof is complete. ∎

Next, we compute the dichotomy spectrum of the free system associated with (6.1).

LEMMA 6.2. *The following statements hold:*

(i) The sequence $A:=(A(n))_{n\in\mathbb{N}}\in\mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{2\times2})$ *and* $\|A\|_{\mathrm{Lya}}\le\sqrt{1+2e^4}$.

(ii) The dichotomy spectrum $\Sigma_{\mathrm{ED}}(A)$ *of the free system is given by* $\Sigma_{\mathrm{ED}}(A)=[-\sqrt{2},\sqrt{2}]$.

Proof. (i) Using (6.3) in Lemma 6.1, we obtain that $\sup_{n\in\mathbb{N}} |\alpha_{n+1} - \alpha_n| < \infty$ and therefore $(A(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{2\times 2})$. In fact, from (6.5) and the fact that $\sup_{n\geq 1} n\ln(1+\frac{1}{n}) = 1$ we also have $|\alpha_{n+1} - \alpha_n| \leq 2$. Then, $\|A\|_{\mathrm{Lya}} \leq \sqrt{1+2e^4}$.
(ii) Obviously, the dichotomy spectrum associated with the first coordinate subsystem $y(n+1) = y(n)$ is $\{0\}$. Then, to complete the proof, it is sufficient to compute the dichotomy spectrum of the second coordinate subsystem

$$y(n+1) = e^{\alpha_{n+1}-\alpha_n} y(n) \qquad \text{for } n \in \mathbb{N}. \tag{6.6}$$

Denote the dichotomy spectrum of the preceding scalar system by $[\alpha, \beta]$. Then, by Proposition 4.5 we have $\Sigma_{\mathrm{ED}}(A) = [\alpha, \beta] \cup \{0\}$. By property (6.4) in Lemma 6.1, for any fixed $j \in \mathbb{N}$, we have $\limsup_{k\to\infty}(\alpha_{k+j} - \alpha_k) = j \limsup_{k\to\infty}(\alpha_{k+1} - \alpha_k)$. Thus,

$$\sup_{k\in\mathbb{N}} \ln\left(\prod_{i=k}^{k+j-1} e^{\alpha_{i+1}-\alpha_i}\right) \geq \limsup_{k\to\infty} j(\alpha_{k+1} - \alpha_k).$$

Using (6.3) in Lemma 6.1, we obtain that

$$\sup_{k\in\mathbb{N}} \ln\left(\prod_{i=k}^{k+j-1} e^{\alpha_{i+1}-\alpha_i}\right) \geq j\sqrt{2},$$

which together with Lemma 4.1 implies that

$$\beta = \lim_{j\to\infty} \frac{1}{j} \sup_{k\in\mathbb{N}} \ln\left(\prod_{i=k}^{k+j-1} e^{\alpha_{i+1}-\alpha_i}\right) \geq \sqrt{2}.$$

On the other hand, using Lemma 4.3 there exists a pair of sequences of natural numbers $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ satisfying that $m_k < n_k < m_{k+1}$ for $k \in \mathbb{N}$, $\lim_{k\to\infty} m_k = \infty$ and $\lim_{k\to\infty}(n_k - m_k) = \infty$ and $\beta = \lim\limits_{k\to\infty} \frac{\alpha_{n_k}-\alpha_{m_k}}{n_k - m_k}$. Thus,

$$\beta = \lim_{k\to\infty} \frac{\alpha_{n_k} - \alpha_{m_k}}{n_k - m_k} \leq \limsup_{n\to\infty}(\alpha_{n+1} - \alpha_n) = \sqrt{2}.$$

Similarly, we have $\alpha = -\sqrt{2}$. Hence, $\Sigma_{\mathrm{ED}}(A) = [-\sqrt{2}, \sqrt{2}]$ and the proof is complete. $\square$

Coming back to the linear control system (6.1), the Kalman controllability matrix $W(n+1, n) = A(n)A(n)^{\mathrm{T}}$ satisfies that

$$\langle \xi, W(n+1, n)\xi\rangle = \|A(n)\xi\|^2 \geq \frac{\|\xi\|^2}{\|A(n)^{-1}\|^2} \geq \frac{\|\xi\|^2}{\|A\|^2_{\mathrm{Lya}}}.$$

Then, using the characterization of controllability in terms of the positivity of the Kalman controllability matrix (see e.g. [18, 37]), the system (6.1) is uniformly completely controllable. Consequently, by Theorem 2.8 the dichotomy spectrum of the associated free system of (6.1) is proportionally locally assignable. Furthermore, in the following theorem, we follow the approach in the proof of Theorem 4.9 to provide an explicit construction of the linear state feedback in the problem of proportional local assignability of dichotomy spectrum of (6.1).

THEOREM 6.3. *Let $\delta := \frac{2\sqrt{2}}{3+2\sqrt{2}}$ and $\ell := (2+\sqrt{2})e(e^4+1)$. Let $[\widehat{\alpha}, \widehat{\beta}]$ be an arbitrary spectral interval with $d_H([\widehat{\alpha}, \widehat{\beta}], [-\sqrt{2}, \sqrt{2}]) \leq \delta$. Define*

$$U(n) = \begin{pmatrix} e^{-\zeta} - 1 & e^{-\zeta} e^{\eta(\alpha_{n+1}-\alpha_n)} - 1 \\ 0 & e^{-\zeta} e^{(1+\eta)(\alpha_{n+1}-\alpha_n)} - e^{\alpha_{n+1}-\alpha_n} \end{pmatrix},$$

where $\eta = \frac{\widehat{\beta}-\widehat{\alpha}-2\sqrt{2}}{2\sqrt{2}}, \zeta = -\frac{\widehat{\alpha}+\widehat{\beta}}{2}$. *Then,* $[\widehat{\alpha}, \widehat{\beta}]$ *is the dichotomy spectrum of the closed-loop system* $x(n+1) = (A(n) + U(n))x(n)$ *and*

$$\|U\|_\infty \le (2+\sqrt{2})e\sqrt{1+2e^4}(e^4+1)d_H([\widehat{\alpha}, \widehat{\beta}], [-\sqrt{2}, \sqrt{2}]). \tag{6.7}$$

Proof. By $\eta = \frac{\widehat{\beta}-\widehat{\alpha}-2\sqrt{2}}{2\sqrt{2}}, \zeta = -\frac{\widehat{\alpha}+\widehat{\beta}}{2}$ (cf. (4.24)), it was proved in the proof of Theorem 4.9 that $[\widehat{\alpha}, \widehat{\beta}]$ is the dichotomy spectrum of the multiplicative perturbation

$$x(n+1) = A(n)R(n)x(n), \qquad R(n) := \begin{pmatrix} e^{-\zeta} & 0 \\ 0 & e^{-\zeta}e^{\eta(\alpha_{n+1}-\alpha_n)} \end{pmatrix}.$$

Furthermore,

$$\sup_{n\in\mathbb{N}} \|R(n) - I\| \le \ell\, d_H([\widehat{\alpha}, \widehat{\beta}], [-\sqrt{2}, \sqrt{2}]). \tag{6.8}$$

Since $U(n) = A(n)(R(n) - I)$ the closed-loop system

$$x(n+1) = (A(n) + U(n))x(n) = A(n)R(n)x(n)$$

has the dichotomy spectrum as $[\widehat{\alpha}, \widehat{\beta}]$. Moreover, by (6.8) and Lemma 6.2(i), we have

$$\|U\|_\infty \le \|A\|_\infty \sup_{n\in\mathbb{N}} \|R(n) - I\| \le (2+\sqrt{2})e\sqrt{1+2e^4}(e^4+1)d_H([\widehat{\alpha}, \widehat{\beta}], [-\sqrt{2}, \sqrt{2}]).$$

∎

We end up this section by giving a remark on the open problem of proportional local assignability of the Lyapunov spectrum for (6.1). This will help to describe in more details the current results of this problem. Also, this will help us to see the difference between the problem of proportional local assignability of Lyapunov spectrum and dichotomy spectrum.

Remark 6.4. (i) We will show that the free linear system has unstable Lyapunov spectrum consisting of two exponents, not regular and not diagonalizable. Note that $\Phi_A(n, 0) = \begin{pmatrix} 1 & \sum_{i=0}^{n-1} e^{\alpha_i} \\ 0 & e^{\alpha_n} \end{pmatrix}$. Thus, the Lyapunov exponents of the two solutions $\Phi_A(n, 0)e_1$ and $\Phi_A(n, 0)e_2$ of the free system is given by

$$\lambda_1 := \limsup_{n\to\infty} \frac{1}{n} \ln \|\Phi_A(n, 0)e_1\| = 0,$$

$$\lambda_2 := \limsup_{n\to\infty} \frac{1}{n} \ln \|\Phi_A(n, 0)e_2\| = \limsup_{n\to\infty} \frac{(n+1)\sin(\ln(n+1))}{n} = 1.$$

Thus, the Lyapunov spectrum of the free system is $\{0, 1\}$. Also, in the formula for λ_2 the limit superior is not a limit and the system is not regular. Finally, recall that from [9, Theorem 1] (cf. [1, Theorem 3.3.3]) the free system would be diagonalizable if only if $\frac{|\det \Phi_A(n,0)|}{\|\Phi_A(n,0)e_1\|\|\Phi_A(n,0)e_2\|} \ge \rho$ for certain $\rho > 0$ and all $n \in \mathbb{N}$. However, a direct computation yields

$$\liminf_{n\to\infty} \frac{|\det \Phi_A(n, 0)|}{\|\Phi_A(n, 0)e_1\|\|\Phi_A(n, 0)e_2\|} = \liminf_{n\to\infty} \frac{e^{\alpha_n}}{\sqrt{\left(\sum_{i=0}^{n-1} e^{\alpha_i}\right)^2 + e^{2\alpha_n}}} = 0,$$

since $\liminf_{n\to\infty} e^{\alpha_n} = 0$. Moreover, since the system (6.1) is not diagonalizable it follows that it does not have the property of integral separateness (see [9, Theorem 2]). On the other hand it is known (see [9, Theorem 6], cf. [1, Theorem 5.4.7]) that integral separateness is equivalent to the fact that the Lyapunov spectrum consists of different numbers and is stable. Thus, the Lyapunov spectrum of (6.1) is unstable.
(ii) It has been proved in [4, Theorem 6.9], that the Lyapunov spectrum is proportionally locally assignable provided that the free system is either regular or diagonalizable or the Lyapunov spectrum is stable and the controled system is uniformly completely controllable. Therefore, in the light of the known literature results, the problem of proportional local assignability of the spectrum of (6.1) remains open.

Remark 6.5. Although we are successful to construct an explicit linear state feedback for local assignment of dichotomy spectrum of (6.1), it is undeniable that the same result for high dimensional linear control systems will be a difficult task. The main problem is that there is no explicit formula for the dichotomy spectrum of an arbitrary linear systems. In fact, we might only hope to obtain the approximation of the spectrum, see [**?**]. So, we might hope to establish a numerical scheme to approximate the suitable linear state feedback. We leave this problem for future research.

7. Conclusions. In this paper we consider a version of the problem of assignability of dichotomy spectrum, namely the proportional local assignability problem. We show that uniform complete controllability is a sufficient condition for the solvability of this problem. It should also be noted that the methods of proof used are constructive, i.e. they enable the construction of a control that ensures the proper location of the dichotomy spectrum, however, the methods used in the proof only give estimates of the constants ℓ and δ from the definition of proportional local assignability of the dichotomy spectrum. This issue is disscussed deeper together with the problem of construction of the feedback, in the case of proportional local assignability of the Lyapunov spectrum in the Section "Discussion of the results" in [4].

8. Appendix. This section is devoted to prove Theorem 2.1. The content of this theorem consists of two things. The first one is the structure of the dichotomy spectrum of one-sided discrete time-varying linear systems and the proof of this result can be seen in [3, Theorem 3.4]. The second one is about block diagonalization of linear systems such that the subsystem of each block corresponds to one spectral interval. The proof of this result for two-sided systems is given in [33]. In what follows, we prove this result for one-sided systems.

Proof of Theorem 2.1. Denote the dichotomy spectrum $\Sigma_{\mathrm{ED}}(M)$ by $\Sigma_{\mathrm{ED}}(M) = \bigcup_{i=1}^{k}[a_i, b_i],$, where $a_k \le b_k < a_{k-1} \le b_{k-1} < \cdots < a_1 \le b_1$. Let $\gamma \in (b_2, a_1)$ be arbitrary. Then, the shifted system

$$x(n+1) = e^{-\gamma} M(n)x(n)$$

exhibits an exponential dichotomy, i.e. there exist K, $\alpha > 0$ and a family of invariant projections $P(\cdot) : \mathbb{N} \to \mathbb{R}^{d\times d}$, $P(n+1)M(n) = M(n)P(n)$ for all $n \in \mathbb{N}$, such that for all $m, n \in \mathbb{N}$ we have

$$\begin{aligned} \|\Phi_M(m,n)P(n)\| &\le Ke^{(\gamma-\alpha)(m-n)} \text{ for } m \ge n, \\ \|\Phi_M(m,n)(I - P(n))\| &\le Ke^{(\gamma+\alpha)(m-n)} \text{ for } m \le n. \end{aligned} \tag{8.1}$$

By invariance of $P(n)$, the dimension of $P(n)$ is independent of n and let $d_1 := \dim \operatorname{im}(I - P(n))$. Let $f_1(n), \ldots, f_{d_1}(n)$ be an orthornormal basis of the subspace

$\mathrm{im}(I - P(n))$ and $f_{d_1+1}(n), \dots, f_d(n)$ be an orthonormal basis of the subspace $\mathrm{im}(P(n))$. For each $n \in \mathbb{N}$, define $L(n) \in \mathbb{R}^{d\times d}$ by

$$L(n)e_i = f_i(n) \quad \text{ for } i = 1, \dots, d,$$

where $e_1, e_2, \dots, e_d$ is the standard Euclidean basis of $\mathbb{R}^d$. Then,

$$\begin{aligned}\|L(n)\| &= \sup_{\|x_1e_1+\cdots+x_de_d\|=1} \|L(n)(x_1e_1 + \cdots + x_de_d)\| \\ &\leq \sup_{x_1^2+\cdots+x_d^2=1} |x_1|\|L(n)e_1\| + \cdots + |x_d|\|L(n)e_d\| \leq d\end{aligned}$$

and

$$\begin{aligned}\|L^{-1}(n)\| &= \sup_{\|x_1f_1(n)+\cdots+x_df_d(n)\|=1} \|L^{-1}(n)(x_1f_1(n) + \cdots + x_df_d(n))\| \\ &= \sup_{\|x_1f_1(n)+\cdots+x_df_d(n)\|=1} \|x_1e_1 + \cdots + x_de_d\| \\ &= \sup_{\|x_1f_1(n)+\cdots+x_df_d(n)\|=1} \sqrt{x_1^2 + \cdots + x_d^2}.\end{aligned}$$

Note that

$$\begin{aligned}\|(I - P(n))(x_1f_1(n) + \cdots + x_df_d(n))\| &= \|x_1f_1(n) + \cdots + x_{d_1}f_{d_1}(n)\| \\ &= \sqrt{x_1^2 + \cdots + x_{d_1}^2}\end{aligned}$$

which together with $\|I - P(n)\| \leq K$ implies that

$$\sup_{\|x_1f_1(n)+\cdots+x_df_d(n)\|=1} \sqrt{x_1^2 + \cdots + x_{d_1}^2} \leq K.$$

Similarly,

$$\sup_{\|x_1f_1(n)+\cdots+x_df_d(n)\|=1} \sqrt{x_{d_1+1}^2 + \cdots + x_d^2} \leq K.$$

Thus, by the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ we arrive at $\|L^{-1}(n)\| \leq 2K$. Hence, $(L(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d\times d})$. By invariance of $P(n)$ the matrices $L(n+1)^{-1}M(n)L(n)$ are of the diagonal form as follows

$$L(n+1)^{-1}M(n)L(n) = \begin{pmatrix} M_1(n) & 0 \\ 0 & \widetilde{M}(n) \end{pmatrix},$$

where $(M_1(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{d_1\times d_1})$ and $(\widetilde{M}(n))_{n\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{(d-d_1)\times(d-d_1)})$. Furthermore, by (8.1) there exists $\widehat{K} > 0$ such that

$$\begin{aligned}\|\Phi_{\widetilde{M}}(m, n)\| &\leq \widetilde{K}e^{(\gamma-\alpha)(m-n)} && \text{for } m \geq n, \\ \|\Phi_{M_1}(m, n)\| &\leq Ke^{(\gamma+\alpha)(m-n)} && \text{for } m \leq n.\end{aligned}$$

Thus,

$$\Sigma_{\mathrm{ED}}(\widetilde{M}) \subset (-\infty, \gamma), \quad \Sigma_{\mathrm{ED}}(M_1) \subset (\gamma, \infty). \tag{8.2}$$

On the other hand, by Proposition 4.5 $\Sigma_{\mathrm{ED}}(\widetilde{M}) \cup \Sigma_{\mathrm{ED}}(M_1) = \bigcup_{i=1}^{k}[a_i, b_i]$, which together with (8.2) implies that

$$\Sigma_{\mathrm{ED}}(M_1) = [a_1, b_1], \quad \Sigma_{\mathrm{ED}}(\widetilde{M}) = \bigcup_{i=2}^{k}[a_i, b_i].$$

Also note that the way to reduce the original system is dimensionless due to the fact that although $P(n)$ might be not unique the $\mathrm{im}P(n)$ and hence $\dim \mathrm{im}P(n)$ is unique. Thanks to the proceeding procedure and reapplying this procedure to subsystems, we complete the proof. ∎

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