POLYNOMIALS POSITIVE SEMI-DEFINITE ON A GENERALIZED STRIP

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Abstract. We study the representation of nonnegative two variable polynomials on a certain class of unbounded closed basic semi-algebraic sets (which are called generalized strips). This class includes the strip $[a, b] \times \mathbb{R}$ which was studied by Marshall in [15]. A denominator-free Nichtnegativstellensatz holds true on a generalized strip when the width of the generalized strip is constant and fails otherwise. Furthermore, we also formulated this result in the ring of matrix polynomials. We need a denominator in the representation of a given nonnegative matrix polynomial for the non-commutative Nichtnegativstellensatz.

1. Introduction

Starting with 17 Hilbert's Problem, many problems have arisen in Real Algebraic Geometry, and many interesting results are known. Given a basic closed semi-algebraic set K in $\mathbb{R}^n$ defined by finitely many polynomial inequalities $\{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$, where each g_i is a real polynomial, Positivstellensätze are results characterizing all polynomials, which are positive on K, in terms of sums of squares and the polynomials g_i used to describe K. Theorems about the existence of such representations have various applications, notably in problems of optimizing polynomial functions on semi-algebraic sets. For a nice survey and details, we refer the reader to [13, 22, 12, 9] with the references therein.

In case K is compact, Schmüdgen [25] has proved that any polynomial, which is positive on K, is in the preordering $T = T(g_1, \dots, g_m)$ generated by the g_i's, i.e., T is the set of finite sums of elements of the form $\sigma_e g_1^{e_1} \cdots g_m^{e_m}$, where $e_i \in \{0, 1\}$ and each σ_e is a sum of squares of polynomials. Schmüdgen's Positivstellensätz holds for polynomials, which are positive on K and satisfy certain extra conditions; see [8, 15, 16, 17, 20, 21], etc. Scheiderer has shown that Schmüdgen's Positivstellensätz does not hold if K is not compact and $\dim K \geq 3$, or $\dim K = 2$ and K contains a 2-dimensional cone, see [19]. We also would like to note that both Schmüdgen's and Putinar's Positivstellensätz (see [25, 18]) were extended from the

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usual real polynomials to the real symmetric matrix polynomials or operator polynomials; see [24, 1, 2, 11]. Here and in the following, by a matrix polynomial we mean a polynomial whose coefficients are matrices of the same order. Equivalently, a matrix polynomial is a matrix whose entries are all polynomials.

A stronger problem which has been attracted much attention: Under which conditions is T saturated? Let's remind that a preordering T is saturated if every polynomial which is nonnegative on K belongs to T. If the dimension of K is greater or equal to 3, then Scheiderer shows that T is not suturated regardless of compactness of K [22, Proposition 3.1.14]. Therefore, we have to look for saturated preorderings in the case where the dimension of K is not greater than 2. Some classes of compact (virtually compact) surfaces (curves, respectively) which have saturated preorderings were given in [20], [21]. A remarkable saturated preordering T for the non-compact case is $T(x(1-x))$ (Marshall's Nichtnegativstellensatz for the strip $[0,1]\times\mathbb{R}$ (see [15])). The main theorem in [15] (or also in [17]) which stated that a real polynomial which is nonnegative on a strip $\mathbb{R}\times[0,1]$ belongs to the preordering $T(y(1-y))$. Recently, Scheiderer and Wenzel [23] has extended [15, Theorem 1.1] on the cylinder $\mathbb{R}\times C$, where C is a nonsingular affine curve over $\mathbb{R}$ with $C(\mathbb{R})$ compact. Note that Schmüdgen's Positivstellensätz fails if K contains a 2-dimensional cone, hence Marshall's Nichtnegativstellensatz is probably an extreme result.

Any semi-algebraic subset of $\mathbb{R}^2$ can be decomposed into a finite union of tentacles and a bounded semi-algebraic set (see [4, Proposition 1.2]). Up to some linear change of coordinates, a tentacle in $\mathbb{R}^2$ is assumed to be of the form:

$$\{(x,y)\in\mathbb{R}^2 \mid \beta_1(x)\le y\le \beta_2(x), x\ge R\},$$

where $R>0$ and β_1,β_2 are convergent Puiseux series at infinity such that the sign of $\beta_1-\beta_2$ is constant on $[R,\infty]$ (see [6, 7]). A tentacle is a semi-algebraic set but need not be *closed basic* and so it is not clear if it has a finitely generated preordering. Therefore, we look for a class of tentacles which are possibly changeable to a closed basic semi-algebraic set. Precisely, we consider a tentacle of the form

$$M=\{(x,y)\in\mathbb{R}^2 \mid \beta_1(x)\le y\le \beta_2(x), x\ge R\},$$

where $\beta_1(x),\beta_2(x)$ have finite terms, that is,

$$\beta_i(x)=\sum_{j=m}^{n} b_{i,j}(\frac{1}{x})^{j/q},\quad m\le n\in\mathbb{Z},\ q\in\mathbb{N},\ b_{i,j}\in\mathbb{R},\ i=1,2.$$

Making the change of variable $z=\sqrt[q]{x}$, we can assume that $q=1$. Then

$$M=\{(x,y)\in\mathbb{R}^2 \mid x^{\max\{n,0\}}\beta_1(x)\le x^{\max\{n,0\}}y\le x^{\max\{n,0\}}\beta_2(x),\ x\ge R\}.$$

Let $g_i(x) = x^{\max\{n,0\}}\beta_i,\ i = 1, 2$. Then $g_1(x), g_2(x)$ are real polynomials in x and

$$M = \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \leq x^{\max\{n,0\}}y \leq g_2(x), x \geq R\}.$$

If M is unbounded, then there exists a positive number N such that $g_2(x) - g_1(x) > 0$ for every $x > N$. Thus, in this paper, we consider a class of clsoed basic semi-algebraic sets of the form:

$$K(g_1, g_2, \alpha) := \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \leq x^{\alpha}y \leq g_2(x)\},$$

where g_1, g_2 are real single variable polynomials. In the case $\alpha = 0$, $g_1 \equiv 0$ and $g_2 \equiv 1$ then $K(0, 1, 0) = \mathbb{R} \times [0, 1]$ is the strip mentioned in [15].

An introduction of a class of (unbounded) closed basic semi-algebraic sets $K(g_1, g_2, \alpha)$ (these sets are called *generalized strips*) is written in Section 2. The main results of this paper are presented in Section 3. The first part of Section 3 presents the representations of polynomials which are nonnegative on these semi-algebraic sets. We define $w(x) := g_2(x)-g_1(x)$ and call it the *width* of $K(g_1, g_2, \alpha)$. In particular, the strip $K(0, 1, 0) = \mathbb{R}\times[0, 1]$ has the width $w(x) = 1$. In this paper, we will point out that the width is a characterization of the saturated property of the preordering T generated by $x^{\alpha}y-g_1(x), g_2(x)-x^{\alpha}y$. Precisely, if the width $w(x)$ is finite, then every polynomial $p(x, y)$ which is nonnegative on $K(g_1, g_2, \alpha)$ belongs to the preordering T provided some technical conditions. In the special case, we obtain the Marshall's Nichtnegativstellensatz [15]. On the other hand, if the width $w(x)$ is infinite, then there exists a polynomial $p(x, y)$ which is positive on $K(g_1, g_2, \alpha)$ but does not belong to the preordering T.

Many works from Hol, Scherer, Putinar, Schmüdgen, Cimpric, Schweighofer, etc. are extended the Positivstellensätzese to noncommutative cases, in particular for the ring of matrix polynomials. As in the commutative case, these works hold for the case where semi-algebraic sets K are compact, see e.g., [3] or [26]. In an attempt to extend these results for noncompact case, Hoa, Pham and Ho obtained the noncommutative Positivstellensätz on a nondegenerate closed basic semi-algebraic set, see [10].

In the last part of Section 3, by applying the diagonalization of matrix polynomial in [26] and the main result of the first part, we can obtain a noncommutative Nichtnegativstellensatz on a generalized strip mentioned above. We need a denominator in the representation of a given nonnegative matrix polynomial for the non-commutative version.

The paper is organized as follows. Some basic notations and the definition of generalized strips in $\mathbb{R}^2$ are presented in Section 2. The main results with proofs are written in Section 3.

2. Preliminaries

Notation. Throughout this paper, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{Z}_{\geq 0}$ the set of nonnegative integer numbers, and $\mathbb{R}^n$ denotes the Euclidean space of dimension n. The corresponding inner product (resp., norm) in $\mathbb{R}^n$ is defined by $\langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$ (resp., $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathbb{R}^n$). We let $\mathbb{R}[x]$ denote the ring of real polynomials in n indeterminates and a polynomial we always mean a real polynomial. Without of confusing, in many cases, x stands also for a single variable and so $\mathbb{R}[x]$ means the ring of single variable polynomials. Denoted by $\mathrm{Mat}_d(\mathbb{R}[x])$ the algebra of matrix polynomials of order d. A matrix polynomial $F \in \mathrm{Mat}_d(\mathbb{R}[x])$ is said to be symmetric on a set $K \subset \mathbb{R}^n$ if $F(x) = F(x)^T$ for every $x \in K$, where $F(x)^T$ is the transpose of $F(x)$. If $K = \mathbb{R}^n$, we simply say F is symmetric. We denote by $\mathrm{Sym}_d(\mathbb{R}[x])$ the space of all symmetric matrix polynomials in $\mathrm{Mat}_d(\mathbb{R}[x])$.

For symmetric matrix polynomials F and G of the same size, we write $F \succeq G$ (resp., $F \succ G$) to express that $F - G$ is positive semidefinite (resp., positive definite). Given a symmetric matrix polynomial $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ and a set $K \subset \mathbb{R}^n$, we write $F \succeq 0$ (resp., $F \succ 0$) on K if for all $x \in K$, the matrix $F(x)$ is positive semidefinite (resp., the matrix $F(x)$ is positive definite). A subset $\mathcal{M}$ of $\mathrm{Sym}_d(\mathbb{R}[x])$ is said to be a *quadratic module* if $I_d \in \mathcal{M}, \mathcal{M} + \mathcal{M} \subset \mathcal{M}$ and $A^T\mathcal{M}A \subset \mathcal{M}$ for every $A \in \mathrm{Mat}_d(\mathbb{R}[x])$. The smallest quadratic module which contains a given subset $\mathcal{G}$ of $\mathrm{Sym}_d(\mathbb{R}[x])$ will be denoted by $\mathcal{M}(\mathcal{G})$. It consists of all finite sums of elements of the form A^TGA where $G \in \mathcal{G} \cup \{I_d\}$ and $A \in \mathrm{Mat}_d(\mathbb{R}[x])$. A polynomial $f(x) \in \mathbb{R}[x]$ can be viewed as an element in $\mathrm{Mat}_d(\mathbb{R}[x])$ via the identification $f(x)$ with $f(x)I_d$. Therefore, if $\mathcal{G}$ is a finite subset of $\mathbb{R}[x]$, the quadratic module $\mathcal{M}(\mathcal{G})$ generated by $\mathcal{G}$ in the ring $\mathrm{Mat}_d(\mathbb{R}[x])$ is the set of finite sums gAA^T for $A \in \mathrm{Mat}_d(\mathbb{R}[x])$ and $g \in \mathcal{G} \cup \{1\}$.

A subset of $\mathbb{R}^n$ is called a basic semi-algebraic set if it is an intersections a finite number of sets of the forms

$$\{x \in \mathbb{R}^n \colon g(x) = 0\} \quad \text{or} \quad \{x \in \mathbb{R}^n \colon h(x) > 0\},$$

where g, h are arbitrary real polynomials in n variables. A semi-algebraic set is a finite union of basic semi-algebraic sets.

Given a finite set $\mathcal{G} = \{g_1, g_2, \ldots, g_m\} \subset \mathbb{R}[x, y]$, the basic closed semi-algebraic set in $\mathbb{R}^2$ generated by $\mathcal{G}$, denoted as $K_{\mathcal{G}}$, is $\{(x, y) \in \mathbb{R}^2 \colon g_1(x, y) \geq 0, \ldots, g_m(x, y) \geq 0\}$. The quadratic module $\mathcal{M}(\mathcal{G}) = \mathcal{M}(g_1, \ldots, g_m)$ generated by $\mathcal{G}$ in the ring $\mathbb{R}[x, y]$ is the set

$$\{r_0 + r_1g_1 + \cdots + r_mg_m \mid r_i \in \sum \mathbb{R}[x, y]^2\},$$

where $\sum \mathbb{R}[x, y]^2$ is the smallest quadratic module in $\mathbb{R}[x, y]$ and is equal to the set of all finite sums of squares of polynomials. The preordering $T(\mathcal{G}) = T(g_1, \ldots, g_m)$ is the quadratic module generated by the set of finitely distinct product of $\{g_1, \ldots, g_m\}$. Hence, $\mathcal{M}(g_1, \ldots, g_m)$

is contained in $T(g_1, \ldots, g_m)$. Some works tried to characterize when $\mathcal{M}(g_1, \ldots, g_m)$ is equal to $T(g_1, \ldots, g_m)$. It is trivial, in the ring $\mathbb{R}[x, y]$, that when $m = 1$ the preordering $T(g_1)$ is the same as the quadratic module $\mathcal{M}(g_1)$.

We say that $\mathcal{M}(\mathcal{G})$ (respectively, $T(\mathcal{G})$) is saturated if for every $f \in \mathbb{R}[x, y]$, f non-negative on $K_{\mathcal{G}}$ implies $f \in \mathcal{M}(\mathcal{G})$ (respectively, in $T(\mathcal{G})$). Marshall's Theorem says that the quadratic module generated by $y - y^2$ in $\mathbb{R}[x, y]$ is saturated.

Tentacle sets. We quote here the definitions and properties of tentacle sets from [4, 5]. By a *Puiseux series at infinity* we will mean a series of the form

$$\beta = \sum_{m}^{\infty} b_j (\frac{1}{X})^{\frac{j}{q}},$$

where $q \in \mathbb{N},\ m \in \mathbb{Z}$, $b_j \in \mathbb{R}$ for $j \geq m$.
If $b_j \in \mathbf{C}$ we call β a *complex* Puiseux series at infinity. The numbers b_j will be called the coefficients of β. If $\beta \neq 0$, we can assume that the first coefficient $b_m \neq 0$. If β is nonzero, we put $\mathrm{ord}_{\infty}\beta := m/q$ and call it *the order at infinity* of β. We also denote $\mathrm{ord}_{\infty} 0 = +\infty$. The set of Puiseux series at infinity with natural addition and multiplication forms a field.

Suppose β is a Puiseux series at infinity. If there exists a closed half-line $I \subset \mathbb{R}$ such that the series $\beta(x)$ is convergent for $x \in I$ we will say that β is *a convergent Puiseux series at infinity.* If this is the case, we will consider $\beta :\ I \longrightarrow \mathbb{R}$ both as a Puiseux series and a real function.

Suppose $\Gamma \subset \mathbb{R}^2$ is an unbounded semi-algebraic curve. The convergent Puiseux series at infinity β is called a *special Puiseux parametrization of the semi-algebraic curve at infinity* if there exists a closed half-line $I \subset \mathbb{R}$ such that

$$\Gamma = \{(x, \beta(x)) \in \mathbb{R}^2 \mid x \in I\}.$$

For convenience, we will sometime call such a series a *Puiseux parametrization.*

Definition 2.1. *An unbounded semi-algebraic set $M \subset \mathbb{R}^2$ is called a tentacle set if for any $r > 0$ the set $M \setminus B(0, r)$ is connected, where $B(0, r)$ is the ball centered at the origin with radius r.*

Any semi-algebraic subset in $\mathbb{R}^2$ has a following decomposition (see [4, Proposition 1.2]):

$$S = K \cup M_1 \cup \ldots \cup M_k,$$

where K is a bounded semi-algebraic set and M_i are pairwise disjoint tentacle sets which are closed in S, i.e. $M_i \cap S = M_i$. The above decomposition is unique in the following sense (see [4, Remark 1.3]): Given two tentacle decompositions

$$S = K \cup M_1 \cup \ldots \cup M_k = \tilde{K} \cup \tilde{M}_1 \cup \ldots \cup \tilde{M}_l,$$

Then $k = l$ and there exists a compact set C such that

$$M_i \setminus C = \tilde{M}_i \setminus C$$

for $i = 1, \ldots, k$ possibly after rearranging the indices of the tentacles.

Suppose that S is a closed, unbounded semi-algebraic set which does not contain a quadrant. The fact that a tentacle of the set S, after some linear change of coordinates and possibly after leaving out a compact subset, is of the form

$$\{(x, y) \in \mathbb{R}^2 \mid \beta_1(x) \leq y \leq \beta_2(x), x \geq R\}$$

where $R > 0$ and β_1, β_2 are convergent Puiseux series at infinity such that the sign of $\beta_1 - \beta_2$ is constant on $[R, \infty]$ (see [6, 7]). A tentacle is a semi-algebraic set. However, it may not be a *closed basic* semi-algebraic set and so it is not easy to find a preordering T which is finitely generated. To avoid such difficulty, we consider a class of tentacles which are possibly changeable to a closed basic semi-algebraic set. Precisely, we consider a tentacle of the form

$$M = \{(x, y) \in \mathbb{R}^2 \mid \beta_1(x) \leq y \leq \beta_2(x), x \geq R\},$$

where $\beta_1(x), \beta_2(x)$ have finite terms, that is,

$$\beta_i(x) = \sum_{j=m}^{n} b_{i,j} (\frac{1}{x})^{j/q}, \quad m \leq n \in \mathbb{Z},\ q \in \mathbb{N},\ b_{i,j} \in \mathbb{R},\ i = 1, 2.$$

Making the change of variable $z = \sqrt[q]{x}$, we can assume that $q = 1$. Then

$$M = \{(x, y) \in \mathbb{R}^2 \mid x^{\max\{n,0\}} \beta_1(x) \leq x^{\max\{n,0\}} y \leq x^{\max\{n,0\}} \beta_2(x),\ x \geq R\}.$$

Let $\alpha = \max\{n, 0\},\ g_i(x) = x^\alpha \beta_i,\ i = 1, 2$. Then $g_1(x), g_2(x)$ are real polynomials in x and

$$M = \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \leq x^\alpha y \leq g_2(x), x \geq R\}. \tag{1}$$

M is unbounded, then there exists a positive number N such that $g_2(x) - g_1(x) > 0$ for every $x > N$. This implies that the polynomial $g_2(x) - g_1(x)$ is either a positive constant or a polynomial of degree at least 1 with the positive highest coefficient.

Definition 2.2. *We will call a closed semi-algebraic set of the form*

$$K(g_1, g_2, \alpha) := \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \leq x^\alpha y \leq g_2(x)\}$$

a generalized strip, where g_1, g_2 are real polynomials in $\mathbb{R}[x]$ and $0 \leq \alpha \in \mathbb{Z}$.

In the case $\alpha = 0$, $g_1 \equiv 0$ and $g_2 \equiv 1$ then $K(0, 1, 0) = \mathbb{R} \times [0, 1]$ is the strip mentioned in [15].

A quadratic module of $K(g_1, g_2, \alpha)$ is the quadratic module generated by $x^\alpha y - g_1(x), g_2(x) - x^\alpha y$. If a tentacle M is determined by (1), then M is a closed basic semi-algebraic and it is called a *half generalized strip.* In this case M has a quadratic module $\mathcal{M}(x^\alpha y - g_1(x), g_2(x) - x^\alpha y, x - R)$.

3. Main results

3.1. **Polynomials nonnegative on a generalized strip.** The remarkable result by Marshall that the quadratic module $M(y(1-y))$ of the strip $K(0,1,0) = \mathbb{R} \times [0,1]$ is saturated, see [15, Theorem 1.1]. We have a generalization on $K(g_1, g_2, 0)$ as the following lemma.

Lemma 3.1. *Let $g_1(x),\ g_2(x)$ be single variable polynomials and the set*

$$K(g_1, g_2) := K(g_1, g_2, 0) = \{(x, y) \in \mathbb{R}^2 \mid g_1(x) \le y \le g_2(x)\}.$$

The following statements hold.

1. *If $deg(g_2 - g_1) > 0$ and the leading coefficient of $g_2 - g_1$ is positive then there exist a polynomial $f(x, y) \in \mathbb{R}[x, y]$ which is positive on $K(g_1, g_2)$ and does not belong to the preordering generated by $y - g_1(x), g_2(x) - y$.*
2. *If $g_2(x) - g_1(x) = c$, where c is a positive constant then the quadratic module $\mathcal{M}([y - g_1(x)][g_2(x) - y])$ is saturated. That is, if $f(x, y)$ is a two variable polynomial which is nonnegative on $K(g_1, g_2)$ then there exist $r_0(x, y), r_1(x, y) \in \sum \mathbb{R}[x, y]^2$ such that*

$$f(x, y) = r_0(x, y) + r_1(x, y)(y - g_1(x))(g_2(x) - y).$$

Proof. (1) We have

$$K(g_1, g_2) = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y - g_1(x) \le g_2(x) - g_1(x)\}.$$

Make the change of variable $z = y - g_1(x)$, we have

$$(x, y) \in K(g_1, g_2) \text{ if and only if } (x, z) \in K(0, w) = \{(x, z) \in \mathbb{R}^2 \mid 0 \le z \le w(x)\},$$

where $w(x) = g_2(x) - g_1(x)$.

Claim: $K(0, w)$ contains an 2-dimesnional cone.

By Claim and [14, Proposition 4.2.3], there exist $\tilde{f}(x, z)$ which is positive on $K(0, w)$ does not belong to $T(z, w(x) - z)$. Set $f(x, y) := \tilde{f}(x, y - g_1(x))$. Then $f(x, y)$ is positive on $K(g_1, g_2)$ but $f(x, y)$ does not belong to $T(y - g_1(x), g_2(x) - y)$.

We need to prove Claim above. Indeed, suppose that a_w is the coefficient of x^s, where $s = \deg(w)$. Then $a_w > 0$ and $w(x) \ge ax^s \ge ax$ for all $x \ge \delta$ where $a = \dfrac{a_w}{2}$ and some sufficiently large δ. Then $K(g_1, g_2)$ contains the cone $(\delta, 0) + C$ where C is the convex cone generated by $(1, 0)$ and $(1, a)$. This completes the proof.

(2) If $g_2 - g_1 = c$, we have $K(g_1, g_2) = \{(x, y) \in \mathbb{R}^2 \mid 0 \le c^{-1}(y - g_1(x)) \le 1\}$. Observe that if $z = c^{-1}(y - g(x))$, then we have

$$(x, y) \in K(g_1, g_2) \Leftrightarrow (x, z) \in \mathbb{R} \times [0, 1].$$

For $f(x, y) \in \mathbb{R}[x, y]$, let's define $\tilde{f}(x, z) := f(x, cz + g_1(x)) \in \mathbb{R}[x, z]$. Then $\tilde{f}(x, c^{-1}(y - g_1(x))) = f(x, y)$. If $f(x, y) \ge 0$ on $K(g_1, g_2)$ then for every $(x, z) \in \mathbb{R} \times [0, 1]$, we have

$\tilde{f}(x,z) = f(x, cz + g_1(x)) \geq 0$. By [15, Theorem 1.1], there exist $\tilde{r}_0(x,z), \tilde{r}_1(x,z) \in \sum \mathbb{R}[x,z]^2$ such that

$$\tilde{f}(x,z) = \tilde{r}_0(x,z) + \tilde{r}_1(x,z)z(1-z).$$

Set $r_0(x,y) := \tilde{r}_0(x, c^{-1}(y - g_1(x)))$ and $r_1(x,y) := c^{-2}\tilde{r}_1(x, c^{-1}(y - g_1(x)))$.
Then $r_0, r_1 \in \sum \mathbb{R}[x,y]^2$ and

$$\begin{aligned} f(x,y) &= \tilde{f}(x, c^{-1}(y - g_1(x))) \\ &= \tilde{r}_0(x, c^{-1}(y - g_1(x))) + \tilde{r}_1(x, c^{-1}(y - g_1(x)))c^{-1}[y - g_1(x)][1 - c^{-1}(y - g_1(x))] \\ &= r_0(x,y) + r_1(x,y)[y - g_1(x)][g_2(x) - y]. \end{aligned}$$

□

Note that if the leading coefficient of $g_2 - g_1$ in Lemma 3.1 is *negative,* then $K(g_1, g_2)$ is compact when the degree of $g_2 - g_1$ is even and comtains an open cone otherwise. In the case the degree of $g_2 - g_1$ is odd, replacing x by $-x$ and y by $-y$, we can assume the leading coefficient of $g_2 - g_1$ is positive.

In the polynomial ring $\mathbb{R}[x,y]$, since $y = y^2 + y(1-y)$ and $1 - y = (1-y)^2 + y(1-y)$, the preordering $T(y, 1-y)$ is the same as the preordering $T(y(1-y))$ and so is equal to the quadratic module $\mathcal{M}(y(1-y))$. Furthermore, let $U = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k]$, where $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$. Then $T(S) = \mathcal{M}(S)$, where $S = \{y - a_1, (y - a_2)(y - b_1), \ldots, (y - a_k)(y - b_{k-1}), y - b_k\}$ (see [16, Proposition 1]). Nguyen and Power [16] extended [15, Theorem 1.1] on $\mathbb{R} \times U$ and on half strip $\mathbb{R}_a \times U$, where a is a nonnegative number and $\mathbb{R}_a$ is the set $\{x \in \mathbb{R} \mid x \geq a\}$. Precisely, the preorderings $T(S)$ and $T(S \cup \{x - a\})$ is saturated [16, Theorem 2, Theorem 4]. Using [16, Theorem 2, Theorem 4] instead of [15, Theorem 1.1] and make the change of variables ($z = y - g(x)$) as in the proof of Lemma 3.1, we obtain the following:

Corollary 3.1. *Let $g(x)$ be a single variable polynomial, $0 \leq c \in \mathbb{R}$ and $a_1, b_1, \ldots, a_k, b_k$ be real numbers with $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$. Then the preorderings $\mathcal{M}(S)$ and $\mathcal{M}(S \cup \{x - c\})$ (in the ring $\mathbb{R}[x,y]$) are saturated, where $S = \{y - g(x) - a_1, (y - g(x) - a_2)(y - g(x) - b_1), \ldots, (y - g(x) - a_k)(y - g(x) - b_{k-1}), y - g(x) - b_k\}$.*

In the case $\alpha > 0$, some further conditions are added and we obtain two following lemmas.

Lemma 3.2. *Let g_1, g_2 and $K(g_1, g_2, \alpha)$ be as in Definition 2.2. Assume that $g_2 - g_1 = c > 0$ and $g_1(0) < 0 < g_2(0)$. Then the quadratic module $\mathcal{M}([x^\alpha y - g_1(x)][g_2(x) - x^\alpha y])$ is saturated.*

Proof. We write

$$K(g_1, g_2, \alpha) = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x^\alpha y - g_1(x) \leq c\}.$$

Observe that if $z = c^{-1}(x^\alpha y - g_1(x))$, then $y = \dfrac{cz + g_1(x)}{x^\alpha}$ for all $x \neq 0$. So, we have

$$\forall x \neq 0 : (x, y) \in K(g_1, g_2, \alpha) \Leftrightarrow (x, z) \in \mathbb{R} \times [0, 1].$$

Take any two variable polynomial $f(x, y)$ which is non-negative on $K(g_1, g_2, \alpha)$. Let's define $\tilde{f}(x, z) := f(x, \dfrac{cz + g_1(x)}{x^\alpha})\ \forall x \neq 0$ then $\tilde{f}(x, c^{-1}(x^\alpha y - g_1(x))) = f(x, y)$. Hence, for every $(x, z) \in \mathbb{R}^* \times [0, 1]$, we have $\tilde{f}(x, z) = f(x, \dfrac{cz + g_1(x)}{x^\alpha}) \geq 0$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. There exists $k \in \mathbb{Z}_{\geq 0}$ such that $x^{2k}\tilde{f}(x, z)$ becomes a polynomial in $\mathbb{R}[x, z]$ and $x^{2k}\tilde{f}(x, z) \geq 0$ on $\mathbb{R}^* \times [0, 1]$. Since the density of $\mathbb{R}^* \times [0, 1]$ in $\mathbb{R} \times [0, 1]$ and the continuity of the polynomial $x^{2k}\tilde{f}(x, z)$, we have $x^{2k}\tilde{f}(x, z) \geq 0$ on $\mathbb{R} \times [0, 1]$.

By [15, Theorem 1.1], there exist $\tilde{r}_0(x, z), \tilde{r}_1(x, z) \in \sum \mathbb{R}[x, z]^2$ such that

$$x^{2k}\tilde{f}(x, z) = \tilde{r}_0(x, z) + \tilde{r}_1(x, z)z(1 - z).$$

Set $r_0(x, y) := \tilde{r}_0(x, c^{-1}(x^\alpha y - g_1(x)))$ and $r_1(x, y) := c^{-2}\tilde{r}_1(x, c^{-1}(x^\alpha y - g_1(x)))$. Then $r_0(x, y), r_1(x, y) \in \sum \mathbb{R}[x, y]^2$ and

$$\begin{aligned} x^{2k}f(x, y) &= x^{2k}\tilde{f}(x, c^{-1}(x^\alpha y - g_1(x))) \\ &= r_0(x, y) + r_1(x, y)[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y]. \end{aligned}$$

If $k \neq 0$ then let $x = 0$, we get $r_0(0, y) + r_1(0, y)[-g_1(0)]g_2(0) = 0\ \forall y \in \mathbb{R}$. By the assumption $g_1(0) < 0 < g_2(0)$, we have $r_0(0, y) = 0, r_1(0, y) = 0\ \forall y \in \mathbb{R}$. So $r_0(x, y) = x^2\bar{r}_0(x, y);\ r_1(x, y) = x^2\bar{r}_1(x, y)$ and therefore

$$x^{2k-2}f(x, y) = \bar{r}_0(x, y) + \bar{r}_1(x, y)[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y].$$

Repeat this procedure, we obtain the following presentation.

$$f(x, y) = s_0(x, y) + s_1(x, y)[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y],$$

where $s_0, s_1 \in \sum \mathbb{R}[x, y]^2$. □

Lemma 3.3. *Let $g_1(x), g_2(x)$ be single variable polynomials and α is a positive integer number. If the degree of $g_2(x) - g_1(x)$ is at least one, the leading coefficient of $(g_2 - g_1)$ is positive and $g_1(0) < g_2(0)$ then there exist a polynomial $f(x, y) \in \mathbb{R}[x, y]$ which is positive on $K(g_1, g_2, \alpha)$ does not belong to the preordering generated by $x^\alpha y - g_1(x), g_2(x) - x^\alpha y$.*

Proof. Make the change of variables $z = x^\alpha y$, we have

$$(x, y) \in K(g_1, g_2, \alpha) \Longrightarrow (x, z) \in K(g_1, g_2, 0) = \{(x, z) \in \mathbb{R}^2 \mid g_1(x) \leq z \leq g_2(x)\}$$

and

$$(x, z) \in K(g_1, g_2, 0) \setminus \{0 \times \mathbb{R}\} \Longrightarrow (x, y) \in K(g_1, g_2, \alpha).$$

By Lemma 3.1, there exist $\tilde{f}(x,z)$ which is positive on $K(g_1,g_2,0)$ does not belong to $T(z-g_1(x), g_2(x)-z)$. Set $f(x,y) := \tilde{f}(x, x^\alpha y)$. We will show that $f(x,y)$ is positive on $K(g_1,g_2,\alpha)$ but $f(x,y)$ does not belong to $T(x^\alpha y - g_1(x), g_2(x) - x^\alpha y)$.

Since $\tilde{f}(x,z) > 0$ on $K(g_1,g_2,0)$, we have $f(x,y) = \tilde{f}(x,x^\alpha y) > 0$ on $K(g_1,g_2,\alpha)$. We suppose that $f(x,y)$ in $T(x^\alpha y - g_1(x), g_2(x) - x^\alpha y)$, that is

$$f(x,y) = r_0(x,y) + r_1(x,y)[x^\alpha y - g_1(x)] + r_2(x,y)[g_2(x) - x^\alpha y] \\ + r_3(x,y)[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y],$$

where $r_i(x,y) \in \sum \mathbb{R}[x,y]^2;\ i = 1,2,3$. Hence, for all $(x,z) \in \mathbb{R}^2, x \neq 0$, we have

$$\tilde{f}(x,z) = f(x, \frac{z}{x^\alpha}) = r_0(x, \frac{z}{x^\alpha}) + r_1(x, \frac{z}{x^\alpha})[z - g_1(x)] + r_2(x, \frac{z}{x^\alpha})[g_2(x) - z] \\ + r_3(x, \frac{z}{x^\alpha})[z - g_1(x)][g_2(x) - z]. \tag{2}$$

So, there exists $k \in \mathbb{Z}_{\geq 0}$ such that $x^{2k} r_i(x, \frac{z}{x^\alpha}) = \tilde{r}_i(x,z)$, where $\tilde{r}_i(x,z) \in \sum \mathbb{R}[x,z]^2;\ i = 1,2,3$ and

$$x^{2k}\tilde{f}(x,z) = \tilde{r}_0(x,z) + \tilde{r}_1(x,z)[z - g_1(x)] + \tilde{r}_2(x,z)[g_2(x) - z] \\ + \tilde{r}_3(x,z)[z - g_1(x)][g_2(x) - z]. \tag{3}$$

By the equality (2) is true for all $x \neq 0$, the equality (3) is also true for all $x \neq 0$. However, the equality (3) holds on $\mathbb{R}^2$ since the continuity of $x^{2k}\tilde{f}(x,z)$ and the density of $\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}$ in $\mathbb{R}^2$. Let $x = 0$, by (3), we have

$$0 = \tilde{r}_0(0,z) + \tilde{r}_1(0,z)[z - g_1(0)] + \tilde{r}_2(0,z)[g_2(0) - z] + \tilde{r}_3(0,z)[z - g_1(0)][g_2(0) - z]. \tag{4}$$

Since the equality (4) is true for all $z \in [g_1(0), g_2(0)]$, we have $\tilde{r}_i(0,z) = 0,\ i = 1,2,3$. So $\tilde{r}_i(x,z) = x^2 r'_i(x,z)$, where $r'_i(x,z) \in \sum \mathbb{R}[x,y]^2;\ i = 1,2,3$. Therefore,

$$x^{2k-2}\tilde{f}(x,z) = r'_0(x,z) + r'_1(x,z)[z - g_1(x)] + r'_2(x,z)[g_2(x) - z] \\ + r'_3(x,z)[z - g_1(x)][g_2(x) - z].$$

Repeat this procedure, we get

$$\tilde{f}(x,z) = s_0(x,z) + s_1(x,z)[z - g_1(x)] + s_2(x,z)[g_2(x) - z] \\ + s_3(x,z)[z - g_1(x)][g_2(x) - z],$$

where s_0, s_1, s_2, s_3 are sums of squares. That is $\tilde{f}(x,z) \in T(z - g_1(x), g_2(x) - z)$, we get a contradiction.

□

Let us denote by $w(x) = g_2(x) - g_1(x)$ and call it the *width* of $K(g_1,g_2,\alpha)$. In particular, the strip $K(0,1,0) = \mathbb{R} \times [0,1]$ has the width $w(x) = 1$. Since $w(x)$ is a polynomial, $\lim_{x\to\infty} w(x)$

is either infinite or constant. The width of $K(x^2, x^3, 0) = \{(x, y) \in \mathbb{R}^2 \colon x^2 \leq y \leq x^3\}$ is $w(x) = x^3 - x^2$ and $\lim\limits_{x\to+\infty} w(x) = \infty$. This set can not be enclosed in any strip with arbitrary width. We see later that the preordering $T(y - x^2, x^3 - y)$ of $K(x^2, x^3, 0)$ is not saturated.

Theorem 3.1. *Let $g_1(x), g_2(x), \alpha$ and $K(g_1, g_2, \alpha)$ be as in Definition 2.2. The following statements hold.*

(1) *Suppose that either $\lim\limits_{x\to+\infty} w(x) = +\infty$ or $\lim\limits_{x\to-\infty} w(x) = +\infty$. Then there exist a polynomial $f(x, y) \in \mathbb{R}[x, y]$ which is positive on $K(g_1, g_2, \alpha)$ does not belong to the preordering generated by $x^\alpha y - g_1(x), g_2(x) - x^\alpha y$ provided that $w(0) > 0$.*

In the case $\alpha = 0$, the hypothesis '$w(0) > 0$' can be removed.

(2) *If $\lim\limits_{x\to\infty} w(x) = c > 0$ and $g_1(0) < 0 < g_2(0)$ then the quadratic module $\mathcal{M}([x^\alpha y - g_1(x)][g_2(x) - x^\alpha y])$ is saturated.*

In the case $\alpha = 0$, the hypothesis '$g_1(0) < 0 < g_2(0)$' can be remove.

Proof. (1) If $\lim\limits_{x\to+\infty} w(x) = +\infty$ then $deg(w) > 0$ and the leading coefficient of w is positive. So if $w(0) > 0$ then by Lemma 3.3 we get the conclusion. Similarly, in the case $\alpha = 0$, we apply Lemma 3.1 (1) instead of Lemma 3.3.

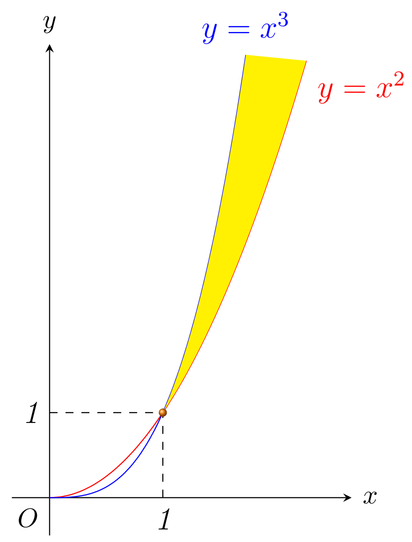
If $\lim\limits_{x\to-\infty} w(x) = +\infty$, we put $t = -x$ and $\bar{w}(t) = w(-t)$ then $\lim\limits_{t\to+\infty} \bar{w}(t) = +\infty$ and the problem becomes to the above case.

(2) If $\lim\limits_{x\to+\infty} w(x) = c > 0$ then $g_2 - g_1 \equiv c > 0$. Now we apply Lemma 3.2 and we get the proof. In the case $\alpha = 0$, we use Lemma 3.1 (2) instead of Lemma 3.2.

□

Example 3.1. *Consider the set*

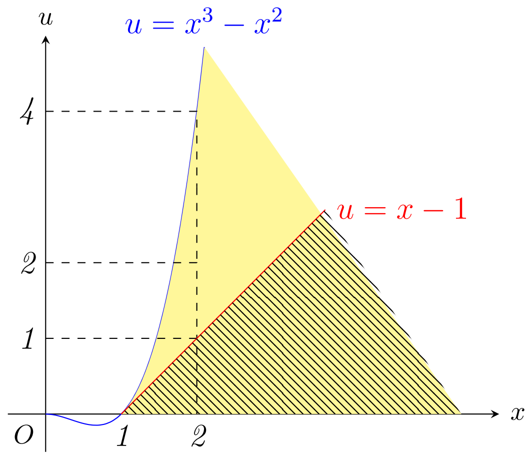
$$K = K(x^2, x^3, 0) = \left\{(x, y) \in \mathbb{R}^2 \colon x^2 \leq y \leq x^3\right\}.$$



According the above theorem, there exists the polynomial $f \geq 0$ *on* K *but* f *does not belong to the preordering* $T(x^3 - y, y - x^2)$.

We know that if a semi-algebraic set S *contains a open cone then the Positivstellensatz fails. However, in this case,* K *dose not contain any open cone. Indeed, assume that* K *contains a convex cone* C*, then* K *contain a half line* d*. This is impossible.*

On the other hand, put $u = y - x^2$*, then the set* $\tilde{K} = \{(x,u) \in \mathbb{R}^2 \colon 0 \leq u \leq x^3 - x^2\}$ *contains the open cone* $\tilde{C} = \{(x,u) \in R^2 \colon 0 \leq u \leq x - 1; x \geq 1\}$



Example 3.2. *Let the set*

$$K(x^2, x^2 + x + 1, 2) = \{(x,y) \in \mathbb{R}^2 |\ x^2 \leq x^2 y \leq x^2 + x + 1\}.$$

We have $w(x) = x + 1$, $\lim\limits_{x \to +\infty} w(x) = +\infty$ *and* $w(0) = 1 > 0$*. Hence, by Theorem 3.1 (1), there exist a polynomial* $f(x,y) \in \mathbb{R}[x,y]$ *which is positive on* $K(x^2, x^2 + x + 1, 2)$ *does not belong to the preordering generated by* $x^2y - x^2, x^2 + x + 1 - x^2 y$.

Example 3.3. *Consider the set*

$$K(x - 1, x + 1, x^2 y) = \{(x,y) \in \mathbb{R}^2 |\ x - 1 \leq x^2 y \leq x + 1\}.$$

In this case, $w(x) = 2 > 0$, $g_1(0) = -1 < 0 < g_2(0) = 1$*. So, according Theorem 3.1 (2), the quadratic module* $\mathcal{M}(x^2 y - x + 1, x + 1 - x^2 y)$ *is saturated.*

3.2. **Matrix polynomials positive semidefinite on a generalized strip.** To obtain a noncommutative version of the Nichtnegativstellensatz on the generalized strip above, we first need to recall the diagonalization of symmetric matrix polynomials (see [26, Section 4.3]).

3.2.1. *Diagonalization of matrices with real polynomial entries.* Let $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ be a matrix polynomial which is positive definite on K. If $i = (i_1, \ldots, i_m)$ and $j = (j_1, \ldots, j_m)$ are m-tuples of positive integers such that $1 \leq i_1 < \cdots < i_m \leq d$ and $1 \leq j_1 < \cdots < j_m \leq d$, then $M_i^j = M_{i^j}(F)$ denotes the principal minor of F with rows i_k and columns j_k. If $i_1 = j_1 = 1, \dots, i_m = j_m = m$, we write M_m instead of M_i^j. Since F is positive definite on K, then $M_1, M_2, \dots, M_d$ are polynomials which are positive on K. Define two lower triangular $d \times d$ matrices $Y_\pm = (y_{ij}^\pm)$, where the entries given by the rational functions

$$\begin{aligned} y_{ij}^\pm &= \pm M_{(1,\dots,j-1,j)}^{(1,\dots,j-1,i)} M_j^{-1} \text{ for } j = 1, \dots, d,\ i = j+1, \dots, d, \\ y_{ii}^\pm &= 1 \text{ for } i = 1, \dots, d, \\ y_{jj} &= 0 \text{ otherwise.} \end{aligned}$$

For $a = (a_1, \dots, a_m)$, where $m \le d$, let $D(a) = D(a_1, \dots, a_m)$ denote the $d \times d$ diagonal matrix with diagonal entries $a_1, \dots, a_m, 0, \dots, 0$. Set $X_\pm = M_1 M_2 \dots M_{d-1} Y_\pm$ and $D = M_1 M_2 \dots M_{d-1} D(M_1, M_2 M_1^{-1}, \dots, M_d M_{d-1}^{-1})$. Then $X_\pm$ and D are matrices with polynomial entries. From $Y_+^{-1} = Y_-$ and $F = Y_+ D(M_1, M_2 M_1^{-1}, \dots, M_d M_{d-1}^{-1}) Y_+^t$, we obtain

$$X_+ X_- = X_- X_+ = (M_1 \dots M_{d-1})^2 I, \tag{5}$$

$$(M_1 \dots M_{d-1})^4 F = X_+ D X_+^t, \tag{6}$$

$$D = X_- F X_-^t. \tag{7}$$

Theorem 3.1 can be formulated for the noncommutative ring $\mathrm{Mat}_d(\mathbb{R}[x,y])$ in this subsubsection. We need the following result quoted from [26, Corollary 9]:

Lemma 3.4 ([26])**.** *Let $F \in \mathrm{Sym}_d(\mathbb{R}[x])$. Suppose that F is positive semidefinite on a set K. There exist a nonzero matrix polynomial $A \in \mathrm{Mat}_d(\mathbb{R}[x])$, a nonzero polynomial $h \in \mathbb{R}[x]$ and a diagonal matrix polynomial $D \in \mathrm{Mat}_d(\mathbb{R}[x])$ such that*

$$h^2 F = ADA^T,$$

where D is positive semidefinite on K. In particular, if F is positive definite on K then h can be chosen as $(M_1 M_2 \dots M_{d-1})^2$ and $D = M_1 M_2 \dots M_{d-1} D(M_1, M_2 M_1^{-1}, \dots, M_d M_{d-1}^{-1})$, where M_j is the (i,i)-principal minor of F.

Theorem 3.2. *Let $g_1(x), g_2(x), \alpha$ and $K(g_1, g_2, \alpha)$ be as in Definition 2.2. Suppose that $\lim_{x\to\infty} w(x) = c > 0$ and $g_1(0) < 0 < g_2(0)$. If $F(x,y)$ is a matrix polynomial in two variables which is positive semi-definite on $K(g_1, g_2, \alpha)$ there exist a nonzero polynomial $h \in \mathbb{R}[x,y]$ such that $h^2 F$ belongs to the quadratic module in $\mathrm{Mat}_d(\mathbb{R}[x,y])$ which is generated by $[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y]$. In particular, If $F(x,y)$ is positive definite on $K(g_1, g_2, \alpha)$ then h can be chosen as $(M_1 M_2 \dots M_{d-1})^2$.*

Proof. Since F is positive semi-definite $K(g_1, g_2, \alpha)$, the matrix polynomial F belongs to $\mathrm{Sym}_d(\mathbb{R}[x,y])$. By Lemma 3.4, there exist a nonzero matrix polynomial $A \in \mathrm{Mat}_d(\mathbb{R}[x,y])$, a nonzero polynomial $h \in \mathbb{R}[x, y]$ and a diagonal matrix polynomial $D \in \mathrm{Mat}_d(\mathbb{R}[x, y])$ such that

$$h^2 F = ADA^T.$$

Since D is diagonal, we have the spectral decomposition of D as below:

$$D(x, y) = f_1(x, y)E_{11} + f_2(x, y)E_{22} + \cdots + f_d(x, y)E_{nn},$$

where $f_1, \ldots, f_d$ are polynomials in $\mathbb{R}[x, y]$ and $E_{ii} \in \mathrm{Mat}_d(\mathbb{R})$ is the constant matrix whose (i, i)-entry is 1 and zero elsewhere for every $i = 1, 2, \ldots, d$. Since D is positive semi-definite on $K(g_1, g_2, \alpha)$, the polynomials $f_i(x, y)$ is nonnegative on $K(g_1, g_2, \alpha)$. Now, apply Theorem 3.1 for the polynomials $f_i(x, y)$, we imply that D belongs to the quadratic module $\mathcal{M}$ in $\mathrm{Mat}_d(\mathbb{R}[x, y])$ which is generated by $[x^\alpha y - g_1(x)][g_2(x) - x^\alpha y]$. Hence, $h^2 F \in \mathcal{M}$.

If F is positive definite on $K(g_1, g_2, \alpha)$, by Lemma 3.4, h can be chosen as $(M_1 M_2 \ldots M_{d-1})^2$ and $D = M_1 M_2 \ldots M_{d-1} D(M_1, M_2 M_1^{-1}, \ldots, M_d M_{d-1}^{-1})$, where M_j is the (i, i)-principal minor of F. This ends the proof. ☐

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