New Results on Proper Efficiency for a Class of Vector Optimization Problems*

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Abstract. This paper presents two new theorems on Geoffrion's properly efficient solutions and seven examples illustrating their applications to linear fractional vector optimization problems with unbounded constraint sets. Provided that all the components of the objective function are properly fractional, the first theorem gives sufficient conditions for the efficient solution set to coincide with the Geoffrion properly efficient solution set. Admitting that the objective function can have some affine components, in the second theorem we give sufficient conditions for an efficient solution to be a Geoffrion's properly efficient solution. The recession cone of the constraint set, the derivatives of the scalar objective functions, but no tangent cone to the constraint set at the efficient point, are used in the second theorem.

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1 Introduction

Linear fractional vector optimization problems (LFVOPs) were treated in a systematic way firstly by Choo and Atkins [5, 6]. As the authors observed in [6, pp. 250–251], *"Linear fractional functions are widely used as performance measures in many management situations, production planning and scheduling, educational administration and the analysis of financial enterprises and undertakings... Thus the multicriteria programming problems with linear fractional criterion functions (MPLF) are important and have potentially wide applications."* Other explanations of the applications of linear fractional functions and LFVOPs were given by Steuer [27, Chapter 9].

In linear fractional vector optimization, necessary and sufficient conditions for a feasible point to be an efficient solution and or a weakly efficient solution, interesting topological properties of the solution sets, stability properties, and solution methods can be found, respectively, in [5, 20, 24], [1, 6, 11, 12, 13, 16, 18, 28, 29, 30], [30, Section 5], and [24, 27].

Considering a general vector optimization problem with a standard ordering cone (the nonnegative orthant in an Euclidean space), Geoffrion [7] proposed the notion of properly efficient solution *to eliminate efficient solutions of a certain anomalous type.* Geoffrion's concept of proper efficiency has been widely recognized in vector optimization. It was developed by Borwein [3] and Benson [2]. Actually, Benson's concept, which coincides with that of Geoffrion in the case of problems with the standard ordering cones, is a true generalization of Geoffrion's concept. Later, Henig [8] and other scholars proposed additional concepts of properly efficient solutions which are centered around that one of Geoffrion and Benson.

It is a well known that each efficient solution of a linear vector optimization problem is properly efficient in the sense of Geoffrion (see [26, Corollary 3.1.1 and Theorem 3.1.4] and [17, Remark 2.4]). For LFVOPs with *bounded constraint sets*, by using the necessary and sufficient conditions for efficiency in those problems [5, Theorem 2.1 and Remark 2.1], Choo [4] proved that there is no difference between the efficiency and Geoffrion's proper efficiency.

Two natural questions arise: *There is a difference between the efficient set and the Geoffrion (resp., Borwein) properly efficient set of a LFVOP with an unbounded constraint set, or not? How to find a minimal (in a sense) set of sufficient conditions for an efficient solution of a LFVOP with an unbounded constraint set to be a Geoffrion's (resp., Borwein's) properly efficient point?*

Very recently, one theorem giving verifiable sufficient conditions for an efficient solution to belong to Borwein's properly efficient solution set has been established in [14]. The recession cone of the constraint set, the derivatives of the scalar objective functions, and the tangent cone of the constraint set at the point in question have been used. An example showing that Borwein's properly efficient solution set can be strictly larger the Geoffrion properly efficient solution set can be found in [14].

In our paper [17], by a direct approach via the recession cone of the constraint set and the derivatives of the scalar objective functions at the point in question, we obtained sufficient conditions for an efficient solution of a LFVOP with an unbounded constraint set to be a Geoffrion's properly efficient solution. Some arguments of Choo [4] were used in that paper. Later, based on a result of Benson [2], sufficient conditions for an efficient solution of a LFVOP with an unbounded constraint set to be a Geoffrion's properly efficient solution have been given in [15]. The conditions use the recession cone of the constraint set, the derivatives of the scalar objective functions, and the tangent cone to the constraint set at the efficient point.

In this paper, by combining the method used in [17] and the proof scheme of Choo [4], we obtain two new theorems on Geoffrion's properly efficient solutions and we will give seven examples to illustrate their applications to LFVOPs with unbounded constraint sets. Assuming that all the components of the objective function are properly fractional, in the first theorem we give sufficient conditions for the coincidence of the Geoffrion properly efficient solution set with the efficient solution set. In the second theorem, which provides sufficient conditions for an efficient solution to be a Geoffrion's properly efficient solution, it is admitted that the objective function can have some affine components. These sufficient conditions are based on the recession cone of the constraint set, the derivatives of the scalar objective functions, but make no use of the tangent cone to the constraint set at the efficient point. Both theorems are very different from the preceding results in [15, 17]. In fact, the results and their proofs shed a new light on the relationships between the Geoffrion properly efficient solution set and the efficient solution set of a LFVOP with an unbounded constraint set.

The variety of the sufficient conditions for Geoffrion's proper efficiency shows the difficulties in distinguishing the Geoffrion properly efficient solution set and the efficient solution set of a linear fractional vector optimization problem with unbounded constraint set. In this connection, three open questions were given in [17]. Other three open questions will be given at the end of this paper.

After giving some notations and definitions in Section 2, we establish the main results in Section 3. Then, we consider several illustrative examples and propose three open questions in Section 4.

2 Notations and Definitions

By $\mathbb{N}$ we denote the set of positive integers. The scalar product and the norm in $\mathbb{R}^n$ are denoted, respectively, by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Vectors in $\mathbb{R}^n$ are represented as rows of real numbers in the text, but they are understood as columns of real numbers in matrix calculations. If A is a matrix, then A^T stands for the transposed matrix. Thus, for any $x, y \in \mathbb{R}^n$, one has $\langle x, y \rangle = x^T y$. Let $\mathbb{R}^m_+$ denote the nonnegative orthant in $\mathbb{R}^m$, whose topological interior is abbreviated to $\mathrm{int}\,\mathbb{R}^m_+$.

A nonzero vector $v \in \mathbb{R}^n$ is said to be [25, p. 61] a *direction of recession* of a nonempty convex set $M \subset \mathbb{R}^n$ if $x + tv \in M$ for every $t \geq 0$ and every $x \in M$. The set composed by $0 \in \mathbb{R}^n$ and all the directions $v \in \mathbb{R}^n \setminus \{0\}$ satisfying the last condition, is called the *recession cone* of M and denoted by 0^+M. If M is closed and convex, then $0^+M = \{v \in \mathbb{R}^n \ : \ \exists x \in M \ \text{ s.t. } \ x + tv \in M \ \text{ for all } \ t > 0\}$.

The recession cone of a polyhedral convex set can be easily computed by using next lemma, which can be proved by a direct verification.

Lemma 2.1 *If $M = \big\{x \in \mathbb{R}^n \ : \ Cx \leq d\big\}$ with $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$, and M is nonempty, then $0^+M = \big\{v \in \mathbb{R}^n \ : \ Cv \leq 0\big\}$.*

We will need the following lemma in our proofs.

Lemma 2.2 (See [17, Lemma 2.5]) *Let $M \subset \mathbb{R}^n$ be closed and convex, $\bar{x} \in M$, and let $\{x^p\}$ be a sequence in $M \setminus \{\bar{x}\}$ with $\lim\limits_{p \to \infty} \|x^p\| = +\infty$. If $\lim\limits_{p \to \infty} \dfrac{x^p - \bar{x}}{\|x^p - \bar{x}\|} = v$, then $v \in 0^+M$.*

Consider *linear fractional functions* $f_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, m$, of the form

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i},$$

where $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Let K be a *polyhedral convex set*, i.e., there exist $p \in \mathbb{N}$, a matrix $C = (c_{ij}) \in \mathbb{R}^{p \times n}$, and a vector $d = (d_i) \in \mathbb{R}^p$ such that $K = \big\{x \in \mathbb{R}^n \ : \ Cx \leq d\big\}$. In the sequel, we always assume that K is nonempty.

Our standing assumption is that $b_i^T x + \beta_i > 0$ for all $i \in I$ and $x \in K$, where $I := \{1, \cdots, m\}$. Put $f(x) = (f_1(x), \ldots, f_m(x))$ and let

$$\Omega = \big\{x \in \mathbb{R}^n \ : \ b_i^T x + \beta_i > 0, \ \ \forall i \in I\big\}.$$

Clearly, Ω is open and convex, $K \subset \Omega$, and f is continuously differentiable on Ω. The *linear fractional vector optimization problem* given by f and K is formally written as

$$\text{(VP)} \qquad\qquad \text{Minimize } \ f(x) \ \ \text{subject to } \ x \in K.$$

Definition 2.3 A point $x \in K$ is said to be an *efficient solution* (or a *Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\mathbb{R}^m_+ \setminus \{0\}\big) = \emptyset$. One calls $x \in K$ a *weakly efficient solution* (or a *weak Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\operatorname{int}\mathbb{R}^m_+\big) = \emptyset$.

The efficient solution set (resp., the weakly efficient solution set) of (VP) are denoted, respectively, by E and E^w.

Theorem 2.4 (See [24] and [20, Theorem 8.1]) *For any $x \in K$, one has $x \in E$ (resp., $x \in E^w$) if and only if there exists a multiplier $\xi = (\xi_1, \dots, \xi_m) \in \operatorname{int}\mathbb{R}^m_+$ (resp., $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m_+ \setminus \{0\}$) such that*

$$\Big\langle \sum_{i=1}^{m} \xi_i \left[\big(b_i^T x + \beta_i\big)a_i - \big(a_i^T x + \alpha_i\big)b_i\right], y - x \Big\rangle \geq 0, \quad \forall y \in K. \tag{2.1}$$

If $b_i = 0$ and $\beta_i = 1$ for all $i \in I$, then (VP) coincides with the classical multi-objective linear optimization problem. By the above optimality conditions, for any $x \in K$, one has $x \in E$ (resp., $x \in E^w$) if and only if there exists a multiplier $\xi = (\xi_1, \dots, \xi_m) \in \operatorname{int}\mathbb{R}^m_+$ (resp., a multiplier $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m_+ \setminus \{0\}$) such that

$$\left\langle \sum_{i=1}^{m} \xi_i a_i, y - x \right\rangle \geq 0, \quad \forall y \in K. \tag{2.2}$$

The following property will be used in our investigations.

Lemma 2.5 (See, e.g., [24] and [20, Lemma 8.1]) *Let $\varphi(x) = \dfrac{a^T x + \alpha}{b^T x + \beta}$ be a linear fractional function defined by $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Suppose that $b^T x + \beta \neq 0$ for every $x \in K_0$, where $K_0 \subset \mathbb{R}^n$ is an arbitrary polyhedral convex set. Then, one has*

$$\varphi(y) - \varphi(x) = \frac{b^T x + \beta}{b^T y + \beta}\,\langle \nabla\varphi(x), y - x\rangle, \tag{2.3}$$

for any $x, y \in K_0$, where $\nabla\varphi(x)$ denotes the Fréchet derivative of φ at x.

The connectedness of K_0 and the condition $b^T x + \beta \neq 0$ for every $x \in K_0$ imply that either $b^T x + \beta > 0$ for all $x \in K_0$, or $b^T x + \beta < 0$ for all $x \in K_0$. Hence, for any $x, y \in K_0$, one has $\dfrac{b^T x + \beta}{b^T y + \beta} > 0$. Given vectors $x, y \in K_0$ with $x \neq y$, we consider two points from the line segment $[x, y]$:

$$z_t = x + t(y - x), \quad z_{t'} = x + t'(y - x) \quad (t \in [0, 1],\ t' \in [0, t)).$$

By (2.3) we can assert that

(i) If $\langle \nabla\varphi(x), y-x\rangle > 0$, then $\varphi(z_{t'}) < \varphi(z_t)$ for every $t' \in [0,t)$;

(ii) If $\langle \nabla\varphi(x), y-x\rangle < 0$, then $\varphi(z_{t'}) > \varphi(z_t)$ for every $t' \in [0,t)$;

(iii) If $\langle \nabla\varphi(x), y-x\rangle = 0$, then $\varphi(z_{t'}) = \varphi(z_t)$ for every $t' \in [0,t)$.

This shows that φ is monotonic on every line segment or ray contained in K_0.

Definition 2.6 (See [7, p. 618]) One says that $\bar{x} \in E$ is a *Geoffrion's properly efficient solution* of (VP) if there exists a scalar $M > 0$ such that, for each $i \in I$, whenever $x \in K$ and $f_i(x) < f_i(\bar{x})$ one can find an index $j \in I$ such that $f_j(x) > f_j(\bar{x})$ and $A_{i,j}(\bar{x}, x) \le M$ with $A_{i,j}(\bar{x}, x) := \dfrac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})}$.

The Geoffrion properly efficient solution set of (VP) is denoted by E^{Ge}. *For any $\bar{x} \in E$, $\bar{x} \notin E^{Ge}$ if and only if for every scalar $M > 0$ there exist $x \in K$ and $i \in I$ with $f_i(x) < f_i(\bar{x})$ such that, for all $j \in I$ satisfying $f_j(x) > f_j(\bar{x})$, one has $A_{i,j}(\bar{x}, x) > M$.* This observation has been made in [7, p. 619] for general vector optimization problems.

3 Sufficient Conditions for the Geoffrion Proper Efficiency

In this section, we will establish two new theorems on the Geoffrion proper efficiency for LFVOPs. The first one is obtained by combining the approach of [17] with some arguments of Choo [4]. By this technique, we will also get the second result that deals with the case where some components of the objective function are affine. Our results complement the result of Choo [4], as well as Theorems 3.1 and 3.3 in [17], and Theorem 3.1 in [15].

The following lemma clarifies some assumptions that will be used later on.

Lemma 3.1 *For any $i \in I$ and $v \in 0^+K$, one has $b_i^T v \ge 0$.*

Proof. If the assertion was false, then there would exist an index $i \in I$ and a vector $v \in (0^+K) \setminus \{0\}$ satisfying $b_i^T v < 0$. Fixing a point $u \in K$, one has $u + tv \in K$ for all $t > 0$. So, for $t > 0$ large enough, one has $b_i^T(u+tv) + \beta_i < 0$. This contradicts the condition $b_i^T x + \beta_i > 0$ for all $x \in K$. Our assertion is proved. □

The assumption of the forthcoming theorem is a strengthened form of the property described by Lemma 3.1. If K is unbounded, then the assumption implies that $b_k \neq 0$ for all $k \in I$, i.e., *all* the denominators of the components $f_k(x)$, $k \in I$, of the objective function of (VP) are not real constants.

Theorem 3.2 *If one has $b_k^T v > 0$ for any $k \in I$ and $v \in (0^+K) \setminus \{0\}$, then every efficient solution of* (VP) *is a properly efficient solution in the sense of Geoffrion, i.e., $E = E^{Ge}$.*

Proof. (This proof is based on some arguments of the proof of Theorem 3.1 in [17] and the proof of the main result of [4].) Suppose on the contrary that $b_i^T v > 0$ for any $i \in I$ and $v \in (0^+K) \setminus \{0\}$, but there is a point $\bar{x} \in E$ which does not belong to E^{Ge}. Then, for every $p \in \mathbb{N}$, there exist $x^p \in K$ and $i(p) \in I$ with $f_{i(p)}(x^p) < f_{i(p)}(\bar{x})$ such that, for all $j \in I$ satisfying $f_j(x^p) > f_j(\bar{x})$, one has $A_{i(p),j}(\bar{x}, x^p) > p$. Since the sequence $\{i(p)\}$ has values in the finite set I, by working with a subsequence (if necessary), we may assume that $i(p) = i$ for all p, where $i \in I$ is a fixed index. For each p, as $\bar{x} \in E$ and $f_i(x^p) < f_i(\bar{x})$, the set $J(p) := \{j \in I : f_j(x^p) > f_j(\bar{x})\}$ is nonempty. Since $J(p) \subset I \setminus \{i\}$ for all p, by applying the Dirichlet principle and considering a subsequence, we may assume that $J(p) = J$ for all p, where J is a nonempty subset of $I \setminus \{i\}$. Thus,

$$A_{i,j}(\bar{x}, x^p) = \frac{f_i(\bar{x}) - f_i(x^p)}{f_j(x^p) - f_j(\bar{x})} > p \quad \forall p \in \mathbb{N}, \forall j \in J. \tag{3.1}$$

Hence, for every $j \in J$, one has $\lim\limits_{p\to\infty} A_{i,j}(\bar{x}, x^p) = +\infty$.

Put $v^p = \dfrac{x^p - \bar{x}}{\|x^p - \bar{x}\|}$. Without loss of generality, we may suppose that $\lim\limits_{p\to\infty} v^p = v$, where v is a unit vector. According to Lemma 2.5,

$$f_i(x^p) - f_i(\bar{x}) = \|x^p - \bar{x}\| \frac{b_i^T \bar{x} + \beta_i}{b_i^T x^p + \beta_i} \langle \nabla f_i(\bar{x}), v^p \rangle. \tag{3.2}$$

Similarly,

$$f_j(x^p) - f_j(\bar{x}) = \|x^p - \bar{x}\| \frac{b_j^T \bar{x} + \beta_j}{b_j^T x^p + \beta_j} \langle \nabla f_j(\bar{x}), v^p \rangle. \tag{3.3}$$

Hence, for all $j \in J$, we have

$$A_{i,j}(\bar{x}, x^p) = -\frac{b_i^T \bar{x} + \beta_i}{b_j^T \bar{x} + \beta_j} \cdot \frac{b_j^T x^p + \beta_j}{b_i^T x^p + \beta_i} \cdot \frac{\langle \nabla f_i(\bar{x}), v^p \rangle}{\langle \nabla f_j(\bar{x}), v^p \rangle}.$$

Since $b_k^T \bar{x} + \beta_k > 0$ for every $k \in I$, we have $\lim\limits_{p\to\infty} A_{i,j}(\bar{x}, x^p) = +\infty$ if and only if the quantity

$$\bar{A}_{i,j}(\bar{x}, x^p) := -\frac{b_j^T x^p + \beta_j}{b_i^T x^p + \beta_i} \cdot \frac{\langle \nabla f_i(\bar{x}), v^p \rangle}{\langle \nabla f_j(\bar{x}), v^p \rangle}$$

tends to $+\infty$ as $p \to \infty$ for every $j \in J$.

First, suppose that $\{x^p\}$ is bounded. Then, for each $j \in J$, the assumption $b_k^T x + \beta_k > 0$ for all $k \in I$ and $x \in K$ implies that there exist positive constants $\gamma_{i,j}^1$ and $\gamma_{i,j}^2$ satisfying

$$\gamma_{i,j}^1 \leq \frac{b_j^T x^p + \beta_j}{b_i^T x^p + \beta_i} \leq \gamma_{i,j}^2 \quad (\forall p \in \mathbb{N}).$$

So, for every $j \in J$, one has $\lim\limits_{p\to\infty} \bar{A}_{i,j}(\bar{x}, x^p) = +\infty$ if and only if

$$\lim_{p\to\infty} \frac{\langle \nabla f_j(\bar{x}), v^p\rangle}{\langle \nabla f_i(\bar{x}), v^p\rangle} = 0. \tag{3.4}$$

As $f_i(\bar{x}) - f_i(x^p) > 0$ for every p, expression (3.2) implies that $\langle \nabla f_i(\bar{x}), v^p\rangle < 0$.

Due to the construction of the index set J at the beginning of this proof, for every $k \notin J$, one has $f_k(x^p) \leq f_k(\bar{x})$ for all $p \in \mathbb{N}$. So, by Lemma 2.5,

$$\langle \nabla f_k(\bar{x}), v^p\rangle \leq 0, \quad k \notin J. \tag{3.5}$$

Since $\bar{x} \in E$, by Theorem 2.4 there exist $\xi_k > 0$, $k \in I$, such that

$$\left\langle \sum_{k\in I} \xi_k \left[\big(b_k^T\bar{x} + \beta_k\big)a_k - \big(a_k^T\bar{x} + \alpha_k\big)b_k\right], y - \bar{x}\right\rangle \geq 0, \quad \forall y \in K.$$

For each p, substituting $y = x^p$ to the last inequality and dividing both sides of the obtained inequality by $\|x^p - \bar{x}\|$, we get

$$\left\langle \sum_{k\in I} \xi_k \left[\big(b_k^T\bar{x} + \beta_k\big)a_k - \big(a_k^T\bar{x} + \alpha_k\big)b_k\right], v^p\right\rangle \geq 0. \tag{3.6}$$

Put $\lambda_k = \xi_k(b_k^T\bar{x} + \beta_k)^2$ for every $k \in I$. One has $\lambda_k > 0$ for all $k \in I$. Since

$$\nabla f_k(\bar{x}) = \frac{\big(b_k^T\bar{x} + \beta_k\big)a_k - \big(a_k^T\bar{x} + \alpha_k\big)b_k}{(b_k^T\bar{x} + \beta_k)^2} \qquad (\forall k \in I),$$

from (3.6) we can deduce that

$$\sum_{k\in I} \lambda_k \langle \nabla f_k(\bar{x}), v^p\rangle \geq 0. \tag{3.7}$$

As $\langle \nabla f_i(\bar{x}), v^p\rangle < 0$, this yields $0 \geq \dfrac{1}{\langle \nabla f_i(\bar{x}), v^p\rangle} \sum\limits_{k\in I} \lambda_k \langle \nabla f_k(\bar{x}), v^p\rangle$. So we have

$$0 \geq \lambda_i + \sum_{k\in I\setminus(J\cup\{i\})} \lambda_k \frac{\langle \nabla f_k(\bar{x}), v^p\rangle}{\langle \nabla f_i(\bar{x}), v^p\rangle} + \sum_{k\in J} \lambda_k \frac{\langle \nabla f_k(\bar{x}), v^p\rangle}{\langle \nabla f_i(\bar{x}), v^p\rangle} \tag{3.8}$$

for all $p \in \mathbb{N}$. On one hand, the second term of the sum in (3.8) is nonnegative by (3.5). On the other hand, (3.4) guarantees that the third term of the sum in (3.8) goes to 0 as p tends to ∞. Therefore, since $\lambda_i > 0$, (3.8) cannot hold for sufficiently large indexes p. We have arrived at a contradiction.

Now, consider the situation where $\{x^p\}$ is unbounded. By passing to a subsequence, we may assume that $\lim_{p\to\infty} \|x^p\| = +\infty$. Recall that $v^p = \dfrac{x^p - \bar{x}}{\|x^p - \bar{x}\|}$ and $\lim_{p\to\infty} v^p = v$. Since K is closed and convex, by Lemma 2.2 we have $v \in (0^+K) \setminus \{0\}$. Note that

$$\frac{b_j^T x^p + \beta_j}{\|x^p - \bar{x}\|} = \frac{b_j^T (x^p - \bar{x})}{\|x^p - \bar{x}\|} + \frac{\beta_j}{\|x^p - \bar{x}\|} + \frac{b_j^T \bar{x}}{\|x^p - \bar{x}\|} \qquad (\forall j \in J)$$

and

$$\frac{b_i^T x^p + \beta_i}{\|x^p - \bar{x}\|} = \frac{b_i^T (x^p - \bar{x})}{\|x^p - \bar{x}\|} + \frac{\beta_i}{\|x^p - \bar{x}\|} + \frac{b_i^T \bar{x}}{\|x^p - \bar{x}\|}.$$

Therefore, for every $j \in J$,

$$\bar{A}_{i,j}(\bar{x}, x^p) \;=\; -\frac{b_j^T v^p + \dfrac{\beta_j}{\|x^p - \bar{x}\|} + \dfrac{b_j^T \bar{x}}{\|x^p - \bar{x}\|}}{b_i^T v^p + \dfrac{\beta_i}{\|x^p - \bar{x}\|} + \dfrac{b_i^T \bar{x}}{\|x^p - \bar{x}\|}} . \frac{\langle \nabla f_i(\bar{x}), v^p\rangle}{\langle \nabla f_j(\bar{x}), v^p\rangle}. \tag{3.9}$$

By the assumption of the theorem, $b_k^T v \neq 0$ for all $k \in I$ and $v \in (0^+K) \setminus \{0\}$. As $\lim_{p\to\infty} \|x^p\| = +\infty$, one has $\lim_{p\to\infty} \|x^p - \bar{x}\| = +\infty$. Thus, from (3.9) it follows that $\lim_{p\to\infty} \bar{A}_{i,j}(\bar{x}, x^p) = +\infty$ if and only if (3.4) holds. So, repeating the arguments already used in the case where $\{x^p\}$ is bounded, we obtain a contradiction.

We have thus proved that $E = E^{Ge}$. $\square$

Corollary 3.3 (See [4, p. 218]) *If K is bounded, then $E = E^{Ge}$.*

Proof. If K is bounded, then the set $(0^+K) \setminus \{0\}$ is empty. Hence, the assumption of Theorem 3.2 is automatically satisfied. Therefore, thanks to that theorem, one has $E = E^{Ge}$. $\square$

We now consider the case where K is unbounded and some objective functions of (VP) may be linear (affine, to be more precise), i.e., we may have $f_i(x) = a_i^T x + \alpha_i$ for some $i \in I$. Let $I_1 := \{i \in I \,:\, b_i \neq 0\}$. Then, $b_i = 0$ and $\beta_i = 1$ for all $i \in I_0$, where $I_0 := I \setminus I_1$. In this case, we have the following result.

Theorem 3.4 *Assume that $\bar{x} \in E$. If*

$$\textit{For any } z \in (0^+K) \setminus \{0\}, \;\; b_i^T z > 0 \;\textit{ for all } i \in I_1, \tag{3.10}$$

and

$$\begin{cases} \textit{There is no } (i,j) \in I_0 \times I_1 \textit{ and } \ z \in (0^+K) \setminus \{0\} \textit{ such that} \\ a_i^T z \leq 0 \quad \textit{and} \quad \langle \nabla f_j(\bar{x}), z \rangle \geq 0, \end{cases} \tag{3.11}$$

then $\bar{x} \in E^{Ge}$.

Proof. Assume the fulfillment of the regularity assumptions (3.10) and (3.11). Arguing by contradiction, suppose that there exists $\bar{x} \in E$ with $\bar{x} \notin E^{Ge}$. Then, as it has been shown in the proof of Theorem 3.2, we would find a sequence $\{x^p\} \subset K \setminus \{\bar{x}\}$, an index $i \in I$, and a nonempty subset $J \subset I$, such that

(i) $f_i(x^p) < f_i(\bar{x})$ for each p;

(ii) $J = \{j \in I \, : \, f_j(x^p) > f_j(\bar{x})\}$ for each p;

(iii) For every $j \in J$, one has $\lim\limits_{p \to \infty} A_{i,j}(\bar{x}, x^p) = +\infty$, where $A_{i,j}(\bar{x}, x^p)$ has been defined in (3.1).

Put $v^p = \dfrac{x^p - \bar{x}}{\|x^p - \bar{x}\|}$. Without loss of generality, we may assume that $\lim\limits_{p \to \infty} v^p = v$ with $\|v\| = 1$.

For the efficient solution $\bar{x} \in E$, we construct the multipliers $\xi_k > 0$, the numbers $\lambda_k > 0$, $k \in I$, as in the proof of Theorem 3.2. Then, equality (3.7) holds for every $p \in \mathbb{N}$

If $\{x^p\}$ is bounded then, by the arguments given in the proof of Theorem 3.2, we can obtain a contradiction without relying on the conditions (3.10) and (3.11). Now, consider the case where $\{x^p\}$ is unbounded. Passing to a subsequence if necessary, we may assume that $\lim\limits_{p \to \infty} \|x^p\| = +\infty$. Then, by Lemma 2.2 we have $v \in (0^+K) \setminus \{0\}$. There are two cases: $i \in I_0$, or $i \in I_1$. Note that $\lim\limits_{p \to \infty} \|x^p - \bar{x}\| = +\infty$ and

$$\lim_{p \to \infty} \frac{b_i^T \bar{x} + \beta_i}{\|x^p - \bar{x}\|} = 0 \quad (\forall i \in I_1). \tag{3.12}$$

Case 1: *One has* $i \in I_0$. Then,

$$f_i(\bar{x}) - f_i(x^p) = -\langle a_i, x^p - \bar{x} \rangle = -\|x^p - \bar{x}\| \big(a_i^T v^p\big).$$

This and the above property (i) imply that

$$a_i^T v^p < 0 \quad (\forall p \in \mathbb{N}). \tag{3.13}$$

For each $j \in J$, by (iii) one has $\lim\limits_{p \to \infty} A_{i,j}(\bar{x}, x^p) = +\infty$. If $j \in J \cap I_0$, then

$$f_j(x^p) - f_j(\bar{x}) = \|x^p - \bar{x}\| \big(a_j^T v^p\big). \tag{3.14}$$

Hence, for every $j \in I_0 \cap J$,

$$\bar{A}_{i,j}(\bar{x}, x^p) = -\frac{f_i(x^p) - f_i(\bar{x})}{f_j(x^p) - f_j(\bar{x})} = -\frac{a_i^T v^p}{a_j^T v^p}.$$

So, $A_{i,j}(\bar{x}, x^p)$ tends to $+\infty$ as $p \to \infty$ only if

$$\lim_{p\to\infty} \frac{a_j^T v^p}{a_i^T v^p} = 0. \tag{3.15}$$

The situation $j \in J \cap I_1$ cannot occur, i.e., $J \subset I_0$. Indeed, if there exists some $j \in J \cap I_1$, then the above property (ii) and (3.3) yield $\langle \nabla f_j(\bar{x}), v^p\rangle > 0$ for every p. So, we have $\langle \nabla f_j(\bar{x}), v\rangle \geq 0$. From (3.13) it follows that $a_i^T v \leq 0$. Thus, for the pair $(i, j) \in I_0 \times I_1$ and the vector $v \in (0^+K) \setminus \{0\}$ under consideration, it holds that $a_i^T v \leq 0$ and $\langle \nabla f_j(\bar{x}), v\rangle \geq 0$. This contradicts (3.11).

Due to the construction of the index set J at the beginning of this proof (see property (ii)), for every $k \in I \setminus J$, one has $f_k(x^p) \leq f_k(\bar{x})$ for all $p \in \mathbb{N}$. So, invoking Lemma 2.5 we can assert that

$$\langle \nabla f_k(\bar{x}), v^p\rangle \leq 0, \quad (\forall k \in I \setminus J,\ \forall p \in \mathbb{N}). \tag{3.16}$$

By (3.7) we have

$$\lambda_i\big(a_i^T v^p\big) + \sum_{k\in I\setminus(J\cup\{i\})} \lambda_k\ \langle \nabla f_k(\bar{x}), v^p\rangle + \sum_{k\in J} \lambda_k\big(a_k^T v^p\big) \geq 0.$$

Dividing both sides of this inequality by $a_i^T v^p$ and using (3.13) yield

$$\lambda_i + \sum_{k\in I\setminus(J\cup\{i\})} \lambda_k\, \frac{\langle \nabla f_k(\bar{x}), v^p\rangle}{a_i^T v^p} + \sum_{k\in J} \lambda_k\, \frac{a_k^T v^p}{a_i^T v^p} \leq 0 \tag{3.17}$$

for all $p \in \mathbb{N}$. The second term of the sum in (3.17) is nonnegative by (3.16) and (3.13). Meanwhile, the validity of (3.15) for all $j \in j$ implies that the third term of the sum in (3.17) goes to 0 as $p \to \infty$. Since $\lambda_i > 0$, this shows that the left-hand-side of (3.17) is positive if p is taken large enough. We have thus arrived at a contradiction.

CASE 2: *One has* $i \in I_1$. Let $j \in J$ be given arbitrarily.

If $j \in J \cap I_0$, then it follows from (3.2) and (3.14) that

$$\begin{aligned} A_{i,j}(\bar{x}, x^p) &= -\frac{\dfrac{b_i^T\bar{x} + \beta_i}{b_i^T x^p + \beta_i} \cdot \langle \nabla f_i(\bar{x}), v^p\rangle}{a_j^T v^p} \\ &= -\frac{\langle \nabla f_i(\bar{x}), v^p\rangle}{a_j^T v^p} \cdot \frac{\dfrac{b_i^T\bar{x} + \beta_i}{\|x^p - \bar{x}\|}}{b_i^T v^p + \dfrac{\beta_i}{\|x^p - \bar{x}\|} + \dfrac{b_i^T\bar{x}}{\|x^p - \bar{x}\|}} \end{aligned} \tag{3.18}$$

Since $i \in I_1$, by (3.10) we have $\lim\limits_{p\to\infty} b_i^T v^p = b_i^T v > 0$. Combining this with (3.12), we can assert that the last fraction in (3.18) tends to 0 as $p \to \infty$. By (3.18), the property $\lim\limits_{p\to\infty} A_{i,j}(\bar{x}, x^p) = +\infty$ forces $\lim\limits_{p\to\infty} \Big(-\dfrac{\langle \nabla f_i(\bar{x}), v^p\rangle}{a_j^T v^p}\Big) = +\infty$. It follows that

$$\lim_{p\to\infty} \frac{a_j^T v^p}{\langle \nabla f_i(\bar{x}), v^p\rangle} = 0 \quad (\forall j \in J \cap I_0). \tag{3.19}$$

If $j \in J \cap I_1$ then, by repeating the arguments in the proof of Theorem 3.2, we obtain the equality (3.4). Thus, we have

$$\lim_{p\to\infty} \frac{\langle \nabla f_j(\bar{x}), v^p\rangle}{\langle \nabla f_i(\bar{x}), v^p\rangle} = 0 \quad (\forall j \in J \cap I_1). \tag{3.20}$$

As before, one has $\langle \nabla f_k(\bar{x}), v^p\rangle \le 0$, for all $p \in \mathbb{N}$ and for all $k \in I \setminus J$. Since $\langle \nabla f_i(\bar{x}), v^p\rangle < 0$, from (3.7) it follows that

$$\begin{aligned} 0 \ge \lambda_i \ &+ \sum_{k\in I\setminus(J\cup\{i\})} \lambda_k \frac{\langle \nabla f_k(\bar{x}), v^p\rangle}{\langle \nabla f_i(\bar{x}), v^p\rangle} \\ &+ \sum_{k\in J\cap I_0} \lambda_k \frac{a_k^T v^p}{\langle \nabla f_i(\bar{x}), v^p\rangle} + \sum_{k\in J\cap I_1} \lambda_k \frac{\langle \nabla f_k(\bar{x}), v^p\rangle}{\langle \nabla f_i(\bar{x}), v^p\rangle} \end{aligned} \tag{3.21}$$

for all $p \in \mathbb{N}$. Observe that the second term of the sum in (3.21) is nonnegative. In addition, thanks to (3.19) and (3.20), the third term of the sum in (3.21) tends to 0 as p goes to ∞. Since $\lambda_i > 0$, we conclude that (3.21) cannot hold for sufficiently large indexes p. Thus, we have arrived at a contradiction.

The proof of the theorem is complete. $\square$

Remark 3.5 Theorem 3.4 is a generalization of Theorem 3.2. Indeed, if $I_0 = \emptyset$, then (3.11) is fulfilled and $I_1 = I$. Since (3.10) is exactly the assumption of Theorem 3.2, our claim is justified.

If all the components of the objective function f are affine, then (VP) is the classical *linear vector optimization problem* (see, e.g., [22, Sec. 2 in Chap. 6] and [23]). Theorem 3.4 encompasses the following well-known result, which has an interesting application to vector variational inequalities with polyhedral constraint sets [21].

Corollary 3.6 (See, e.g., [26, Corollary 3.1.1 and Theorem 3.1.4] and [17, Remark 2.4]) *For a linear vector optimization problem, one has* $E = E^{Ge}$.

Proof. Consider the situation where (VP) is a linear vector optimization problem. Then, all the functions $f_i, i \in I$ are affine. Hence, $I_1 = \emptyset$. So, both regularity conditions (3.10) and (3.11) are automatically satisfied; and by Theorem 3.4 one has $E = E^{Ge}$. $\square$

4 Illustrative Examples

First, we apply Theorem 3.2 to three well-known examples. In comparison with the analysis given in [17], where other sufficient conditions for the Geoffrion proper efficiency were used, the checking of the equality $E = E^{Ge}$ herein is much simpler and easier.

Example 4.1 (See [6, Example 2]) Consider problem (VP) with

$$K = \big\{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 2,\ 0 \leq x_2 \leq 4\big\},$$

$$f_1(x) = \frac{-x_1}{x_1 + x_2 - 1}, \quad f_2(x) = \frac{-x_1}{x_1 - x_2 + 3}.$$

The fact that $E = E^w = \big\{(x_1, 0) \,:\, x_1 \geq 2\} \cup \{(x_1, 4) \,:\, x_1 \geq 2\big\}$ can be verified by using Theorem 2.4. Since $0^+K = \{v = (v_1, 0) \,:\, v_1 \geq 0\}$, $b_1 = (1, 1)$, and $b_2 = (1, -1)$, we have $b_1^T v = b_2^T v = v_1 > 0$ for any $v \in (0^+K) \setminus \{0\}$. Thus, by Theorem 3.2, the equality $E = E^{Ge}$ holds.

Example 4.2 (See [12, p. 483]) Consider problem (VP) where $n = m = 3$,

$$\begin{aligned} K = \big\{x \in \mathbb{R}^3 \,:\, & x_1 + x_2 - 2x_3 \leq 1,\ x_1 - 2x_2 + x_3 \leq 1, \\ & -2x_1 + x_2 + x_3 \leq 1,\ x_1 + x_2 + x_3 \geq 1\big\}, \end{aligned}$$

and

$$f_i(x) = \frac{-x_i + \dfrac{1}{2}}{x_1 + x_2 + x_3 - \dfrac{3}{4}} \qquad (i = 1, 2, 3).$$

In [12], the authors have proved that

$$\begin{aligned} E = E^w = \ & \{(x_1, x_2, x_3) \,:\, x_1 \geq 1,\ x_3 = x_2 = x_1 - 1\} \\ & \cup \{(x_1, x_2, x_3) \,:\, x_2 \geq 1,\ x_3 = x_1 = x_2 - 1\} \\ & \cup \{(x_1, x_2, x_3) \,:\, x_3 \geq 1,\ x_2 = x_1 = x_3 - 1\}. \end{aligned}$$

Using Lemma 2.1, one can show that $0^+K = \{v = (\tau, \tau, \tau) \in \mathbb{R}^3 \,:\, \tau \geq 0\}$. Since $b_1 = b_2 = b_3 = (1, 1, 1)$, for any $i \in I$ and $v = (\tau, \tau, \tau) \in (0^+K) \setminus \{0\}$, one has $b_i^T v = 3\tau > 0$. Thus, by Theorem 3.2 we can assert that $E = E^{Ge}$.

The number of criteria in the following LFVOP can be any integer $m \geq 2$.

Example 4.3 (See [12, pp. 479–480]) Consider problem (VP) where $n = m$, $m \geq 2$,

$$K = \Big\{x \in \mathbb{R}^m \,:\, x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_m \geq 0,\ \sum_{k=1}^{m} x_k \geq 1\Big\},$$

and

$$f_i(x) = \frac{-x_i + \dfrac{1}{2}}{\displaystyle\sum_{k=1}^{m} x_k - \dfrac{3}{4}} \qquad (i = 1, \ldots, m).$$

According to [12, p. 483], one has

$$\begin{aligned} E = E^w = \quad & \{(x_1, 0, \ldots, 0) \ : \ x_1 \geq 1\} \\ & \cup\{(0, x_2, \ldots, 0) \ : \ x_2 \geq 1\} \\ & \ldots \ \ldots \ \ldots \\ & \cup\{(0, \ldots, 0, x_m) \ : \ x_m \geq 1\}. \end{aligned}$$

Since $0^+K = \mathbb{R}^m_+$ and $b_i = (1, 1, 1, \ldots, 1)$ for $i = 1, \ldots, m$, one has $b_i^T v > 0$ for all $i \in I$ and $v \in (0^+K) \setminus \{0\}$. So, by Theorem 3.2 we get $E = E^{Ge}$.

As Theorem 3.4 is a generalization of Theorem 3.2 (see Remark 3.5), we can apply it to the above three examples to show that $E = E^{Ge}$. The advantage of Theorem 3.4 is that it can treat problems with mixed objective criteria: both affine and non-affine functions are allowed.

If at least one of the two regularity assumptions in Theorem 3.4 is violated, then we may not have $\bar{x} \in E^{Ge}$. The forthcoming two examples will justify this claim.

Example 4.4 (See [17, Example 2.6]) Consider problem (VP) where $m = 3$, $n = 2$, $K = \mathbb{R}^2_+$, and

$$f_1(x) = -x_2, \quad f_2(x) = \frac{x_2}{x_1 + x_2 + 1}$$

for every $x = (x_1, x_2)$. In [17], we have shown that $E = \{(x_1, 0) \ : \ x_1 \geq 0\}$ and $E^{Ge} = \emptyset$. Clearly, $0^+K = K$, $I_0 = \{1\}$, $I_1 = \{2\}$, $a_1 = (0, -1)$, and $b_2 = (1, 1)$. For every vector $\bar{x} = (\bar{x}_1, 0)$ from E, one has $\nabla f_2(\bar{x}) = \Big(0, \dfrac{1}{\bar{x}_1 + 1}\Big)$. Condition (3.10) is fulfilled because $b_2^T v > 0$ for any $v \in (0^+K) \setminus \{0\}$. Meanwhile, choosing

$$(i, j) = (1, 2) \quad \text{and} \quad z = (1, 0) \in (0^+K) \setminus \{0\},$$

we have $a_i^T z = 0$ and $\langle \nabla f_j(\bar{x}), z\rangle = 0$. So, condition (3.11) is not fulfilled. The reason for $\bar{x} \notin E^{Ge}$ is that the two regularity assumptions in Theorem 3.4 do not hold simultaneously.

Example 4.5 (See [17, Example 4.7]) Consider problem (VP) with $m = 3$, $n = 2$, $K = \mathbb{R}^2_+$, and

$$f_1(x) = -x_1 - x_2, \quad f_2(x) = \frac{x_2}{x_1 + x_2 + 1}, \quad f_3(x) = x_1 - x_2$$

for every $x = (x_1, x_2)$. In [17], we have shown that

$$E = \{x = (x_1, x_2) \,:\, x_1 \geq 0,\ x_2 \geq 0,\ x_2 < x_1 + 1\},$$

and any efficient solution $\bar{x}$ of the form $\bar{x} = (\bar{x}_1, 0)$, $\bar{x}_1 \geq 0$, is improper in the sense of Geoffrion. Improving the last fact, the authors of [15] have proved that $E^{Ge} = \emptyset$. Let us check the conditions of Theorem 3.4. We have $0^+K = \mathbb{R}^2_+$, $I_0 = \{1, 3\}$, $I_1 = \{2\}$, $b_2 = (1, 1)$, and $a_1 = (-1, -1)$. So, condition (3.10) is satisfied. In addition, taking a point $\bar{x} = (\bar{x}_1, 0) \in E$, one has $\nabla f_2(\bar{x}) = \left(0, \ \dfrac{1}{\bar{x}_1 + 1}\right)$ and $\bar{x}_1 \geq 0$. By selecting $(i, j) = (1, 2) \in I_0 \times I_1$ and $v = (1, 1) \in (0^+K) \setminus \{0\}$, one gets $a_i^T v < 0$ and $\langle \nabla f_j(\bar{x}), v\rangle = \dfrac{1}{\bar{x}_1 + 1} > 0$. This means that condition (3.11) is violated.

In the forthcoming two examples, Theorem 3.2 (resp., Theorem 3.4) can be applied, but Theorems 3.1 and 3.3 in [17] cannot be used.

Example 4.6 Consider problem (VP) where $m = 2$, $n = 2$, $K = \mathbb{R}^2_+$, and

$$f_1(x) = \frac{x_1 - x_2}{x_1 + x_2 + 1}, \quad f_2(x) = \frac{x_2 - x_1}{x_1 + x_2 + 1}$$

for every $x = (x_1, x_2)$. Taking any $x \in K$ and applying Theorem 2.4 with $\xi_2 = \xi_1 = \frac{1}{2}$, we can assert that $x \in E$, because $\sum_{i=1}^{2} \xi_i \left[\big(b_i^T x + \beta_i\big)a_i - \big(a_i^T x + \alpha_i\big)b_i\right] = 0$. Since $0^+K = \mathbb{R}^2_+$ and $b_k = (1, 1)$ for all $k \in I$, Theorem 3.2 assures that $E = E^{Ge}$. Given any $\bar{x} = (\bar{x}_1, \bar{x}_2) \in E$, we claim that Theorem 3.1 in [17] cannot be used because the first regularity condition there is not satisfied. To justify the claim, observe that

$$\nabla f_1(\bar{x}) = \frac{1}{q(\bar{x})}\big(2\bar{x}_2 + 1, -2\bar{x}_1 - 1\big), \quad \nabla f_2(\bar{x}) = \frac{1}{q(\bar{x})}\big(-2\bar{x}_2 - 1, 2\bar{x}_1 + 1\big),$$

where $q(\bar{x}) := \big(\bar{x}_1 + \bar{x}_2 + 1\big)^2$. So, for $v_1 := \dfrac{2\bar{x}_1 + 1}{2\bar{x}_2 + 1}$ and $v_2 := 1$, one sees that $v = (v_1, v_2)$ belongs to $0^+K \setminus \{0\}$, but $\langle \nabla f_1(\bar{x}), v\rangle = 0$ and $\langle \nabla f_2(\bar{x}), v\rangle = 0$. Thus, the first regularity condition of [17, Theorem 3.1] is not fulfilled.

Example 4.7 Consider problem (VP) where $m = 3$, $n = 2$, $K = \mathbb{R}^2_+$, $f_1(x) = x_1 - x_2$, $f_2(x) = x_2$, and $f_3(x) = -x_1$ for every $x = (x_1, x_2)$. For any $x \in K$, applying Theorem 2.4 with $\xi_3 = \xi_2 = \xi_1 = \frac{1}{3}$ gives $x \in E$, because (2.1) collapses to (2.2) and $\sum_{i=1}^{3} \xi_i a_i = 0$. Since $I_1 = \emptyset$, the assumptions of Theorem 3.4 for any $\bar{x} \in E$. So, by that theorem, $E = E^{Ge}$. Choosing $(i, j, k) = (3, 1, 2) \in I^3$ and $v = (1, 1)$, one has $v \in (0^+K) \setminus \{0\}$, $\langle \nabla f_i(\bar{x}), v\rangle < 0$, $\langle \nabla f_j(\bar{x}), v\rangle = 0$, and $\langle \nabla f_k(\bar{x}), v\rangle > 0$. Hence, the third regularity condition of [17, Theorem 3.3] is not fulfilled. This shows that the latter cannot be used for the problem in this example.

Three open questions on Geoffrion's proper efficiency for LFVOPs have been stated in [17, Section 5]. We end this section with the following new open questions.

(Q1) *Does there exist a linear fractional vector optimization problem for which both inclusions in the expression* $\emptyset \subset E^{Ge} \subset E$ *are strict (i.e., Geoffrion's properly efficient solution set is nonempty, but it does not coincide with the efficient solution set)?*

(Q2) *Does there exist a linear fractional vector optimization problem whose Geoffrion properly efficient solution set is different from the efficient solution set, but disconnected?*

(Q3) *The Geoffrion properly efficient solution set is a semialgebraic set?*

Concerning (Q3), we observe that semialgebraic sets and some results from semialgebraic geometry were used in linear fractional vector optimization for the first time in [16]. Applications of the concept of semialgebraic set to polynomial vector optimization problems can be found in the papers by Huong *et al.* [10], Kim *et al.* [19], and Hieu [9].

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