

# THOM ISOTOPY THEOREM FOR NON PROPER MAPS AND COMPUTATION OF SETS OF STRATIFIED GENERALIZED CRITICAL VALUES

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ABSTRACT. Let  $X \subset \mathbb{C}^n$  be an affine variety and  $f : X \rightarrow \mathbb{C}^m$  be the restriction to  $X$  of a polynomial map  $\mathbb{C}^n \rightarrow \mathbb{C}^m$ . In this paper, we construct an affine Whitney stratification of  $X$ . The set  $K(f)$  of stratified generalized critical values of  $f$  can be also computed. We show that  $K(f)$  is a nowhere dense subset of  $\mathbb{C}^m$ , which contains the set  $B(f)$  of bifurcation values of  $f$  by proving a version of the Thom isotopy lemma for non-proper polynomial maps on singular varieties.

## 1. INTRODUCTION

Ehresmann’s fibration theorem [3] states that a proper smooth surjective submersion  $f : X \rightarrow N$  between smooth manifolds is a locally trivial fibration. With some extra assumptions, this result has been considered in different contexts.

Firstly, if we remove the assumption of properness or submersiveness, in general, Ehresmann’s fibration theorem does not hold since  $f$  might have “local singularities” or “singularities at infinity”. The set of points in  $N$  where  $f$  fails to be trivial, denoted by  $B(f)$ , is called the **bifurcation set** of  $f$ , which is the union of the set  $K_0(f)$  of **critical values** and the set  $B_\infty(f)$  of **bifurcation values at infinity** of  $f$ . So far, characterizing  $B_\infty(f)$  is still an open problem. In general, people use a bigger set (but easier to describe), the **set of asymptotic critical values of  $f$**  (see Definition 3.1), denoted by  $K_\infty(f)$ , to control  $B_\infty(f)$ . The set  $K_\infty(f)$  is always a nowhere dense subset of  $\mathbb{C}^m$  and it is a good approximation of the set  $B_\infty(f)$ . For a dominant map  $f : X \rightarrow \mathbb{C}^m$  on smooth complex affine variety  $X$ , the computation of  $K_\infty(f)$ , and hence of the **set of generalized critical values**,  $K(f) := K_0(f) \cup K_\infty(f)$ , is given in [8], [9], [10].

Now if  $X$  is singular, we need to partition  $X$  into disjoint smooth manifolds and then apply Ehresmann’s fibration theorem on each part. However, if we do not require any extra assumption, then the trivialization on the parts may not match. This obstacle can be overcome by introducing the Whitney conditions [21, 22]. Indeed, if  $f$  is proper and  $X$  admits a Whitney stratification, then  $f$  is locally trivial if it is a submersion on strata [18, 13, 20]. Moreover, if  $f$  is non proper and non submersive, we can also define the bifurcation set of  $f$  such that  $f$  is locally trivial outside  $B(f)$ .

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*Date:* February 23, 2019.

1991 *Mathematics Subject Classification.* Primary 32B20; Secondary 14P.

<sup>†</sup>This author is partially supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) grant 101.04-2017.12, <sup>‡</sup>This author is partially supported by the grant of Narodowe Centrum Nauki, grant number 2015/17/B/ST1/02637.

However, so far, to our knowledge, no connection between  $B(f)$  and the **set of stratified generalized critical values of  $f$** , defined by  $K(f, \mathcal{S}) := \bigcup_{X_\alpha \in \mathcal{S}} K(f, X_\alpha)$ , for a Whitney stratification  $\mathcal{S}$  of  $X$ , has been established. Here  $K(f, X_\alpha) = K_0(f, X_\alpha) \cup K_\infty(f, X_\alpha)$ , where  $K_0(f, X_\alpha)$  is the closure of the set of critical values of  $f|_{X_\alpha}$  and  $K_\infty(f, X_\alpha) = \{y \in \mathbb{C}^m : \text{there is a sequence } x^k \rightarrow \infty; x^k \in X_\alpha : \|x^k\| \nu(d_{x^k}(f|_{X_\alpha})) \rightarrow 0 \text{ and } f(x^k) \rightarrow y\}$  ( $\nu$  denotes the Rabier function, see section 5).

Let  $X \subset \mathbb{C}^n$  be a singular algebraic set of dimension  $n - r$  with  $I(X) = (g_1, \dots, g_\omega)$  and let  $f := (f_1, \dots, f_m) : X \rightarrow \mathbb{C}^m$  be a polynomial dominant map, i.e.,  $\overline{f(X)} = \mathbb{C}^m$ . Now restricting ourselves to the cases of dominant polynomial maps on singular affine varieties, the main goals of this paper are the following:

- Construct an affine Whitney stratification  $\mathcal{S}$  of  $X$ .
- Establish some version of the Thom isotopy lemma for  $f$  which yields the inclusion  $B(f) \subset K(f, \mathcal{S})$ .
- Calculate the set  $K(f, \mathcal{S})$  of stratified generalized critical values of  $f$ .

The paper is organized as follows. In Section 2, we recall the definitions of Whitney regularity and Whitney stratification, then we construct an affine stratification from a filtration of  $X$  by means of some hypersurfaces, and refine it to get an affine Whitney stratification. Some versions of the Thom isotopy lemma for non-proper polynomial maps (Theorem 3.1 and Corollary 3.1) will be given in Section 3. Then we compute the set of stratified generalized critical values of  $f$ , which contains the bifurcation values of  $f$ , where  $f := (f_1, \dots, f_m) : X \rightarrow \mathbb{C}^m$  is a polynomial dominant map, in the last Sections 4 and 5.

For the remainder of the paper, the differential of  $f$  at a point  $x$  is identified with its (row) matrix, so we write  $d_x f = \left( \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$ . Let

$$\nabla f(x) := \begin{bmatrix} \overline{\frac{\partial f}{\partial x_1}(x)} \\ \vdots \\ \overline{\frac{\partial f}{\partial x_n}(x)} \end{bmatrix},$$

the Hermitian transpose of  $d_x f$ . For  $v, w \in \mathbb{C}^n$ , denote by  $\langle v, w \rangle = \sum_{i=1}^n \bar{v}_i w_i$  the Hermitian product, and let  $v \cdot w = \sum_{i=1}^n v_i w_i$ . For a set  $A \subset \mathbb{C}^n$ , let  $\bar{A}$  and  $\bar{A}^Z$  denote respectively the topological closure and the Zariski closure of  $A$ . For an algebraic variety  $X$ , the singular part and the regular part of  $X$  are denoted respectively by  $\text{sing}(X)$  and  $\text{reg}(X)$ .

## 2. AFFINE WHITNEY STRATIFICATIONS

**2.1. Preliminaries.** For any two different points  $x, y \in \mathbb{C}^n$ , define the secant  $\overline{xy}$  to be the line passing through the origin which is parallel to the line through  $x$  and  $y$ .

A **stratification**  $\mathcal{S}$  of  $X$  is a decomposition of  $X$  into a locally finite disjoint union  $X = \bigsqcup_{\alpha \in I} X_\alpha$  of non-empty, non-singular, connected, locally closed subvarieties, called strata, such that the

boundary  $\partial X_\alpha := \overline{X_\alpha} \setminus X_\alpha$  of any stratum  $X_\alpha$  is a union of strata. If, in addition, for each  $\alpha$ , the closure  $\overline{X_\alpha}$  and the boundary  $\partial X_\alpha$  are affine varieties, then we call  $\mathcal{S}$  an **affine stratification**. It is obvious that any affine stratification is finite.

For linear subspaces  $F, G \subseteq \mathbb{C}^n$ , let

$$\delta(F, G) := \sup_{\substack{x \in F \\ \|x\|=1}} \text{dist}(x, G),$$

where  $\text{dist}(x, G)$  is the Hermitian distance.

Let  $(X_\alpha, X_\beta)$  be a pair of strata of  $\mathcal{S}$  such that  $X_\beta \subset \overline{X_\alpha}$  and let  $x \in X_\beta$ . We recall some regularity conditions:

- (a) The pair  $(X_\alpha, X_\beta)$  is said to be **Whitney (a) regular** at  $x \in X_\beta$  if it satisfies the following Whitney condition (a) at  $x$ : if  $x^k \in X_\alpha$  is any sequence such that  $x^k \rightarrow x$  and  $T_{x^k} X_\alpha \rightarrow T$ , then  $T \supset T_x X_\beta$ .
- (w) The pair  $(X_\alpha, X_\beta)$  is said to be **(w) regular** at  $x \in X_\beta$  (or **strictly Whitney (a) regular at  $x$  with exponent 1**) if it satisfies the following condition (w) at  $x$ : there exist a neighborhood  $U$  of  $x$  in  $\mathbb{C}^n$  and a constant  $c > 0$  such that for any  $y \in X_\alpha \cap U$  and  $x' \in X_\beta \cap U$ , we have  $\delta(T_{x'} X_\beta, T_y X_\alpha) \leq c \|y - x'\|$ .
- (b) The pair  $(X_\alpha, X_\beta)$  is said to be **Whitney regular** at  $x \in X_\beta$  if it satisfies the following Whitney condition (b) at  $x$ : for any sequences  $x^k \in X_\alpha$  and  $y^k \in X_\beta$ ,  $y^k \neq x^k$ , such that  $x^k \rightarrow x$ ,  $y^k \rightarrow x$ ,  $T_{x^k} X_\alpha \rightarrow T$  and  $\overline{x^k y^k}$  converges to a line  $\ell$  in the projective space  $\mathbb{P}^{n-1}$ , we have  $\ell \subset T$ .

We say that the pair  $(X_\alpha, X_\beta)$  is **Whitney (a) regular** (resp. **Whitney regular**) if it is Whitney (a) regular (resp. Whitney regular) at every point of  $X_\beta$ . We say that  $\mathcal{S}$  is a **Whitney (a) stratification** (resp. a **Whitney stratification**) if any pair of strata  $(X_\alpha, X_\beta)$  of  $\mathcal{S}$  with  $X_\beta \subset \overline{X_\alpha}$  is Whitney (a) regular (resp. Whitney regular). It is well-known that Whitney regularity implies Whitney (a) regularity [21, 22]. Moreover, in light of [17, V.1.2], the Whitney condition (b) is equivalent to the condition (w) for the category of complex analytic sets, so to check Whitney regularity, we can verify either condition (w) or condition (b).

For the purpose of this paper, we also need the following notion of Whitney (resp. Whitney (a)) regularity along a stratum. Let  $X_\beta$  be a stratum of  $\mathcal{S}$  and let  $x \in X_\beta$ . We say that  $X_\beta$  is **Whitney regular** (resp. **Whitney (a) regular**) at  $x$  if for any stratum  $X_\alpha$  such that  $X_\beta \subset \overline{X_\alpha}$ , the pair  $(X_\alpha, X_\beta)$  is Whitney (resp. Whitney (a)) regular at  $x$ . The stratum  $X_\beta$  is **Whitney regular** (resp. **Whitney (a) regular**) if it is Whitney (resp. Whitney (a)) regular at every point of  $X_\beta$ . It is clear that  $\mathcal{S}$  is a Whitney (resp. a Whitney (a)) stratification if and only if each stratum of  $\mathcal{S}$  is Whitney (resp. Whitney (a)) regular.

**2.2. Construction of affine stratifications.** Let us, first of all, fix an affine stratification of  $X$  whose construction is based on the following proposition.

**Proposition 2.1.** *Let  $X \subset \mathbb{C}^n$  be an affine subvariety of pure codimension  $r$ . Assume that  $I(X) = (g_1, \dots, g_\omega)$ , where  $\deg g_i \leq D$ . Let  $W$  be an affine subvariety of positive codimension in  $X$  with  $I(W) = (g_1, \dots, g_\omega, u_1, \dots, u_\tau)$  where  $u_i \notin I(X)$  and  $\deg u_i \leq D'$ . Then there exists a polynomial  $p_{X,W}$  on  $\mathbb{C}^n$  of degree less than or equal to  $r(D-1) + D'$  such that  $W \subseteq V(p_{X,W}) := \{x \in \mathbb{C}^n : p_{X,W}(x) = 0\}$  and  $X \setminus V(p_{X,W})$  is a smooth, dense subset of  $X$ . Moreover, the polynomial  $p_{X,W}$  can be constructed effectively.*

*Proof.* Let  $X = \bigcup_{i=1}^m Y_i$ , where  $Y_i$  are irreducible (hence  $r$ -codimensional) components of  $X$ . Take sufficiently generic (random) rational numbers  $\alpha_{ij} \in \mathbb{Q}$ ,  $i = 1, \dots, r$ ,  $j = 1, \dots, \omega$  and set

$$G_i = \sum_{j=1}^{\omega} \alpha_{ij} g_j, \quad i = 1, \dots, r.$$

Here and in the following, to obtain a generic number, it is sufficient to take a random rational number and verify the genericity condition, the procedure is repeated until the genericity condition is satisfied. Note that the set  $Z := V(G_1, \dots, G_r)$  has pure codimension  $r$  and  $X \subset Z$ . Let  $\gamma_1, \dots, \gamma_\tau$  be some (random) generic rational numbers and set

$$H := \begin{cases} 1 & \text{if } W = \emptyset, \\ \sum_{i=1}^{\tau} \gamma_i u_i & \text{otherwise.} \end{cases}$$

Clearly  $\dim(X \cap V(H)) < \dim X$ . Moreover, for a sufficiently general linear  $r$ -dimensional subspace  $L^r \subset \mathbb{C}^n$  the intersection  $L^r \cap Z$  has only isolated smooth points and  $L^r \cap Y_i \neq \emptyset$  for every  $i = 1, \dots, m$ . We can assume that  $L^r$  is determined by the linear forms  $l_i = \sum_{j=1}^n \beta_{ij} x_j$ ,  $i = 1, \dots, n-r$ , where  $\beta_{ij}$  are sufficiently generic (random) rational numbers. Now take

$$p_{X,W} = |\text{Jac}(G_1, \dots, G_r, l_1, \dots, l_{n-r})| \cdot H,$$

where  $\text{Jac}(\cdot)$  denotes the Jacobian matrix. Then  $p_{X,W}$  is a polynomial with the required properties.  $\square$

**Remark 2.1.** Theoretically, a random rational number is a generic rational number, but practically by random numbers we mean rational numbers produced by special random algorithms.

The polynomial  $p_{X,W}$  can be found by using a probabilistic algorithm. First recall the following.

**Definition 2.1.** *Let  $I$  be an ideal in  $\mathbb{C}[x_1, \dots, x_n]$ . We define the homogenization of  $I$  to be the ideal  $I^h$  generated by  $\{f^h : f \in I\} \subset \mathbb{C}[x_0, \dots, x_n]$ , where  $f^h$  is the homogenization of  $f$ .*

**Theorem 2.1.** [2, Theorem 4, §4, Chapter 8, page 388] *Let  $I$  be an ideal in  $k[x_1, \dots, x_n]$  and let  $G = \{g_1, \dots, g_s\}$  be a Gröbner basis for  $I$  with respect to a graded lexicographic order in  $k[x_1, \dots, x_n]$  (i.e., the lexicographic order that first compares the total degree:  $x^\alpha > x^\beta$  whenever  $|\alpha| > |\beta|$ ). Then  $G^h = \{g_1^h, \dots, g_s^h\}$  is a basis for  $I^h \subset k[x_0, x_1, \dots, x_n]$ .*

This theorem allows us to compute the set of points at infinity of an affine variety given by the ideal  $I$ , to this aim it is enough to compute the Gröbner basis  $\{g_1, \dots, g_s\}$  of the ideal  $I$  and then to consider the ideal  $I_\infty = (x_0, g_1^h, \dots, g_s^h)$ . In particular we can check effectively, whether  $L^r \cap \overline{X} \cap \{x_0 = 0\} = \emptyset$ , which implies that  $L^r \cap Y_i \neq \emptyset$  for  $i = 1, \dots, m$  (see the proof of Proposition 2.1). This is crucial for our computations.

Now we sketch the algorithm to compute the polynomial  $p_{X,W}$ . Note that for a given ideal  $I$  we can compute  $\dim V(I)$  by [19].

INPUT: The ideal  $I = I(X) = (g_1, \dots, g_\omega)$  and the ideal  $J = I(W) = (g_1, \dots, g_\omega, u_1, \dots, u_\tau)$

1) **repeat**

choose random rational numbers  $\alpha_{i1}, \dots, \alpha_{i\omega}$ ,  $i = 1, \dots, r$ ;

put  $G_i := \sum_{k=1}^{\omega} \alpha_{ik} g_k$ ,  $i = 1, \dots, r$ ;

put  $I = (G_1, \dots, G_r)$ ;

**until**  $\dim V(I) = n - r$ .

2) **repeat**

choose random rational numbers  $\beta_{i1}, \dots, \beta_{in}$ ,  $i = 1, \dots, n - r$ ;

put  $l_i := \sum_{k=1}^n \beta_{ik} x_k$ ,  $i = 1, \dots, n - r$ ;

put  $I = (G_1, \dots, G_r, l_1, \dots, l_{n-r})$ ;

compute the ideal  $I_\infty = (H_1, \dots, H_m) \subset k[x_0, \dots, x_n]$ ;

**if**  $\dim V(I_\infty) = 0$  **then**

**begin**

compute  $V(G_1, \dots, G_r, l_1, \dots, l_r) := \{a_1, \dots, a_p\}$

**end**

**until**  $\dim V(I_\infty) = 0$  and  $|Jac(G_1, \dots, G_r, l_1, \dots, l_{n-r})(a_i)| \neq 0$  for  $i = 1, \dots, p$ .

3) **repeat**

choose random rational numbers  $\gamma_1, \dots, \gamma_\tau$ ;

put  $H := \sum_{k=1}^{\tau} \gamma_k u_k$ ;

put  $J = (G_1, \dots, G_r, H)$ ;

**until**  $\dim V(J) < n - r$ .

OUTPUT:  $p_{X,W} = |Jac(G_1, \dots, G_r, l_1, \dots, l_{n-r})| \cdot H$

**Remark 2.2.** Let us assume that  $I(X)$  and  $I(W)$  are generated by polynomials from the ring  $\mathbb{F}[x_1, \dots, x_n]$ , where  $\mathbb{F}$  is a subfield of  $\mathbb{C}$ . Then we can choose a polynomial  $p_{X,W}$  such that  $p_{X,W} \in \mathbb{F}[x_1, \dots, x_n]$ .

From the proof of Proposition 2.1, with no loss of generality, we can assume that  $\text{rank } Jac(g_1, \dots, g_r) = r$  on some non-empty regular open subset  $X^0$  of  $X$  and that  $X = \overline{X^0}$ . It is clear that  $V(p_{X,W})$  contains  $\text{sing}(X) \cup W$  and the singular points of the projection  $(l_1, \dots, l_{n-r}) : X \rightarrow \mathbb{C}^{n-r}$ . Now to

construct an affine stratification of  $X$ , it is enough to construct an affine filtration  $X = X_0 \supset X_1 \supset \cdots \supset X_{n-r} \supset X_{n-r+1} = \emptyset$  by induction with  $X_{i+1} := X_i \cap V(p_{X_i, \emptyset})$ ,  $i = 0, \dots, n-r$ . The degree of each  $X_i$  can be calculated and depends only on  $D$ .

**2.3. Construction of affine Whitney stratifications.** In this section, we will construct an affine Whitney stratification of a given affine variety  $X$ , with  $I(X) = (g_1, \dots, g_r)$  and  $\deg g_i \leq D$ , by refining the affine stratification given in Subsection 2.2 so that the resulting stratification is still affine and moreover satisfies the Whitney condition.

First of all, inspired by the construction in [5, 17], let us describe the Whitney condition (b) algebraically. Assume that  $Y \subset X$  is an affine subvariety of  $X$  with  $\dim Y < \dim X$  defined by

$$Y := X \cap \{\tilde{g}_{r+1} = \cdots = \tilde{g}_p = 0\}.$$

Set

$$\Gamma_1 := \left\{ \begin{array}{l} (x, y, w, v, \gamma, \lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^r : \\ g_1(x) = \cdots = g_r(x) = 0 \\ g_1(y) = \cdots = g_r(y) = \tilde{g}_{r+1}(y) = \cdots = \tilde{g}_p(y) = 0 \\ w = \gamma(x - y) \\ v = \sum_{i=1}^r \lambda_i d_x g_i \end{array} \right\},$$

and let

$$\pi_1 : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^r \rightarrow \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$$

be the projection on the first  $4n$  coordinates. Let  $C(X, Y) = \overline{\pi_1(\Gamma_1)}^Z \subset (X \times Y \times \mathbb{C}^n \times \mathbb{C}^n)$ , where the closure is taken in the Zariski topology (which coincides with the topological closure, see e.g., [16, Proposition 7]). Of course,  $C(X, Y)$  is an affine variety. We have the following.

**Lemma 2.1.** *For each  $(x, x, w, v) \in C(X, Y)$ , there are sequences  $x^k \in X^0$ ,  $y^k \in Y$ ,  $\gamma^k \in \mathbb{C}$  and  $\lambda^k \in \mathbb{C}^r$  such that*

- $x^k \rightarrow x$ ,
- $y^k \rightarrow x$ ,
- $w^k := \gamma^k(x^k - y^k) \rightarrow w$ ,
- $v^k := \sum_{i=1}^r \lambda_i^k d_{x^k} g_i \rightarrow v$ .

*Proof.* By construction, there are sequences  $\bar{x}^k \in X$ ,  $y^k \in Y$ ,  $\bar{\gamma}^k \in \mathbb{C}$  and  $\lambda^k \in \mathbb{C}^r$  such that  $\bar{x}^k, y^k \rightarrow x$ ,  $\bar{w}^k := \bar{\gamma}^k(\bar{x}^k - y^k) \rightarrow w$  and  $\sum_{i=1}^r \lambda_i^k d_{\bar{x}^k} g_i \rightarrow v$ . It is clear that by taking subsequences if necessary, we may suppose that:

- either  $\bar{x}^k = y^k$  for every  $k$  or  $\bar{x}^k \neq y^k$  for every  $k$ ,
- for each  $i$ , either  $\lambda_i^k \neq 0$  for every  $k$  or  $\lambda_i^k = 0$  for every  $k$ .

Set

$$\gamma^k = \begin{cases} 0 & \text{if } \bar{x}^k = y^k \text{ for every } k, \\ \bar{\gamma}^k & \text{if } \bar{x}^k \neq y^k \text{ for every } k. \end{cases}$$

Suppose that  $\lambda_i^k \neq 0$  for  $i = 1, \dots, r' \leq r$ ,  $k \in \mathbb{N}$  and  $\lambda_i^k = 0$  for  $i = r' + 1, \dots, r$ ,  $k \in \mathbb{N}$ . Since  $\bar{x}^k \in \overline{X^0}$ , there exists a sequence  $x^k \in X^0$  such that

$$\|x^k - \bar{x}^k\| \leq \begin{cases} \frac{1}{k} & \text{if } \bar{x}^k = y^k \text{ for every } k, \\ \frac{\|\bar{x}^k - y^k\|}{k} & \text{if } \bar{x}^k \neq y^k \text{ for every } k, \end{cases}$$

so  $x^k \rightarrow x$ . By continuity, we can also choose  $x^k$  so that  $\|d_{x^k}g_i - d_{\bar{x}^k}g_i\| < \frac{1}{k\lambda_i^k}$  if  $\lambda_i^k \neq 0$ . Set  $v^k := \sum_{i=1}^r \lambda_i^k d_{x^k}g_i$ . Then

$$\begin{aligned} \|v^k - \sum_{i=1}^r \lambda_i^k d_{\bar{x}^k}g_i\| &= \left\| \sum_{i=1}^{r'} \lambda_i^k (d_{x^k}g_i - d_{\bar{x}^k}g_i) \right\| \\ &\leq \sum_{i=1}^{r'} |\lambda_i^k| \|d_{x^k}g_i - d_{\bar{x}^k}g_i\| < \frac{r'}{k} \rightarrow 0, \end{aligned}$$

i.e.,  $v^k \rightarrow v$ . Set  $w^k := \gamma^k(x^k - y^k)$ . Now if  $\bar{x}^k = y^k$  for every  $k$ , then  $\gamma^k = 0$  and  $w = \bar{w}^k = 0$ , so we have  $w^k = 0 = w$ . If  $\bar{x}^k \neq y^k$  for every  $k$ , then

$$\|w^k - \bar{w}^k\| = |\gamma^k| \cdot \|x^k - \bar{x}^k\| \leq |\gamma^k| \cdot \frac{\|\bar{x}^k - y^k\|}{k} = \frac{\|\bar{w}^k\|}{k} \rightarrow 0.$$

Hence  $w^k \rightarrow w$ . The lemma is proved.  $\square$

The following algebraic criterion permits us to check Whitney regularity on  $Y^0 = Y \setminus V(p_{Y,W})$ , where the notation  $V(p_{Y,W})$  is from Proposition 2.1, and the affine set  $W$  will be determined later.

**Lemma 2.2.** *Let  $x \in Y^0$ . Then the pair  $(X^0, Y^0)$  satisfies the Whitney condition (b) at  $x$  if and only if for any  $(x, x, w, v) \in C(X, Y)$ , we have  $v \cdot w = 0$ .*

*Proof.* Suppose that  $(X^0, Y^0)$  is Whitney regular at  $x$  and assume for contradiction that there is  $(x, x, w, v) \in C(X, Y)$  such that  $v \cdot w \neq 0$ . In view of Lemma 2.1, there are sequences  $x^k \in X^0$ ,  $y^k \in Y$ ,  $\gamma^k \in \mathbb{C}$  and  $\lambda^k \in \mathbb{C}^r$  such that

- $x^k \rightarrow x$ ,  $y^k \rightarrow x$ ,
- $w^k := \gamma^k(x^k - y^k) \rightarrow w$ ,
- $v^k := \sum_{i=1}^r \lambda_i^k d_{x^k}g_i \rightarrow v$ .

Note that  $w \neq 0$ , so  $w$  determines the limit of the sequence of secants  $\overline{x^k y^k}$  and it follows that  $x^k \neq y^k$  for  $k$  large enough. By taking a subsequence if necessary, we may assume that  $T_{x^k}X^0 \rightarrow T$ . By assumption,  $w \in T$ . For each  $k$ , let  $\{b_1^k, \dots, b_r^k\}$  be an orthonormal basis of  $N_{x^k}X^0$ ; recall that  $N_{x^k}X^0 := \text{span}\{\overline{d_{x^k}g_1}, \dots, \overline{d_{x^k}g_r}\}$  is the normal space of  $X^0$  at  $x^k$ . Obviously  $\langle \overline{b_1^k}, \dots, \overline{b_r^k} \rangle^\perp = T_{x^k}X^0$ . By compactness, each sequence  $b_i^k$  has an accumulation point  $b_i$ . Without loss of generality, suppose that  $b_i^k \rightarrow b_i$ . It is clear that the system  $\{b_1, \dots, b_r\}$  is also orthonormal and  $\langle \overline{b_1}, \dots, \overline{b_r} \rangle^\perp = T$ . Let  $\tilde{\lambda}^k = (\tilde{\lambda}_1^k, \dots, \tilde{\lambda}_r^k)$  be such that  $v^k := \sum_{i=1}^r \tilde{\lambda}_i^k b_i^k$ . Then  $\tilde{\lambda}^k$  is convergent to a limit  $\tilde{\lambda}$  and it is clear that  $v = \sum_{i=1}^r \tilde{\lambda}_i b_i$ . Finally, we have  $w \in T = \langle \overline{b_1}, \dots, \overline{b_r} \rangle^\perp \subset \langle \tilde{v} \rangle^\perp$ , i.e.,  $v \cdot w = 0$ , which is a contradiction.

Now suppose that  $v \cdot w = 0$  for any  $(x, x, w, v) \in C(X, Y)$  and assume, that  $(X^0, Y^0)$  is not Whitney regular at  $x$ . So there are sequences  $x^k \in X^0$  and  $y^k \in Y^0$  with the following properties:

- $x^k \neq y^k$ ,  $x^k \rightarrow x$ ,  $y^k \rightarrow y$ ;

- $T_{x^k}X^0 \rightarrow T$ ;
- the sequence of secants  $\overline{x^k y^k}$  tends to a line  $\ell \not\subset T$ .

For each  $k$ , let  $\{b_1^k, \dots, b_r^k\}$  be an orthonormal basis of  $N_{x^k}X^0$  so  $\langle \overline{b_1^k}, \dots, \overline{b_r^k} \rangle^\perp = T_{x^k}X^0$ . As above, we may assume that  $b_i^k \rightarrow b_i$ . Then the system  $\{b_1, \dots, b_r\}$  is also orthonormal and  $\langle \overline{b_1}, \dots, \overline{b_r} \rangle^\perp = T$ . Let  $w^k := \frac{x^k - y^k}{\|x^k - y^k\|}$ ; we can assume that the limit  $w := \lim w^k$  exists and clearly  $w$  is a direction vector of  $\ell$ . By assumption,  $w \notin T = \langle \overline{b_1}, \dots, \overline{b_r} \rangle^\perp$ , i.e., there exists an index  $j$  such that  $b_j \cdot w \neq 0$ . To get a contradiction, it is enough to show that there is a sequence  $v^k := \sum_{i=1}^r \lambda_i^k d_{x^k} g_i$  such that  $v^k \rightarrow b_j$ , but this is clear since  $b_j^k \in \text{span}\{d_{x^k} g_1, \dots, d_{x^k} g_r\}$  so such a sequence always exists. The lemma is proved.  $\square$

Now according to [11, Algorithm 3.3], [7, Algorithm 4.5.3], (see also [4, 6]), it is possible to calculate a basis for the ideal  $I(\Gamma_1)$  by calculating the radical of the following ideal in  $\mathbb{C}[x, y, w, v, \gamma, \lambda]$ :

$$\left( \begin{array}{l} g_1(x), \dots, g_r(x), \\ g_1(y), \dots, g_r(y), \tilde{g}_{r+1}(y), \dots, \tilde{g}_p(y), \\ w - \gamma(x - y), \\ v - \sum_{i=1}^r \lambda_i d_x g_i \end{array} \right).$$

Then by Buchberger's algorithm, we can calculate a Gröbner basis of  $I(\Gamma_1)$ . So in view of [10, Theorem 5.1], [14], we can compute a Gröbner basis of the ideal  $I(C(X, Y))$ . Now we give another criterion for Whitney regularity.

**Lemma 2.3.** *Let  $\{h_1(x, y, w, v), \dots, h_q(x, y, w, v)\}$  be a Gröbner basis of  $I(C(X, Y))$  and set*

$$\Gamma_2 := \left\{ \begin{array}{l} (x, x, w, v, \gamma, \lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : \\ h_1(x, x, w, v) = \dots = h_q(x, x, w, v) = 0 \\ \gamma \sum_{j=1}^n v_j w_j = 1 \\ \lambda p_{Y, \emptyset}(x) = 1 \end{array} \right\},$$

where  $p_{Y, \emptyset}(x)$  is the polynomial determined in Proposition 2.1. Let  $Y^0 := Y \setminus V(p_{Y, \emptyset})$ . Then the pair  $(X^0, Y^0)$  is not Whitney regular at  $x$  if and only if there exists  $(w, v, \gamma, \lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$  such that  $(x, x, w, v, \gamma, \lambda) \in \Gamma_2$ .

*Proof.* Note that  $x \in Y^0$  if and only if  $p_{Y, \emptyset}(x) \neq 0$ , i.e., there exists  $\lambda \in \mathbb{C}$  such that  $\lambda p_{Y, \emptyset}(x) = 1$ . In view of Lemma 2.2, the pair  $(X^0, Y^0)$  is not Whitney regular at  $x$  if and only if there exist  $w, v$  with  $v \cdot w \neq 0$  such that  $(x, x, w, v) \in C(X, Y)$ . The lemma follows easily.  $\square$

Now we determine an algebraic set  $W = W(X, Y)$  in  $Y$  with  $\dim W < \dim Y$  and  $V(p_{Y, \emptyset}) \subset W$  such that the pair  $(X^0, Y \setminus W)$  is Whitney regular. Let

$$\pi_2 : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^n$$

be the projection on the first  $n$  coordinates. By Lemma 2.3,  $\pi_2(\Gamma_2)$  is the set of points where the Whitney condition (b) fails to be satisfied. Let  $\overline{\pi_2(\Gamma_2)}^Z$  be the Zariski closure of  $\pi_2(\Gamma_2)$ , then



$\overline{\pi_2(\Gamma_2)}^{\mathcal{Z}}$  is affine. It follows from [21, Theorem 8.5],[22, Lemma 19.3] that  $\dim \pi_2(\Gamma_2) < \dim Y$ , so  $\dim \overline{\pi_2(\Gamma_2)}^{\mathcal{Z}} < \dim Y$ . Set

$$W = W(X, Y) := \overline{\pi_2(\Gamma_2)}^{\mathcal{Z}};$$

then obviously  $\dim W < \dim Y$ . Again, applying [11, Algorithm 3.3] or [7, Algorithm 4.5.3], (see also [4, 6]) to find a system of generators of  $I(W)$ , then applying [10, Theorem 5.1], [14], we can compute a Gröbner basis of the ideal  $I(W)$ .

Finally, let

- $X_0 := X$ ,
- $X_1 := X_0 \cap V(p_{X_0, \emptyset})$ ,
- $X_2 := X_1 \cap V(p_{X_1, W(X_0, X_1)})$ ,
- $X_3 := X_2 \cap V(p_{X_2, W(X_0, X_2) \cup W(X_1, X_2)})$ ,  $\dots$ ,
- $X_i := X_{i-1} \cap V(p_{X_{i-1}, \bigcup_{j=0}^{i-2} W(X_j, X_{i-1})})$ ,  $\dots$

By induction, we can construct a finite filtration of algebraic sets  $X = X_0 \supset X_1 \supset \dots \supset X_{n-r} \supset X_{n-r+1} = \emptyset$  with  $\dim X_i > \dim X_{i+1}$ . Let  $B_i := X_i \setminus X_{i+1}$ . Then  $\mathcal{S} := \{B_i\}_{i=1, \dots, q}$  is a Whitney stratification of  $X$ . Note that the degree of  $X_i$  can be determined explicitly and depends only on  $D$ .

### 3. THOM ISOTOPY LEMMA FOR NON-PROPER MAPS

We start this section with:

**Definition 3.1.** Let  $f : X \rightarrow \mathbb{C}^m$  be a polynomial dominant map where  $X$  is an algebraic set. Let  $\mathcal{S} = \{X_\alpha\}_{\alpha \in I}$  be a stratification of  $X$ . By  $K_\infty(f, X_\alpha)$  we mean the set  $\{y \in \mathbb{C}^m : \text{there is a sequence } x^k \rightarrow \infty; x^k \in X_\alpha : \|x^k\| \nu(d_{x^k}(f|_{X_\alpha})) \rightarrow 0 \text{ and } f(x^k) \rightarrow y\}$  (here  $\nu$  denotes the Rabier function, for details see [10] and section 5). Now let  $C(f, X_\alpha)$  denote the set of points where  $f|_{X_\alpha}$  is not a submersion. By  $\text{sing}(f, \mathcal{S})$  we denote the set of stratified singular values of  $f$ , i.e.,

$$\text{sing}(f, \mathcal{S}) = \bigcup_{\alpha \in I} K_0(f, X_\alpha), \tag{1}$$

where  $K_0(f, X_\alpha) = \overline{f(C(f, X_\alpha))}$ .

By [10, Theorem 3.3], we have that for every  $\alpha$  the set  $K_\infty(f|_{X_\alpha})$  is closed and has measure 0 in  $\mathbb{C}^m$ . In particular the set  $K(f)$  defined below is also closed and has measure 0.

**Definition 3.2.** Let  $K(f, \mathcal{S})$  be the set of stratified generalized critical values of  $f$  given by

$$K(f, \mathcal{S}) := \bigcup_{\alpha \in I} (K_0(f, X_\alpha) \cup K_\infty(f, X_\alpha)) \tag{2}$$

**Remark 3.1.** The set  $K(f, \mathcal{S})$  is closed. Indeed, it is enough to see that  $K_\infty(f, X_\alpha)$  is closed for every  $\alpha$ . Assume that  $y^k \in K_\infty(f, X_\alpha)$  and  $y^k \rightarrow y$ . So for every  $k$ , there are suitable sequences  $x^{kj} \in X_\alpha$ ,  $j = 1, 2, \dots$ , such that  $\lim_{j \rightarrow \infty} f(x^{kj}) = y^k$ . Hence for any  $k$ , we can choose  $x^{kjk}$  such that:

- $\|f(x^{kj_k}) - y^k\| < 1/k$ ,
- $\|x^{kj_k}\| > k$ ,
- $\|x^{kj_k}\| \nu(d_{x^{kj_k}} f) < 1/k$ .

Set  $z^k := x^{kj_k}$ . Thus  $z^k \in X_\alpha$ ,  $z^k \rightarrow \infty$ ,  $\|z^k\| \nu(d_{z^k}(f|_{X_\alpha})) \rightarrow 0$  and  $f(z^k) \rightarrow y$ . Consequently  $y \in K_\infty(f, X_\alpha)$ .

Assuming that  $\mathcal{S}$  is an affine Whitney stratification of  $X$ , we prove that  $K(f, \mathcal{S})$  contains the set of bifurcation values of  $f$ .

**Theorem 3.1** (First isotopy lemma for non-proper maps). *Let  $X \subset \mathbb{C}^n$  be an affine variety with an affine Whitney stratification  $\mathcal{S}$  and let  $f = (f_1, \dots, f_m) : X \rightarrow \mathbb{C}^m$  be a polynomial dominant map. Let  $K(f, \mathcal{S})$  be the set of stratified generalized critical values of  $f$  given by (2). Then  $f$  is locally trivial outside  $K(f, \mathcal{S})$ .*

Before proving Theorem 3.1, recall that the Whitney condition (b) is equivalent to the condition (w) (see [17, V.1.2]), so it is more convenient to use the condition (w) since we will need to construct rugose vector fields in the sense of [20]. In what follows, it is more convenient to work with the underlying real algebraic set of  $X$  in  $\mathbb{R}^{2n}$ , denoted also by  $X$ ; the affine Whitney stratification  $\mathcal{S}$  of  $X$  induces a semialgebraic Whitney stratification of the underlying set with the corresponding strata denoted by the same notations  $X_\beta$ . We also identify the polynomial map  $f$  with the real polynomial map  $(\operatorname{Re}f_1, \dots, \operatorname{Re}f_m, \operatorname{Im}f_1, \dots, \operatorname{Im}f_m) : X \rightarrow \mathbb{R}^{2m}$ .

Let us recall the definitions pertaining to rugosity. Let  $\psi : X \rightarrow \mathbb{R}$  be a real function. We say that  $\psi$  is a **rugose function** if the following conditions are fulfilled:

- The restriction  $\psi|_{X_\beta}$  to any stratum  $X_\beta$  is a smooth function.
- For any stratum  $X_\beta$  and for any  $x \in X_\beta$ , there exist a neighborhood  $U$  of  $x$  in  $\mathbb{R}^{2n}$  and a constant  $c > 0$  such that for any  $y \in X \cap U$  and  $x' \in X_\beta \cap U$ , we have  $|\psi(y) - \psi(x')| \leq c\|y - x'\|$ .

A **rugose map** is a map whose components are rugose functions. A vector field  $v$  on  $X$  is called a **rugose vector field** if  $v$  is a rugose map and  $v(x)$  is tangent to the stratum containing  $x$  for any  $x \in X$ .

*Proof of Theorem 3.1.* Let  $z \in \mathbb{C}^m \setminus K(f, \mathcal{S})$  where we identify  $\mathbb{C}^m$  with  $\mathbb{R}^{2m}$  and let  $B$  be an open box centered at  $z$  such that  $\overline{B} \cap K(f, \mathcal{S}) = \emptyset$ . To prove the theorem, it is enough to prove that  $f$  is trivial on  $B$ . Without loss of generality, we may suppose that  $z = 0$  and  $B = (-1, 1)^{2m}$ . Let  $\partial_1, \dots, \partial_{2m}$  be the restrictions of the coordinate vector fields (on  $\mathbb{R}^{2m}$ ) to  $\overline{B}$ . Set  $U := f^{-1}(B)$ ,  $U_\beta := U \cap X_\beta$  and

$$I' := \{\beta \in I : U_\beta \neq \emptyset\}.$$

Clearly  $\overline{U} = f^{-1}(\overline{B})$  and  $I' = \{\beta \in I : \overline{U} \cap X_\beta \neq \emptyset\}$ . First of all, let us give a sufficient condition for trivializing a rugose vector field.

**Lemma 3.1.** For  $i = 1, \dots, 2m$ , let  $v_i$  be vector fields on  $X$  which are rugose in  $U$ . Assume that  $df(v_i) = \partial_i$  and there is a positive constant  $c > 0$  such that  $\|v_i(x)\| \leq \frac{\|x\|+1}{c}$  for any  $x \in U$ . Then  $f$  is a topologically trivial fibration over  $B$ .

*Proof.* It is enough to prove that there is a homeomorphism  $\phi : f^{-1}(B) \rightarrow f^{-1}(0) \times B$  such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(B) & \xrightarrow{\phi} & f^{-1}(0) \times B \\ f \searrow & & \swarrow \pi \\ & B & \end{array}$$

where  $\pi$  denotes the projection on the second factor. We note the following facts:

- (i) The flow of  $v_i$  preserves the stratification. This is a consequence of the rugosity. For more detail, see [20, Proposition 4.8].
- (ii) For each  $i$  and any  $x \in U$ , there is a unique integral curve of  $v_i$  passing through  $x$ . This follows from the uniqueness of integral curves of smooth vector fields and the fact that  $v_i$  preserves the stratification.

Set  $Y_t^i := (y_1, \dots, y_{i-1}, t, y_{i+1}, \dots, y_n)$  and  $Y^i = \{Y_t^i : t \in (-1, 1)\}$ . First of all, we will prove that the flow of  $v_i$  induces a homeomorphism  $\phi_i : f^{-1}(Y^i) \rightarrow f^{-1}(Y_0^i) \times (-1, 1)$  such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(Y^i) & \xrightarrow{\phi_i} & f^{-1}(Y_0^i) \times (-1, 1) \\ p_i \circ f \searrow & & \swarrow \pi_i \\ & (-1, 1) & \end{array}$$

where  $\pi_i$  denotes the projection on the second factor and  $p_i$  denotes the projection on the  $i^{\text{th}}$  coordinate. This follows from the following claim which states that there is no trajectory of  $v_i$  going to infinity.

**Claim 3.1.** For each  $x \in f^{-1}(Y_0^i)$ , let  $\gamma$  be the integral curve of  $v_i$  such that  $\gamma(0) = x$ . Then  $\gamma$  reaches any level  $f^{-1}(Y_t^i)$  at time  $t$  for  $t \in (-1, 1)$ .

*Proof.* By assumption,  $\|\dot{\gamma}(t)\| \leq \frac{\|\gamma(t)\|+1}{c}$ . Without loss of generality, suppose that  $t > 0$ . In light of the Gronwall Lemma, by repeating the calculation of [1, Theorem 3.5], we get

$$\begin{aligned} \|\gamma(t)\| &\leq \|\gamma(0)\| + \int_0^t \frac{\|\gamma(s)\| + 1}{c} ds \\ &= \|x\| + \frac{t}{c} + \int_0^t \frac{\|\gamma(s)\|}{c} ds \\ &\leq \left(\|x\| + \frac{t}{c}\right) \exp \int_0^t \frac{ds}{c} = \left(\|x\| + \frac{t}{c}\right) e^{\frac{t}{c}} < +\infty, \end{aligned}$$

which implies that the trajectory  $\gamma$  does not go to infinity at time  $t$ . Now we have

$$f(\gamma(t)) - f(x) = \int_0^t d[f(\gamma(t))] = \int_0^t d_{\gamma(t)} f(\dot{\gamma}(t)) dt = \int_0^t \partial_i dt = (0, \dots, 0, t, 0, \dots, 0).$$

Since  $f(x) \in Y_0^i$ , clearly  $f(\gamma(t)) \in Y_t^i$  and the claim follows.  $\square$

For any  $x \in f^{-1}(Y_0^i)$ , let  $h_i(x, t) = x + \int_0^t \hat{\gamma}(s) ds$ . Then  $h_i$  defines a homeomorphism  $f^{-1}(Y_0^i) \times (-1, 1) \rightarrow f^{-1}(Y^i)$ . Then  $\phi_i = h_i^{-1}$  is the required homeomorphism. Now for  $x \in f^{-1}(0)$ , let  $h : f^{-1}(0) \times B \rightarrow f^{-1}(B)$  be defined by

$$h(x, t_1, \dots, t_{2m}) = h_{2m}(\dots (h_2(h_1(x, t_1), t_2), \dots, t_{2m})).$$

Then  $\phi := h^{-1}$  is a homeomorphism, as required. The lemma is proved.  $\square$

Now let us prove the following.

**Lemma 3.2.** *There is a constant  $c > 0$  such that for any  $\beta \in I'$  and any  $x \in \bar{U} \cap X_\beta$ , we have*

$$(\|x\| + 1)\nu(d_x(f|_{X_\beta})) \geq c.$$

*Proof.* Assume for contradiction that there exist an index  $\beta \in I'$  and a sequence  $x^k \in \bar{U} \cap X_\beta$  such that  $(\|x^k\| + 1)\nu(d_{x^k}(f|_{X_\beta})) \rightarrow 0$ . Taking a subsequence if necessary, we can suppose that  $x^k \rightarrow x$ ,  $T_{x^k}X_\beta \rightarrow T$  and  $f(x^k) \rightarrow y \in \bar{B}$  with  $x \in \mathbb{C}^n$  or  $x = \infty$ . If  $x = \infty$ , then by definition,  $y \in K_\infty(f|_{X_\beta}) \subset K(f, \mathcal{S})$ . This is a contradiction since  $\bar{B} \cap K(f, \mathcal{S}) = \emptyset$ . Thus  $x \in \mathbb{C}^n$  and we get  $\nu(d_{x^k}(f|_{X_\beta})) \rightarrow 0$ . In the case  $x \in X_\beta$ , in view of [15, Lemma 2.2], we have  $\nu(d_x(f|_{X_\beta})) = 0$ , i.e.,  $y \in K_0(f, X_\beta)$ , which is also a contradiction. Therefore  $x \in \bar{X}_\beta \setminus X_\beta$ . Denote by  $X_\alpha$  the stratum containing  $x$ . Let  $F = (f_1, \dots, f_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m$  be the polynomial extending  $f$  on  $\mathbb{C}^n$ . Obviously  $\nu(d_{x^k}F|_{T_{x^k}X_\beta}) = \nu(d_{x^k}(F|_{X_\beta})) = \nu(d_{x^k}(f|_{X_\beta})) \rightarrow 0$ . Moreover, since  $f|_{X_\alpha}$  is a submersion at  $x$ , so  $F$  is a submersion at  $x$ . Hence  $\nu(d_x(F|_{X_\alpha})) \neq 0$ . Since the stratification is Whitney, it implies that  $T \supset T_x X_\alpha$ . Consequently  $\nu(d_x F|_T) \geq \nu(d_x(F|_{X_\alpha})) \neq 0$ . To get a contradiction, we will need the following claim.

**Claim 3.2.** *Let  $A_k : \mathbb{R}^q \rightarrow \mathbb{R}^p$  be a sequence of linear maps such that  $A_k \rightarrow A$  as  $k \rightarrow +\infty$  (i.e., the terms of the matrix of  $A_k$  tends to the corresponding terms of the matrix of  $A$ ). Let  $H_k \subset \mathbb{R}^q$  be a sequence of linear subspaces of same dimension such that  $H_k \rightarrow H$  (i.e.,  $\delta(H_k, H) \rightarrow 0$  where  $\delta(H_k, H) := \sup_{y \in H_k, \|y\|=1} \text{dist}(y, H)$  is the distance between  $H_k$  and  $H$ ,  $\text{dist}(\cdot, \cdot)$  is the Euclidean distance). Then  $\nu(A_k|_{H_k}) \rightarrow \nu(A|_H)$ .*

*Proof.* It is clear that

$$\|\nu(A_k|_{H_k}) - \nu(A|_H)\| \leq \|\nu(A_k|_{H_k}) - \nu(A|_{H_k})\| + \|\nu(A|_{H_k}) - \nu(A|_H)\|.$$

In light of [15, Lemma 2.1 (iv)], we have

$$\|\nu(A_k|_{H_k}) - \nu(A|_{H_k})\| \leq \|A_k|_{H_k} - A|_{H_k}\| = \|(A_k - A)|_{H_k}\| \leq \|A_k - A\| \rightarrow 0. \quad (3)$$

Note that  $\nu(A)$  is the length of a minimal semiaxis of the ellipsoid  $A(\mathbb{B}^q)$  where  $\mathbb{B}^q$  is the unit ball in  $\mathbb{R}^q$ . Since  $H_k \rightarrow H$  we have  $\mathbb{B}^q \cap H_k \rightarrow \mathbb{B}^q \cap H$  and  $A(\mathbb{B}^q \cap H_k) \rightarrow A(\mathbb{B}^q \cap H)$  by continuity of  $A$ . Hence also  $\nu(A|_{H_k}) \rightarrow \nu(A|_H)$ .  $\square$

Applying Claim 3.2 with  $A_k = d_{x^k}F$  and  $H_k = T_{x^k}X_\beta$ , we get

$$0 = \lim_{k \rightarrow \infty} \nu(d_{x^k}F|_{T_{x^k}X_\beta}) = \nu(d_xF|_T),$$

which is a contradiction. The lemma follows.  $\square$

For each  $\beta \in I'$ , it is clear that  $f|_{X_\beta}$  is a submersion on  $(f|_{X_\beta})^{-1}(\overline{B})$ , so for  $x \in \overline{U} \cap X_\beta$ , the differential  $d_x(f|_{X_\beta}) : T_xX_\beta \rightarrow \mathbb{R}^{2m}$  is surjective, which induces an isomorphism of vector spaces

$$\widetilde{d}_x(f|_{X_\beta}) : T_xX_\beta / \ker d_x(f|_{X_\beta}) \cong \mathbb{R}^{2m}.$$

Thus for each  $i = 1, \dots, 2m$ , the vector field  $\partial_i$  can be lifted uniquely and smoothly on each stratum  $X_\beta$  with  $\beta \in I'$ , to the vector field called the horizontal lift of  $\partial_i$  and denoted by  $v_i^\beta$ . Clearly,  $v_i^\beta(x)$  is the unique vector in  $T_xX_\beta$  which lifts  $\partial_i$  and is orthogonal to  $\ker d_x(f|_{X_\beta})$ . Each  $v_i^\beta$  has the following important properties.

**Lemma 3.3.** *Let  $c > 0$  be the constant in Lemma 3.2. Then for each  $x \in \overline{U} \cap X_\beta$  with  $\beta \in I'$ , we have*

$$\|v_i^\beta(x)\| \leq \frac{\|x\| + 1}{c}.$$

*Proof.* Let  $\mathbb{B}_\beta$  be the closed unit ball centered at the origin in  $T_xX_\beta$ . Then  $d_x(f|_{X_\beta})(\mathbb{B}_\beta)$  is an ellipsoid in  $\mathbb{R}^{2m}$  with  $\nu(d_x(f|_{X_\beta}))$  as the length of shortest semiaxis. Let  $\mathbb{B}^{2m}$  be the closed unit ball centered at the origin in  $\mathbb{R}^{2m}$ . Then  $(\widetilde{d}_x(f|_{X_\beta}))^{-1}(\nu(d_x(f|_{X_\beta}))\mathbb{B}^{2m})$  is an ellipsoid in  $T_xX_\beta$  with 1 as the length of longest semiaxis. Therefore the longest semiaxis of the ellipsoid  $(\widetilde{d}_x(f|_{X_\beta}))^{-1}(\mathbb{B}^{2m})$  is  $1/\nu(d_x(f|_{X_\beta}))$ . Consequently,

$$\|v_i^\beta(x)\| \leq \frac{1}{\nu(d_x(f|_{X_\beta}))} \leq \frac{\|x\| + 1}{c},$$

which yields the lemma.  $\square$

Note that, for fixed  $i$ , the vector field on  $U$  which coincides with  $v_i^\beta$  on each  $U_\beta$  is not necessarily a rugose vector field. In what follows, we will try to deform these vector fields to produce a rugose vector field, which satisfies the assumption of Lemma 3.1. The process is carried out by induction on dimension.

For  $2m \leq d \leq 2 \dim_{\mathbb{C}} X$ , let  $I'_d := \{\beta \in I' : 2m \leq \dim X_\beta \leq d\}$  and  $U_d := \bigcup_{\beta \in I'_d} X_\beta \cap U$ . By induction on  $d$ , we construct a rugose vector field on  $U_{2 \dim_{\mathbb{C}} X}$  with the property of Lemma 3.1. For  $d = 2m$ , let  $v_i^{2m}$  be the restriction to  $U_{2m}$  of the smooth vector field on  $\bigcup_{\beta \in I'_{2m}} X_\beta$  which coincides with each  $v_i^\beta$  on  $X_\beta$  for  $\beta \in I'_{2m}$ . Then  $v_i^{2m}$  is clearly rugose,  $df(v_i^{2m}) = \partial_i$  and by Lemma 3.3,  $\|v_i^{2m}(x)\| \leq \frac{\|x\| + 1}{c}$  for any  $x \in U_{2m}$ .

For each  $i$ , assume that we have constructed a rugose vector field, denoted by  $v_i^d$ , on  $U_d$  such that  $d_x f(v_i^d(x)) = \partial_i$  and  $\|v_i^d(x)\| \leq \frac{\|x\| + 1}{c_d}$  for every  $x \in U_d$ , where  $c_d$  is a positive constant. We need to extend each  $v_i^d$  to a rugose vector field  $v_i^{d+2}$  on  $U_{d+2}$  such that  $\|v_i^{d+2}(x)\| \leq \frac{\|x\| + 1}{c_{d+2}}$  for every  $x \in U_{d+2}$ , where  $c_{d+2}$  is also a positive constant (recall that the strata of  $\mathcal{S}$  have even dimension).

Note that to construct  $v_i^{d+2}$ , it is enough to construct  $v_i^{d+2}$  separately on each stratum  $X_\alpha$  with  $\alpha \in I'_{d+2} \setminus I'_d$ . Without loss of generality, suppose that  $I'_{d+2} \setminus I'_d = \{\alpha\}$ .

By [20, Lemma 4.4], for each  $i = 1, \dots, 2m$ , there is a rugose vector field on  $U_{d+2}$ , denoted by  $\tilde{w}_i^{d+2}$ , which extends  $v_i^d$ , so the restriction  $\tilde{w}_i^{d+2}|_{U_{d+2} \cap X_\alpha}$  is a smooth vector field. We need to adjust  $\tilde{w}_i^{d+2}$  to get a new rugose vector field  $w_i^{d+2}$  on  $U_{d+2}$  such that for any  $y \in X_\alpha \cap U_{d+2}$ , we have  $d_y f(w_i^{d+2}(y)) = \partial_i$ .

**Lemma 3.4.** *For  $y \in U_{d+2} \cap X_\alpha$ , write*

$$\tilde{w}_i^{d+2}(y) = \sum_{j=1}^{2m} a_j(y) v_j^\alpha(y) + P(y),$$

where  $P(y) \in \ker d_y f$ . Define

$$w_i^{d+2}(x) := \begin{cases} v_i^\alpha(x) + P(x) & \text{if } x \in X_\alpha \cap U_{d+2} \\ v_i^d(x) & \text{if } x \in U_d. \end{cases}$$

Then  $w_i^{d+2}$  is a rugose vector field on  $U_{d+2}$  and  $d_x f(w_i^{d+2}(x)) = \partial_i$  for  $x \in U_{d+2}$ .

*Proof.* For  $x' \in U_d$ , let

$$v_i^d(x') = \sum_{j=1}^{2m} b_j(x', y) v_j^\alpha(y) + Q(x', y) + S(x', y),$$

where  $Q(x', y) \in \ker d_y f$  and  $S(x', y) \in (T_y X_\alpha)^\perp$ . Since  $\tilde{w}_i^{d+2}$  is rugose, for each  $x \in U_d$ , there is a neighborhood  $W_x$  of  $x$  such that for any  $y \in W_x \cap X_\alpha$  and any  $x' \in W_x \cap X_\beta$ , we have:

- $\|P(y) - Q(x', y)\| < C\|y - x'\|$ ,
- $\|S(x', y)\| < C\|y - x'\|$ ,

for some  $C > 0$ , where  $X_\beta$  is the stratum containing  $x$ . Shrink  $W_x$  and increasing  $C$  if necessary, we can suppose that  $dF : x \mapsto d_x F$  is Lipschitz on  $W_x$ , where  $F$  is a polynomial extension of  $f$  to  $\mathbb{C}^n$ . Hence  $\|d_y F - d_{x'} F\| < C\|y - x'\|$ . In particular,

$$\|d_y F(v_i^d(x')) - d_{x'} F(v_i^d(x'))\| < C\|y - x'\| \cdot \|v_i^d(x')\|,$$

i.e.,

$$\left\| \sum_{j=1}^{2m} b_j(x', y) \partial_j - \partial_i \right\| - \|d_y F(S(x', y))\| < C\|y - x'\| \cdot \|v_i^d(x')\|,$$

Shrink  $W_x$ , if necessary, so that  $\overline{W_x} \cap X_\gamma \subset U_{d+2}$  for any  $\gamma \in I'_{d+2}$ . Let  $M := \sup_{z \in U_d} \|v_i^d\|$  and  $N := \sup_{z \in W_x} \|d_z F\|$ . Then we have

$$\sum_{j \neq i} |b_j(x', y)| + |b_i(x', y) - 1| < C(M + N)\|y - x'\|,$$

and

$$\|w_i^{d+2}(y) - v_i^d(x')\| < \left( \sum_{j \neq i} |b_j(x', y)| + |b_i(x', y) - 1| \right) D + \|P(y) - Q(x', y)\| + \|S(x', y)\|,$$

where  $D := \sup_{z \in W_x \cap X_\alpha} \|v_i^\alpha\|$ . Thus

$$\|w_i^{d+2}(y) - v_i^d(x')\| < (2C + CD(M + N))\|y - x'\|.$$

Hence  $w_i^{d+2}$  is rugose and of course  $d_x f(w_i^{d+2}(x)) = \partial_i$  for  $x \in U_{d+2}$ .  $\square$

Since  $w_i^{d+2}$  is rugose (in view of Lemma 3.4), it is continuous. Then by shrinking  $W_x$ , where  $W_x$  is determined in the proof of Lemma 3.4, if necessary, we may assume that  $W_x \subset \{z \in \mathbb{C}^n : \|z - x\| \leq 1\}$  and

$$\|w_i^{d+2}(y)\| < 2\|v_i^d(x)\|, \quad (4)$$

for any  $y \in W_x \cap X_\alpha$ . Let  $W_d := \bigcup_{x \in U_d} W_x$ , then  $W_d$  is an open neighborhood of radius not bigger than 1 of  $U_d$ . Let  $F = (f_1, \dots, f_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m$  be the polynomial extending  $f$  on  $\mathbb{C}^n$ . Let  $K := F^{-1}(B)$ , which is considered as a subset of  $\mathbb{R}^{2n}$  under the identification of  $\mathbb{C}^n$  with  $\mathbb{R}^{2m}$ . Then  $U_d$  is a closed set in  $K$  for the induced topology from the Euclidean topology. By a smooth version of Urysohn's lemma, there is a smooth function  $\varphi : K \rightarrow [0, 1]$  such that  $\varphi^{-1}(0) = K \setminus W_d$  and  $\varphi^{-1}(1) = U_d$ . (Remark that there may not exist such a function  $\varphi$  defined on the whole of  $\mathbb{R}^{2n}$  since  $U_d$  is not closed in  $\mathbb{R}^{2n}$ , namely, there may be no smooth extensions of  $\varphi$  on  $\mathbb{R}^{2n}$ ). For  $x \in U_{d+2}$ , set

$$v_i^{d+2}(x) := (1 - \varphi(x))v_i^\alpha(x) + \varphi(x)w_i^{d+2}(x).$$

Clearly, the restriction of  $v_i^{d+2}$  on each stratum is a smooth vector field. Moreover, we have

$$\begin{aligned} d_x f(v_i^{d+2}(x)) &= d_x f\left((1 - \varphi(x))v_i^\alpha(x) + \varphi(x)w_i^{d+2}(x)\right) \\ &= (1 - \varphi(x))d_x f(v_i^\alpha(x)) + \varphi(x)d_x f(w_i^{d+2}(x)) \\ &= (1 - \varphi(x))\partial_i + \varphi(x)\partial_i = \partial_i. \end{aligned}$$

Let us prove that  $v_i^{d+2}$  is a rugose vector field. For any  $x \in U_d$ , let  $X_\beta$  be the stratum containing  $x$ . For  $x' \in W_x \cap X_\beta$  and  $y \in W_x \cap X_\alpha$ , we have

$$\begin{aligned} \|v_i^{d+2}(y) - v_i^{d+2}(x')\| &= \|(1 - \varphi(y))v_i^\alpha(y) + \varphi(y)w_i^{d+2}(y) - v_i^d(x')\| \\ &= \|(1 - \varphi(y))v_i^\alpha(y) - (1 - \varphi(y))w_i^{d+2}(y) + w_i^{d+2}(y) - v_i^d(x')\| \\ &\leq (1 - \varphi(y))\|v_i^\alpha(y) - w_i^{d+2}(y)\| + \|w_i^{d+2}(y) - v_i^d(x')\| \\ &\leq (1 - \varphi(y))(\|v_i^\alpha(y)\| + \|w_i^{d+2}(y)\|) + \|w_i^{d+2}(y) - w_i^{d+2}(x')\|. \end{aligned}$$

We note the following facts:

- Since  $1 - \varphi(y)$  is a smooth function, it is locally Lipschitz; with no loss of generality, assume that  $1 - \varphi(y)$  is Lipschitz on  $W_x$  with constant  $c_1$ . Then

$$1 - \varphi(y) = (1 - \varphi(y)) - (1 - \varphi(x')) \leq c_1\|y - x'\|.$$

- By Lemma 3.3 and by the continuity of  $w_i^{d+2}$ , there is a positive constant  $c_2$  depending only on  $x$  such that  $\|v_i^\alpha(y)\| + \|w_i^{d+2}(y)\| \leq c_2$  (we can take  $c_2 := \sup_{z \in \overline{W_x \cap X_\alpha}} \frac{\|z\| + 1}{c} +$

$$\sup_{z \in \overline{W_x \cap X_\alpha}} \|w_i^{d+2}(z)\|, \text{ note that } \overline{W_x \cap X_\alpha} \subset U_{d+2}).$$

- Since  $w_i^{d+2}$  is rugose, it follows that there is a positive constant  $c_3$  depending only on  $x$  such that  $\|w_i^{d+2}(y) - w_i^{d+2}(x')\| \leq c_3\|y - x'\|$ .

Hence

$$\|v_i^{d+2}(y) - v_i^{d+2}(x')\| \leq (c_1c_2 + c_3)\|y - x'\|,$$

i.e.,  $v_i^{d+2}$  is rugose. Now it remains to show that there is a positive constant  $c_{d+2}$  such that  $\|v_i^{d+2}(y)\| \leq \frac{\|y\|+1}{c_{d+2}}$  for every  $y \in U_{d+2}$ . Obviously, the statement holds for  $y \in U_d$  by the induction assumption and for  $y \in U_{d+2} \setminus W_d$  by Lemma 3.3, so we can suppose that  $y \in W \setminus U_d$ , which clearly implies that  $y \in X_\alpha$ . In light of Lemma 3.3, we get

$$\|v_i^\alpha(y)\| \leq \frac{\|y\| + 1}{c}, \quad (5)$$

where  $c$  is the constant in the same lemma. In view of (4) and the induction assumption, we have

$$\|w_i^{d+2}(y)\| < 2\|v_i^d(x)\| \leq 2\frac{\|x\| + 1}{c_d} \leq 2\frac{\|y\| + \|x - y\| + 1}{c_d} \leq 2\frac{\|y\| + 2}{c_d} \leq 4\frac{\|y\| + 1}{c_d}. \quad (6)$$

Thus (5) and (6) yield

$$\begin{aligned} \|v_i^{d+2}(y)\| &= \|(1 - \varphi(y))v_i^\alpha(y) + \varphi(y)w_i^{d+2}(y)\| \\ &\leq (1 - \varphi(y))\|v_i^\alpha(y)\| + \varphi(y)\|w_i^{d+2}(y)\| \\ &\leq (1 - \varphi(y))\frac{\|y\|+1}{c} + \varphi(y)4\frac{\|y\|+1}{c_d} \\ &< \left(\frac{1}{c} + \frac{4}{c_d}\right)(\|y\| + 1). \end{aligned}$$

Set  $c_{d+2} = \min\left\{\frac{1}{\frac{1}{c} + \frac{4}{c_d}}, c, c_d\right\}$ , then  $\|v_i^{d+2}(y)\| \leq \frac{\|y\|+1}{c_{d+2}}$  for every  $y \in U_{d+2}$ . By induction, there exists a rugose vector field on  $U_{2 \dim_{\mathbb{C}} X}$  with the property of Lemma 3.1. Then the theorem follows by applying Lemma 3.1.  $\square$

The following corollary follows immediately from Theorem 3.1.

**Corollary 3.1.** *Let  $X \subset \mathbb{C}^n$  be an affine variety with an affine Whitney stratification  $\mathcal{S}$  and let  $f : X \rightarrow \mathbb{C}^m$  be a polynomial dominant map. Assume that for any stratum  $X_\beta \in \mathcal{S}$ , the restriction  $f|_{X_\beta}$  is a submersion and  $K_\infty(f, X_\beta) = \emptyset$ . Then  $f$  is a locally trivial fibration.*

#### 4. COMPUTATION OF THE SETS OF STRATIFIED GENERALIZED CRITICAL VALUES

In this section, we will compute the set  $K(f, \mathcal{S})$  of stratified generalized critical values of  $f$ , for which we need to construct an affine Whitney stratification of  $X$  and then apply [10] for each stratum of this stratification. The process is a bit different from the construction in Subsection 2.3 since we only need to construct such an affine Whitney stratification “partially”, by remarking the following facts:

- As the construction of Whitney stratifications is by induction on dimension, we only need to proceed until the dimension shrinks below  $m$  since the restriction of  $f$  to any stratum of dimension  $< m$  is always singular.



- For any algebraic set  $Z \subseteq X$ , let

$$r_Z := \max_{x \in Z \setminus V(p_{Z, \emptyset})} \text{rank} \text{Jac}_x(f|_Z) \quad \text{and} \quad H(Z) := \overline{\{x \in Z \setminus V(p_{Z, \emptyset}) : \text{rank} \text{Jac}_x(f|_Z) < r_Z\}}^Z.$$

Then at any step of the induction process, the construction in Subsection 2.3 can be omitted if  $r_Y < m$ .

Let us now construct such a stratification. With the same notations as in Lemma 2.3, let

$$\Gamma_3 := \bigcup_{k=1}^t \left\{ \begin{array}{l} (x, x, w, v, \gamma, \lambda, \mu) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^t : \\ h_1(x, x, w, v) = \dots = h_q(x, x, w, v) = 0 \\ \gamma \sum_{j=1}^n v_j w_j = 1 \\ \lambda p_{Y, \emptyset}(x) = 1 \\ \mu_k M_k^{(m, p)}(x) = 1 \end{array} \right\},$$

where each  $M_k^{(m, p)}(x)$  is a minor of the matrix

$$A(x) := \begin{bmatrix} d_x f_1 \\ \vdots \\ d_x f_m \\ d_x g_1 \\ \vdots \\ d_x g_r \\ d_x \tilde{g}_{r+1} \\ \vdots \\ d_x \tilde{g}_p \end{bmatrix},$$

obtained by deleting  $n - m - p$  columns. So  $\Gamma_3$  differs from  $\Gamma_2$  in the last  $t$  equations since we are only interested in finding the points where the Whitney condition (b) is not satisfied, outside  $P(Y, \emptyset)$ . Let

$$\pi_3 : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^t \rightarrow \mathbb{C}^n$$

be the projection on the first  $n$  coordinates. By Lemma 2.3,  $\pi_3(\Gamma_3)$  is the set of points where the Whitney condition (b) fails. Obviously  $\pi_3(\Gamma_3) \subset \text{reg}(f|_{Y \setminus P(Y)})$  and  $\dim \pi_3(\Gamma_3) < \dim Y$ . Set  $\widetilde{W} := \overline{\pi_3(\Gamma_3)}^Z$ . Then obviously  $\dim \widetilde{W} < \dim Y$ . Again, we can compute a Gröbner basis of the ideal  $I(\widetilde{W})$ .

Finally, set

- $X_0 := X$ ,
- $X_1 := X_0 \cap V(p_{X_0, \emptyset})$ ,  $S_1 = K_0(f, X_0 \setminus X_1), \dots$ ,
- $X_i := X_{i-1} \cap V(p_{X_{i-1}, \bigcup_{j=0}^{i-2} \widetilde{W}(X_j, X_{i-1})})$ ,  $S_i = K_0(f, X_{i-1} \setminus X_i), \dots$

By induction, we can construct a finite filtration of algebraic sets  $X = X_0 \supset X_1 \supset \dots \supset X_q \supset X_{q+1} \supseteq \emptyset$  with  $\dim X_i > \dim X_{i+1}$  and  $r_{X_{q+1}} < m$ . It is clear that this filtration does not induce an

affine Whitney stratification  $\mathcal{S}$  of  $X$ . However, it shows that there is an affine Whitney stratification  $\mathcal{S}$  such that

$$K_0(f, \mathcal{S}) = \bigcup_{i=1}^q S_i \cup \overline{f(X_{q+1})}.$$

Let  $Z_i := X_i \setminus X_{i+1}$ . Then  $\{Z_i\}_{i=0, \dots, q}$  is an affine Whitney stratification of  $X \setminus X_{q+1}$ . Every variety  $Z_i$  can be realized as a closed affine variety  $\tilde{Z}_i$  in  $\mathbb{C}^{n+1}$ , by the embedding  $Z_i \ni x \mapsto (x, 1/P_{X_i \cup \bigcup_{j=0}^{i-1} \tilde{W}(X_j, X_i)}(x)) \in \mathbb{C}^{n+1}$  for  $i > 0$  or the embedding  $Z_0 \ni x \mapsto (x, 1/P_{X_0, \emptyset}(x)) \in \mathbb{C}^{n+1}$ . Let  $K_\infty(f, Z_i)$  be the set of asymptotic critical values of  $f|_{Z_i}$ , which now can be computed analogously as in [8],[10] - this will be done in the next section. Then from the construction, it is clear that the set of stratified generalized critical values of  $f$  is given by

$$K(f, \mathcal{S}) := \bigcup_{i=1}^q (K_\infty(f, Z_i) \cup K_0(f, Z_i)) \cup \overline{f(X_{q+1})}$$

and  $K(f, \mathcal{S})$  can be computed effectively. Note that Remark 2.2 and elementary properties of Gröbner bases imply:

**Corollary 4.1.** *Let  $X \subset \mathbb{C}^n$  be an affine variety of pure dimension and let  $f = (f_1, \dots, f_m) : X \rightarrow \mathbb{C}^m$  be a polynomial mapping. Let  $\mathbb{F} \subset \mathbb{C}$  be a subfield generated by coefficients of generators of  $I(X)$  and all coefficients of polynomials  $f_i$ ,  $i = 1, \dots, m$ . Then there is a nowhere dense affine variety  $K(f, \mathcal{S}) \subset \mathbb{C}^m$ , which is described by polynomials from  $\mathbb{F}[x_1, \dots, x_m]$ , such that all bifurcation values  $B(f)$  of  $f$  are contained in  $K(f, \mathcal{S})$ . In particular, for  $m = 1$ , if  $X$  and  $f$  are described by polynomials from  $\mathbb{Q}[x_1, \dots, x_n]$ , then all bifurcation values of  $f$  are algebraic numbers.*

## 5. COMPUTATION OF $K_0(f, Z_i) \cup K_\infty(f, Z_i)$

Let  $k = \mathbb{R}$  or  $k = \mathbb{C}$ . Let  $X \cong k^n$ ,  $Y \cong k^m$  be finite dimensional vector spaces (over  $k$ ). We consider those spaces equipped with the canonical scalar (hermitian) products. Let us denote by  $\mathcal{L}(X, Y)$  the set of linear mappings from  $X$  to  $Y$  and by  $\Sigma = \Sigma(X, Y) \subset \mathcal{L}(X, Y)$  the set of non-surjective mappings. In this section we give several different expressions for a distance of an  $A \in \mathcal{L}(X, Y)$  to the space  $\Sigma$  of singular operators. Let us recall first the following ([15]):

**Definition 5.1.** *Let  $A \in \mathcal{L}(X, Y)$ . Set*

$$\nu(A) = \inf_{\|\phi\|=1} \|A^*(\phi)\|,$$

where  $A^* : \mathcal{L}(Y^*, X^*)$  is the adjoint operator and  $\phi \in Y^*$ .

Let  $\alpha, \beta : \mathcal{L}(X, Y) \rightarrow \mathbb{R}_+$  be two non-negative functions. We shall say that  $\alpha$  and  $\beta$  are *equivalent* (we write  $\alpha \sim \beta$ ) if there are constants  $c, d > 0$  such that

$$c\alpha(A) \leq \beta(A) \leq d\alpha(A)$$

for any  $A \in \mathcal{L}(X, Y)$ . We shall give below several functions equivalent to  $\nu$ . Let  $A = (A_1, \dots, A_m) \in \mathcal{L}(X, Y)$  and let  $\overline{A_i} = \text{grad } A_i$ . Denote by  $\langle (\overline{A_j})_{j \neq i} \rangle$  the linear space generated by vectors

$(\overline{A_j})$ ,  $j \neq i$ . Let

$$\kappa(A) = \min_{1 \leq i \leq m} \text{dist}(\overline{A_i}, \langle (\overline{A_j})_{j \neq i} \rangle),$$

be the *Kuo number* of  $A$ .

**Proposition 5.1** ([12]). *The Kuo function  $\kappa$  is equivalent to  $\nu$  of Rabier. More precisely*

$$\nu(A) \leq \kappa(A) \leq \sqrt{m}\nu(A).$$

**Definition 5.2.** *Let  $A \in \mathcal{L}(X, Y)$  and let  $H \subset X$  be a linear subspace. We set*

$$\nu(A, H) = \nu(A|_H), \quad \kappa(A, H) = \kappa(A|_H),$$

where  $A|_H$  denotes the restriction of  $A$  to  $H$ .

From Proposition 5.1 we get immediately the following corollary.

**Corollary 5.1.** *We have  $\nu(A, H) \sim \kappa(A, H)$ .*

In fact we have also the following explicit expression for  $\kappa(A, H)$  (see [9], [10]).

**Proposition 5.2.** *Let  $A = (A_1, \dots, A_m) \in \mathcal{L}(X, Y)$  and let  $H \subset X$  be a linear subspace. Assume that  $H$  is given by a system of linear equations  $B_j = 0$ ,  $j = 1, \dots, r$ . Then*

$$\kappa(A, H) = \min_{1 \leq i \leq m} \text{dist}(\overline{A_i}, \langle (\overline{A_j})_{j \neq i}; (\overline{B_j})_{j=1, \dots, r} \rangle),$$

where  $\overline{A_i} = \text{grad } A_i$  and  $\overline{B_j} = \text{grad } B_j$ .

Finally we introduce a function  $g'$  which will be useful in the explicit description of the set of generalized critical values.

**Definition 5.3.** *Let  $A \in \mathcal{L}(k^n, k^m)$ , where  $n \geq m + r$ , and let  $H \subset k^n$  be a linear subspace given by a system of independent linear equations  $B_l = \sum b_{lk}x_k$ ,  $l = 1, \dots, r$ . By abuse of notation we denote by  $A$  the matrix (in the canonical bases in  $k^n$  and  $k^m$ ) of the mapping  $A$ . Let  $C$  be an  $(m + r) \times n$  matrix given by rows  $A_1, \dots, A_m; B_1, \dots, B_r$  (we identify  $A_i = \sum a_{ik}x_k$  with the vector  $(a_{j1}, \dots, a_{jn})$ , similarly for  $B_l$ ). Let  $M_I$ , where  $I = (i_1, \dots, i_{m+r})$ , denote an  $((m + r) \times (m + r))$  minor of  $C$  given by columns indexed by  $I$ . Let  $M_J(j)$  denote an  $(m + r - 1) \times (m + r - 1)$  minor given by columns indexed by  $J$  and by deleting the  $j^{\text{th}}$  row, where  $1 \leq j \leq m$ . Note that we delete only  $A_j$  rows! We set*

$$g'(A, H) = \max_I \left\{ \min_{\{J \subset I, 1 \leq j \leq m\}} \frac{|M_I|}{|M_J(j)|} \right\},$$

(where we consider only numbers with  $M_J(j) \neq 0$ , if all numbers  $M_J(j)$  are zero, we put  $g'(A) = 0$ ).

In particular we have the following (see [9], [10]).

**Proposition 5.3.** *We have  $g'(A, H) \sim \nu(A, H)$ .*

Now we can prove the following theorem.

**Theorem 5.1.** *Let  $Z_i$  be a stratum of  $X$  as in Section 4. Then the set  $K(f, Z_i) = K_0(f, Z_i) \cup K_\infty(f, Z_i)$  is a nowhere dense algebraic subset of  $\mathbb{C}^m$ .*

*Proof.* It is a standard fact that  $K_0(f, Z_i)$  is algebraic and nowhere dense (for details see the end of subsection 5.1). Hence it is enough to focus on  $K_\infty(f, Z_i)$ .

By construction the set  $X := Z_i \subset \mathbb{C}^n$  is a subset of complete intersection,  $X \subset \{b_1 = 0, \dots, b_s = 0\}$  and  $\text{rank}\{\nabla b_k, k = 1, \dots, s\} = s$  ( $X$  has codimension  $s$ ). Let us recall notation of Definition 5.3. For  $x \in \mathbb{C}$  let  $A = d_x f$ , and  $B_l = d_x b_l$ ,  $l = 1, \dots, s$ . Let  $A \in \mathcal{L}(k^n, k^m)$ , where  $n \geq m + s$ , and let  $T_x X = H \subset k^n$  be a linear subspace given by a system of independent linear equations  $B_l = \sum b_{lk} x_k$ ,  $l = 1, \dots, s$ . By abuse of notation we denote by  $A$  the matrix (in the canonical bases in  $k^n$  and  $k^m$ ) of the mapping  $A$ . Let  $C$  be an  $(m + s) \times n$  matrix given by rows  $A_1, \dots, A_m; B_1, \dots, B_s$  (we identify  $A_i = \sum a_{ik} x_k$  with the vector  $(a_{j1}, \dots, a_{jn})$ , similarly for  $B_l$ ).

For an index  $I = (i_1, \dots, i_{m+s}) \subset \{1, \dots, n\}$  let  $M_I(x)$  denote the  $((m + s) \times (m + s))$  minor of  $C$  given by columns indexed by  $I$ . For integers  $j \in I, 1 \leq k \leq m$  we denote by  $M_{I(k,j)}(x)$  the  $(m + s - 1) \times (m + s - 1)$  minor obtained by deleting the  $j^{\text{th}}$  column and the  $k^{\text{th}}$  row. Note that we delete only  $A_k, 1 \leq k \leq m$ , rows!

Hence  $M_I$  and  $M_{I(k,j)}$  are regular (restriction of polynomials) functions on  $X$ . We define now a family of rational functions on  $X$ :

$$W_{I(k,j)}(x) = M_I(x)/M_{I(k,j)}(x)$$

where for  $M_{I(k,j)} \equiv 0$ , we put  $W_{I(k,j)} \equiv 0$ . We write  $b = (b_1, \dots, b_s)$  and  $(f, b) : \mathbb{C}^n \rightarrow \mathbb{C}^m \times \mathbb{C}^s$ , here we consider  $f_1, \dots, f_m$ , and  $b_1, \dots, b_s$  as polynomials on  $\mathbb{C}^n$ .

Let  $w = \binom{n}{m+s}$  and let  $M_{I_1}, \dots, M_{I_w}$  be all possible main minors of a matrix of  $d_x(f, b)$ . For every index  $I_l$  take a pair  $(k_l, j_l)$  which determines an  $(m + s - 1) \times (m + s - 1)$  minor of  $M_{I_l}$  (we consider here only minors which are not identically zero). We denote a sequence  $(k_1, j_1), \dots, (k_w, j_w)$  by  $(k, j) \in \mathbb{N}^w \times \mathbb{N}^w$  and we consider a rational function:

$$\Phi_{(k,j)} = \Phi((k_1, j_1), \dots, (k_w, j_w)) : X \rightarrow \mathbb{C}^m \times \mathbb{C}^N$$

where the first component of  $\Phi_{(k,j)}$  is  $f$  and next components are  $W_{I_p(k_p, j_p)}, p = 1, \dots, w$  and all products  $x_l W_{I_p(k_p, j_p)}, p = 1, \dots, w; l = 1, \dots, n$ .

We can assume that for some choice of  $l$  we have  $W_{I_l(k_l, j_l)} \neq 0$ , and consequently  $\dim \text{cl}(\Phi_{(k,j)}(X)) = \dim X = n - s$ . Here  $\text{cl}(Y)$  stands for the closure of  $Y$  in the strong (or which is the same, in the Zariski topology). Let  $\Gamma(k, j) = \text{cl}(\Phi_{(k,j)}(X))$  (by  $\Phi_{(k,j)}(X)$  we mean the set  $\Phi_{(k,j)}(X \setminus P)$ , where  $P$  is a set of poles of  $\Phi_{(k,j)}$ ).

Now for a given  $r \in \{1, \dots, n\}$ , consider the set  $X_r := X \setminus \{x_r = 0\}$ . Finally let  $\Phi_{(k,j),r}(x) := (\Phi_{(k,j)}(x), 1/x_r)$  and  $\Gamma((k, j), r) := \text{cl}(\Phi_{(k,j),r}(X_r))$ .

Let us recall that  $y \in K_\infty(f, Z_i)$  if there exists a sequence  $x \rightarrow \infty; x \in Z_i$ , such that

$$f(x) \rightarrow y \text{ and } \|x\|g'(x) \rightarrow 0,$$

where  $g'(x) = g'(d_x f, T_x Z_i)$ . We have

**Lemma 5.1.**

$$K_\infty(f, X) = \mathbb{C}^m \cap \bigcup_{(k,j),r} \Gamma((k, j), r),$$

where we identify  $\mathbb{C}^m$  with  $\mathbb{C}^m \times (0, \dots, 0)$ .

*Proof.* We identify  $X$  with  $\tilde{Z}_i \subset \mathbb{C}^{n+1}$ , hence we can assume that  $X$  is closed in  $\mathbb{C}^{n+1}$ . Let  $y \in K_\infty(f, X)$ . Hence there is a sequence  $x^l \rightarrow \infty$ , such that  $x^l \in X$  and  $f(x^l) \rightarrow y$  and  $\|x^l\|g'(x^l) \rightarrow 0$ . Moreover, if  $x = (x_1, \dots, x_n)$ , then there is at least one  $r; 1 \leq r \leq n$  such that  $x_r^l \rightarrow \infty$ . If  $\{x^l, l = 1, 2, \dots\} \subset C(f, X)$  ( $C(f, X)$  denotes the set of critical points of  $f|_X$ ), then it is easy to see that  $y \in \mathbb{C}^m \cap \Gamma((k, j), r)$  for every  $(k, j)$  ( we can choose a close sequence  $x^l$  such that  $f(x^l) \rightarrow y$  and  $\|x^l\|g'(x^l) \rightarrow 0$  and functions  $W_{I(k,j)}$  are defined). Consequently we can assume that  $\{x^l, l = 1, 2, \dots\} \cap C(f, X) = \emptyset$ .

Thus there is a sequence  $x^l \rightarrow \infty$ , such that for every  $I_i$  there are integers  $(k_i, j_i)$ , such that  $\|x^l\|M_{I_i}/M_{I_i(k_i, j_i)}(x^l) \rightarrow 0$  and  $f(x^l) \rightarrow y$ . This also gives  $y \in \Gamma((k, j), r) \cap \mathbb{C}^m$  with  $((k, j), r) = ((k_1, j_1), \dots, (k_w, j_w), r)$ .

Conversely, if  $y \in \Gamma((k, j), r) \cap \mathbb{C}^m$ , then we can choose a sequence  $x^l \rightarrow \infty, x^l \in X_r$ , such that  $f(x^l) \rightarrow y$  and  $\|x^l\|M_{I_i}/M_{I_i(k_i, j_i)}(x^l) \rightarrow 0$ . It is easy to observe that this implies  $\|x^l\|g'(x^l) \rightarrow 0$  and  $f(x^l) \rightarrow y$ , i.e.  $y \in K_\infty(f, X)$ .  $\square$

Now in light of [10, Theorem 3.3], we have that  $K_\infty(f, X) \neq \mathbb{C}^m$  hence  $\mathbb{C}^m \cap \bigcup_{((k,j),r)} \Gamma((k, j), r) \neq \mathbb{C}^m$ . By Lemma 5.1,  $K_\infty(f, X)$  is an algebraic set. The theorem follows.  $\square$

**5.1. A sketch of an algorithm.** Let  $X := Z_i \subset \mathbb{C}^n$  be a smooth affine variety of dimension  $n - s$ . Let  $f = (f_1, \dots, f_m) : X \rightarrow \mathbb{C}^m$  be a polynomial dominant mapping. Then the set  $K_\infty(f, X)$  can be computed as follows.

By construction  $X$  is a subset of complete intersection, hence we can choose polynomials  $b_1, \dots, b_s \in I(X)$  such that  $\text{rank} \{\text{grad } b_1, \dots, \text{grad } b_s\} = s$  on  $X$ . Let us consider the rational mapping:

$$\begin{aligned} \Phi((k_1, j_1), \dots, (k_w, j_w), r) : X \ni x \mapsto & (f(x), W_{I_1(k_1, j_1)}(x), x_1 W_{I_1(k_1, j_1)}(x), \dots, x_n W_{I_1(k_1, j_1)}(x), \\ & \dots, W_{I_s(k_w, j_w)}(x), x_1 W_{I_s(k_w, j_w)}(x), \dots, x_n W_{I_s(k_w, j_w)}(x), 1/x_r) \in \mathbb{C}^m \times \mathbb{C}^N, \end{aligned}$$

which is constructed exactly as in the proof of Theorem 5.1. Recall that

$$\Gamma((k_1, j_1), \dots, (k_w, j_w), r) = \text{cl}(\Phi((k_1, j_1), \dots, (k_w, j_w), r)(X)).$$

We know also that

$$K_\infty(f, X) = L \cap \left( \bigcup_{((k_1, j_1), \dots, (k_w, j_w)), r} \Gamma((k_1, j_1), \dots, (k_w, j_w), r) \right),$$

where  $L = \mathbb{C}^m \times (0, \dots, 0)$ . First we compute the ideal of the set  $\Gamma((k_1, j_1), \dots, (k_w, j_w), r)$ . Now we can consider a variety  $X$  as a closed affine variety in  $\mathbb{C}^{n+1}$  (see the end of section 4). Let  $I(X) = (b_1, \dots, b_q)$ .

To this end we restrict the mapping  $\Phi((k, j), r)$  to an open dense subset  $U \subset X$  on which this mapping is regular. In particular we can choose the set  $U = X \setminus (\bigcup_{l=1}^w \{M_{I_l(k_l, j_l)} = 0\} \cup \{x_r = 0\})$ . The set  $U$  can be identified with the set

$$V((k_1, j_1), \dots, (k_w, j_w), r) := \\ := \{(x, t, z_1, \dots, z_w) \in X \times \mathbb{C} \times \mathbb{C}^w : j = 1, \dots, w; x_r t = 1; M_{I_p(k_p, j_p)} z_p = 1; p = 1, \dots, w\}.$$

Now we can consider a morphism

$$\Psi((k_1, j_1), \dots, (k_w, j_w)) : V((k_1, j_1), \dots, (k_w, j_w), r) \rightarrow \mathbb{C}^m \times \mathbb{C}^N.$$

defined by

$$(x, t, z) \rightarrow (f(x), z_1 M_{I_1}(x), x_1 z_1 M_{I_1}(x), \dots, x_n z_1 M_{I_1}(x), \\ \dots, z_p M_{I_w}(x), x_1 z_w M_{I_w}(x), \dots, x_n z_w M_{I_w}(x), t).$$

Denote  $\Psi((k_1, j_1), \dots, (k_w, j_w), r) := (\psi_1(x, z), \dots, \psi_{m+N}(x, z))$ . It is easy to see that

$$\Gamma((k_1, j_1), \dots, (k_w, j_w), r)$$

is the closure of

$$\Psi((k_1, j_1), \dots, (k_w, j_w), q)(V((k_1, j_1), \dots, (k_w, j_w), r)).$$

Let  $G((k_1, j_1), \dots, (k_w, j_w), r) = \text{graph}(\Psi((k_1, j_1), \dots, (k_w, j_w), r))$ . A basis of the ideal  $I$  of the set  $G((k_1, j_1), \dots, (k_w, j_w), r)$  in the ring  $\mathbb{C}[x_1, \dots, x_n, x_{n+1}, t, z_1, \dots, z_w; y_1, \dots, y_{m+N}]$  is given by the polynomials

$$\{b_j, j = 1, \dots, w\} \cup \{z_r M_{I_r(k_r, j_r)}(x) - 1, r = 1, \dots, s\} \cup \{t x_r - 1\} \cup \{y_i - \psi_i(x, z), i = 1, \dots, m + N\}.$$

To compute a basis  $\mathcal{B}((k_1, j_1), \dots, (k_w, j_w), r)$  of the ideal of the set  $cl(\Gamma((k_1, j_1), \dots, (k_w, j_w), r))$ , it is enough to compute a Gröbner basis  $\mathcal{A}((k_1, j_1), \dots, (k_w, j_w), r)$  of the ideal  $I$  in  $\mathbb{C}[x, t, z, y]$  with respect to the lexicographic order in which  $y < x, t, z$  (see e.g. [14]) and then to take

$$\mathcal{B}((k_1, j_1), \dots, (k_w, j_w), r) = \mathcal{A}((k_1, j_1), \dots, (k_w, j_w), r) \cap \mathbb{C}[y_1, \dots, y_{m+N}].$$

Consequently,

$$K_\infty(f, X) = \bigcup_{((k_1, j_1), \dots, (k_w, j_w)), r} \{y \in \mathbb{C}^m : h(y, 0, \dots, 0) = 0 \text{ for every } h \in \mathcal{B}((k_1, j_1), \dots, (k_w, j_w), r)\}.$$

The computation of the set  $K_0(f, X)$  is standard. Let  $I(X) = (b_1, \dots, b_q)$ . Consider the set

$$U := \{x \in \mathbb{C}^{n+1} : b_j = 0, j = 1, \dots, w; M_{I_r} = 0; r = 1, \dots, s\}.$$

Now we can consider a morphism  $f : U \rightarrow \mathbb{C}^m$ . We have  $K_0(f, X) = \overline{f(U)}$ . Let  $\Gamma$  be a graph of  $f|_U$  and  $I = I(\Gamma)$ .

A basis of the ideal  $I$  is given by the polynomials

$$\{b_j; j = 1, \dots, w\} \cup \{M_{I_r}(x); r = 1, \dots, p\} \cup \{y_i - f_i; i = 1, \dots, m\}.$$

To compute a basis  $\mathcal{B}$  of the ideal  $I$  it is enough to compute a Gröbner basis  $\mathcal{A}$  of the ideal  $I$  in  $\mathbb{C}[x_1, \dots, x_{n+1}; y_1, \dots, y_m]$  and then to take

$$\mathcal{B} = \mathcal{A} \cap \mathbb{C}[y_1, \dots, y_m].$$

Consequently,  $K_0(f, X) = \bigcup \{y \in \mathbb{C}^m : h(y, 0, \dots, 0) = 0 \text{ for every } h \in \mathcal{B}\}$ .

**Acknowledgment.** We would like to thank Nguyen Xuan Viet Nhan and Nguyen Hong Duc for some helpful discussions during the preparation of this paper. We also would like to thank the referees for their careful reading, valuable comments and suggestions.

#### REFERENCES

- [1] D. D’Acunto, V. Grandjean, *On gradient at infinity of semialgebraic functions*, Ann. Pol. Math. 87 (2005), 39–49.
- [2] D. Cox, J. Little, D. O’Shea, *Ideals, Varieties, and Algorithms*, Springer Verlag, New York, 2007.
- [3] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*, Colloque de Topologie, Bruxelles (1950), 29–55.
- [4] D. Eisenbud, C. Huneke, W. Vasconcelos, *Direct methods for primary decomposition*, Invent. Math. 110 (1992), no. 2, 207–235.
- [5] A. G. Flores and B. Teissier, *Local polar varieties in the geometric study of singularities*, Ann. Fac. Sci. Toulouse 27, no. 4, (2018), 679–695.
- [6] P. Gianni, B. Trager, G. Zacharias, *Gröbner Bases and Primary Decomposition of Polynomial Ideals*, J. Symb. Comp. 6, (1988), 149–167 .
- [7] G-M. Greuel, G. Pfister, *A singular introduction to commutative algebra*, Second, extended edition. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, Springer, Berlin, 2008. xx+689 pp.
- [8] Z. Jelonek, *On the generalized critical values of polynomial mappings*, Manuscripta Math. 110, (2003), 145–157.
- [9] Z. Jelonek, *On asymptotic critical values and the Rabier Theorem*, Banach Center Publication 65, Polish Academy of Science, Warsaw, (2004), 125–133.
- [10] Z. Jelonek and K. Kurdyka, *Quantitative generalized Bertini–Sard theorem for smooth affine varieties*, Discrete Comput. Geom. 34, no. 4 (2005), 659–678.
- [11] T. Krick and A. Logar, *An algorithm for the computation of the radical of an ideal in the ring of polynomials*, Applied algebra, algebraic algorithms and error-correcting codes (New Orleans, LA, 1991), 195–205, Lecture Notes in Comput. Sci., 539, Springer, Berlin, 1991.
- [12] K. Kurdyka, P. Orro, S. Simon, *Semialgebraic Sard theorem for generalized critical values*, Journal of Differential Geometry 56, 67-92, 2000.
- [13] J. Mather, *Notes on topological stability*, Bull. Amer. Math. Soc. (N.S.) 49 (2012), no. 4, 475–506.
- [14] F. Pauer and M. Pfeifhofer, *The theory of Gröbner bases*, L’Enseignement Mathématique 34 (1988), 215–232.
- [15] P. Rabier, *Ehresmann fibrations and Palais–Smale conditions for morphisms of Finsler manifolds*, Ann. of Math. (2) 146 (1997), no. 3, 647–691.
- [16] J-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble 6 (1956), 1–42.
- [17] B. Teissier, *Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney*, Algebraic geometry (La Ràbida, 1981), 314–491, Lecture Notes in Math., 961, Springer, Berlin, 1982.
- [18] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. 75 (1969), 240–284.
- [19] M. S. Uddin, *Computing dimension of affine varieties using Gröbner basis approach*, IOSR Journal of Mathematics, vol. 8, issue 3, (2013), 36–39.
- [20] J-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard*, Invent. Math. 36 (1976), 295–312.

- [21] H. Whitney, *Local properties of analytic varieties*, Differential and Combinatorial Topology, Princeton University Press, Princeton, (1965), 205–244.
- [22] H. Whitney, *Tangents to an analytic variety*, Ann. of Math. 81 (1965), 496–549.

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