

ON DEFINABLE OPEN CONTINUOUS MAPPINGS

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ABSTRACT. For a definable continuous mapping f from a definable connected open subset Ω of \mathbb{R}^n into \mathbb{R}^n , we show that the following statements are equivalent:

- (i) The mapping f is open.
- (ii) The fibers of f are finite and the Jacobian of f does not change sign on the set of points at which f is differentiable.
- (iii) The fibers of f are finite and the set of points at which f is not a local homeomorphism has dimension at most $n - 2$.

As an application, we prove that Whyburn's conjecture is true for definable mappings: A definable open continuous mapping of one closed ball into another which maps boundary homeomorphically onto boundary is necessarily a homeomorphism.

1. INTRODUCTION

We are interested in the possible necessary and sufficient conditions for a continuous mapping to be open. Recall that a mapping is *open* if it maps open sets onto open sets.

By *Remmert's open mapping theorem* (see, for example, [19, Theorem 2, p. 297] or [23, Proposition 4, p. 132]), it is well known that a holomorphic mapping $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is open if and only if its fibers are discrete.

From the work of Church [2] (see also [4]) we know that if a mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is C^n and light (i.e., the fibers of f are totally disconnected), then f is open if and only if the Jacobian (i.e., the determinant of the Jacobian matrix) of f does not change sign, equivalently, the set of points at which f is not a local homeomorphism has dimension at most $n - 2$.

For a real polynomial mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, Gamboa and Ronga [11] proved that f is open if and only if the fibers of f are finite and the Jacobian of f does not change sign. After that, Hirsch [15] affirmed that the Jacobian of a real analytic open mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ does not change sign. Recently in [9, Theorem 3.14], Denkowski and Loeb showed that a subanalytic (or definable in some o-minimal structure) mapping f of class

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C^1 is open if and only if the fibers of f are discrete and the Jacobian of f does not change sign.

In the non-smooth setting, a result of Scholtes [27] (see Corollary 3.1 below) stated that a piecewise affine mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is open if and only if it is coherently oriented (meaning that the Jacobian of affine mappings associated to f have the same nonzero sign). See also [1, 12, 17, 22, 24, 37] for related works.

We present here a *definable non-smooth* version of the above results. Namely, let $f: \Omega \rightarrow \mathbb{R}^n$ be a definable continuous mapping, where Ω is a definable connected open set in \mathbb{R}^n . Denote by D_f the set of points in Ω at which f is differentiable; this set is dense in Ω (see, e.g., [6, 31]). We also denote by B_f the set of points in Ω at which f fails to be a local homeomorphism. Then the following statements are equivalent:

- (i) The mapping f is open.
- (ii) The fibers of f are finite and the Jacobian of f does not change sign on D_f .
- (iii) The fibers of f are finite and the set B_f has dimension at most $n - 2$.

The idea of the proof of the equivalence (i) \Leftrightarrow (ii) is similar to those in [9, 11, 15]. However, more arguments need to be taken into account in the non-smooth case.

In [35] Whyburn stated the following conjecture and verified it for the case $n = 2$: *Suppose that f is a light open continuous mapping of one closed ball of dimension n onto another. If f maps the boundary homeomorphically, then f is a homeomorphism.*

Whyburn's conjecture has been proved for several class of continuous mappings by McAuley [21]. The conjecture has been verified for differentiable mappings with an additional hypothesis by Cronin and McAuley [7]. It is also valid for C^2 mappings due to Marx [20]. On the other hand, it has been shown by Wilson [36] that the conjecture is false for continuous mappings on higher dimensional spaces. However, the second result of this paper shows that the conjecture is still true for definable continuous mappings.

The rest of this paper is organized as follows. Section 2 contains some properties of definable sets and mappings in o-minimal structures. For the convenience of the reader, the classical invariance of domain theorem and some properties of the Brouwer degree are also recalled here. Finally, results are given in Section 3.

2. PRELIMINARIES

2.1. Notation and definitions. We suppose $1 \leq n \in \mathbb{N}$ and abbreviate (x_1, \dots, x_n) by x . The space \mathbb{R}^n is equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. The open ball and sphere of radius r centered at the origin in \mathbb{R}^n

will be denoted by \mathbb{B}_r^n and \mathbb{S}_r^{n-1} , respectively. If $x \in \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a non-empty set then the distance from x to Ω is defined by $\text{dist}(x, \Omega) := \inf_{y \in \Omega} \|x - y\|$. The closure and boundary of a set $\Omega \subset \mathbb{R}^n$ will be written as $\bar{\Omega}$ and $\partial\Omega$, respectively.

Let f be a mapping from a subset Ω of \mathbb{R}^n into \mathbb{R}^m . Then f is an *open mapping* if $f(U)$ is an open subset of \mathbb{R}^m whenever U is an open subset of Ω ; f has *finite fibers* if for each $y \in \mathbb{R}^m$, the fiber $f^{-1}(y)$ is a finite (possibly empty) set. Suppose that $m = n$ and Ω is open. Let D_f denote the set of points in Ω at which f is differentiable. If $x \in D_f$ then we denote the Jacobian matrix of f at x by $df(x) = \left[\frac{\partial f_i}{\partial x_j}(x) \right]$, and the determinant of $df(x)$ is the *Jacobian* of f at x , denoted by $Jf(x)$. Let R_f denote the set of points $x \in \Omega$ such that f is of class C^1 in a neighborhood of x and the Jacobian $Jf(x)$ is nonzero. Observe that R_f is an open set but it is not necessarily connected. As in [2], the *branch* set of f , denoted by B_f , is the set of points in Ω at which f fails to be a local homeomorphism. Note that $B_f \subset \Omega \setminus R_f$ by the inverse mapping theorem.

2.2. The invariance of domain theorem and the Brouwer degree. For the convenience of the reader, we recall here the classical invariance of domain theorem and some properties of the Brouwer degree.

Lemma 2.1 (Invariance of domain). *Let Ω be an open subset of \mathbb{R}^n . Then every injective continuous mapping from Ω into \mathbb{R}^n is open.*

Proof. See, for example, [10, Chapter 4, Proposition 7.4]. □

Suppose Ω is an open and bounded set in \mathbb{R}^n , $f: \bar{\Omega} \rightarrow \mathbb{R}^n$ is a continuous mapping, and $y \notin f(\partial\Omega)$. The *Brouwer degree* of f on Ω with respect to y , denoted by $\text{deg}(f, \Omega, y)$, is a function of integer values which enjoys several important properties (normalization, domain decomposition, local constancy, homotopy invariance, etc.). The Brouwer degree is a power tool used in analysis and topology, in particular, it gives an estimation and the nature of the solution(s) of the equation $f(x) = y$ in Ω . For more details, the reader may refer to [8, 18] and the references therein.

The two lemmas below provide some useful properties of the Brouwer degree.

Lemma 2.2. *Let Ω be an open and bounded set in \mathbb{R}^n , $f: \bar{\Omega} \rightarrow \mathbb{R}^n$ be a continuous mapping, and $y \notin f(\partial\Omega)$. Then the following statements hold:*

- (i) *If $\text{deg}(f, \Omega, y) \neq 0$, then there is $x \in \Omega$ such that $f(x) = y$.*
- (ii) *For all $y' \in \mathbb{R}^n$ with $\|y' - y\| < \text{dist}(y, f(\partial\Omega))$ we have*

$$\text{deg}(f, \Omega, y') = \text{deg}(f, \Omega, y).$$

- (iii) If $H: \overline{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$ is continuous, $\gamma: [0, 1] \rightarrow \mathbb{R}^n$ is continuous, and $\gamma(t) \notin H(\partial\Omega, t)$ for every $t \in [0, 1]$, then $\deg(H(\cdot, t), \Omega, \gamma(t))$ is independent of $t \in [0, 1]$.

Proof. See, for example, [8, 18]. □

Lemma 2.3. *Let Ω be an open and bounded set in \mathbb{R}^n and $f: \overline{\Omega} \rightarrow \mathbb{R}^n$ be a continuous mapping. Then the following statements hold:*

- (i) *Let $x \in \Omega$ be such that the Jacobian matrix $df(x)$ exists and is nonsingular. Then there exists a neighborhood W of x such that $\overline{W} \cap f^{-1}(f(x)) = \{x\}$ and*

$$\deg(f, W, f(x)) = \text{sign} Jf(x).$$

- (ii) *Let $y \notin f(\partial\Omega)$ be such that for every $x \in f^{-1}(y)$, the Jacobian matrix $df(x)$ exists and is nonsingular. Then*

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \text{sign} Jf(x).$$

Proof. See [8, 18] for the smooth case and see [13, 26, 28] for the non-smooth case. □

2.3. O-minimal structures and definable mappings. The notion of o-minimality was developed in the late 1980s after it was noticed that many proofs of analytic and geometric properties of semi-algebraic sets and mappings can be carried over verbatim for subanalytic sets and mappings. We refer the reader to [6, 31, 32] for the basic properties of o-minimal structures used in this paper.

Definition 2.1. An *o-minimal structure* on $(\mathbb{R}, +, \cdot)$ is a sequence $\mathcal{D} := (\mathcal{D}_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$:

- (a) \mathcal{D}_n is a Boolean algebra of subsets of \mathbb{R}^n .
- (b) If $X \in \mathcal{D}_m$ and $Y \in \mathcal{D}_n$, then $X \times Y \in \mathcal{D}_{m+n}$.
- (c) If $X \in \mathcal{D}_{n+1}$, then $\pi(X) \in \mathcal{D}_n$, where $\pi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the projection on the first n coordinates.
- (d) \mathcal{D}_n contains all algebraic subsets of \mathbb{R}^n .
- (e) Each set belonging to \mathcal{D}_1 is a finite union of points and intervals.

A set belonging to \mathcal{D} is said to be *definable* (in that structure). *Definable mappings* in structure \mathcal{D} are mappings whose graphs are definable sets in \mathcal{D} .

Examples of o-minimal structures are

- the semi-algebraic sets (by the Tarski–Seidenberg theorem),

- the globally subanalytic sets, i.e., the subanalytic sets of \mathbb{R}^n whose (compact) closures in the real projective space $\mathbb{R}\mathbb{P}^n$ are subanalytic (using Gabrielov’s complement theorem).

In this note, we fix an arbitrary o-minimal structure on $(\mathbb{R}, +, \cdot)$. The term “definable” means definable in this structure. We recall some useful facts which we shall need later.

Lemma 2.4 (Monotonicity). *Let $f: (a, b) \rightarrow \mathbb{R}$ be a definable function and p be a positive integer. Then there are finitely many points $a = t_0 < t_1 < \dots < t_k = b$ such that the restriction of f to each interval (t_i, t_{i+1}) is of class C^p , and either constant or strictly monotone.*

Lemma 2.5 (Path connectedness). *The following statements hold:*

- Every definable set has a finite number of connected components and each such component is definable.*
- Every definable connected set X is path connected, i.e., for every points x, y in X , there exists a definable continuous curve $\gamma: [0, 1] \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

By the cell decomposition theorem (see, for example [32, 4.2]), for any $p \in \mathbb{N}$ and any nonempty definable subset X of \mathbb{R}^n , we can write X as a disjoint union of finitely many definable C^p -manifolds of different dimensions. The *dimension* $\dim X$ of a nonempty definable set X can thus be defined as the dimension of the manifold of highest dimension of such a decomposition. This dimension is well defined and independent on the decomposition of X . By convention, the dimension of the empty set is taken to be negative infinity. A point $x \in X$ is *generic* if there exists a neighborhood U of x in \mathbb{R}^n such that $X \cap U$ is a definable C^1 -manifold of dimension $\dim X$.

Lemma 2.6. *Let $X \subset \mathbb{R}^n$ be a nonempty definable set. Then the following statements hold:*

- The set X has measure zero if and only if $\dim X < n$.*
- The interior of X is nonempty if and only if $\dim X = n$.*
- $\dim(\overline{X} \setminus X) < \dim X$. In particular, $\dim \overline{X} = \dim X$.*
- Let $Y \subset \mathbb{R}^n$ be a definable set containing X . If X is dense in Y then $\dim(Y \setminus X) < \dim Y$.*
- If $Y \subset \mathbb{R}^n$ is definable, then $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$.*
- Let $f: X \rightarrow \mathbb{R}^m$ be a definable mapping. Then $\dim f(X) \leq \dim X$.*
- The complement in X of the set of generic points in X is a definable set of dimension less than $\dim X$.*

Lemma 2.7. *Let $X \subset \mathbb{R}^n$ be a definable open set and $f: X \rightarrow \mathbb{R}^m$ be a definable mapping. Then for each positive integer p , the set of points where f is not of class C^p is a definable set of dimension less than n .*

In the sequel we will make use of Hardt's triviality (see [14, 31]).

Theorem 2.1 (Hardt's triviality). *Consider a definable continuous mapping $f: X \rightarrow Y$ where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are definable sets. Then there exists a finite partition $Y = Y_1 \cup \dots \cup Y_k$ of Y into definable sets Y_i such that f is definably trivial over each Y_i , that is, there exists a definable set $F_i \subset \mathbb{R}^{n_i}$, for some n_i , and a definable homeomorphism $h_i: f^{-1}(Y_i) \rightarrow Y_i \times F_i$ such that the composition h_i with the projection $Y_i \times F_i \rightarrow Y_i$, $(y, z) \mapsto y$, is equal to the restriction of f to $f^{-1}(Y_i)$.*

3. RESULTS AND PROOFS

The following result provides necessary and sufficient conditions for a definable continuous mapping to be open. For related results, we refer the reader to [2–5, 9, 11, 15, 29, 30, 33, 34].

Theorem 3.1. *Let $f: \Omega \rightarrow \mathbb{R}^n$ be a definable continuous mapping, where Ω is a definable connected open set in \mathbb{R}^n . Then the following two conditions are equivalent:*

- (i) *The mapping f is open.*
- (ii) *The fibers of f are finite and the Jacobian Jf does not change sign on D_f .*
- (iii) *The fibers of f are finite and the Jacobian Jf does not change sign on R_f .*
- (iv) *The fibers of f are finite and the branch set B_f has dimension at most $n - 2$.*

In order to prove Theorem 3.1, we need some lemmas. The first one is [25, Fact 2.2], which is a consequence of [16, Proposition 2]; we recall it here together with a direct proof.

Lemma 3.1. *Let Ω be a definable connected open subset of \mathbb{R}^n and R be a definable open dense subset of Ω . Consider the graph Γ whose vertices are the connected components of R with two components R_i and R_j connected by an edge if and only if $\dim(\Omega \cap \bar{R}_i \cap \bar{R}_j) \geq n - 1$. Then Γ is connected.*

Proof. Suppose for contradiction that the graph Γ is not connected. There exists a connected component of Γ whose vertices are R_1, \dots, R_p for some $p \geq 1$, and let R_{p+1}, \dots, R_q with $q \geq p + 1$ be the remaining vertices of Γ . By assumption, we have for all $i = 1, \dots, p$, and all $j = p + 1, \dots, q$,

$$\dim(\Omega \cap \bar{R}_i \cap \bar{R}_j) < n - 1.$$

Set

$$X_1 := \bigcup_{i=1}^p (\Omega \cap \bar{R}_i) \quad \text{and} \quad X_2 := \bigcup_{j=p+1}^q (\Omega \cap \bar{R}_j).$$

Observe that

$$X_1 \cap X_2 = \left(\bigcup_{i=1}^p (\Omega \cap \bar{R}_i) \right) \cap \left(\bigcup_{j=p+1}^q (\Omega \cap \bar{R}_j) \right) = \bigcup_{\substack{i=1, \dots, p \\ j=p+1, \dots, q}} (\Omega \cap \bar{R}_i \cap \bar{R}_j).$$

Hence $\dim(X_1 \cap X_2) < n - 1$ and so $\Omega \setminus (X_1 \cap X_2)$ is path connected. On the other hand, since $R = \cup_{i=1, \dots, q} R_i$ is dense in Ω , we have $\Omega = X_1 \cup X_2$ and so

$$\Omega \setminus (X_1 \cap X_2) = (X_1 \setminus X_2) \cup (X_2 \setminus X_1).$$

Therefore, there is a continuous path

$$\gamma: [0, 1] \rightarrow \Omega \setminus (X_1 \cap X_2)$$

such that $\gamma(0) \in X_1 \setminus X_2$ and $\gamma(1) \in X_2 \setminus X_1$. Set

$$t_* := \sup\{t \in [0, 1] : \gamma(s) \in X_1 \setminus X_2 \text{ for all } s \in [0, t]\}.$$

Then it is easy to check that $\gamma(t_*) \in X_1 \cap X_2$, which is a contradiction. The lemma is proved. \square

The following result is taken from [25, Lemma 3.1].

Lemma 3.2. *Let $W \subset \mathbb{R}^{n-1}$ and $U \subset \mathbb{R}^n$ be definable open sets such that the set $\widetilde{W} := W \times \{0\}$ is contained in the boundary of U . Let $f: U \cup \widetilde{W} \rightarrow \mathbb{R}$ be a definable function continuous on $U \cup \widetilde{W}$ and C^1 on U . Let $g: W \rightarrow \mathbb{R}$ be the function $y \mapsto f(y, 0)$ and $a \in W$ be a generic point. Then g is differentiable at a and, for all $i = 1, \dots, n - 1$,*

$$\frac{\partial g}{\partial x_i}(a) = \lim_{x \rightarrow (a, 0), x \in U} \frac{\partial f}{\partial x_i}(x).$$

The following fact is simple but useful.

Lemma 3.3. *Let $f: \Omega \rightarrow \mathbb{R}^n$ be a definable continuous mapping, where Ω is a definable open set in \mathbb{R}^n . Assume that the fibers of f are finite. Then the following two statements hold:*

- (i) *If X is a definable subset of Ω , then $\dim X = \dim f(X)$.*
- (ii) *The set R_f is dense in Ω .*

Proof. (i) Let X be a definable subset of Ω . Applying Hardt's Trivial Theorem 2.1 to the restriction mapping $f|_X$ of f to X , we obtain a finite definable partition of $f(X)$ onto Y_1, \dots, Y_k such that $f|_X$ is definably trivial over each Y_i . Since the fibers of f are finite, it follows that

$$\dim(f|_X)^{-1}(Y_i) = \dim Y_i \quad \text{for } i = 1, \dots, k.$$

Hence

$$\dim X = \max_{i=1, \dots, k} \dim(f|_X)^{-1}(Y_i) = \max_{i=1, \dots, k} \dim Y_i = \dim f(X),$$

which yields (i).

(ii) Let B be the set of points $x \in \Omega$ such that f is not C^1 in a neighborhood of x . Then B is closed in Ω . Further, in view of Lemma 2.7, B is a definable set of dimension less than n . Consider the definable set

$$C := \{x \in \Omega \setminus B : Jf(x) = 0\}.$$

Applying Sard's theorem to the C^1 -mapping

$$\Omega \setminus B \rightarrow \mathbb{R}^n, \quad x \mapsto f(x),$$

we have that $f(C)$ has measure zero, and so it has dimension less than n in view of Lemma 2.6(i). This, together with the statement (i), implies that

$$\dim C = \dim f(C) < n.$$

Consequently, $\dim(B \cup C) < n$. Now (ii) follows immediately since $R_f = \Omega \setminus (B \cup C)$. This ends the proof of the lemma. \square

We can prove the following lemma by the same argument as in [11, Proposition, page 298].

Lemma 3.4. *Let $f: \Omega \rightarrow \mathbb{R}^d$ be a definable open continuous mapping, where Ω is a definable subset of \mathbb{R}^n . Then*

$$\dim f^{-1}(y) \leq \dim \Omega - d \quad \text{for all } y \in \mathbb{R}^d.$$

In particular, if $\dim \Omega = d$, then the fibers of f are finite.

Proof. The proof is the same as that of [11, Proposition, page 298]. For the convenience of the reader we repeat the proof here. We proceed by induction on d ; the case $d = 0$ is trivial. If $d > 0$, let $y \in \mathbb{R}^d$ and let $H \subset \mathbb{R}^d$ be a hyperplane containing y . Since f is open, the restriction of f on $f^{-1}(H)$ is also open and the induction hypothesis implies that

$$\dim f^{-1}(y) \leq \dim f^{-1}(H) - (d - 1).$$

If $\dim f^{-1}(H) = \dim \Omega$ then the set of generic points in $f^{-1}(H)$ is open in Ω , but its image by f is contained in H , contradicting that f is open, and the desired inequality follows. \square

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. (cf. [9, Theorem 3.14], [11, Theorem, page 297], [25, Theorem 3.2]).

(ii) \Rightarrow (iii): Obviously.

(iii) \Rightarrow (i): To see that f is open, take any $\bar{x} \in \Omega$ and let W be a neighborhood of \bar{x} . Since the fiber $f^{-1}(f(\bar{x}))$ is finite, there exists an open ball U centered at \bar{x} with $\bar{U} \subset W$ such that $\bar{U} \cap f^{-1}(f(\bar{x})) = \{\bar{x}\}$. Since ∂U is compact and $f(\bar{x}) \notin f(\partial U)$, we can find an open ball $V \subset \mathbb{R}^n$ centered at $f(\bar{x})$ such that $f(\partial U) \cap V = \emptyset$. Now the conclusion follows if we can show that $f(U) \supset V$.

Recall that R_f denotes the set R_f of points $x \in \Omega$ such that f is of class C^1 on a neighborhood of x and the Jacobian $Jf(x)$ is nonzero. In view of Lemma 3.3(ii), R_f is dense in Ω . Since U is an open subset of Ω , there exists a point u in $U \cap R_f$. Then f is of class C^1 on a neighborhood of u and the Jacobian matrix of f at u is nonsingular. By the inverse mapping theorem, the image $f(U)$ must contain an open set $Y \subset V$. On the other hand, it follows from Lemmas 3.3 and 2.6(iv) that

$$\dim f(\Omega \setminus R_f) = \dim(\Omega \setminus R_f) < n.$$

So there must exist some point \bar{y} in Y with $f^{-1}(\bar{y}) \subset R_f$. Therefore, $U \cap f^{-1}(\bar{y})$ is nonempty and for every $x \in U \cap f^{-1}(\bar{y})$, the Jacobian matrix $df(x)$ exists and is nonsingular. Furthermore, by construction, \bar{y} is not in $f(\partial U)$. For simplicity of notation, we let g stand for the restriction of f to \bar{U} . Then the Brouwer degree $\deg(g, U, \bar{y})$ is defined. In light of Lemma 2.3(ii), we have

$$\deg(g, U, \bar{y}) = \sum_{x \in g^{-1}(\bar{y})} \text{sign} Jg(x).$$

On the other hand, by assumption, the Jacobian Jf is positive (or negative) on R_f , and so

$$\sum_{x \in g^{-1}(\bar{y})} \text{sign} Jg(x) \neq 0.$$

Therefore $\deg(g, U, \bar{y}) \neq 0$.

Finally, to prove that $V \subset f(U)$, take any $y \in V$ and consider the continuous curve

$$\gamma: [0, 1] \rightarrow V, \quad t \mapsto (1-t)y + t\bar{y},$$

connecting y to \bar{y} . Since $f(\partial U) \cap V = \emptyset$, we have $\gamma(t) \notin g(\partial U)$ for all $t \in [0, 1]$. This, together with Lemma 2.2(iii), implies that

$$\deg(g, U, y) = \deg(g, U, \bar{y}) \neq 0.$$

By Lemma 2.2(i), $g(x) = y$ for some x in U , and this proves the openness of f .

(i) \Rightarrow (ii): Assume that f is open. If $n = 1$, then f is strictly monotone and there is nothing to prove. So for the rest of the proof we assume that $n > 1$.

By Lemma 3.4, the fibers of f are finite.

We first show that the Jacobian Jf has constant sign on R_f which means that Jf is positive (or negative) on R_f .

Obviously, the set R_f is definable open, and according to Lemma 3.3(ii), it is dense in Ω . By Lemma 2.5, R_f has a finite number of connected components, say R_1, \dots, R_k , and these components are path connected. By definition, Jf has constant sign on each R_i . According to Lemma 3.1, we need to show that for any two components R_i, R_j with $\dim(\Omega \cap \bar{R}_i \cap \bar{R}_j) \geq n - 1$, the sign of Jf on R_i is the same as on R_j . We consider two such components R_i, R_j and assume that they are R_1 and R_2 . Let $\Sigma := \Omega \cap \bar{R}_1 \cap \bar{R}_2$. In view of Lemma 2.6(iii), it is easy to see that $\dim \Sigma = n - 1$.

Let \bar{x} be a generic point in Σ ; then Σ is a C^1 -submanifold of \mathbb{R}^n of dimension $n - 1$ near \bar{x} . Hence, there is a definable connected open neighborhood $U \subset \Omega$ of \bar{x} and a definable diffeomorphism Φ from U onto an open subset of \mathbb{R}^n such that $\Phi(U \cap \Sigma) \subset \{x_n = 0\}$. Shrinking U and composing Φ with the reflection with respect to the hyperplane $\{x_n = 0\}$ if necessary, we may assume that $\Phi(U \cap R_1) \subset \{x_n > 0\}$ and $\Phi(U \cap R_2) \subset \{x_n < 0\}$.

Since Φ is a diffeomorphism and U is connected, the sign of $J\Phi$ is constant on U . Furthermore $f \circ \Phi^{-1}$ is open on $\Phi(U)$. Hence we may replace Ω by $\Phi(U)$, f by $f \circ \Phi^{-1}$ and assume that

$$R_1 \subset \{x_n > 0\}, \quad R_2 \subset \{x_n < 0\} \quad \text{and} \quad \Sigma \subset \{x_n = 0\}.$$

On the other hand, since the fibers of f are finite, it follows from Lemma 3.3(i) that

$$\dim f(\Sigma) = \dim \Sigma = n - 1.$$

Let \bar{y} be a generic point in $f(\Sigma)$; then $f(\Sigma)$ is a C^1 -submanifold of \mathbb{R}^n of dimension $n - 1$ near \bar{y} . As before, by applying an appropriate definable diffeomorphism on an open neighborhood of \bar{y} we may assume that $f(\Sigma) \subset \{y_n = 0\}$. Applying Hardt's Trivial Theorem 2.1 to the restriction of f to $f^{-1}(f(\Sigma))$, we obtain a finite definable partition of $f(\Sigma)$ onto Y_1, \dots, Y_k such that this restriction mapping is definably trivial over each Y_i . By Lemma 2.5, $f^{-1}(Y_i)$ has a finite number of connected components and each such

component is homeomorphic to Y_i because f has finite fibers. Observe that there exists an index i such that the set $f^{-1}(Y_i) \cap \Sigma$ is of dimension $n - 1$. Let $\tilde{\Sigma}$ be a connected component of $f^{-1}(Y_i) \cap \Sigma$ of dimension $n - 1$. Now, by shrinking Ω we may assume that $\Omega \cap f^{-1}(f(\tilde{\Sigma})) = \tilde{\Sigma}$.

By construction, it is easy to see that each of the (connected open) sets $f(R_1)$ and $f(R_2)$ is contained in either $\{y_n > 0\}$ or $\{y_n < 0\}$ but not in both. Furthermore, since f is open, the sets $f(R_1)$ and $f(R_2)$ cannot lie in the same half-space $\{y_n > 0\}$ or $\{y_n < 0\}$. Hence, without loss of generality, we may assume that

$$f(R_1) \subset \{y_n > 0\} \quad \text{and} \quad f(R_2) \subset \{y_n < 0\}.$$

Write $f := (f_1, \dots, f_n)$ and let $(a, 0) \in \mathbb{R}^{n-1} \times \{0\}$ be a generic point in $\tilde{\Sigma}$. Observe that

$$f_n(a, -t) < f_n(a, 0) = 0 < f_n(a, t)$$

for all $t > 0$ sufficiently small. By Lemma 2.4, it is easy to see that $f_n(a, t)$ is strictly increasing in t near 0. Consequently, we can find $\epsilon > 0$ small enough such that $\frac{\partial f_n}{\partial x_n}(a, t) > 0$ for all $t \in (-\epsilon, \epsilon)$ different from 0. Since $(a, 0)$ is generic in $\tilde{\Sigma}$, we may assume, by shrinking Ω if needed, that for all $(x_1, \dots, x_n) \in \Omega$, if $x_n \neq 0$, then $\frac{\partial f_n}{\partial x_n}(x_1, \dots, x_n) > 0$. If this last change of Ω makes the point $(a, 0)$ not generic in $\tilde{\Sigma}$, we replace $(a, 0)$ by another generic point and continue to assume that $(a, 0)$ is generic in $\tilde{\Sigma}$.

Define the definable continuous mapping $\Psi: \Omega \rightarrow \mathbb{R}^n$ by

$$\Psi(x_1, \dots, x_n) := (x_1, \dots, x_{n-1}, f_n(x_1, \dots, x_n)).$$

Obviously, Ψ is identity on $\tilde{\Sigma}$ and differentiable on $\Omega \setminus \tilde{\Sigma}$. Furthermore, Ψ is injective since for every $(x_1, \dots, x_{n-1}, 0) \in \tilde{\Sigma}$, the function $x_n \mapsto f_n(x_1, \dots, x_n)$ is strictly increasing. In light of Lemma 2.1, Ψ is a homeomorphism from Ω onto $\Psi(\Omega)$. Observe that $J\Psi(x) = \frac{\partial f_n}{\partial x_n}(x)$ for all $x \in \Omega \setminus \tilde{\Sigma}$, hence $J\Psi$ is positive outside of $\tilde{\Sigma}$. Consequently, we can replace the mapping f by the mapping $f \circ \Psi^{-1}$ without changing the sign of the Jacobian of f on $\Omega \setminus \tilde{\Sigma}$. Without loss of generality, assume from now on that $f_n(x_1, \dots, x_n) = x_n$ on Ω .

For $x = (x_1, \dots, x_n) \in \Omega \setminus \tilde{\Sigma}$ we have

$$Jf(x) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_{n-1}}(x) & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1}(x) & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}}(x) & \frac{\partial f_{n-1}}{\partial x_n}(x) \\ 0 & \cdots & 0 & 1 \end{vmatrix} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_{n-1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1}(x) & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}}(x) \end{vmatrix}. \quad (1)$$

As f is open, the restriction of f to $f^{-1}(\mathbb{R}^{n-1} \times \{0\})$ is an open mapping from $f^{-1}(\mathbb{R}^{n-1} \times \{0\})$ into $\mathbb{R}^{n-1} \times \{0\}$. Hence the mapping

$$g: W \rightarrow \mathbb{R}^{n-1}, \quad x' \mapsto (f_1(x', 0), \dots, f_{n-1}(x', 0)),$$

is definable, open and continuous, where $W := \{x' \in \mathbb{R}^{n-1} : (x', 0) \in \widetilde{\Sigma}\}$ is a definable open subset of \mathbb{R}^{n-1} . By Lemma 3.4, the fibers of g are finite. Applying Lemma 3.3(ii) to the mapping g , we have that the set R_g is dense in W . Thus, replacing a by a point in R_g if necessary, we may assume that $Jg(a) \neq 0$.

By Lemma 3.2, for $1 \leq i, j \leq n-1$, we have

$$\lim_{x \rightarrow (a, 0), x \in \Omega} \frac{\partial f_i}{\partial x_j}(x) = \frac{\partial g_i}{\partial x_j}(a).$$

It follows then from (1) that

$$\lim_{x \rightarrow (a, 0), x \in \Omega} Jf(x) = Jg(a).$$

Hence, for $x \in \Omega$ close enough to $(a, 0)$, whether in R_1 or in R_2 , the sign of $Jf(x)$ is the same as the sign of $Jg(a)$. In particular, the sign of Jf is the same in R_1 and in R_2 .

We have thus proved that the Jacobian Jf has constant sign on R_f . So, without loss of generality, we may assume that the Jacobian Jf is positive on R_f . It remains to show that

$$Jf(x) \geq 0 \quad \text{for all } x \in D_f.$$

To see this, let $x \in D_f$ be such that $Jf(x) \neq 0$. In view of Lemma 2.3(i), there exists a definable open and bounded neighborhood W of x such that $\overline{W} \cap f^{-1}(f(x)) = \{x\}$ and

$$\deg(f, W, f(x)) = \text{sign} Jf(x). \quad (2)$$

On the other hand, by Lemma 3.3, the set $f(R_f)$ is dense in $f(\Omega)$. Note that $f(W)$ is an open subset of $f(\Omega)$ because the mapping f is open. Consequently, there exists a point $y \in f(W)$ with $\|y - f(x)\| < \text{dist}(f(x), f(\partial W))$ such that $f^{-1}(y) \subset R_f$. In particular, for every $w \in f^{-1}(y)$, the Jacobian matrix $df(w)$ exists and is nonsingular. It follows from Lemmas 2.2(ii) and 2.3(ii) that

$$\deg(f, W, f(x)) = \deg(f, W, y) = \sum_{w \in f^{-1}(y)} \text{sign} Jf(w) > 0. \quad (3)$$

Combining (2) and (3), we get $Jf(x) > 0$, which completes the proof of the implication (i) \Rightarrow (ii).

(i) \Rightarrow (iv): If $n = 1$, then f is strictly monotone and there is nothing to prove. So for the rest of the proof we assume that $n > 1$.

By Lemma 3.4, the fibers of f are finite. Hence, it suffices to show that $\dim B_f \leq n-2$. (Recall that B_f denotes the set of points at which f is not a local homeomorphism.) By the inverse mapping theorem, we have $B_f \subset \Omega \setminus R_f$. This, together with Lemmas 2.6 and 2.7, implies

$$\dim B_f \leq \dim(\Omega \setminus R_f) \leq n-1.$$

Suppose for contradiction that $\dim B_f = n-1$. By analysis similar to that in the proof of the implication (i) \Rightarrow (ii), we may assume the following conditions hold:

- (a) $\Omega \setminus R_f = B_f$;
- (b) $B_f \subset \{x_n = 0\}$ and $f(B_f) \subset \{y_n = 0\}$;
- (c) $\Omega \setminus B_f$ has two connected components, denoted by R_1 and R_2 with

$$R_1 \subset \{x_n > 0\}, \quad R_2 \subset \{x_n < 0\}, \quad f(R_1) \subset \{y_n > 0\} \quad \text{and} \quad f(R_2) \subset \{y_n < 0\};$$

- (d) $f_n(x_1, \dots, x_n) = x_n$ on Ω ;
- (e) the mapping

$$g: W \rightarrow \mathbb{R}^{n-1}, \quad x' \mapsto (f_1(x', 0), \dots, f_{n-1}(x', 0)),$$

is definable, open and continuous, where $W := \{x' \in \mathbb{R}^{n-1} : (x', 0) \in B_f\}$ is a definable open subset of \mathbb{R}^{n-1} ;

- (f) $Jg(x') \neq 0$ for all $x' \in W$.

Let $\bar{x} := (a, 0) \in B_f$. Then f is not a local homeomorphism at \bar{x} , and so it is not injective. Hence there are sequences $x^k \rightarrow \bar{x}$ and $y^k \rightarrow \bar{x}$ such that $x^k \neq y^k$ and $f(x^k) = f(y^k)$ for all k . Taking subsequences if needed, we can suppose that the sequences x^k and y^k belong to only one of the sets R_1 , R_2 and B_f . Note that by Item (f), the restriction of f on B_f is a local diffeomorphism at \bar{x} . So $x^k, y^k \notin B_f$ for all k large enough. With no loss of generality, assume that $x^k, y^k \in R_1$. Furthermore, by construction, we can assume that the segment joining x^k and y^k is contained in R_1 for all k .

Clearly f is C^1 on R_1 . So for each $i = 1, \dots, n-1$ and for each k , by the mean value theorem, there is a point z^{ik} in the segment joining x^k and y^k such that

$$0 = f_i(y^k) - f_i(x^k) = [df_i(z^{ik})](y^k - x^k). \quad (4)$$

Let

$$v^k := \frac{y^k - x^k}{\|y^k - x^k\|}.$$

By Item (d), we have $f_n(x) = x_n$ for all $x \in R_1$, so the condition $f(x^k) = f(y^k)$ implies $x_n^k = y_n^k$, i.e., $v_n^k = 0$ for all k . Furthermore, in view of (4), we have

$$[df_i(z^{ik})](v^k) = \frac{1}{\|y^k - x^k\|} [df_i(z^{ik})](y^k - x^k) = 0.$$

Equivalently

$$\sum_{j=1}^{n-1} \frac{\partial f_i}{\partial x_j}(z^{ik}) v_j^k = 0. \quad (5)$$

On the other hand, by Lemma 3.2, for $1 \leq i, j \leq n-1$, we have

$$\lim_{x \rightarrow (a,0), x \in \Omega} \frac{\partial f_i}{\partial x_j}(x) = \frac{\partial g_i}{\partial x_j}(a).$$

In addition, since $x^k \rightarrow \bar{x}$ and $y^k \rightarrow \bar{x}$, it follows that $z^{ik} \rightarrow \bar{x}$. Hence

$$\lim_{k \rightarrow +\infty} \frac{\partial f_i}{\partial x_j}(z^{ik}) = \frac{\partial g_i}{\partial x_j}(a).$$

Furthermore, taking a subsequence if necessary, we can assume that the sequence v^k converges to a limit $v = (v', 0)$. It follows then from (5) that

$$\sum_{j=1}^{n-1} \frac{\partial g_i}{\partial x_j}(a) v_j = 0.$$

Equivalently $dg(a)(v') = 0$. On the other hand, $dg(a)$ is a linear isomorphism by Item (f). In addition, since $v^k \rightarrow v = (v', 0)$ and $\|v^k\| = 1$ for all k , it follows that $\|v'\| = \|v\| = 1$. These imply $dg(a)(v') \neq 0$, which is a contradiction. Therefore the dimension of B_f must be smaller than $n-1$.

(iv) \Rightarrow (iii): Let $x^0, x^1 \in R_f$. By the inverse mapping theorem, $x^0, x^1 \in \Omega \setminus B_f$. On the other hand, the set $\Omega \setminus B_f$ is path connected because of our assumption that $\dim B_f \leq n-2$. Hence, there exists a continuous curve $\alpha: [0, 1] \rightarrow \Omega \setminus B_f$ such that $\alpha(0) = x^0$ and $\alpha(1) = x^1$. For each $t \in [0, 1]$, we can find an open neighbourhood U_t of $\alpha(t)$ with $\bar{U}_t \subset \Omega$ such that the restriction of f to U_t , denoted by $f|_{U_t}$, is a homeomorphism. Using properties of the Brouwer degree, it is not hard to check that the function

$$[0, 1] \rightarrow \mathbb{Z}, \quad t \mapsto \deg(f|_{U_t}, U_t, f(\alpha(t))),$$

is constant. In particular, we have

$$\deg(f|_{U_0}, U_0, f(x^0)) = \deg(f|_{U_1}, U_1, f(x^1)).$$

This relation, together with Lemma 2.3(i), gives

$$\text{sign} Jf(x^0) = \text{sign} Jf(x^1).$$

Since x^0, x^1 are two arbitrary points in R_f , we get the desired conclusion. \square

Remark 3.1. The equivalence (i) \Leftrightarrow (iv) also can be deduced from Lemma 3.4 and [4, Theorem 2], but we include a different proof here for self-containment of the paper.

Recall that a continuous mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *piecewise affine* if there exists a set of triples $(\Omega_i, A_i, b_i), i = 1, \dots, k$, such that each Ω_i is a polyhedral set in \mathbb{R}^n with non-empty interior, each A_i is an $n \times n$ -matrix, each b_i is a vector in \mathbb{R}^n , and

- (a) $\mathbb{R}^n = \cup_{i=1}^k \Omega_i$;
- (b) for $i \neq j$, $\Omega_i \cap \Omega_j$ is either empty or a proper common face of Ω_i and Ω_j ;
- (c) $f(x) = A_i x + b_i$ on $\Omega_i, i = 1, \dots, k$.

We say that f is *coherently oriented* if all the matrices A_i have the same nonzero determinant sign. The following result is well-known; for more details, please refer to [27, Theorem 2.3.1] and the references therein.

Corollary 3.1. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a piecewise affine mapping. Then f is open if and only if it is coherently oriented.*

Proof. This is a direct consequence of Theorem 3.1. □

Finally we prove that Whyburn's conjecture is true for definable mappings.

Theorem 3.2. *Let $f: \overline{\mathbb{B}_r^n} \rightarrow \overline{\mathbb{B}_s^n}$ be a definable open continuous mapping such that $f^{-1}(\mathbb{S}_s^{n-1}) = \mathbb{S}_r^{n-1}$ and the restriction of f to \mathbb{S}_r^{n-1} is a homeomorphism. Then f is a homeomorphism.*

Proof. By the invariance of domain theorem (Lemma 2.1), it suffices to show that f is injective. To this end, define the mapping $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$g(x) := \begin{cases} f(x) & \text{if } \|x\| < r, \\ \frac{\|x\|}{r} f\left(\frac{r}{\|x\|}x\right) & \text{otherwise.} \end{cases}$$

Then it is easy to see that g is a definable open continuous mapping. By Theorem 3.1, the fibers of g are finite and the Jacobian Jg does not change sign on R_g . Moreover, in view of Lemma 3.3(ii), R_f is dense in \mathbb{R}^n . Fix $r' > r$. There exists a point $x \in R_g$ with $r < \|x\| < r'$. By construction, $g^{-1}(g(x)) = \{x\}$. It follows from Lemma 2.3 that

$$\deg(g|_{\overline{\mathbb{B}_r^n}}, \overline{\mathbb{B}_{r'}^n}, g(x)) = \text{sign } Jg(x) = \pm 1,$$

where $g|_{\overline{\mathbb{B}_r^n}}$ stands for the restriction of g to $\overline{\mathbb{B}_r^n}$. This, together with Lemma 2.2(iii), yields

$$\deg(g|_{\overline{\mathbb{B}_r^n}}, \overline{\mathbb{B}_{r'}^n}, y) = \pm 1 \quad \text{for all } y \in \overline{\mathbb{B}_s^n}. \quad (6)$$

We now show that f is injective, or equivalently, the restriction of g to $\overline{\mathbb{B}_r^n}$ is injective. By contradiction, suppose that there exist two distinct points $x^0, x^1 \in \overline{\mathbb{B}_r^n}$ whose

image point $y \in g(\mathbb{B}_r^n) = \mathbb{B}_s^n$. Let U^0 and U^1 be disjoint open sets containing x^0 and x^1 , respectively. Then $g(U^0) \cap g(U^1)$ is a nonempty open set and thus contains a point y' of $\mathbb{B}_s^n \setminus g(\mathbb{R}^n \setminus R_g)$. This means that $g^{-1}(y')$ is a subset of $\mathbb{B}_r^n \cap R_g$ and it contains at least two points. Since the Jacobian Jg does not change sign on R_g , it follows from Lemma 2.3 that

$$\left| \deg(g|_{\mathbb{B}_r^n}, \mathbb{B}_r^n, y') \right| = \left| \sum_{x \in g^{-1}(y')} \text{sign} Jg(x) \right| \geq 2,$$

which contradicts (6). The theorem is proved. \square

Remark 3.2. Another proof of Theorem 3.2 can be obtained by applying Theorem 3.1 and [30, Theorem 5.5]; the detail is left to the reader.

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